10.1 Bending of a Cantilever Loaded at its End

Consider a cantilever beam with rectangular cross-section of unit width. The beam is loaded by a concentrated force P applied at its tip in the manner shown in Figure 10.1.

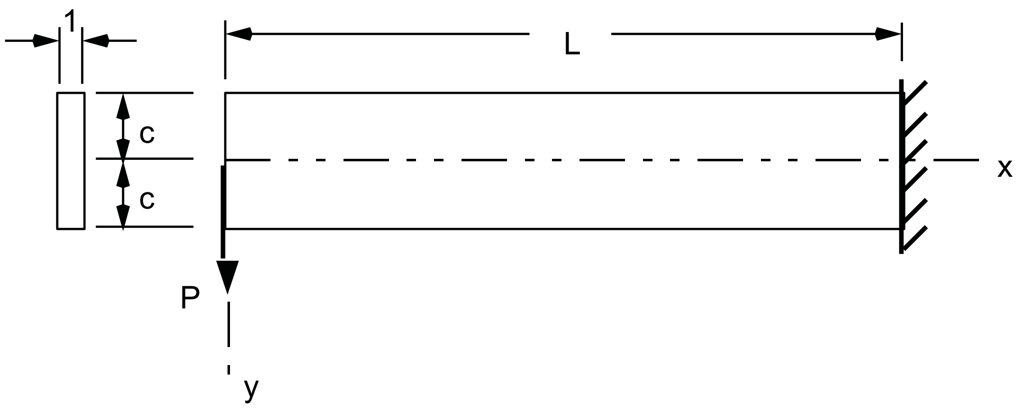

Figure 10.1: Cantilever Beam Loaded by Concentrated Load

• **Elasticity Solution**

The elasticity solution for this problem is again obtained using a Airy stress function [1]. In this case,

$$\Phi = b_2 xy - \frac{d_4}{6} xy^3 \tag{10.1}$$

where b_2 and d_4 are constants. Recalling the relation between stresses in the $x-y$ plane and Φ, it follows that

$$\sigma_{11} = \frac{\partial^2 \Phi}{\partial y^2} = -d_4 xy \quad , \quad \sigma_{22} = \frac{\partial^2 \Phi}{\partial x^2} = 0 \quad , \quad \sigma_{12} = -\frac{\partial^2 \Phi}{\partial x \partial y} = -b_2 + \frac{d_4}{2} y^2 \tag{10.2}$$

The constants b_2 and d_4 are evaluated by first noting that the longitudinal sides $y = \pm c$ are free from force, implying that $\sigma_{12} = 0$. It therefore follows that

$$-b_2 + \frac{d_4}{2} y^2 = 0 \quad \Rightarrow \quad b_2 = \frac{d_4}{2} c^2 \tag{10.3}$$

Thus, equations (10.2) become

$$\sigma_{11} = -d_4 xy \quad , \quad \sigma_{22} = 0 \quad , \quad \sigma_{12} = \frac{d_4}{2}\left(y^2 - c^2\right) \tag{10.4}$$

To evaluate d_4 we note that at the loaded end the shearing forces must sum to P. Thus, accounting for the proper sign for σ_{12}, leads to

$$-\int_A \sigma_{12}\, dA = P \tag{10.5}$$

Specializing this equation for the cantilever beam under consideration and integrating the resulting expression gives

$$-\int_{-c}^{c} \frac{d_4}{2}\left(y^2 - c^2\right)(1)dy = -d_4\left(\frac{1}{3}y^3 - c^2 y\right)\Big|_0^c = \frac{2}{3}d_4 c^3 = P \tag{10.6}$$

But $2c^3/3$ is equal to the moment of inertia I of the cross-section, thus $d_4 = P/I$. The stress state in the beam is thus described by

$$\sigma_{11} = -\frac{P}{I}xy \tag{10.7}$$

$$\sigma_{22} = 0 \tag{10.8}$$

$$\sigma_{12} = \frac{P}{2I}\left(y^2 - c^2\right) \tag{10.9}$$

Remark: These expressions for stress are *identical* to those associated with the elementary solution given in strength of materials textbooks; i.e., with standard Bernoulli-Euler beam theory.

Remark: The present solution is an exact one only if the shearing forces are distributed according to the same parabolic law as assumed herein. If the distribution of forces is different from this parabolic law but is equivalent statically, then the above expressions for σ_{11} and σ_{12} do not represent a correct solution at the ends of the beam. Away from the ends (say a distance on the order of the depth of the beam), Saint-Venant's principle[1] assures that the solution will be satisfactory.

The strains associated with the problem are are related to the stresses through the constitutive relations. Assuming a linear, isotropic elastic material gives

$$\varepsilon_{11} = \frac{1}{E}\left(\sigma_{11} - \nu\sigma_{22}\right) = -\frac{Pxy}{EI} \tag{10.10}$$

$$\varepsilon_{22} = \frac{1}{E}\left(\sigma_{22} - \nu\sigma_{11}\right) = \frac{\nu Pxy}{EI} \tag{10.11}$$

$$\gamma_{12} = \frac{\sigma_{12}}{G} = \frac{P}{2IG}\left(y^2 - c^2\right) \tag{10.12}$$

The strains are next related to the displacements through the kinematic relations. Assuming that the displacements and displacement gradients are infinitesimal, it follows that

$$\varepsilon_{11} = \frac{\partial u}{\partial x} = -\frac{Pxy}{EI} \tag{10.13}$$

$$\varepsilon_{22} = \frac{\partial v}{\partial y} = \frac{\nu Pxy}{EI} \tag{10.14}$$

$$\gamma_{12} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{P}{2GI}\left(y^2 - c^2\right) \tag{10.15}$$

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[1] Saint-Venant's principle states that the stresses some distance from the point of application of the load are not affected by the precise behavior of the body close to the point of application of the load. This principle is named in recognition of the French engineer and mechanician Adhémar Jean Claude Barré de Saint-Venant (1797-1886). The original statement of this principle was published in French by Saint-Venant in 1855 [2]. Beginning with the work of von Mises in 1945 [3], the mathematical literature gives a more rigorous interpretation of Saint-Venant's principle in the context of partial differential equations.

The displacements u and v are obtained by suitably integrating equations (10.13) and (10.14), subjected to appropriate boundary conditions. In particular,

$$u = -\frac{Px^2y}{2EI} + f_1(y) \tag{10.16}$$

$$v = \frac{\nu Pxy^2}{2EI} + f_2(x) \tag{10.17}$$

where $f_1(y)$ and $f_2(x)$ are as yet unknown functions of y and x, respectively.

Differentiating equations (10.16) and (10.17) with respect to y and x, respectively, gives

$$\frac{\partial u}{\partial y} = -\frac{Px^2}{2EI} + \frac{df_1(y)}{dy} \tag{10.18}$$

$$\frac{\partial v}{\partial x} = \frac{\nu Py^2}{2EI} + \frac{df_2(x)}{dx} \tag{10.19}$$

Substituting equations (10.18) and (10.19) into equation (10.15) gives

$$-\frac{Px^2}{2EI} + \frac{df_2(x)}{dx} + \frac{\nu Py^2}{2EI} + \frac{df_1(y)}{dy} = \frac{P}{2GI}\left(y^2 - c^2\right) \tag{10.20}$$

Equation (10.20) contains terms that are functions of x only and of y only, and one term that is independent of both x and y. It is thus re-written as

$$F(x) + G(y) = H \tag{10.21}$$

where

$$F(x) = -\frac{Px^2}{2EI} + \frac{df_2(x)}{dx} \tag{10.22}$$

$$G(y) = \frac{\nu Py^2}{2EI} - \frac{Py^2}{2GI} + \frac{df_1(y)}{dy} \tag{10.23}$$

$$H = -\frac{Pc^2}{2GI} \tag{10.24}$$

The form of equation (10.21) means that $F(x)$ must be some constant C_1 and $G(y)$ must be some constant C_2. Equations (10.22) and (10.23) are thus re-written as

$$\frac{df_1(y)}{dy} = \frac{Py^2}{2GI} - \frac{\nu Py^2}{2EI} + C_2 \tag{10.25}$$

$$\frac{df_2(x)}{dx} = \frac{Px^2}{2EI} + C_1 \tag{10.26}$$

Integrating equations (10.25) and (10.26) gives

$$f_1(y) = \frac{Py^3}{6GI} - \frac{\nu Py^3}{6EI} + C_2y + C_3 \tag{10.27}$$

$$f_2(x) = \frac{Px^3}{6EI} + C_1x + C_4 \tag{10.28}$$

where C_3 and C_4 are constants of integration. Substituting equations (10.27) and (10.28) into equations (10.16) and (10.17) gives

$$u = -\frac{Px^2y}{2EI} + \frac{Py^3}{6GI} - \frac{\nu Py^3}{6EI} + C_2y + C_3 \tag{10.29}$$

$$v = \frac{\nu Pxy^2}{2EI} + \frac{Px^3}{6EI} + C_1x + C_4 \tag{10.30}$$

The constants C_1 to C_4 are determined from equation (10.21) and from three boundary conditions that prevent the beam from moving as a rigid body in the $x - y$ plane. Assuming that the centroid of the cross-section is fixed, it follows that for all boundary conditions $u = v = 0$ at $x = L$ and $y = 0$. It follows from equations (10.29) and (10.30) that

$$C_3 = 0 \quad , \quad C_4 = -\frac{PL^3}{6EI} - C_1L \tag{10.31}$$

Equations (10.29) and (10.30) thus reduce to

$$u = -\frac{Py}{6EI}\left(3x^2 + \nu y^2\right) + \frac{Py^3}{6GI} + C_2y \tag{10.32}$$

$$v = \frac{P}{6EI}\left(x^3 + 3\nu xy^2 - L^3\right) - C_1(L - x) \tag{10.33}$$

The general expression for the *elastic curve* is obtained by setting $y = 0$ in equation (10.33). This gives

$$v\big|_{y=0} = \frac{P}{6EI}\left(x^3 - L^3\right) - C_1(L - x) \tag{10.34}$$

For determining the constant C_1, at least *three* sets of boundary condition are possible at the built-in end [1]. In particular,

• **Boundary Condition 1**: The first possible constraint condition assumes that an element of the longitudinal beam axis is fixed; i.e., $\dfrac{\partial v}{\partial x} = 0$ at $x = L$ and $y = 0$. Differentiating equation (10.33) and evaluating it at $x = L$ and $y = 0$ gives

$$\left.\frac{\partial v}{\partial x}\right|_{x=L,\, y=0} = \frac{PL^2}{2EI} + C_1 = 0 \quad \Rightarrow \quad C_1 = -\frac{PL^2}{2EI} \tag{10.35}$$

From equation (10.21) it follows that

$$C_2 = H - C_1 = \frac{P}{2I}\left(\frac{L^2}{E} - \frac{c^2}{G}\right) \tag{10.36}$$

where equation (10.24) has been used. Equations (10.32) and (10.33) thus become

$$u = -\frac{Px^2y}{2EI} + \frac{Py^3}{6I}\left(\frac{1}{G} - \frac{\nu}{E}\right) + \frac{Py}{2I}\left(\frac{L^2}{E} - \frac{c^2}{G}\right) \tag{10.37}$$

$$v = \frac{P}{6EI}\left(3\nu xy^2 + x^3 - 3L^2x + 2L^3\right) \tag{10.38}$$

The general equation for the elastic curve (equation 10.34) becomes

$$v\big|_{y=0} = \frac{P}{6EI}\left(x^3 - 3L^2x + 2L^3\right) \tag{10.39}$$

which is *identical* to the Bernoulli-Euler beam theory solution. The corresponding tip deflection is

$$v|_{x=0,\,y=0} = \frac{PL^3}{3EI} \tag{10.40}$$

• **Boundary Condition 2**: The second possible constraint condition at the fixed end assumes that a vertical element of the cross-section is fixed; i.e., $\dfrac{\partial u}{\partial y} = 0$ at $x = L$ and $y = 0$. Differentiating equation (10.32) and evaluating it at $x = L$ and $y = 0$ gives

$$\left.\frac{\partial u}{\partial y}\right|_{x=L,\,y=0} = -\frac{PL^2}{2EI} + C_2 = 0 \quad \Rightarrow \quad C_2 = \frac{PL^2}{2EI} \tag{10.41}$$

From equation (10.21) it follows that

$$C_1 = H - C_2 = -\frac{P}{2I}\left(\frac{L^2}{E} + \frac{c^2}{G}\right) \tag{10.42}$$

where equation (10.24) has again been used. Equations (10.32) and (10.33) thus become

$$u = \frac{Py}{2EI}\left(L^2 - x^2\right) + \frac{Py^3}{6I}\left(\frac{1}{G} - \frac{\nu}{E}\right) \tag{10.43}$$

$$v = \frac{P}{6EI}\left(3\nu xy^2 + x^3 - 3L^2x + 2L^3\right) + \frac{Pc^2}{2IG}\left(L - x\right) \tag{10.44}$$

The corresponding equation of the elastic curve (equation 10.34) and the tip deflection are thus

$$v|_{y=0} = \frac{P}{6EI}\left(x^3 - 3L^2x + 2L^3\right) + \frac{Pc^2}{2IG}\left(L - x\right) \tag{10.45}$$

$$v|_{x=0,\,y=0} = \frac{PL^3}{3EI} + \frac{Pc^2L}{2GI} \tag{10.46}$$

The latter is seen to differ from equation (10.40) because of the presence of the shear term $Pc^2L/2IG$.

• **Boundary Condition 3**: The third possible constraint condition at the fixed end assumes that three points of the boundary lie on a vertical line. As such, in addition to the common condition of $u = v = 0$ at $x = L$ and $y = 0$, $u = v = 0$ at $x = L$ and $y = \pm c$.

Evaluating equation (10.32) at $x = L$ and $y = c$ gives

$$u = -\frac{Pc}{6EI}\left(3L^2 + \nu c^2\right) + \frac{Pc^3}{6GI} + C_2 c = 0 \tag{10.47}$$

which leads to

$$C_2 = \frac{Pc}{6EI}\left(3L^2 + \nu c^2\right) - \frac{Pc^3}{6GI} \tag{10.48}$$

From equation (10.21) it follows that

$$C_1 = H - C_2 = -\frac{Pc^2}{3EI} - \frac{P}{6EI}\left(3L^2 + \nu c^2\right) \tag{10.49}$$

where equation (10.24) has again been used. Equations (10.32) and (10.33) thus become

$$u = -\frac{Px^2y}{2EI} + \frac{Py^3}{6I}\left(\frac{1}{G} - \frac{\nu}{E}\right) + \frac{Py}{6I}\left(\frac{3L^2}{E} + \nu\frac{c^2}{E} - \frac{c^2}{G}\right) \tag{10.50}$$

$$v = \frac{P}{6EI}\left(3\nu xy^2 + x^3 - 3L^2x + 2L^3\right) + \frac{Pc^2}{6I}(L - x)\left(\frac{\nu}{E} + \frac{2}{G}\right) \tag{10.51}$$

The corresponding equation of the elastic curve is thus

$$v|_{y=0} = \frac{P}{6EI}\left(x^3 - 3L^2x + 2L^3\right) + \frac{Pc^2}{6I}(L - x)\left(\frac{\nu}{E} + \frac{2}{G}\right) \tag{10.52}$$

Finally, the tip deflection is given by

$$v|_{x=0,\, y=0} = \frac{PL^3}{3EI} + \frac{Pc^2L}{6I}\left(\frac{\nu}{E} + \frac{2}{G}\right) \tag{10.53}$$

The tip deflection is seen to differ from equation (10.40) because of the presence of the term $Pc^2L\left(\nu/E + 2/G\right)/6I$, which contains both bending and shear contributions.

10.1.1 Insight Into Shear Deformations

To gain some insight into the role and importance of shear deformations, recall that for the cross-section in question $I = 2c^3/3$. Equation (10.46) is thus re-written as

$$v|_{x=0,\,y=0} = \frac{PL^3}{2Ec^3} + \frac{3PL(1+\nu)}{2Ec} \tag{10.54}$$

where the relation $G = E/2(1+\nu)$ has been used. Based on equation (10.54) we make the following observations:

- For given values of P, L, E and ν, *increases* in c decrease the flexural contribution to v by c^{-3}; the shear contribution is reduced only by c^{-1}.

- For given values of P, c, E and ν, *decreases* in L decrease the flexural contribution to v by L^3; the shear contribution is reduced only by L.

The following conclusions can thus be drawn:

1. The effect of shear deformations tends to be more pronounced for *short*, *deep* beams. Transverse displacements in such beams will thus be *greater* than those predicted using standard Bernoulli-Euler beam theory.

2. For "usual" beams with span-to-depth ratios greater than approximately 10, shear deformations will be *negligible*. Use of standard Bernoulli-Euler beam theory is thus appropriate in such cases.

Consider a specific example with $E = 30\,\mathrm{x}\,10^6$, $\nu = 0.30$ and $P = 1000$.The following cases are next investigated:

- For $c = 1$ and $L = 40$: The span-to-depth ratio is 20:1. The associated tip displacement is $v|_{x=0,\,y=0} = 1.067 + 0.0026 = 1.069$. The shear deformation thus comprises only approximately 0.24 % of the total tip displacement.

- For $c = 1$ and $L = 20$: The span-to-depth ratio is now 10:1. The associated tip displacement is $v|_{x=0,\,y=0} = 0.1333 + 0.0013 = 0.1346$. The shear deformation thus comprises only approximately 1.0 % of the total tip displacement.

- For $c = 1$ and $L = 5$: The span-to-depth ratio is 2.5:1. The associated tip displacement is $v|_{x=0,\,y=0} = 0.00208 + 0.00033 = 0.00241$. The shear deformation now comprises 13.7 % of the total tip displacement.
- For $c = 5$ and $L = 20$: The span-to-depth ratio is 2:1. The associated tip displacement is $v|_{x=0,\,y=0} = 0.00107 + 0.00026 = 0.00133$. The shear deformation thus comprises 19.6 % of the total tip displacement.

BIBLIOGRAPHY

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