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Lecture 5B — Friday, August 23.

Bambordé Baldé

Recap of the basic notions covered during the last session:

(a) **Topology on a set**

(b) **Open sets**

(c) **Closed sets**

Questions:

(a) **Did anyone find it hard to grasp these basic concepts?**

(b) **Did anyone find it hard to solve the assignments?**

If the answer to the above is yes, then you should go back and review the set theoretic concepts already covered!

Definition 1.0

Let $\mathcal{T}$ be a topology on X and $A \subseteq X$. We define $\mathcal{T}_A$ on A as $\mathcal{T}_A = \{A \cap O \mid O \in \mathcal{T}\}$.

(1) It's clear that $\varnothing, A \in \mathcal{T}_A$ because $A = A \cap X$ and $\varnothing = A \cap \varnothing$. It's not hard to prove that $\mathcal{T}_A$ is indeed a topology on A.

(2) The pair $(A, \mathcal{T}_A)$ is called the subspace (or subset) topology in respect to the topological space $(X, \mathcal{T})$.

Proposition 1.0

Let $(X, \mathcal{T})$ be a topological space and $(A, \mathcal{T}_A)$ be a subspace. If $B \in \mathcal{T}_A$ and $A \in \mathcal{T}$ then $B \in \mathcal{T}$.

Proof: Homework!

Remark 1: Hence, there is a link between open sets of a subspace topology if its underlying set A happens to be an open set in the parent topological space!

Curiosity question (homework): Is it true that, if A is open in X then $\mathcal{T}_A$ consists of all open sets of X contained in A? What if A happens to be closed?

Exercise 1.0

Let $X = \{\beta_1, \beta_2, \beta_3\}$ and $\mathcal{T} = \{\varnothing, X, \{\beta_1\}, \{\beta_1, \beta_2\}\}$. Now consider the subset $A = \{\beta_1, \beta_3\}$. Which of the following is the true subspace topology for A:

(a) $\mathcal{T}_A = \{\varnothing, A, \{\beta_2\}\}$.

(b) $\mathcal{T}_A = \{\varnothing, A, \{\beta_1\}\}$.

(c) $\mathcal{T}_A = \{\varnothing, A, \{\beta_3\}\}$.

Exercise 1.1

Let again $X = \{\beta_1, \beta_2, \beta_3\}$ and $\mathcal{T} = \{\varnothing, X, \{\beta_1\}, \{\beta_1, \beta_2\}\}$. Find the subspace topology for the following cases:

(a) $A = \{\beta_2, \beta_3\}$.

(b) $A = \{\beta_1, \beta_2\}$.

(c) $A = \{\beta_1\}$.

(d) $A = \{\beta_2\}$.

(e) $A = \{\beta_3\}$.

Definition 1.1

Let $(X, \mathcal{T})$ be a topological space. A subset $N \subseteq X$ is said to be a neighbourhood of a point $p \in X$ if there is an open set $O \subseteq N$ such that $p \in O$.

Remark 2:

(a) Note that N doesn't need to necessarily be open i.e. we don't necessarily have $N \in \mathcal{T}$. However, if $N \in \mathcal{T}$, then N is called an open neighbourhood of p.

(b) What if N is closed? What would you name it in respect to our p?

(c) Some authors often assume $N \in \mathcal{T}$ in their definition of neighbourhood!

Proposition 1.1

Let $(X, \mathcal{T})$ be a topological space. Then a subset $A \subseteq X$ is open if and only if A is a neighborhood of all its points $p \in A$.

Proof : Homework!

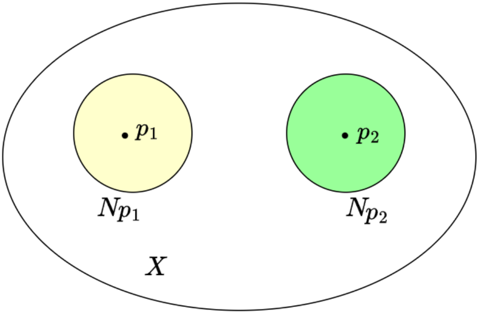
Proposition 1.2

Let $(X, \mathcal{T})$ be a topological space and $p \in X$. If $N_1 \subseteq N_2$ is a neighbourhood of p then N_2 is also a neighbourhood of p.

Proof : Homework!

Definition 1.2

A topological space $(X, \mathcal{T})$ is called a Hausdorff space (or separated) if for any two $p_1, p_2 \in X$ such that $p_1 \neq p_2$, there are open neighbourhoods N_{p_1} and N_{p_2} of the respective points such that $N_{p_1} \cap N_{p_2} = \emptyset$.



Remark 3:

(a) In the early days of Topology, the definition above was part of the definition of a topological space. It was called the 'Hausdorff Axiom' or 'Separation Axiom'.

(b) Alternatively, mathematicians use the terms 'separated spaces' or 'T2 spaces'.

Proposition 1.3

If a topological space $(X, \mathcal{T})$ is Hausdorff and $A \subseteq X$. Then the subspace topology $(A, \mathcal{T}_A)$ is also Hausdorff.

Proof : Homework!

Remark 4:

(a) The proposition above is already hinting that Hausdorff spaces have nice structural properties, they are well-behaved.

(b) You'll see later that a space being Hausdorff is a 'topological property' i.e. the Hausdorff condition is preserved under homeomorphisms.

(a) We'll later see that topological spaces induced by a metric are always Hausdorff.

(b) In a Hausdorff space X, every singleton $\{p\} \subset X$ is closed. This property will become handy when constructing certain types of n-dimensional topological manifolds such as the n-Sphere.

(c) Many important topological spaces are Hausdorff. Very often, constructions made on topological spaces (e.g. manifolds) require the spaces to be Hausdorff. Why is this the case? Well, Hausdorf spaces make life easier for mathematicians to prove theorems and make useful constructions.

****Takeaway for the developers (naive analogy):** Hausdorff spaces are like APIs for mathematicians—they streamline complex proofs, making it easier to achieve more with less effort! ⌣̈ ☺

Curiosity note: Some topological spaces are non-Hausdorff and therefore not metrizable (meaning no metric can be defined on them), but they are still useful in mathematics! The Zariski topology is a good example of this.

Congratulations for making it this far!