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Lecture 4B — Friday, July 26.

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Recap of the basic notions covered in lecture 4A:

(a) **The notion of indexed sets**

(b) **Indexed family of sets**

(c) **Union and intersection of indexed families**

Questions:

(a) **Did anyone find it hard to grasp these basic concepts?**

(b) **Did anyone find it hard to solve the assignments?**

If the answer to the above is yes, then you should not proceed with the content on lecture 4B until you feel comfortable with the basics of lecture 4A!

Definition 1.0

Let $\{A_i\}_{i\in I}$ be a family of indexed nonempty sets with $I = \{1, 2, 3, \ldots, k\}$ for some very very large $k \in \mathbb{N}$. The Cartesian product between the sets in the family is defined as $\prod_{i\in I} A_i = A_1 \times A_2 \times A_3 \times \ldots \times A_k = \{(a_1, a_2, a_3, \ldots, a_k) \mid a_i \in A_i\}$.

(a) There is a more abstract definition of products over abitrary family of indexed sets. However, for practicality, the above is what is most common.

(a) As we add structure to sets, the Cartesian product of sets with a given structure generally inherits the underlying structure. For example, the Cartesian product of two topological spaces is also a topological space. Similarly, the Cartesian product of sets with manifold structures, vector space structures, and other mathematical structures will inherit those structures.

Exercise 1.0

$\{A_i\}_{i\in I}$ and $\{B_j\}_{j\in J}$ be two indexed family of sets with $I = \{1, 2, 3, \ldots, m\}$ and $J = \{1, 2, 3, \ldots, n\}$. Prove or disprove the following:

(a) $(\bigcup\limits_{i\in I} A_i) \times (\bigcup\limits_{j\in J} B_i) = (\bigcup\limits_{i\in I} A_i) \bigcap (\bigcup\limits_{j\in J} B_j)$.

(b) $(\bigcup\limits_{i\in I} A_i) \times (\bigcup\limits_{j\in J} B_i) = \bigcup\limits_{(i,j)\in I\times J} (A_i \times B_j)$.

(c) $(\bigcap\limits_{i\in I} A_i) \times (\bigcup\limits_{j\in J} B_i) = \bigcup\limits_{(i,j)\in I\times J} (A_i \times B_j)$.

(d) $(\bigcap\limits_{i\in I} A_i) \times (\bigcap\limits_{j\in J} B_i) = \bigcap\limits_{(i,j)\in I\times J} (A_i \times B_j)$.

Curiosity question: Suppose we have a collection of nonempty sets $A_1, A_2, \ldots, A_k$. Wouldn't it be nice if we can form a new set made of exactly one element of each set from our collection?

Natural follow up question: Why not extend the question above to an arbitratry collection of sets?

Definition 1.1

Let $\mathcal{C} = \{A_i\}_{i\in I}$ be an indexed family of nonempty sets i.e. $A_i \neq \varnothing$ for all $i \in I$. A map $f : \mathcal{C} \longrightarrow \bigcup_{i\in I} A_i$ is called a 'choice function' if $f(A_i) \in A_i$ for all $A_i \in \mathcal{C}$.

Concrete intuitive example 1:

(a) According to the United nations, there are 195 countries in the world. So let $I = \{1, 2, 3, \ldots, 195\}$ and $\mathcal{C} = \{A_i\}_{i\in I}$ where each A_i is a country made of all its cities. For example let A_1 be the United states, then A_1 is made of all the cities in the US.

(b) According to the above, $\bigcup_{i\in I} A_i$ is the collection of all the cities in the world right?

(c) We can define a map $f : \mathcal{C} \longrightarrow \bigcup_{i\in I} A_i$ as $f(A_i) = Cap(A_i)$, where $Cap(A)$ denotes the capital city of A_i. Hence, $f(A_1) =$ Washington DC. It's clear that f is a choice function right?

Concrete intuitive example 2:

(a) Let $\mathcal{C} = \{A_i\}_{i\in I}$ where each A_i is a pair of shoes that has been made over the past three centuries. Then $\bigcup_{i\in I} A_i$ is the collection of all the pair of shoes made over the past three centuries right?

(c) We can define a map $f : \mathcal{C} \longrightarrow \bigcup_{i\in I} A_i$ as $f(A_i) = Left(A_i)$, where $Left(A)$ denotes the left shoe of A_i. It's clear that f is a choice function right?

Definition 1.2 (Axiom of Choice)

For any collection of indexed family of nonemppty sets $\mathcal{C} = \{A_i\}_{i \in I}$, there is a choice function $f : \mathcal{C} \longrightarrow \bigcup_{i \in I} A_i$.

Remarks:

(a) The axiom simply guarantees the existing of the choice function, we don't need to bother constructing it! The existence of the choice function is taken as an axiom!

(b) There a other equivalent ways of stating the axiom of choice. One alternative is to say for example, the Cartesian product of any collection of nonempty sets is nonempty!

(c) The axiom of choice is virtually omnipresent in branches of mathematics that are built around sets, including; Analysis, Linear algebra, Abstract algebra, Measure theory, and General topology. Many proofs rely on arbitrary choices to establish general statements and theorems, which are often self-evident and intrinsic.

Definition 1.3

Let X be a nonempty set. The power set of X denoted $\mathcal{P}(X)$ is defined as the set of all the subsets of X i.e. $\mathcal{P}(X) = \{A \mid A \subseteq X\}$.

(a) It's obvious that both X and the empty set $\varnothing$ are in $\mathcal{P}(X)$ right?

(b) In practice, an indexing family of sets is often a powersets of some set.

Toy example: If $X = \{a, b\}$, then the subsets of X are: $\varnothing$, X, $\{a\}$ and $\{b\}$. Hence, $\mathcal{P}(X) = \{\varnothing, X, \{a\}, \{b\}\}$.

Proposition 1.0

If X is a finite set with cardinality k, then the powerset $\mathcal{P}(X)$ has cardinality 2^k.

Theorem (Cantor's powerset theorem)

Let X be a set, then $|X| < |\mathcal{P}(X)|$ i.e. the cardinality of the set X is always less than the cardinality of $\mathcal{P}(X)$.

Remark: We will release a video on Cantor's diagonal proof, as well as one on the theorem above.

Exercise 1.1

Try prove or disprove the following:

(a) $\{0, \pi\} \in \mathcal{P}(\mathbb{Q})$, where $\mathbb{Q}$ is the set of rationals.

(b) $\{0, \sqrt{2}\} \in \mathcal{P}(\mathbb{R})$, where $\mathbb{R}$ is the set of reals.

(c) $\mathbb{Q} \cap (0,1) \in \mathcal{P}(\mathbb{Q})$ where $(0,1) = \{x \in \mathbb{R} \mid 0 < x < 1\}$.

(d) $\mathbb{Q} \cup (0,\sqrt{2}) \in \mathcal{P}(\mathbb{R})$ where $(0,\sqrt{2}) = \{x \in \mathbb{R} \mid 0 < x < \sqrt{2}\}$.

(e) $\mathbb{Q} \cup [0,\sqrt{2}] \in \mathcal{P}(\mathbb{R})$ where $[0,\sqrt{2}] = \{x \in \mathbb{R} \mid 0 \leq x \leq \sqrt{2}\}$.

(f) $(\bigcap_{i \in I} \mathcal{P}(A_i))^c = \bigcap_{i \in I} \mathcal{P}(A_i)^c$.

(g) $(\bigcup_{i \in I} \mathcal{P}(A_i))^c = \bigcup_{i \in I} \mathcal{P}(A_i)^c$.

(h) $|\mathcal{P}(\mathbb{N})| = |\mathcal{P}(\mathbb{R})|$ i.e. the cardinality of the powerset of the naturals $\mathbb{N}$ is the same as the powerset of the reals.

Congratulations for completing the 'Naive set theory' section!