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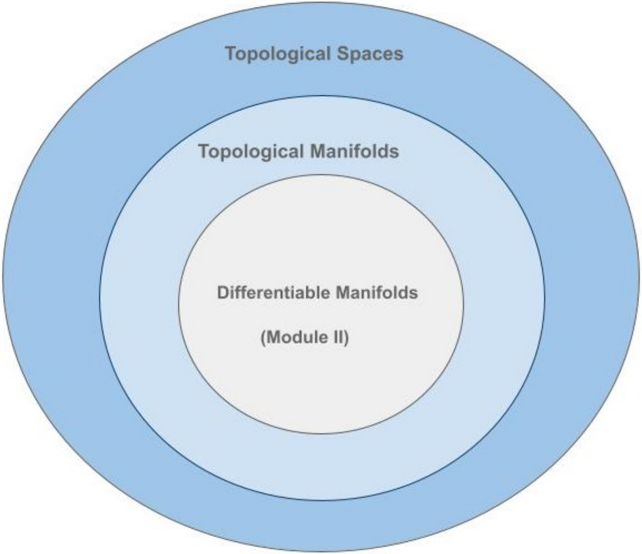
Lecture 5A — Friday, August 9.

Bambordé Baldé

An important objective of this bootcamp is to introduce you the following key notions:



For those interested in the background and history of Topology, the following is a good starting point:

MT MacTutor

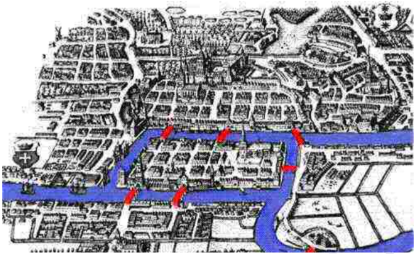
HOME BIOGRAPHIES HISTORY TOPICS MAP CURVES SEARCH

A history of Topology

Topological ideas are present in almost all areas of today's mathematics. The subject of topology itself consists of several different branches, such as point set topology, algebraic topology and differential topology, which have relatively little in common. We shall trace the rise of topological concepts in a number of different situations.

Perhaps the first work which deserves to be considered as the beginnings of topology is due to Euler. In 1736 Euler published a paper on the solution of the *Königsberg bridge problem* entitled *Solutio problematis ad geometriam situs pertinentis* Ⓣ. The title itself indicates that Euler was aware that he was dealing with a different type of geometry where distance was not relevant.

Here is **a diagram of the Königsberg bridges**



Definition 1.0

Let X be a nonempty set. A topology on X is a collection of subsets $\mathcal{T} \subseteq \mathcal{P}(X)$ satisfying the following conditions:

(1) $\varnothing \in \mathcal{T}$ and $X \in \mathcal{T}$.

(2) If $\{O_1, O_2, \ldots, O_k\} \subseteq \mathcal{T}$, then $O_1 \cap O_2 \cap \cdots \cap O_k \in \mathcal{T}$.

(3) If $\{O_i\}_{i \in I} \subseteq \mathcal{T}$, then $\bigcup_{i \in I} O_i \in \mathcal{T}$.

Remark 1:

(a) The pair $(X, \mathcal{T})$ is called a *topological space*. By convention, whenever the topology $\mathcal{T}$ is understood from the context, we'll just write X and call it a topological space. We'll also call the elements of set X as 'points'.

(b) The elements of the collection $\mathcal{T}$ are called *open sets*. Consequently, the notion of an *open set* is relative to the chosen topology. In other words, a subset of X can be designated as open only if a specific topology on X is specified.

Toy examples and counterexamples:

(a) Is $\mathcal{T} = \mathcal{P}(X)$ a topology on X? What about $\mathcal{T} = \{X, \varnothing\}$?

(b) Let $X = \{\beta_1, \beta_2, \beta_3\}$. Is $\mathcal{T} = \{\varnothing, X, \{\beta_1\}, \{\beta_1, \beta_2\}\}$ a topology on X? What if $\mathcal{T} = \{\varnothing, X, \{\beta_2\}, \{\beta_3\}, \{\beta_1, \beta_2\}\}$?

An interesting topology: If $X = \{0, 1\}$, then $\mathcal{T} = \{\varnothing, X, \{1\}\}$ is a topology on X (Sierpinski topology). At first glance, this topology might not seem particularly interesting, but it actually is!

On the Number of Possible Topologies

For a finite set X, here is something interesting:

Cardinality of X	**Possible number of topologies**
1	1
2	4
3	29
4	356
5	6942
6	209527

Table: Number of topologies based on the cardinality of X.

Natural follow up question:
Can we generalise the above table to any cardinality n i.e. is there a general formula that computes how many topologies are there for any cardinality n?

Exercise 1.0

Let $X = \{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6\}$. You are challenged to make the following:

(a) Construct a topology $\mathcal{T}$ with no open sets containing the elements β_1 and β_4.

(b) Construct a topology $\mathcal{T}$ with four open sets containing the elements β_1 and β_4.

(c) Construct a topology $\mathcal{T}$ with two open sets having exactly two elements in common.

Exercise 1.1

Let $X = \{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6\}$. Which of the following are topologies on X:

(a) $\mathcal{T}_1 = \{X, \varnothing, \{\beta_3\}, \{\beta_2, \beta_4, \beta_5\}, \{\beta_2, \beta_3, \beta_4, \beta_5\}, \{\beta_2\}\}$.

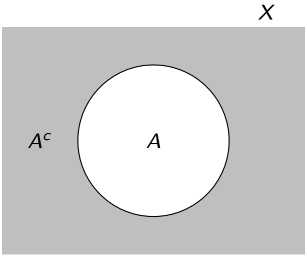
(b) $\mathcal{T}_2 = \{X, \varnothing, \{\beta_1\}, \{\beta_2, \beta_4, \beta_5\}, \{\beta_1, \beta_2, \beta_4\}, \{\beta_1, \beta_2, \beta_4, \beta_5\}\}$.

(c) $\mathcal{T}_3 = \{X, \varnothing, \{\beta_2\}, \{\beta_1, \beta_2, \beta_3\}, \{\beta_4, \beta_5, \beta_6\}, \{\beta_2, \beta_4, \beta_5, \beta_6\}\}$.

Definition 1.1

A subset $A \subseteq X$ is said to be a closed set in respect to a topology $\mathcal{T}$ on X if $A^c \in \mathcal{T}$.

Absolute complement reminder:



Remark 2: In general, a subset $A \subseteq X$ can be the following in respect to $\mathcal{T}$:

(a) Open
(b) closed
(c) Open and closed (aka clopen)
(d) Open and not closed
(e) Not open and closed
(f) Not open and not closed

Exercise 1.2

Let $\mathcal{T}$ be a topology on X . Prove or disprove the following:

(a) $A_1, A_2, \dots, A_k$ are closed sets in respect to $\mathcal{T}$ then $\bigcap\limits_{i=1}^{k} A_i$ is also closed.

(b) $A_1, A_2, \dots, A_k$ are closed sets in respect to $\mathcal{T}$ then $\bigcup\limits_{i=1}^{k} A_i$ is also closed.

(c) If $\{A_i\}_{i \in I}$ is an indexed family of closed sets in respect to $\mathcal{T}$ with I being an arbitrary index. Then $\bigcup\limits_{i \in I} A_i$ is also closed.

(d) If $\{A_i\}_{i \in I}$ is an indexed family of closed sets in respect to $\mathcal{T}$ with I being an arbitrary index. Then $\bigcap\limits_{i \in I} A_i$ is also closed.

(e) If $\mathcal{T} = \mathcal{P}(X)$, then every open set is clopen.

Congratulations for making it to this important section of this bootcamp!