Lecture 3 Assignment Feedback

Introduction

This document provides feedback on the assignment for Lecture 3. The feedback is divided into individual question feedback and overall feedback. Please review the comments carefully to improve your understanding and performance in future assignments.

For each of the proof assignments, in the subsection titled 'model solution', one of your submissions has been selected to display a particularly excellent answer to the question.

Feedback on Individual Questions

Proof Assignments

Proposition 1.0

In this question, some people said something along the lines of 'if there is a bijection between $\mathbb{N}_{k_1}$ and $\mathbb{N}_{k_2}$, then this bijection is one-to-one, and thus pairs their elements, proving their cardinality is the same'. This is not correct logic in and of itself. The issue here is that the term 'one-to-one' in mathematics is commonly used as a synonym for 'injective' And if there is an injection f from a set X to a set Y, this only tells you that Y has *at least* as many elements as X. Thus, you need to use something more to complete the proof. For example, that f^{-1} is an injection from Y to X and hence X also has at least as many elements as Y.

Editor's note : I avoid the term 'one-to-one' for this reason. Although it means injective, one could easily misinterpret it to mean bijective.

The following submission nailed this idea, except that they used surjectivity instead of injectivity as the property to complete the proof. The proof is very clearly written. To be pedantic, the only thing that is missing is that on the final line it should be specified '$\forall i \in \mathbb{N}_{k_1}$'.

Model Solution:

Proof.
[only if: $\mathbb{N}_{k1} \simeq \mathbb{N}_{k2} \implies k_1 = k_2$]
$\mathbb{N}_{k1} \simeq \mathbb{N}_{k2} \implies \exists$ a bijection $f : \mathbb{N}_{k1} \to \mathbb{N}_{k2}$.
f is surjective $\implies k_2 = |\mathbb{N}_{k2}| \leq |\mathbb{N}_{k1}| = k_1$ i.e. $k_2 \leq k_1$.
Since f is a bijection, $f^{-1} : \mathbb{N}_{k2} \to \mathbb{N}_{k1}$ is also surjective, so $k_1 = |\mathbb{N}_{k1}| \leq |\mathbb{N}_{k2}| = k_2$ i.e $k_1 \leq k_2$, so $k_1 = k_2$.

[if: $k_1 = k_2 \implies \mathbb{N}_{k1} \simeq \mathbb{N}_{k2}$]

$id : \mathbb{N}_{k1} \to \mathbb{N}_{k2}, i \mapsto i$ is an isomorphism.

□

Proposition 1.1

For this question, one of the main mistakes that I saw people make was that they assumed that '$\times$' is a symmetric operation, essentially assuming that $A \times B = B \times A$ for every pair of sets A and B, and thus they assumed that they only had to show that $X \times \emptyset = \emptyset$, and $\emptyset \times X = \emptyset$ would be immediate from this. However, the operation '$\times$' is not symmetric. To see this, think of the subsets $[0,1] \times [0,100]$ and $[0,100] \times [0,1]$ of $\mathbb{R}^2$. One of these rectangles is very tall but not wide, the other is very wide but not tall. They are not equal. Proposition 1.2 puts this more formally.

The following submission to the question chooses proof by contradiction to give a well-written solution. The only (again, pedantic) mistake that this person made is to forget the edge case that $X = \emptyset$.

Model Solution:

Proof:

$X \times \emptyset = \emptyset$ | Let $X \neq \emptyset$ and assume on the contrary that $X \times \emptyset = Y$ where $Y \neq \emptyset$. Thus $\exists y \in Y$ s.t. $y = (x, a)$ for $x \in X$ and $a \in \emptyset$. This is a contradiction since there does not exist an $a \in \emptyset$ as it is empty, thus $X \times \emptyset = \emptyset$.

$\emptyset \times X = \emptyset$ | Once again let $X \neq \emptyset$ and assume $\emptyset \times X = Y$ for $Y \neq \emptyset$. thus $\exists y \in Y$ such that $y = (a, x)$ for $a \in \emptyset$ and $x \in X$. This is a contradiction and $\emptyset \times X = \emptyset$. $\square$

Proposition 1.2

For this question, the 'only if' claim, that '$X = Y$ or $X = \emptyset$ or $Y = \emptyset$' implies that $X \times Y = Y \times X$, is trivial given Proposition 1.1, and the fact that $X = Y \implies X \times Y = X \times X = Y \times X$.

In the other direction, the easiest way to prove the claim is to prove the contrapositive result, i.e., show that 'if X and Y are nonempty sets and $X \neq Y$, then $X \times Y \neq Y \times X$'. In this case, since $X \neq Y$, there are two possibilities: that there is some $x \in X \setminus Y$ or some $y \in Y \setminus X$. Without loss of generality, assume the first (in case the second possibility holds, use symmetric logic). Let $x \in X \setminus Y$, and take any $y \in Y$. Then $(x, y) \in X \times Y$, but $(x, y) \notin Y \times X$ because $x \notin Y$. So, $(x, y) \in X \times Y \setminus Y \times X$, hence $X \times Y \neq Y \times X$.

A key mistake that I saw a few times is saying something along the lines of 'if $X \times Y = Y \times X \neq \emptyset$, then $(x, y) = (y, x)$ for every $(x, y) \in X \times Y$'. This is not the case. To see this, note that the square $[0,1] \times [0,1]$ is not equal to the diagonal line segment $\{(x, x) : x \in [0,1]\}$.

Unfortunately I think nobody completely nailed the solution to this one. It was a challenging question and its logicc has many places to slip up!

Exercise Assignments

Exercise 1.0

People generally did very well on this question so I will showcase your solutions in the feedback!

The only recurring mistake seemed to be when people said something like 'let $(x, y) \in$**LHS**, ..., then $(x, y) \in$**RHS**. Therefore **LHS**=**RHS**.' This is not complete. All this shows is that **LHS**$\subseteq$**RHS**. You would need to prove the reverse direction to complete the proof.

Again, remember that to *disprove* something, all you have to do is find a counterexample, i.e., display one instance where the claim fails to hold. On the other hand, to *prove* something to be correct, it is not enough to just find one example in which the claim holds.

(a) (a) X × (Y ∩ Z) = (X × Y) ∪ (X × Z)
Disproof: This statement is false. Let X = {1}, Y = {2}, and Z = {3}. Then X × (Y ∩ Z) = {1} × ∅ = ∅ But (X × Y) ∪ (X × Z) = {(1,2)} ∪ {(1,3)} = {(1,2), (1,3)}
Therefore, X × (Y ∩ Z) ≠ (X × Y) ∪ (X × Z)

(b) True

$(X \times Y) \cap (X \times Z) = \{(x, y) | \ x \in X \textbf{ AND } y \in Y\} \cap \{(x, z) | \ x \in X \textbf{ AND } z \in Z\} = \{(x, t) | \ x \in X \textbf{ AND } (t \in Y \textbf{ AND } t \in Z)\} = X \times (Y \cap Z)$

(c) c $X \times (Y \cup Z) = (X \times Y) \cup (X \times Z)$ TRUE

1.1) Let's take $(x,w) \in X \times (Y \cup Z)$

1.2) This means $w \in Y$ or $w \in Z$ (or both)

1.3) If $w \in Y$, then $(x,w) \in (X \times Y)$, and therefore $(x,w) \in (X \times Y) \cup (X \times Z)$

1.4) If $w \in Z$, then $(x,w) \in (X \times Z)$, and therefore $(x,w) \in (X \times Y) \cup (X \times Z)$

$\rightarrow X \times (Y \cup Z) \subseteq (X \times Y) \cup (X \times Z)$

2.1) Let's take $(x,w) \in (X \times Y) \cup (X \times Z)$

2.2) This means $(x,w) \in (X \times Y)$ or $(X \times Z)$ (or both)

2.3) If $(x,w) \in (X \times Y)$, $x \in X$ and $w \in Y$

2.4) Therefore $w \in (Y \cup Z)$ and $(x,w) \in X \times (Y \cup Z)$

2.5) If $(x,w) \in (X \times Z)$, $x \in X$ and $w \in Z$

2.6) Therefore $w \in (Y \cup Z)$ and $(x,w) \in X \times (Y \cup Z)$

$\rightarrow (X \times Y) \cup (X \times Z) \subseteq X \times (Y \cup Z)$

(d) (d) $X \times (Y \cup Z) = (X \times Y) \cap (X \times Z)$ FALSE
Let X = {1}, Y = {1}, Z = {2}.
Then LHS = {(1,1), (1,2)} while RHS = {(1,1)} ∩ {(1,2)} = ∅

(e) e) $(X \cap Y) \times Z = (X \times Z) \cap (Y \times Z)$ (TRUE)

Proof. Let $(a,b) \in (X \cap Y) \times Z$, thus $a \in X \cap Y$ and $b \in Z$. Thus $a \in X$ and $a \in Y$. Thus we certainly have $(a,b) \in (X, Z)$ and $a \in (Y, Z)$. Thus $(a,b) \in (X \times Z) \cap (Y \times Z)$ and $LHS \subseteq RHS$.

Now let $(a,b) \in (X \times Z) \cap (Y \times Z)$. Then $(a,b) \in X \times Z$ and $(a,b) \in Y \times Z$, thus $a \in X$ and $a \in Y$ and $b \in Z$. Thus $a \in X \cap Y$ and $b \in Z \Rightarrow (a,b) \in (X \cap Y) \times Z$. and $RHS \subseteq LHS$. Thus $(X \cap Y) \times Z = (X \times t) \cap (Y \times Z)$.

(f) *6.* $(X \cup Y) \times Z = (X \times Z) \cup (Y \times Z)$
Solution. $(X \cup Y) \times Z = \{(v,z) : v \in X \cup Y, z \in Z\}$
$= \{(x,z) : x \in X, z \in Z\} \cup \{(y,z) : y \in Y, z \in Z\}$
$= (X \times Z) \cup (Y \times Z)$
TRUE

Exercise 1.1

With sets A and B, the set $A \times B$ is simply the set $\{(a,b) : a \in A, b \in B\}$. If A and B are finite, then the number of elements of $A \times B$ is the product of the number of elements of A and the number of elements of B. If you are asked to calculate $A \times B$ like in this exercise, make sure to check that the number of elements is correct! Understanding the differences between sets and ordered pairs (tuples) is the key to solving these exercises.

(a) Because Y consists of integers, and Z consists of tuples, they have no common elements. So $Y \cap Z = \emptyset$, and hence $X \times (Y \cap Z) = X \times \emptyset = \emptyset$ (see Proposition 1.1 for why this holds).

(b) $Y \cup X = \{1, 2, 3, \{1,2\}, \{3\}\}$. Hence,

$$X \times (Y \cup X) = \{(\{1,2\},1), (\{3\},1), (\{1,2\},2), (\{3\},2), (\{1,2\},3), (\{3\},3) \\ (\{1,2\},\{1,2\}), (\{3\},\{1,2\}), (\{1,2\},\{3\}), (\{3\},\{3\})\} .$$

(c) $Y \cup Z = \{1, 2, 3, (1,2), (3,4)\}$. Hence,

$$X \times (Y \cup Z) = \{(\{1,2\},1), (\{3\},1), (\{1,2\},2), (\{3\},2), (\{1,2\},3), (\{3\},3) \\ (\{1,2\},(1,2)), (\{3\},(1,2)), (\{1,2\},(3,4)), (\{3\},(3,4))\} .$$

(d) For this exercise, simply reverse the order of the tuples in the above solution to obtain

$$(Y \cup Z) \times X = \{(1,\{1,2\}), (1,\{3\}), (2,\{1,2\}), (2,\{3\}), (3,\{1,2\}), (3,\{3\}) \\ ((1,2),\{1,2\}), ((1,2),\{3\}), ((3,4),\{1,2\}), ((3,4),\{3\})\} .$$

(e) X is a set containing sets, whereas Y is a set containing natural numbers. They therefore share no common elements (To know here: '3' is a number, but '$\{3\}$' is the set containing the number 3. Importantly, '$3 \neq \{3\}$'!). Hence $X \cap Y = \emptyset$ and by Proposition 1.1 again, $(X \cap Y) \times Z = \emptyset$.

(f) Since the operation '$\cup$' is symmetric, we have that $X \cup Y = Y \cup X$, so as in part (b) we have that $X \cup Y = \{1, 2, 3, \{1,2\}, \{3\}\}$. Hence,

$$(X \cup Y) \times Z = \{(1,(1,2)), (1,(3,4)), (2,(1,2)), (2,(3,4)), (3,(1,2)), (3,(3,4)) \\ (\{1,2\},(1,2)), (\{1,2\},(3,4)), (\{3\},(1,2)), (\{3\},(3,4))\} .$$

Exercise 1.2

Some details have been omitted from the following proof. As you will come to realise with more practice, when writing proofs it is sometimes hard to know whether a detail can be reasonably assumed. This intuition comes with time. In particular, we have omitted the proofs that π and τ are bijective, as well as the proof that domain restrictions of injective functions are injective, and the proof that whenever there is an injection $X \to \mathbb{N}$, we have that X is countable. Try to spend some time to convince yourself of these facts.

When given a countably infinite set X, the ability to enumerate its elements as $X = \{x_i : i \in \mathbb{N}\}$ as we do in this proof, can be performed by taking any bijection $f : X \to \mathbb{N}$, and for each $i \in \mathbb{N}$, defining $x_i = f^{-1}(i)$. In practice, the ability to enumerate in this way is a really handy trick to know!

Proof. **True -** if X and Y are countable, suppose first that they are countably infinite. Then we may enumerate their elements to write $X = \{x_i : i \in \mathbb{N}\}$, and $Y = \{y_i : i \in \mathbb{N}\}$.

Cantor's Pairing Function is the bijection $\pi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, defined by

$$(m, n) \mapsto \frac{(m+n)(m+n+1)}{2} + n \ .$$

Define a new function $\tau : X \times Y \to \mathbb{N}$ to have action $(x_i, y_j) \mapsto \pi(i, j)$ for every $i, j \in \mathbb{N}$. Then τ is a bijection.

If one (or both) of X and Y are finite, then enumerate X and Y as $X = \{x_i : i \in A\}$, $Y = \{y_i : i \in B\}$ for some subsets A and B of $\mathbb{N}$. Then the domain restriction of π to the set $A \times B$ defines an injection from $A \times B \to \mathbb{N}$, proving the claim. □

Overall Feedback

General Comments: Thanks to everyone who submitted. This was a challenging assignment so don't be disheartened if you made some mistakes. The more practice you get, the better you will become. Well done if one of your solutions was accepted as a model answer!

Areas of Improvement: There were some instances of wordy proofs which lack the precision of a true mathematical proof. Again, the purpose of writing mathematically is to remove all ambiguity in what you are trying to say. Try to go through your answers at the end of each assignment and ask yourself if they can be made more precise. Quite often, understanding the concepts is easier than conveying your ideas!

The most difficult question this week seemed to be Proposition 1.2. Try to go over this one at home. Don't be shy to ask us questions in the classroom too - although it is not currently possible for us to give personalised feedback to all of your work, we are always there to help.

Strengths: People were very strong on the exercises, particularly 1.0 and 1.1. The interactions between Cartesian products and set relations can be difficult to begin with so it is encouraging to see so few mistakes on these parts.

People were generally defining variables properly for every relevant question. This makes your work make sense mathematically, and it shows me that you are taking on board the feedback from the previous assignments!