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Lecture 3 — Friday, June 21.

Bambordé Baldé

Recap of the basic notions covered in lecture 2:

(a) ***Informal definition of maps between sets**

(b) **Domains and images**

(c) **Special type of maps: surjective, injective and bijective maps**

(d) **Composition of maps**

(e) **Identity and inverse maps**

Questions:

(a) **Did anyone find it hard to grasp these basic concepts?**

(b) **Did anyone find it hard to solve the assignments?**

If the answer to the above is yes, then you should not proceed with the content on lecture 3 until you feel comfortable with the basics of lecture 2!

Definition 1.0

A non-empty set X is finite if there exists a natural number $k \geq 1$ such that $X \simeq \mathbb{N}_k = \{1, \dots, k\}$. We call such k the cardinality of X and write $|X| = k$.

- **Toy example :** Let $X = \{a, b, c\}$. Then X is finite because if we make $k = 3$, then $X \simeq \mathbb{N}_3 = \{1, 2, 3\}$ right?

Proposition 1.0 (homework)

For all $k_1, k_2 \in \mathbb{N}$, $\mathbb{N}_{k_1} \simeq \mathbb{N}_{k_2}$ if and only if $k_1 = k_2$.

Definition 1.1

A set X is infinite if it contains a proper subset Λ such that $\Lambda \simeq X$.

Remark 1 : We could have also said that X is infinite if no such k in definition 1.0 exists. But definition 1.1 is somewhat better, do you agree?
Live challenge: Looking at the definition above of an infinite set, which concrete examples pop up in your mind?

Definition 1.2

A set X is countably infinite if $X \simeq \mathbb{N}$. Otherwise, if X is infinite but not isomorphic to $\mathbb{N}$, we say X is uncountably infinite or just uncountable.

▶ **Concrete examples :** Let's identify which of the following sets are countably infinite or uncountable:

(a) The set of integers $\mathbb{Z}$.

(b) The set of rationals $\mathbb{Q}$.

(c) The set of real numbers $\mathbb{R}$.

Definition 1.3

If X is infinite and countable, then the cardinality of X is denoted $\aleph_0$ (read as aleph-null). The cardinality of $\mathbb{R}$ is on the other hand denoted $\mathfrak{c}$ (read as continuum).

Question : Is there a set X such that the cardinality of X is between $\aleph_0$ and $\mathfrak{c}$?

▶ **Continuum Hypothesis (CH) (classical version):** Every subset of $\mathbb{R}$ is either

countable or has cardinality $\mathfrak{c}$ i.e. there is no cardinality in-between.

Remark 2 : Paul Cohen's work showed that it is impossible to either prove or disprove

the CH within the framework of the ZFC axiomatic system!

CH Reading Reference

https://www.ias.edu/ideas/2011/kennedy-continuum-hypothesis

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Definition 1.4

The Cartesian product between two sets X and Y is defined as
$X \times Y = \{(x, y) : x \in X \text{ and } y \in Y\}$.

Remark 3 : Cartesian products play a crucial role in classical mechanics, especially when studying multiple systems. The Cartesian product of the corresponding phase spaces of these systems is used to analyse the composite system. In quantum mechanics, however, a different concept is required: the tensor product, which we will define in Module II. The property of tensor products is a fundamental aspect that distinguishes quantum computing from classical computing. Tensor products enable the creation of complex multi-qubit systems, leading to intriguing phenomena such as entanglement.

Proposition 1.1 (homework)

For any set X, $X \times \emptyset = \emptyset \times X = \emptyset$.

Proposition 1.2 (homework)

For any sets X and Y, $X \times Y = Y \times X$ iff $X = Y$, or $X = \emptyset$ or $Y = \emptyset$.

Remark 4 : Hence, the Cartesian product is generally noncommutative.

Exercise 1.0

For any sets X, Y and Z be sets. Prove or disprove the following:

(a) $X \times (Y \cap Z) = (X \times Y) \cup (X \times Z)$.
(b) $X \times (Y \cap Z) = (X \times Y) \cap (X \times Z)$.
(c) $X \times (Y \cup Z) = (X \times Y) \cup (X \times Z)$.
(d) $X \times (Y \cup Z) = (X \times Y) \cap (X \times Z)$.
(e) $(X \cap Y) \times Z = (X \times Z) \cap (Y \times Z)$.
(f) $(X \cup Y) \times Z = (X \times Z) \cup (Y \times Z)$.

Exercise 1.1

Let $X = \{\{1, 2\}, \{3\}\}$, $Y = \{1, 2, 3\}$ and $Z = \{(1, 2), (3, 4)\}$. Compute the following:

(a) $X \times (Y \cap Z)$.
(b) $X \times (Y \cup X)$.
(c) $X \times (Y \cup Z)$.
(d) $(Y \cup Z) \times X$.
(e) $(X \cap Y) \times Z$.
(f) $(X \cup Y) \times Z$.

Exercise 1.2

Is it true that if X and Y are countable, then $X \times Y$ is countable? If you think yes, can you prove it?

Definition 1.5

Let X and Y be sets. A map from X to Y is a subset $f \subset X \times Y$ such that for all $x \in X$ there is exactly one $y \in Y$ such that $(x, y) \in X \times Y$.

Remark 5 : It is a convention to write $f : X \longrightarrow Y$ to denote the map $f \subset X \times Y$. Picture f as a rule that associates an element $x \in X$ to a unique element $f(x) \in Y$. We also have the following naming conventions:

(a) f is injective if distinct elements of X have distinct images in Y.

(b) f is surjective if all elements in Y are images of elements in X.

(c) f is bijective if both injective and surjective i.e. if any element of Y is the image of precisely one element of X.

Discussion point for Zulip: Between our previous naive definition of a map and the refined one above, which do you prefer? Please share your opinion with everyone!

Thank you for listening!