LCU Example

Implement a circuit to model the evolution of the unitary e^{itX} by using a linear combination of unitaries (LCU) of the second-order approximation of its Taylor-series expansion:

$$e^{itX} = I + itX - \frac{1}{2!}t^2X^2 - \frac{i}{3!}t^3X^3 + \dots$$

$$\tilde{U} = I + itX - \frac{1}{2}t^2X^2 \leftarrow \text{Second-order approximation}$$

- The circuit below will probabilistically apply the unitary $U^0 + itU^1 - \frac{1}{2}t^2U^2$ up to a normalization factor.

- In the unitary V is a state-preparation gate that for a general input $|0\rangle^{\otimes n}$, generates the state $|x\rangle$:

$$|x\rangle = \sum_{k=0}^{N \leftarrow 2^n-1} x_k |k\rangle, \quad \text{where:} \quad x_k = \begin{cases} \sqrt{c_k/C}, & \text{for } k \leq M \\ 0, & \text{for } k > M \end{cases}$$

- Here, c_k are the magnitudes of the coefficients in the Taylor expansion approximation, given by: $c_k = \frac{t^k}{k!}$

  Note: The phase i^k must be "absorbed" into the unitaries U^k for this approach to work.

- M is the total number of coefficients c_k, which can be smaller or equal to the total number of basis states $N = 2^n - 1$.

- And C is a normalization factor given by the 1-norm of the coeffients c_k: $C = \sum_{j=0}^{M-1} |c_k|$

→ So, in this particular example:

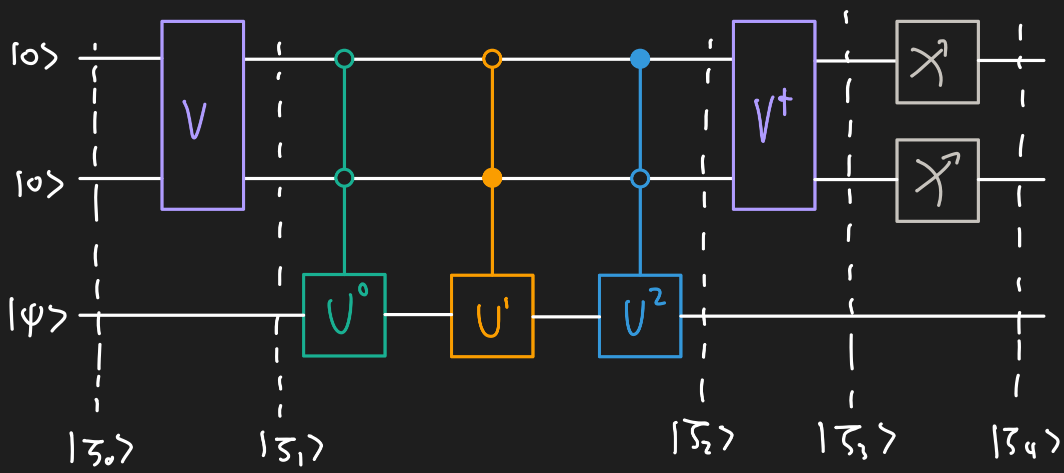
$$c_0 = 1, \quad c_1 = t, \quad c_2 = \frac{1}{2}t^2 \quad \text{with: } C = 1 + t + \frac{1}{2}t^2 \quad (t \geq 0)$$

$$x_0 = \frac{1}{\sqrt{1+t+\frac{1}{2}t^2}}, \quad x_1 = \frac{\sqrt{t}}{\sqrt{1+t+\frac{1}{2}t^2}}, \quad x_3 = \frac{t/\sqrt{2}}{\sqrt{1+t+\frac{1}{2}t^2}}, \quad x_4 = 0$$

→ The unitaries then are:

$$U^0 = I, \quad U^1 = iX, \quad U^2 = i^2X^2 = -I$$

Circuit implementation



- $|\zeta_0\rangle = |00\rangle \otimes |\psi\rangle$
- $|\zeta_1\rangle = \frac{1}{\sqrt{1+t+\frac{1}{2}t^2}}\left(1|00\rangle + \sqrt{t}\,|01\rangle + \frac{t}{\sqrt{2}}|10\rangle + 0\,|11\rangle\right)$
- $|\zeta_2\rangle = \frac{1}{\sqrt{1+t+\frac{1}{2}t^2}}\left(1\,|00\rangle U^0|\psi\rangle + \sqrt{t}\,|01\rangle U^1|\psi\rangle + \frac{t}{\sqrt{2}}|10\rangle U^2|\psi\rangle\right)$
- $|\zeta_3\rangle = \frac{1}{1+t+\frac{1}{2}t^2}\left[|00\rangle\left(U^0 + tU^1 + \frac{1}{2}t^2U^2\right)|\psi\rangle + \underbrace{\dots}_{\text{terms for } |01\rangle, |10\rangle}\right]$

- After measurement, we get state $|00\rangle$ with probability:

$$P(00) = \frac{\langle\psi|\tilde{U}^\dagger\tilde{U}|\psi\rangle}{(1+t+\frac{1}{2}t^2)^2}, \quad \text{where} \quad \tilde{U} = U^0 + tU^1 + \frac{1}{2}t^2U^2$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + ti\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \frac{1}{2}t^2 \cdot -1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-\frac{1}{2}t^2 & it \\ it & 1-\frac{1}{2}t^2 \end{bmatrix}$$

- For $|\psi\rangle = |0\rangle$: $P(00) = \dfrac{(1 - \frac{1}{2}t^2)^2 + t^2}{(1 + t + \frac{1}{2}t^2)^2} = \dfrac{1 + t^4/4}{(1 + t + \frac{1}{2}t^2)^2}$

→ So the normalized state post-measurement (after measuring $|00\rangle$):

- $|\zeta_4\rangle = \dfrac{1}{\sqrt{1 + t^4/4}} |00\rangle \otimes \left[(1 - t^2/2)|0\rangle + it|1\rangle \right]$

⊛ For the third-order approximation:

$$c_0 = 1,\ c_1 = t,\ c_2 = \frac{1}{2}t^2,\ c_3 = \frac{1}{6}t^3 \text{ with: } C = 1 + t + \frac{1}{2}t^2 + \frac{1}{6}t^3 \quad (t \geq 0)$$

$$x_0 = \frac{1}{\sqrt{1+t+\frac{1}{2}t^2+\frac{1}{6}t^3}},\quad x_1 = \frac{\sqrt{t}}{\sqrt{1+t+\frac{1}{2}t^2+\frac{1}{6}t^3}},\quad x_3 = \frac{t/\sqrt{2}}{\sqrt{1+t+\frac{1}{2}t^2+\frac{1}{6}t^3}},\quad x_4 = \frac{\sqrt{t^3/6}}{\sqrt{1+t+\frac{1}{2}t^2+\frac{1}{6}t^3}}$$

$$U^0 = I,\quad U^1 = iX,\quad U^2 = i^2X^2 = -I,\quad U^3 = i^3X^3 = -iX$$

$$\tilde{U} = U^0 + tU^1 + \frac{1}{2}t^2U^2 + \frac{1}{6}t^3U^3$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + ti\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \frac{1}{2}t^2 \cdot -1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{6}t^3 \cdot -i \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1-\frac{1}{2}t^2 & i(t-t^3/6) \\ i(t-t^3/6) & 1-\frac{1}{2}t^2 \end{bmatrix}$$

- Prob of measuring $|00\rangle$ for $|\psi\rangle = |0\rangle$: $$\frac{(1-\frac{1}{2}t^2)^2 + (t - t^3/6)^2}{(1+t+\frac{1}{2}t^2+\frac{1}{6}t^3)^2} = \frac{\frac{1}{36}t^6 - \frac{1}{12}t^4 + 1}{(1+t+\frac{1}{2}t^2+\frac{1}{6}t^3)^2}$$

- $$|\zeta_4\rangle = \frac{1}{\sqrt{\frac{1}{36}t^6 - \frac{1}{12}t^4 + 1}}\, |00\rangle \otimes \left[(1 - t^2/2)|0\rangle + i(t - t^3/6)|1\rangle \right]$$