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Abstract Mathematics Bootcamp 2024: General Feedback
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Lecture 2 Assignment Feedback

Introduction

This document provides feedback on the assignment for Lecture 2. The feedback is divided into individual question feedback and overall feedback. Please review the comments carefully to improve your understanding and performance in future assignments.

For each of the proof assignments, one particularly excellent submission has been selected as a model solution.

Feedback on Individual Questions

Proof Assignments

Proposition 1.0

This is a classic exercise. Let X and Y be sets. Then $X \cong Y$ means that there is a bijection between X and Y. The exercise can therefore be completed by showing that a bijection from X to Y, composed with a bijection from Y to Z, is itself a bijection.

To prove that a function is a bijection, you have to prove that it is simultaneously an injection and a surjection. A combination of Exercise Assignment 1.0, (1) and (4) thus gives a proof to this proposition.

Model Solution:

① If f and g are injective, then $g \circ f$ is injective TRUE

1) Let's assume $(g \circ f)(x_1) = (g \circ f)(x_2)$ for some given $x_1, x_2 \in X$

2) This is the same as writting $g(f(x_1)) = g(f(x_2))$, or $g(y_1) = g(y_2)$ for $y_1, y_2 \in Im(f)$. (i.e., $y_1 = f(x_1)$, $y_2 = f(x_2)$)

3) Because g is injective, $g(y_1) = g(y_2)$ implies $y_1 = y_2$ $\forall y_1, y_2 \in Dom(g)$ therefore $f(x_1) = f(x_2)$.

4) Since f is injective, $f(x_1) = f(x_2)$ implies $x_1 = x_2$ $\forall x_1, x_2 \in X$.

5) Therefore, $(g \circ f)(x_1) = (g \circ f)(x_2)$ implies $x_1 = x_2$.

④ If f and g are surjective, $g \circ f$ is surjective TRUE

1) Since g is surjective, $\forall z \in Z$, $\exists y \in Dom(g)$ s.t. $z = g(y)$

2) Since f is surjective, $\forall y \in Y$, $\exists x \in Dom(f)$ s.t. $y = f(x)$

3) Substituting: $z = g(f(x))$, which is the composition $z = (g \circ f)(x)$

4) So this shows that for an arbitrary z in Z $\exists x \in Dom(f)$ s.t. $(g \circ f)(x) = z$

Proposition 1.1

This is another exercise in proving something which is in a sense 'obvious'. So long as you are clear in your writing, the proof should be straightforward. The main mistakes here came from the writing being ambiguous or confusing. Oftentimes, giving very abrupt, short answers means ommiting necessary detail. Try not to abbreviate everything you write in mathematics - you won't be saving as much time as you think!

Model Solution:

1. $f \circ id_X = f$ *and* $id_Y \circ f = f$

   Proof.

   Part 1 $\forall x \in X$, *we have* $id_X(x) = x$
   $\implies (f \circ id_X)(x) = f(id_X(x)) = f(x)$
   $\implies f \circ id_X = f$

   Part 2 $\forall y \in Y$, *we have* $id_Y(y) = y$
   $\implies (id_Y \circ f)(x) = id_Y(f(x)) = id_Y(y) = y = f(x)$
   $\implies id_Y \circ f = f$

2. $h \circ (g \circ f) = (h \circ g) \circ f$

   Proof. $(h \circ (g \circ f))(x) = h((g \circ f)(x)) = h(g(f(x)))$
   $((h \circ g) \circ f)(x) = (h \circ g)(f(x)) = h(g(f(x)))$
   $\therefore h \circ (g \circ f) = (h \circ g) \circ f$

Proposition 1.2

The statement of this proposition uses 'iff', which is short-hand for 'if and only if'. This means that the left hand side statemend is logically equivalent to the right hand side statement, and what this means for the proof is that you have to show two things. Firstly that every invertible map is bijective with unique inverse, and secondly, that every bijective map with a unique inverse is invertible.

The second case is trivial because if you are told that a map has an inverse, it by definition must be invertible. So it remains only to prove the first case. You must show that all invertible functions are bijective (injective and surjective), and further prove that the inverse to any invertible function is unique. To prove uniqueness, the standard method is to assume that there are two such inverses, and then prove that they must be equal.

Model Solution:

Proof

For the first part of the proof, we show that if a map $f : X \to Y$ is invertible, then f is bijective and the inverse f^{-1} is unique.

Let $f : X \to Y$ be an invertible map.
Then the inverse map $f^{-1} : Y \to X$ exists, satisfying the following two conditions:
(1) $f^{-1} \circ f = \text{id}_X$
(2) $f \circ f^{-1} = \text{id}_Y$

Assume f is not injective.
Then there exists $f(x_1)$ and $f(x_2)$ such that $f(x_1) = f(x_2)$ and $x_1 \neq x_2$.

But then $(f^{-1} \circ f)(x_1) = (f^{-1} \circ f)(x_2)$, and by the first identity supposition it follows that $x_1 = x_2$.
However, this contradicts the assumption that f is not injective.
Therefore f is injective.

Assume f is not surjective.
Then there exists $y \in Y$ such that $y \neq f(x)$ for any $x \in X$, which implies $(f \circ f^{-1})(y) \neq y$.
But then we have $f \circ f^{-1} \neq \text{id}_Y$, which is against the second identity supposition.
Therefore f is surjective.
Thus f is bijective.

To complete the first part of the proof, it remains to show f^{-1} is unique.
Let $g^{-1} : Y \to X$, satisfying the following two conditions:
(1) $g^{-1} \circ f = \text{id}_X$
(2) $f \circ g^{-1} = \text{id}_Y$
Now assume $g^{-1} \neq f^{-1}$.
Then for some $y \in Y$, it is not the case that $f^{-1}(y) = g^{-1}(y)$.
But then, it is also not the case that $f(f^{-1}(y)) = f(g^{-1}(y))$.
However, this contradicts the identity suppositions $f(f^{-1}(y)) = f(g^{-1}(y)) = y$.
Therefore, our assumption is contradicted, and $g^{-1} = f^{-1}$.
Thus the inverse f^{-1} is unique.

We must now prove the converse.

Let $f : X \to Y$ be bijective, and let the inverse f^{-1} be unique.
Since f has an inverse, then the inverse $f^{-1} : Y \to X$ exists, satisfying the following two conditions:
(1) $f^{-1} \circ f = \text{id}_X$
(2) $f \circ f^{-1} = \text{id}_Y$
Thus f is invertible.

Therefore, a map $f : X \to Y$ is invertible iff f is bijective and the inverse f^{-1} is unique.

Exercise Assignments

Notice here that the counter examples chosen are all very simple. I have omitted the proofs that these counter examples are indeed counter examples. Try to show this at home.

Exercise 1.0

1. See Proposition 1.0 feedback.

2. **False** - counter example: $X = \{1\}$, $Y = \{1, 2\}$, $Z = \{1\}$. Then set $f(1) = 1$, define g with action $g(1) = 1$ and $g(2) = 1$.

3. **True** - if $x, y \in X$, then $f(x) = f(y) \implies g(f(x)) = g(f(y))$. Thus if f is not injective, neither is $g \circ f$. The result is the contrapositive.

4. See Proposition 1.0 feedback.

5. **True** - The image of $g \circ f$ must be a subset of the image of g *(Exercise : show this!)*. So if $g \circ f$ is surjective, it has image equal to Z, meaning that the image of g must contain Z. It must therefore be equal to its entire codomain Z.

6. **False** - counter example: $X = \{1\}$, $Y = \{1, 2\}$, $Z = \{1\}$. Define f and g to be the constant functions mapping everything to 1.

7. **True** - If both f and g are bijective, they are injective and surjective. Combining (1) and (4) then tells us that $g \circ f$ is both injective and surjective, completing the proof.

8. **False** - (2) gives a counter example.

9. **False** - (6) gives a counter example.

Exercise 1.1

A mistake that I saw a few times here was that some people were constructing maps to and from some sets such as $\{x_0, x_1, x_2\}$ and $\{y_1, y_2\}$. The question was to construct a map from X to Y, not from some subset of X to some subset of Y.

Another common mistake was defining the map from X to Y by something like '$x_n \mapsto y_n$'. This is not a definition of a map *unless* you say what values n can take. More broadly, if you want to introduce a new variable such as n, then you need to define n formally, otherwise n has no meaning.

1. Such a map can *not* be constructed. To see this, let $f : X \to Y$.

   If f is injective, then every element of X is mapped to a unique element of Y by f. It follows that $\{f(x) : x \in X\}$ is a 9-element subset of Y, so must be equal to Y. The range of f is equal to Y is another way of saying that f is a surjection.

2. Such a map can *not* be constructed. To see this, let $f : X \to Y$.

   If f was not injective, then there would be some pair $j_1, j_2 \in \{1, \dots, 9\}$ with $j_1 \neq j_2$ but $f(x_{j_1}) = f(x_{j_2}) = y_k$ for some $k \in \{1, \dots, 9\}$. So only a maximum of 8 elements of Y could belong to the range of f, which would mean that f is not surjective. This proves the claim contrapositively.

3. $f : X \to Y$ with action $f(x_i) = y_i$ for every $i \in \{1, 2, \dots, 9\}$.

4. Set $f : X \to Y$ to have action $x_1 \mapsto y_2$ and $x_i \mapsto y_i$ for all $i \in \{2, \dots, 9\}$.

Exercise 1.2

This question highlights one of the strange things about subsets in the infinite - an infinite set X can be 'smaller' than a set Y in the sense that X is a strict subset of Y, but still be 'the same size' as Y in the sense that a bijection between X and Y can be defined!

1. Consider the sequence $(x_i)_{i=1}^{\infty} = (0, 1, -1, 2, -2, 3, -3, 4, -4, \dots) \subseteq \mathbb{Z}$. It is easy to see that every element of $\mathbb{Z}$ belongs to $(x_i)_{i=1}^{\infty}$ exactly once.

   Therefore, the map $f : \mathbb{N} \to \mathbb{Z}$ defined by $f(i) = x_i$ for every $i \in \mathbb{N}$ defines a bijection. *(Exercise: prove this claim formally!)*

2. The identity map $\mathbb{N} \to \mathbb{Z} : i \mapsto i$ is injective but not surjective.

Overall Feedback

General Comments: A major thing to understand about functions is that they consist of three main parts. A domain, a codomain, and some 'instruction' to connect the two. Missing any of these three parts, the function is not well-defined. We hope that these exercises illustrate why each of these three parts are important. Altering any one of them can drastically change the properties of the function.

Areas of Improvement: A mistake that is very common for those beginning to write mathematically, is introducing variables and not defining them, as I alluded to in the feedback of exercise 1.1. A really easy way to improve your mathematical writing ability is to make sure that everything in your work is defined as it should be. This will have the added effect of making the method clearer in your own head.

Strengths: In many of the submissions there was very good mathematical writing, with precision and clarity.

Overall, the standard was high this time! I think people have grasped the key concepts very well.