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Lecture 2 — Friday, June 7.

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Recap of the basic notions covered in lecture 1:

(a) **Naive definition of a set**

(b) **Empty set**

(c) **Subset including proper subset**

(d) **Unions and intersection of sets**

(e) **Relative and absolute complements**

Questions:

(a) **Did anyone find it hard to grasp these basic concepts?**

(b) **Did anyone find it hard to solve the assignments?**

If the answer to the above is yes, then you should not proceed with the content on lecture 2 until you feel comfortable with the basics of lecture 1!

Definition 1.0

Let X and Y be sets. A map (or function) from X to Y written $f : X \longrightarrow Y$ is a prescription that associates an element of X with an element of Y.

- **Remark 1:**

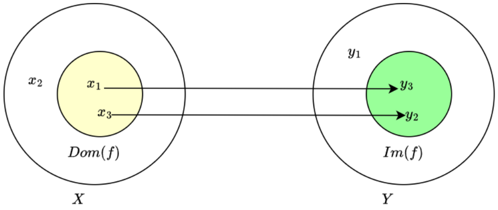
(a) If f is a map from X to Y, then we write $f : X \to Y$, or $X \xrightarrow{f} Y$. The element y of Y assigned by f to an element x of X is denoted by $f(x)$ and called the *image* of x under f.

(b) The set of all the elements of X that are covered under the map f is called the domain of f, and we normally denote it as $Dom(f)$.

(c) The image of f is defined as $Im(f) = \{f(x) \mid x \in Dom(f)\}$.

(d) Very often we write Im_f instead of $Im(f)$.

- **Abstract toy example:** Let $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3\}$. We can construct a map $f : X \longrightarrow X$ using the prescription; $f(x_1) = y_3$ and $f(x_3) = y_2$. This means that $Dom(f) = \{x_1, x_3\}$ and $Im(f) = \{y_3, y_2\}$ right?



Definition 1.1

Let X and Y be sets. A map $f : X \longrightarrow Y$ is called:

(a) **Surjective** (or onto) if $Im(f) = Y$ i.e. $\forall y \in Y$ there exists a $x \in X$ such that $y = f(x)$.

(b) **Injective** if $\forall x_1, x_2 \in Dom(f)$, $f(x_1) = f(x_2)$ if only if $x_1 = x_2$.

(c) **Bijective** if it's both surjective and injective.

Definition 1.2

We say X is (set)-isomorphic to Y and write $X \simeq Y$ if there is a map $f : X \longrightarrow Y$ such that $Dom(f) = X$ and f is a bijection i.e. f is both surjective and injective.

- **Remark 2:**

(a) Two (set)-isomorphic sets are considered to be the same. A very common alternative term used is equinumerous or equipotent instead of set-isomorphic.

(b) During the bootcamp we will encounter more interesting extensions of (set)-isomomorphims as we add structures to the underlying e.g, homeomorphisms of topological spaces and group isomomorphisms.

Proposition 1.0

Let X, Y, and Z be sets. If $X \simeq Y$ and $Y \simeq Z$, then $X \simeq Z$.

Definition 1.3

The **identity map** on a set X is the map written $id_X : X \to X : x \mapsto x$ i.e. $id_X(x) = x$ $\forall x \in X$.

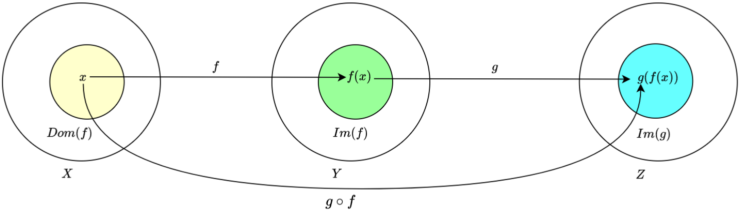
(1) When the set X is understood from the context, we just write id instead of id_X.

(2) Obviously if we consider another set Y, then it's identity is writen id_Y to distinguish it from id_X. The same applies if we consider a third set Z, where we would then write id_Z.

(3) It's clear that the identity map id_X is unique right?

Definition 1.4

Let X, Y and Z be sets. The *composition* of maps $f : X \to Y$ and $g : Y \to Z$ is the map written $g \circ f : X \to Z$ and defined as $(g \circ f)(x) = g(f(x))$.



Proposition 1.1

Given any maps $f : X \longrightarrow Y$, $g : Y \longrightarrow Z$ and $h : Z \longrightarrow U$. The following identities hold:

(1) $f \circ id_X = f$ and $id_Y \circ f = f$.

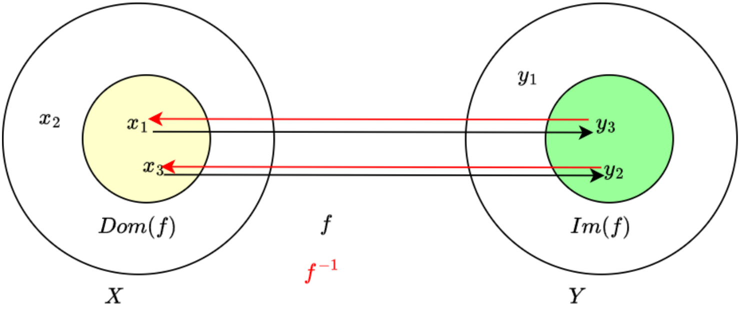
(2) $h \circ (g \circ f) = (h \circ g) \circ f$ (associativity).

Definition 1.5

Given a map $f : X \longrightarrow Y$, the inverse of f (if it exists) is a map $g : Y \longrightarrow X$ satisfying the following two conditions:

(1) $g \circ f = id_X$.
(2) $f \circ g = id_Y$.

- **Remark 3**: By convention we write f^{-1} to denote the inverse of f.



Proposition 1.2

A map $f : X \longrightarrow Y$ is invertible iff f is bijective and the inverse f^{-1} is unique.

Homework exercises (1)

Exercise 1.0

Consider the maps $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$. Prove or disprove the following:

(1) If f and g are injective, then $g \circ f$ is injective.

(2) If $g \circ f$ is injective, then g is injective.

(3) If $g \circ f$ is injective, then f is injective.

(4) If f and g are surjective, then $g \circ f$ is surjective.

(5) If $g \circ f$ is surjective, then g is surjective.

(6) If $g \circ f$ is surjective, then f is surjective.

(7) If f and g are bijective, then $g \circ f$ is bijective.

(8) If $g \circ f$ is bijective, then g is bijective.

(9) If $g \circ f$ is bijective, then f is bijective.

Exercise 1.1

Let $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9\}$ and $Y = \{y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9\}$.
Construct the following examples of maps:

(1) A map $f : X \longrightarrow Y$ that is injective but not surjective.

(2) A map $f : X \longrightarrow Y$ that is surjective but not injective.

(3) A map $f : X \longrightarrow Y$ that is bijective i.e that is both surjective and injective.

(4) A map $f : X \longrightarrow Y$ that is neither injective nor surjective.

Exercise 1.2

Let $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$ be the set of all integers and $\mathbb{N} = \{0, 1, 2, 3, 4, 5, \dots\}$ be the set of all natural numbers.

(1) Even though $\mathbb{N}$ is a proper subset of $\mathbb{Z}$, is it true that $\mathbb{N} \simeq \mathbb{Z}$? Meaning that there is a bijection $f : \mathbb{N} \longrightarrow \mathbb{Z}$! If true, can you construct such a bijection?

(2) Construct a map $f : \mathbb{N} \longrightarrow \mathbb{Z}$ that is injective but not surjective.

Thank you for listening!