CS-512 Homework 0

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A let: $A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $B = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, $C = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$

① $$2A - B = 2\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \boxed{\begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}}$$

② $$\|A\| = \sqrt{1^2 + 2^2 + 3^2} = \boxed{\sqrt{14}}$$

Angle relative to positive X axis = $\cos\theta = \frac{1}{\sqrt{14}} \rightarrow \boxed{\theta = \text{ArCos}\left(\frac{1}{\sqrt{14}}\right) \approx 74'5°}$

③ $$\boxed{\hat{A} = \left(1/\sqrt{14}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}} \right)}$$

④ Direction of cosines of A:

$$\alpha = \cos a = \frac{A.e_x}{\|A\|} = \frac{1}{\sqrt{14}}$$

$$\beta = \cos b = \frac{A.e_y}{\|A\|} = \frac{2}{\sqrt{14}}$$

$$\gamma = \cos c = \frac{A e_z}{\|A\|} = \frac{3}{\sqrt{14}}$$

⑤ $$A.B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 15 \\ 30 \\ 45 \end{bmatrix}$$

$$B.A = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 24 \\ 30 \\ 36 \end{bmatrix}$$

matriz form

or

$\hat{A}\hat{B} = (1, 2, 3)(4, 5, 6) = 32$

$\hat{A}\hat{B} = \hat{B}\hat{A}$

$\hat{B}\hat{A} = (4, 5, 6)(1, 2, 3) = 32$

vector form (escalar product)

(6) Angle between A and B = θ

$$\cos\theta = \frac{\hat{A}\cdot\hat{B}}{\|\hat{A}\|\cdot\|\hat{B}\|} = \frac{(1,2,3)(4,5,6)}{\sqrt{1^2+2^2+3^2}\sqrt{4^2+5^2+6^2}} = \frac{4+10+18}{\sqrt{14}\sqrt{77}} = 0'97$$

$$\boxed{\theta = \text{ArcCos}(0'97) = 12'93^\circ}$$

(7) $\perp$ to vector A = P

$\hat{A} = (1,2,3)$ $\qquad 0 = \underbrace{(x,y,z)}_{\hat{P}} \cdot \underbrace{(1,2,3)}_{A} \rightarrow x+2y+3z=0$

For instance : $\begin{cases} x=-3 \\ y=0 \\ z=1 \end{cases} \rightarrow \boxed{\hat{P} = (-3,0,1) \rightarrow P = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}}$

(8) $\boxed{\hat{A}\times\hat{B}} = (1,2,3)\times(4,5,6) = (-3,6,-3) = \boxed{(-1,2,-1)}$

$$\begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} i - \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} j + \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} k = -3i+6j-3k$$

$\boxed{\hat{B}\times\hat{A}} = (4,5,6)\times(1,2,3) = (3,-6,3) = \boxed{(1,-2,1)}$

$$\begin{vmatrix} i & j & k \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 5 & 6 \\ 2 & 3 \end{vmatrix} i - \begin{vmatrix} 4 & 6 \\ 1 & 3 \end{vmatrix} j + \begin{vmatrix} 4 & 5 \\ 1 & 2 \end{vmatrix} k = 3i-6j+3k$$

$$\hat{A}\times\hat{B} = -(\hat{B}\times\hat{A})$$

(9) Directly could be $\hat{A}\times\hat{B} = (-1,2,-1)$ a $\perp$ vector to both $\hat{A}$ and $\hat{B}$

(10) A, B, C dependency $\rightarrow \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix} a_1 + \begin{Bmatrix} 4 \\ 5 \\ 6 \end{Bmatrix} a_2 + \begin{Bmatrix} -1 \\ 1 \\ 3 \end{Bmatrix} a_3 = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$

$$\begin{bmatrix} 1 & 4 & -1 \\ 2 & 5 & 1 \\ 3 & 6 & 3 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \longrightarrow \begin{bmatrix} 1 & 4 & -1 \\ 0 & -3 & 3 \\ 0 & -6 & 6 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} 1 & 4 & -1 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = -a_3 \begin{Bmatrix} -5 \\ -1 \end{Bmatrix}$$

$a_1 = 5a_3$, $a_2 = a_3$, a_3 is arbitrary so A, B, C are linearly dependent

(11) $\boxed{A^T B} = [1, 2, 3] \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \boxed{32}$

$\boxed{AB^T} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 4 & 5 & 6 \end{bmatrix} = \boxed{\begin{bmatrix} 4 & 5 & 6 \\ 8 & 10 & 12 \\ 12 & 15 & 18 \end{bmatrix}}$

B let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & -2 & 3 \\ 0 & 5 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -4 \\ 3 & -2 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -1 & 1 & 3 \end{bmatrix}$

(1) $\boxed{2A - B} = 2 \begin{bmatrix} 1 & 2 & 3 \\ 4 & -2 & 3 \\ 0 & 5 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -4 \\ 3 & -2 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & 2 & 5 \\ 6 & -5 & 10 \\ -3 & 12 & -3 \end{bmatrix}}$

(2) $\boxed{AB} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & -2 & 3 \\ 0 & 5 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -4 \\ 3 & -2 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 14 & -2 & -4 \\ 9 & 0 & 15 \\ 7 & 7 & -21 \end{bmatrix}}$

$\boxed{BA} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -4 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & -2 & 3 \\ 0 & 5 & -1 \end{bmatrix} = \boxed{\begin{bmatrix} 9 & 3 & 8 \\ 6 & -18 & 13 \\ -5 & 7 & 2 \end{bmatrix}}$

3

$$\boxed{(AB)^T = \begin{bmatrix} 14 & 9 & 7 \\ -2 & 0 & 7 \\ -4 & 15 & -21 \end{bmatrix}}$$

$$\boxed{B^T A^T = (AB)^T}$$

4

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & -2 & 3 \\ 0 & 5 & -1 \end{vmatrix} = \begin{vmatrix} -2 & 3 \\ 5 & -1 \end{vmatrix} - 4 \cdot \begin{vmatrix} 2 & 3 \\ 5 & -1 \end{vmatrix} = -13 + 68 = \boxed{55}$$

$$|C| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -1 & 1 & 3 \end{vmatrix} = \begin{vmatrix} 5 & 6 \\ 1 & 3 \end{vmatrix} - 4 \begin{vmatrix} 2 & 3 \\ 1 & 3 \end{vmatrix} - 1 \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} = 9 - 12 + 3 = \boxed{0}$$

5 orthogonal set $\Leftrightarrow AA^T = I$

$$A \cdot A^T = \begin{pmatrix} 1 & 2 & 3 \\ 4 & -2 & 3 \\ 0 & 5 & -1 \end{pmatrix} \begin{pmatrix} 1 & 4 & 0 \\ 2 & -2 & 5 \\ 3 & 3 & -1 \end{pmatrix} = \begin{pmatrix} 14 & 9 & 7 \\ 9 & 29 & -13 \\ 7 & -13 & 26 \end{pmatrix}$$

$$B \cdot B^T = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & -4 \\ 3 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & -2 \\ 1 & -4 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 21 & 0 \\ 0 & 0 & 14 \end{pmatrix}$$

Orthogonal set of vectors

$$C \cdot C^T = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -1 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 4 & -1 \\ 2 & 5 & 1 \\ 3 & 6 & 3 \end{pmatrix} = \begin{pmatrix} 14 & 32 & 10 \\ 32 & 77 & 19 \\ 10 & 19 & 11 \end{pmatrix}$$

⑥ $A^{-1} = \frac{1}{|A|} \cdot Adj(A) = \frac{1}{\begin{vmatrix}1&2&3\\4&-2&3\\0&5&-1\end{vmatrix}} \cdot \begin{pmatrix} \begin{vmatrix}-2&3\\5&-1\end{vmatrix} & -\begin{vmatrix}2&3\\5&-1\end{vmatrix} & \begin{vmatrix}2&3\\-2&3\end{vmatrix} \\ -\begin{vmatrix}4&3\\0&-1\end{vmatrix} & \begin{vmatrix}1&3\\0&-1\end{vmatrix} & -\begin{vmatrix}1&3\\4&3\end{vmatrix} \\ \begin{vmatrix}4&-2\\0&5\end{vmatrix} & -\begin{vmatrix}1&2\\0&5\end{vmatrix} & \begin{vmatrix}1&2\\4&-2\end{vmatrix}\end{pmatrix} = \frac{1}{55}\begin{pmatrix}-13&17&12\\4&-4&9\\22&-5&-2\end{pmatrix}$

$B^{-1} = \frac{1}{\begin{vmatrix}1&2&1\\2&1&-4\\3&-2&1\end{vmatrix}} \begin{pmatrix} \begin{vmatrix}1&-4\\-2&1\end{vmatrix} & -\begin{vmatrix}2&1\\-2&1\end{vmatrix} & \begin{vmatrix}2&1\\1&-4\end{vmatrix} \\ -\begin{vmatrix}2&-4\\3&1\end{vmatrix} & \begin{vmatrix}1&1\\3&1\end{vmatrix} & -\begin{vmatrix}1&1\\2&-4\end{vmatrix} \\ \begin{vmatrix}2&1\\3&-2\end{vmatrix} & -\begin{vmatrix}1&2\\3&-2\end{vmatrix} & \begin{vmatrix}1&2\\2&1\end{vmatrix}\end{pmatrix} = \frac{1}{-42}\begin{pmatrix}-7&-4&-9\\-14&-2&6\\-7&8&-3\end{pmatrix}$

$\boxed{②}$ let $A = \begin{bmatrix}1&2\\3&2\end{bmatrix}$, $B = \begin{bmatrix}2&-2\\-2&5\end{bmatrix}$

① $|A-\lambda I| = \left|\begin{bmatrix}1&2\\3&2\end{bmatrix} - \lambda\begin{bmatrix}1&0\\0&1\end{bmatrix}\right| = \begin{vmatrix}1-\lambda&2\\3&2-\lambda\end{vmatrix} = \lambda^2 - 3\lambda - 4$

$\lambda^2-3\lambda-4=0 \to \begin{cases}\lambda = 4\\ \lambda=-1\end{cases}$ Eigenvalues of A

$(\lambda-4)(\lambda+1)=0$

• For $\lambda=-1$: $(A-I)_{v_{\lambda=-1}} = \begin{bmatrix}2&2\\3&3\end{bmatrix}\begin{bmatrix}v_1\\v_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$ $\quad v_1=-v_2$ $\quad \boxed{V_{\lambda=-1} = \begin{bmatrix}1\\-1\end{bmatrix}}$

Eigenvectors of A

• for $\lambda=4$: $(A-I)_{V_{\lambda=4}} = \begin{bmatrix}-3&2\\3&-2\end{bmatrix}\begin{bmatrix}v_1\\v_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$ $\quad v_1 = \frac{-2}{3}v_2$ $\quad \boxed{V_{\lambda=4} = \begin{bmatrix}1\\-\frac{3}{2}\end{bmatrix}}$

② $V^{-1}AV = \begin{bmatrix}1&1\\-1&-\frac{3}{2}\end{bmatrix}^{-1}\begin{bmatrix}1&2\\3&2\end{bmatrix}\begin{bmatrix}1&1\\-1&-\frac{3}{2}\end{bmatrix} = 2\begin{bmatrix}-\frac{3}{2}&-1\\1&1\end{bmatrix}\begin{bmatrix}-1&-2\\1&0\end{bmatrix} =$

$= \begin{bmatrix}-3&-2\\2&2\end{bmatrix}\begin{bmatrix}-1&-2\\1&0\end{bmatrix} = \begin{bmatrix}1&6\\0&-4\end{bmatrix}$

③ dot producto of eigenvectors of A :

being $\begin{pmatrix} a \\ b \end{pmatrix}$ and $\begin{pmatrix} c \\ d \end{pmatrix}$ eigenvectors $\rightarrow$ dot product $= ac + bd$

$\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -3/2 \end{pmatrix}$ $\rightarrow$ dot product $= 1 + 3/2 = \boxed{\frac{5}{2}}$

④ $|B - \lambda I| = \left| \begin{bmatrix} 2 & -2 \\ 2 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = \begin{vmatrix} 2-\lambda & -2 \\ 2 & 5-\lambda \end{vmatrix} = \lambda^2 - 7\lambda + 14$

$\lambda = \frac{7 \pm \sqrt{49 - 4 \cdot 14}}{2} = \frac{7 \pm \sqrt{-8}}{2} = \frac{7 \pm \sqrt{8}\, i}{2} = \frac{7}{2} \pm \sqrt{2}\, i$

Imaginary!

Because it has imaginary eigenvectors we treat them as one :

$$\begin{cases} \left(-\frac{3}{2} - \sqrt{2}\, i\right) v_1 - 2 v_2 = 0 \\ 2 v_1 + \left(\frac{3}{2} - \sqrt{2}\, i\right) v_2 = 0 \end{cases}$$

$$\frac{\left(\frac{3}{2} + \sqrt{2}\, i\right) v_1}{2} = v_2 \qquad \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{3}{4} + \frac{\sqrt{2}}{2} i \end{pmatrix}$$

There is another eigenvector corresponding to the conjugate $\begin{pmatrix} 1 \\ \frac{3}{4} - \frac{\sqrt{2}}{2} i \end{pmatrix}$

$$\boxed{\text{dot Product}} = 1 + \left(\frac{3}{4} + \frac{\sqrt{2}}{2} i\right)\left(\frac{3}{4} - \frac{\sqrt{2}}{2} i\right) = 1 + \frac{9}{16} + \frac{1}{2} = \boxed{\frac{33}{16}}$$

⑤ As we have seen in ④, the eigenvectors of matrix B are imaginary. They have the property of being complex-conjugate.

D $f(x) = x^2 + 3$, $g(x,y) = x^2 + y^2$

① $\boxed{f'(x) = 2x}$ $\boxed{f''(x) = 2}$

② $\frac{\partial g}{\partial x} = 2x$ $\frac{\partial g}{\partial y} = 2y$

③ $\nabla g(x,y) = \begin{pmatrix} 2x \\ 2y \end{pmatrix}$

④ $$f(x) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right) \qquad -\infty < x < +\infty$$