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## Lecture 2 — Friday, June 7.

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## Recap of the basic notions covered in lecture 1:

- (a) **Naive definition of a set**
- (b) **Empty set**
- (c) **Subset including proper subset**
- (d) **Unions and intersection of sets**
- (e) **Relative and absolute complements**

## Questions:

- (a) **Did anyone find it hard to grasp these basic concepts?**
- (b) **Did anyone find it hard to solve the assignments?**

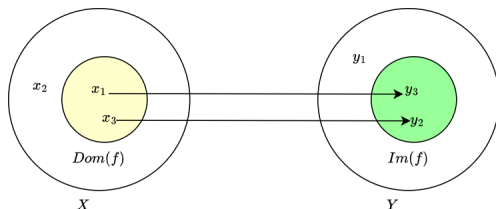
If the answer to the above is yes, then you should not proceed with the content on lecture 2 until you feel comfortable with the basics of lecture 1!

## Definition 1.0

Let  $X$  and  $Y$  be sets. A map (or function) from  $X$  to  $Y$  written  $f : X \longrightarrow Y$  is a prescription that associates an element of  $X$  with an element of  $Y$ .

### ► Remark 1:

- (a) If  $f$  is a map from  $X$  to  $Y$ , then we write  $f : X \rightarrow Y$ , or  $X \xrightarrow{f} Y$ . The element  $y$  of  $Y$  assigned by  $f$  to an element  $x$  of  $X$  is denoted by  $f(x)$  and called the *image* of  $x$  under  $f$ .
  - (b) The set of all the elements of  $X$  that are covered under the map  $f$  is called the domain of  $f$ , and we normally denote it as  $Dom(f)$ .
  - (c) The image of  $f$  is defined as  $Im(f) = \{f(x) \mid x \in Dom(f)\}$ .
  - (d) Very often we write  $Im_f$  instead of  $Im(f)$ .
- **Abstract toy example:** Let  $X = \{x_1, x_2, x_3\}$  and  $Y = \{y_1, y_2, y_3\}$ . We can construct a map  $f : X \longrightarrow Y$  using the prescription;  $f(x_1) = y_3$  and  $f(x_3) = y_2$ . This means that  $Dom(f) = \{x_1, x_3\}$  and  $Im(f) = \{y_3, y_2\}$  right?



## Definition 1.1

Let  $X$  and  $Y$  be sets. A map  $f : X \longrightarrow Y$  is called:

- (a) **Surjective** (or onto) if  $Im(f) = Y$  i.e.  $\forall y \in Y$  there exists a  $x \in X$  such that  $y = f(x)$ .
- (b) **Injective** if  $\forall x_1, x_2 \in Dom(f)$ ,  $f(x_1) = f(x_2)$  if only if  $x_1 = x_2$ .
- (c) **Bijjective** if it's both surjective and injective.

## Definition 1.2

We say  $X$  is (set)-isomorphic to  $Y$  and write  $X \simeq Y$  if there is a map  $f : X \longrightarrow Y$  such that  $Dom(f) = X$  and  $f$  is a bijection i.e.  $f$  is both surjective and injective.

### ► Remark 2:

- (a) Two (set)-isomorphic sets are considered to be the same. A very common alternative term used is equinumerous or equipotent instead of set-isomorphic.
- (b) During the bootcamp we will encounter more interesting extensions of (set)-isomomorphisms as we add structures to the underlying e.g, homeomorphisms of topological spaces and group isomomorphisms.

## Proposition 1.0

Let  $X$ ,  $Y$ , and  $Z$  be sets. If  $X \simeq Y$  and  $Y \simeq Z$ , then  $X \simeq Z$ .

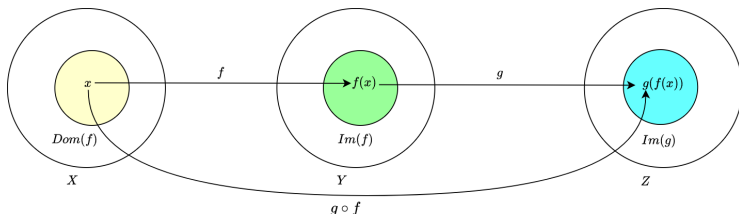
### Definition 1.3

The **identity map** on a set  $X$  is the map written  $id_X : X \rightarrow X : x \mapsto x$  i.e.  $id_X(x) = x \forall x \in X$ .

- (1) When the set  $X$  is understood from the context, we just write  $id$  instead of  $id_X$ .
- (2) Obviously if we consider another set  $Y$ , then it's identity is written  $id_Y$  to distinguish it from  $id_X$ . The same applies if we consider a third set  $Z$ , where we would then write  $id_Z$ .
- (3) It's clear that the identity map  $id_X$  is unique right?

## Definition 1.4

Let  $X$ ,  $Y$  and  $Z$  be sets. The *composition* of maps  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  is the map written  $g \circ f : X \rightarrow Z$  and defined as  $(g \circ f)(x) = g(f(x))$ .



## Proposition 1.1

Given any maps  $f : X \rightarrow Y$ ,  $g : Y \rightarrow Z$  and  $h : Z \rightarrow U$ . The following identities hold:

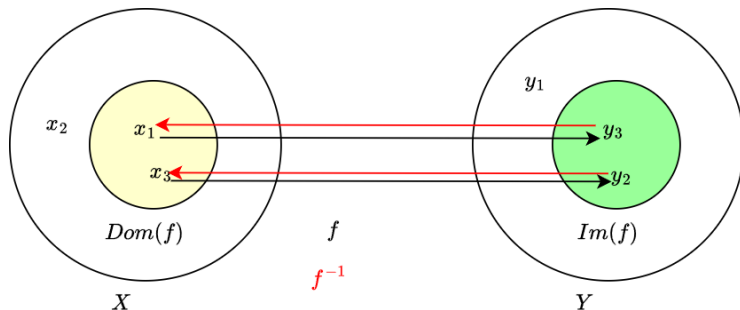
- (1)  $f \circ id_X = f$  and  $id_Y \circ f = f$ .
- (2)  $h \circ (g \circ f) = (h \circ g) \circ f$  (associativity).

## Definition 1.5

Given a map  $f : X \longrightarrow Y$ , the inverse of  $f$  (if it exists) is a map  $g : Y \longrightarrow X$  satisfying the following two conditions:

- (1)  $g \circ f = id_X$ .
- (2)  $f \circ g = id_Y$ .

► **Remark 3:** By convention we write  $f^{-1}$  to denote the inverse of  $f$ .



## Proposition 1.2

A map  $f : X \longrightarrow Y$  is invertible iff  $f$  is bijective and the inverse  $f^{-1}$  is unique.

## Exercise 1.0

Consider the maps  $f : X \longrightarrow Y$  and  $g : Y \longrightarrow Z$ . Prove or disprove the following:

- (1) If  $f$  and  $g$  are injective, then  $g \circ f$  is injective.
- (2) If  $g \circ f$  is injective, then  $g$  is injective.
- (3) If  $g \circ f$  is injective, then  $f$  is injective.
- (4) If  $f$  and  $g$  are surjective, then  $g \circ f$  is surjective.
- (5) If  $g \circ f$  is surjective, then  $g$  is surjective.
- (6) If  $g \circ f$  is surjective, then  $f$  is surjective.
- (7) If  $f$  and  $g$  are bijective, then  $g \circ f$  is bijective.
- (8) If  $g \circ f$  is bijective, then  $g$  is bijective.
- (9) If  $g \circ f$  is bijective, then  $f$  is bijective.

## Exercise 1.1

Let  $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9\}$  and  $Y = \{y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9\}$ . Construct the following examples of maps:

- (1) A map  $f : X \longrightarrow Y$  that is injective but not surjective.
- (2) A map  $f : X \longrightarrow Y$  that is surjective but not injective.
- (3) A map  $f : X \longrightarrow Y$  that is bijective i.e that is both surjective and injective.
- (4) A map  $f : X \longrightarrow Y$  that is neither injective nor surjective.



## Exercise 1.2

Let  $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$  be the set of all integers and  $\mathbb{N} = \{0, 1, 2, 3, 4, 5, \dots\}$  be the set of all natural numbers.

- (1) Even though  $\mathbb{N}$  is a proper subset of  $\mathbb{Z}$ , is it true that  $\mathbb{N} \simeq \mathbb{Z}$ ? Meaning that there is a bijection  $f : \mathbb{N} \rightarrow \mathbb{Z}$ ? If true, can you construct such a bijection?
- (2) Construct a map  $f : \mathbb{N} \rightarrow \mathbb{Z}$  that is injective but not surjective.

# Thank you for listening!