Time warping invariants and quasisymmetric functions

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Slides at: `https://diehlj.github.io`

I hope to answer:

- What are time warping invariants?
- What (and why) are quasisymmetric functions?

To pique interest in the two people I know are in the audience (the other speakers)

- There is a map from matroids into quasisymmetric functions.[1] Is there a matroid structure visible in this talk? Does it matter for what we do?
- What is the *strength* of quasisymmetric polynomials/functions? Does it matter for what we do?

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Time warping invariants

Often the speed at which a time series is run is unknown or does not matter.

Example

GPS measurements of people running around a lake. (1D-Example)

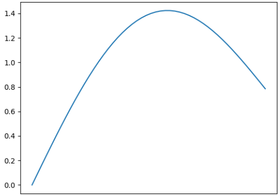


Fig: Runner A

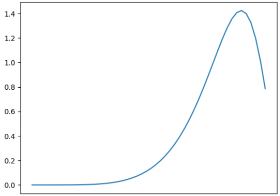


Fig: Runner B

They are both running the same route, just at different speeds.

Examples

- Heartbeat
- Different sampling speeds.

Continuous function

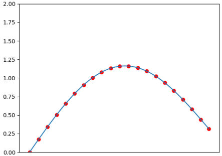


Fig: Slow sampling.

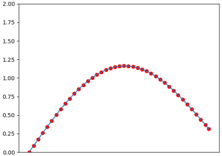
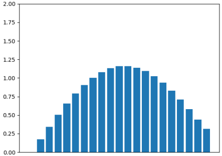
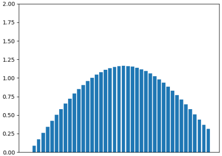


Fig: Fast sampling.

Discretely sampled





Invariance to speed is usually dealt with via the
dynamic time warping (DTW) distance
(immense amount of literature: Sakoe, Chiba, Keogh, Berndt, Clifford, Petitjean, Salvador, Chan, Müller, Jain, . . .)

This talk:
features that are invariant to time warping. (.. to be defined ..)

Mathematical formulation

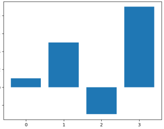
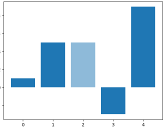
For us a time series is just a sequence of real numbers

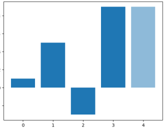
$$x = (x_1, x_2, .., x_{N-1}, x_N) \in \mathbb{R}^N.$$

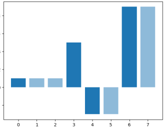
For convenience we work with an infinite time horizon and sequences that are eventually constant, i.e.

$$x = (x_1, x_2, .., x_{N-1}, x_N, x_N, x_N, \dots) \in \mathbb{R}^{\mathbb{N}}_{\text{ev. const.}}.$$

We say a function $F : \mathbb{R}^{\mathbb{N}}_{\text{ev. const.}} \to \mathbb{R}$ is time warping invariant (or *invariant to stuttering*) if, for example,

$F\Big($  $\Big) = F\Big($  $\Big)$

$= F\Big($  $\Big)$

$= F\Big($  $\Big)$.

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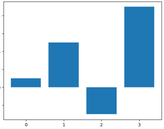
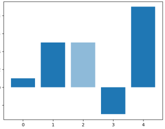
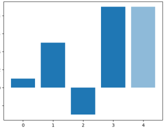
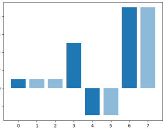
$F\Big($  $\Big) = F\Big($  $\Big)$

$= F\Big($  $\Big)$

$= F\Big($  $\Big).$

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$$F(x) := 17$$

Not very descriptive ...

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This works for any $p = 1, 2, 3, ..$:

$$F(x) = \sum_{n \leq N} (\delta x_n)^p.$$

These are the (power-sum) symmetric polynomials in the increments.

Since we work with time-series that are eventually constant, we can write

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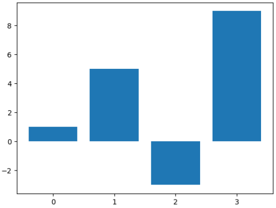
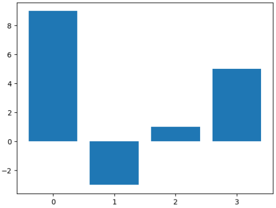
Clearly, these can <u>not</u> distinguish these two time series:





How to get the order of events ? Let us try

$$F(x) := \sum_{i_1 < i_2} (\delta x_{i_1})^2 \delta x_{i_2}.$$

This is not a symmetric function. Nonetheless:

$$F\Big((x_0, x_1, x_2, x_3, x_4, \dots)\Big)$$

$$= (\delta x_1)^2 \cdot \delta x_2$$

$$+ \Big[(\delta x_1)^2 + (\delta x_2)^2\Big] \cdot \delta x_3$$

$$+ \Big[(\delta x_1)^2 + (\delta x_2)^2 + (\delta x_3)^2\Big] \cdot \delta x_4$$

$$+ \dots$$

$$F\Big(z := (x_0, x_1, x_1, x_2, x_3, \dots)\Big)$$

$$= \underbrace{(\delta z_1)^2}_{(\delta x_1)^2} \cdot \underbrace{\delta z_2}_{=0}$$

$$+ \underbrace{\Big[(\delta z_1)^2 + (\delta z_2)^2\Big]}_{(\delta x_1)^2} \cdot \underbrace{\delta z_3}_{\delta x_2}$$

$$+ \underbrace{\Big[(\delta z_1)^2 + (\delta z_2)^2 + (\delta z_3)^2\Big]}_{(\delta x_1)^2 + (\delta x_2)^2} \cdot \underbrace{\delta z_4}_{\delta x_3}$$

$$+ ..$$

$\Rightarrow F$ is time warping invariant.

Along the same vain, all expressions of the form

$$\sum_{i_1 < \cdots < i_n} (\delta x_{i_1})^{\alpha_1} \ldots (\delta x_{i_n})^{\alpha_n},$$

are invariant to time warping.

QUESTIONS

- **A** Are these all of the (polynomial) invariants?
- **B** There are algebraic dependencies, like

$$\left(\sum_i \delta x_i\right)^2 = 2 \sum_{i_1 < i_2} \delta x_{i_1} \delta x_{i_2} + \sum_i (\delta x_i)^2.$$

How to disentangle them?

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These are far from generalizations for generalization's sake: they are most important objects in their own right [1].

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Hazewinkel, Gubareni, Kirichenko. "Algebras, Rings and Modules - Lie Algebras and Hopf Algebras", AMS, 2010.

Def A formal power series in commuting $X_1, X_2, X_3, ..$ is a quasisymmetric function if for all $i_1 < \cdots < i_n$, $j_1 < \cdots < j_n$ and all $\alpha_1, \ldots, \alpha_n \geq 1$ the cofficients of these monomials coincide,

$$X_{i_1}^{\alpha_1} \ldots X_{i_n}^{\alpha_n} \quad \text{and} \quad X_{j_1}^{\alpha_1} \ldots X_{j_n}^{\alpha_n}.$$

Examples

$$\sum_i X_i \qquad \sum_i (X_i)^2 \qquad \sum_{i_1 < i_2} X_{i_1} X_{i_2} \qquad \sum_{i_1 < i_2} (X_{i_1})^2 X_{i_2} \qquad \ldots$$

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Lemma (D,Ebrahimi-Fard,Tapia '19)

All polynomial invariants to time warping are given by quasisymmetric functions (evaluated at the increments)

$$\sum_{i_1<\cdots<i_n} (\delta x_{i_1})^{\alpha_1} \cdot \cdots \cdot (\delta x_{i_n})^{\alpha_n}, \quad n \geq 1, \alpha \in \mathbb{N}^n_{\geq 1}.$$

This answers question A.

To answer question B, it is convenient to store all these invariants as a character on a (Hopf) algebra.[2]

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Define the iterated-sums signature $(y := \delta x)$

$$
\begin{aligned}
\mathrm{ISS}(x) := 1 &+ \sum_{i_1} y_{i_1} \cdot 1 + \sum_{i_1} y_{i_1}^2 \cdot 2 + \sum_{i_1} y_{i_1}^3 \cdot 3 + \ldots \\
&+ \sum_{i_1 < i_2} y_{i_1} y_{i_2} \cdot 1|1 + \sum_{i_1 < i_2} y_{i_1}^2 y_{i_2} \cdot 2|1 + \ldots \\
&\in H := \mathbb{R} \oplus \mathbb{R}^{\mathbb{N}} \oplus (\mathbb{R}^{\mathbb{N}})^{\otimes 2} \oplus (\mathbb{R}^{\mathbb{N}})^{\otimes 3} \oplus {}^{3}..
\end{aligned}
$$

$$
\Big\langle 573, \mathrm{ISS}(x) \Big\rangle = \sum_{i_1 < i_2 < i_3} y_{i_1}^5 y_{i_2}^5 y_{i_3}^3 .
$$

In words: H is the space of formal (infinite) $\mathbb{R}$-linear combinations of words in $1, 2, \ldots$.
The product on H is concatenation of words, e.g.

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5|7 \bullet 3|1|2 = 5|7|3|1|2.
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NONCOMMUTATIVE

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For $x, x' \in \mathbb{R}^{\mathbb{N}}_{ev.\ const.}$ let denote $x \sqcup x' \in \mathbb{R}^{\mathbb{N}}_{ev.\ const.}$ their concatenation. Then:

$$\mathrm{ISS}(x \sqcup x') = \mathrm{ISS}(x) \bullet \mathrm{ISS}(x') \qquad \text{"Chen's identity".}$$

Remark

This leads to a (naive) implementation, since ISS('one increment') is easy to calculate.

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Lemma (Malvenuto, Reutenauer '94; DEFT '19)

$$\Big\langle \alpha, \mathrm{ISS}(x) \Big\rangle \cdot \Big\langle \beta, \mathrm{ISS}(x) \Big\rangle = \Big\langle \alpha \sqcup\!\sqcup\, \beta, \mathrm{ISS}(x) \Big\rangle \quad \text{“shuffle identity”.}$$

Here $\sqcup\!\sqcup$ is the quasishuffle on H^*. For example

$$1 \sqcup\!\sqcup\, 1 = 2\ 1|1 + 2 \qquad \leftrightarrow \left(\sum_i \delta x_i \right)^2 = 2 \sum_{i_1 < i_2} \delta x_{i_1} \delta x_{i_2} + \sum_i (\delta x_i)^2.$$

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COMMUTATIVE

$$1|2 \sqcup\!\sqcup 3 = 1|2|3 + 1|3|2 + 3|1|2 + 1|5 + 4|2 = 3 \sqcup\!\sqcup 1|2.$$

Just as for the iterated-integrals signature (Chen, Lyons, Reutenauer, ..), we get

Theorem (D,Ebrahimi-Fard,Tapia '19)

The iterated-sums signature is a character on the Hopf algebra, endowed with quasishuffle product and deconcatenation coproduct.

And this works for multidimensional time-series analogously.

Without going into details, this result gives that $\mathrm{ISS}(x)$ is grouplike, and in bijection to its logarithm

$$\log \mathrm{ISS}(x) = \mathrm{ISS}(x) + \frac{1}{2!}\mathrm{ISS}(x)^{\bullet 2} + \frac{1}{3!}\mathrm{ISS}(x)^{\bullet 3} + \dots$$

This logarithm lives in a Lie algebra and should yield the disentanglement of Question B.

	$\dim \mathrm{ISS}$ of weight n	$\dim \log \mathrm{ISS}$ of weight n
$n = 1$	1	1
$n = 2$	2	1
$n = 3$	3	2
$n = 4$	4	2

To really show that this is the *minimal* amount of information contained in $\mathrm{ISS}(x)$, one step is still missing though ... (Chow's theorem)

SUMMARY

- Time warping invariants are *very* relevant for time series analysis.
- All *polynomial* invariants are given by (evaluation of) quasisymmetric functions.
- They are conveniently stored in a Hopf algebra framework and provide a concrete view on these abstract structures.

THANK YOU!

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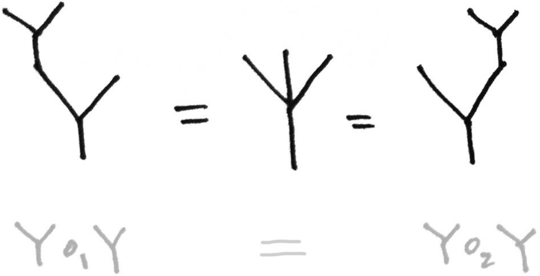
Noémie Combe (MPI Leipzig) and I are organizing an online reading group on operads .



If you are interested in joining: please get in touch!