# Lecture 1 Assignment Feedback

## Introduction

This document provides feedback on the assignment for Lecture 1. The feedback is divided into individual question feedback and overall feedback. Please review the comments carefully to improve your understanding and performance in future assignments.

For each of the proof assignments, one particularly excellent submission has been selected as a model solution.

## Feedback on Individual Questions

### Proof Assignments

#### Proposition 1.0

To prove that $X$ is a subset of $Z$, you need to show that if $x \in X$, then $x \in Z$. The solution follows simply from a couple of applications of the definition of subsets.

Occasionally, people mistook the notation $A \subseteq B$, thinking that it meant that $B$ necessarily contains an element not in $A$. This is not the case. In fact there is another notation '$\subsetneq$' for when this stronger condition is true. We always have that $A \subseteq A$.

**Model Solution:**

**Proposition 1.0** *Let $X, Y$ and $Z$ be sets. If $X \subseteq Y$ and $Y \subseteq Z$ then $X \subseteq Z$*

***Proof.*** $Y \subseteq Z \implies \forall y \in Y, y \in Z$

$X \subseteq Y \implies \forall x \in X, x \in Y$

*Since* $x \in X \implies x \in Y$ *and* $y \in Y \implies y \in Z$, *therefore* $\forall x \in X, x \in Z$

$\implies X \subseteq Z$

#### Proposition 1.1

For showing that two sets $A$ and $B$ are equal, you need to show that $x \in A \iff x \in B$. This is by definition equivalent to showing that $A \subseteq B$ and $B \subseteq A$. Once you know this method, and you break down each side of each equation in terms of the definitions of unions and intersections, the results become relatively simple to show.

For the third and fourth questions, there were cases of people assuming the answer to show the result. The results you are asked to prove here do seem trivial, but that is by design! The purpose of these questions is to give proofs to things that you intuitively already believe.

**Model Solutions:**

1. ***Proof.*** Let $X = \{x : x \in X\}$. Since $X \cap \emptyset = \{x : x \in X \wedge x \in \emptyset\}$ and $\emptyset$ has no elements, then $\nexists x \in (X \cap \emptyset)$.Therefore, $X \cap \emptyset = \emptyset$.

$\square$

2. First we show that $X \cap X \subseteq X$.

$$\begin{aligned} \alpha \in X \cap X &\implies \alpha \in X \text{ and } \alpha \in X && \text{(intersection)} \\ &\implies \alpha \in X \\ \therefore \alpha \in X \cap X,\ \alpha \in X &\implies X \cap X \subseteq X && \text{(subset)} \end{aligned}$$

Then we show that $X \subseteq X \cap X$.

$$\begin{aligned} \alpha \in X &\implies \alpha \in X \text{ and } \alpha \in X && \text{(redundant)} \\ &\implies \alpha \in X \cap X && \text{(intersection)} \\ \therefore \alpha \in X,\ \alpha \in X \cap X &\implies X \subseteq X \cap X && \text{(subset)} \end{aligned}$$

Finally, using the definition of set equality:

$$\begin{aligned} X \cap X &\subseteq X \\ X &\subseteq X \cap X \\ &\implies X \cap X = X \quad \blacksquare \end{aligned}$$

3. (3) $X \cap Y = Y \cap X$ MEANS $X \cap Y \subseteq Y \cap X$ AND $Y \cap X \subseteq X \cap Y$

[$X \cap Y \subseteq Y \cap X$]

SAY $x \in X \cap Y$. THIS MEANS $\alpha \in X$ AND $\alpha \in Y$, WHICH IS THE SAME AS SAYING $\alpha \in Y$ AND $\alpha \in X$ WHICH MEANS $\alpha \in Y \cap X$, WHICH WAS TO BE SHOWN.

[$Y \cap X \subseteq X \cap Y$] SINCE $X$ AND $Y$ ARE ARBITRARY, THE SAME ARGUMENT HOLDS WITH $X$ AND $Y$ INTERCHANGED.

4. 4. Let $x \in (X \cap Y) \cap Z$. Then $x \in X \cap Y$ and $x \in Z$. By the definition of set intersection this implies $x \in X$, $x \in Y$, and $x \in Z$. Similarly, let $x \in X \cap (Y \cap Z)$. Then $x \in X$, $x \in Y$, and $x \in Z$. Thus, $x \in (X \cap Y) \cap Z$ if and only if $x \in X \cap (Y \cap Z)$. Hence, $(X \cap Y) \cap Z = X \cap (Y \cap Z)$.

### Proposition 1.2

Again, the questions depend on proving set equality which means proving that two sets have identically the same elements. And again, make sure not to invoke that these properties are trivial or 'follow straight from the definitions' when the task is to prove them.

**Model Solutions:**

1.

$$\begin{aligned} X \cup \emptyset &= \{\alpha : \alpha \in X \text{ or } \alpha \in \emptyset\} \\ &= \{\alpha : \alpha \in X\} \\ &= X \end{aligned}$$

$$\begin{aligned} X \cup X &= \{\alpha : \alpha \in X \text{ or } \alpha \in X\} \\ &= \{\alpha : \alpha \in X\} \\ &= X \end{aligned}$$

2. (2) $X \cup Y = Y \cup X$ (commutativity).

$$\begin{aligned} \text{Let } \gamma \in X \cup Y &\iff (\gamma \in X \text{ or } \gamma \in Y) \\ &\iff (\gamma \in Y \text{ or } \gamma \in X) \\ &\iff (\gamma \in Y \cup X) \end{aligned}$$

So, $X \cup Y = Y \cup X$.

3. ***Part 1*** $x \in (X \cup Y) \cup Z \implies (x \in X \text{ or } x \in Y) \text{ or } x \in Z$
$\implies x \in X \text{ or } x \in Y \text{ or } x \in Z$
$\implies x \in X \text{ or } (x \in Y \text{ or } x \in Z)$
$\implies x \in X \cup (Y \cup Z)$
$\implies (X \cup Y) \cup Z \subseteq X \cup (Y \cup Z)$

***Part 2*** $x \in X \cup (Y \cup Z) \implies x \in X \text{ or } (x \in Y \text{ or } x \in Z)$
$\implies x \in X \text{ or } x \in Y \text{ or } x \in Z$
$\implies (x \in X \text{ or } x \in Y) \text{ or } x \in Z$
$\implies x \in (X \cup Y) \cup Z$
$\implies X \cup (Y \cup Z) \subseteq (X \cup Y) \cup Z$

$\therefore (X \cup Y) \cup Z = X \cup (Y \cup Z)$

## Exercise Assignments

### Exercise 1.0

These can be answered out by similar methods to the proof exercises. The true statements are (1): $(X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z))$ and (2): $(X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z))$.

### Exercise 1.1

Remember that the complement operation is defined by the relationship between a set and its subsets. So, the superscript notation '$c$' denotes complementation *within* $X$, and not in some undefined overarching 'universal set'. Be careful using the notation as the context needs to be made clear (perhaps we should have been clearer here too!). The true statements are (2): $(A \cap A^c = \emptyset)$, and (5): $(X^c = \emptyset)$.

### Exercise 1.2

These questions will be familiar to anyone who has seen De Morgan's laws, which are similar in boolean logic and set theory. The true statements are (2): $((A \cup B)^c = A^c \cap B^c)$ and (4): $((A \cap B)^c = A^c \cup B^c)$.

# Overall Feedback

**General Comments:** For the true/false questions, always give a reason alongside your answer. For the true results, this will look like a short proof, and for the false results, this will be a counter-example. You will thank yourself later for the extra practice in writing.

Proving things will naturally be a big part of this course, so familiarise yourself with some of the main strategies such as proof by contradiction, and contrapositivity.

**Areas of Improvement:** Venn diagrams, whilst being a useful visual tool, do not constitute a proof in and of themselves. We want you to complete the exercises in set theoretical language. To structure your proof and make it well organised, try to adhere to the writing style that you see in lecture notes or textbooks, or even make a bullet-pointed list of implications.

Also, there were some instances of obvious ChatGPT use in the responses. Aside from the fact that this type of AI is currently very poor at mathematical reasoning, there is no point in even submitting the exercises if you didn't complete them yourself. The exercises are there to help you to learn!

An important thing to know is that an example is *not* a proof (e.g., if we are asked to show that every multiple of 4 is also a multiple of 2, it is not sufficient to just show that the number 8 is a multiple of 2). A lot of people proved that the intersection and union equalities hold for some chosen examples of set, when what you needed to show was that they hold for *any* given sets.

**Strengths:** Some very good presentation, both in Latex and handwritten. We do not require that you use Latex, but we always appreciate when effort has been put into style and readability.