Asset pricing chapters of Cochrane (2005)

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This paper reviews 5 chapters in Asset Pricing book of Cochrane (2005). For brevity, we skip all complicated proofs and provide the underlying theories only. All proofs are assumed to be solved by readers or we could provide it in appendix later.

The first chapter provides a very brief overview of the consumption-based model: basic model, discount factor, and some classical issues in finance (risk free, risk correction, equity premium).

The second chapter goes to details of the stochastic discount factor. Two theorems in this chapter is Law of one price and No arbitrage. Both are important for SDF existence.

The third chapter goes to details of mean-variance frontier, which is the big foundation of the CAPM and factor models.

Chapter 1

Comsumption-based model overview

1.1 *Basic pricing equation*

The basic comsumption-based model is:

$$p_t = E_t\left[\beta \frac{u'(c_{t+1})}{u'(c_t)} x_{t+1}\right] \tag{1.1}$$

Explain the model: In this equation, $x_{t+1}$is the *payoffs* in next period, which is $x_{t+1} = p_{t+1} + d_{t+1}$. Investor has an *utility* function, which captures what investor wants. Utility function often has the power utility form:

$$u(c_t) = \frac{1}{1-\gamma} c_t^{1-\gamma}$$

From this, we have its first derivative $u'(c_t) = c_t^{-\gamma} > 0$ so increasing utility function (i.e., more consumption is more desirable); and second derivative $u''(c_t) = -\gamma c_t^{-\gamma-1} < 0$ so the utility function is concave (i.e., investor is risk averse and declining marginal value of additional consumption). The curvature of the utility function generates the risk aversion and intertemporal substitution: the investor prefers a stable/steady consumption stream over time and across states of nature.

In addition, β is *subjective disount factor*.

Problem of investor Denote e_t as original consumption level (or original wealth) and ξ is amount of the asset he choose to buy at price p_t. His problem is:

$$\underset{\{\xi\}}{max}\, u(c_t) + E_t[\beta u(c_{t+1})] \quad s.t.$$

$$c_t = e_t - p_t \xi,$$

$$c_{t+1} = e_{t+1} + x_{t+1}\xi.$$

Solve the first-order condition, we could get

$$p_t u'(c_t) = E_t[\beta u'(c_{t+1}) x_{t+1}] \tag{1.2}$$

then the Equation 1.1 above. Investors buy or sell until they could get the standard marginal condition in Equation 1.2, where the left side is the loss in utility if investors buy one more unit of asset now and the right side is the *discounted* utility he optains from an extra payoff (from asset) at $t+1$. So, the optimum is when marginal loss equals marginal gain.

1.2 Stochastic discount factor

Rewrite basic pricing equation:

$$p_t = E(m_{t+1} x_{t+1})$$

$$m_{t+1} = \beta \frac{u'(c_{t+1})}{u'(c_t)}$$

where m_{t+1} is *stochastic discount factor* (SDF). Sometimes, it is also called *marginal rate of substitution*, meaning that rate at which the investor is willing to substitute consumption at time $t+1$ for consumption at time t. Further, it is also called *pricing kernel*.

If no uncertainty, we can write: $p_t = \frac{1}{R^f} x_{t+1}$ where R^f is gross risk-free rate. For a risky asset *i*, it should be discounted using an asset-specific risk-adjusted discount factor $1/R^i$ so $p_t^i = \frac{1}{R^i} E(x_{t+1}^i)$.

1.3 Further notation

The *gross return*: $R_{t+1} = \frac{x_{t+1}}{p_t}$ so

$$1 = E[m_{t+1} R_{t+1}] \tag{1.3}$$

Returns are used much more often in empirical work, rather than price, because it is typically *stationary* over time.

Other types of returns such as:

- *net return* $r_{t+1} = R_{t+1} - 1$ such that if gross return is 1.05 then net return is 5%.
- *excess return* R^e such as $R^e = R - R^f$, which in case *zero-cost portfolio* or long-short assets.

1.4 Classical issues in Finance

This section use basic pricing equation to introduce some important classical topics/issues in Finance such as interest rate, risk adjustement, systematic versus idiosyncratic risk, return-beta, mean-variance frontier, time-varying expected return, present value relations.

Risk-free rate The risk free rate and SDF relation:

$$R^f = 1/E(m) \tag{1.4}$$

with lognormal consumption growth:

$$r_t^f = \delta + \gamma E_t(\triangle ln\, c_{t+1}) - \frac{\gamma^2}{2}\sigma_t^2(\triangle ln\, c_{t+1}) \tag{1.5}$$

How to prove Equation 1.4

Because we have $p = E(mx)$ so for risk-free asset (such as Treasury) we assume price is $p = 1$ and the payoff is R^f. Thus, $1 = E(mR^f) = E(m)R^f$ because R^f is constant.

Apply to the case with utility function is $u'(c) = c^{-\gamma}$ so we have $R^f = \frac{1}{\beta}\left(\frac{c_{t+1}}{c_t}\right)^{\gamma}$. This result has some intuitions:

1. Real interest rate is high when people are impatient (they want to consume now) or when β is small. So they need higher interest rates to convince them to save.
2. The real interest rate is high when consumption growth is high. Because the interest rate is high, people save more and consume less now. As a result, they will get more payoff and consume more in the future, or consumption growht will be high.
3. The real interest rate is more sensitive to consumption growth when power parameter γ is large. If the utility is highly curved, investors care more about maintaining a smooth consumption over time and less willing to rearrange consumption over time in response to interest rate incentives. Thus, he requires a larger change in interest rate to induce him to given consumption growth.

Cochrane proposes a case that if log risk-free is r_t^f; subjective discount rate δ is by $\beta = e^{-\delta}$; and consumption growth is lognormally distributed $\triangle ln\, c_{t+1} = ln\, c_{t+1} - ln\, c_t$. For normal z variable, we can derive: $E(e^z) = e^{E(z)+1/2\sigma^2(z)}$.

From the function: $R^f = 1/E_t\left[\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}\right]$ of utility function $u'(c) = c^{-\gamma}$, he derives Equation 1.5. We need an appendix to derive this result.

Intuitions of Equation 1.5:

1. Real interest rate is high when impatience δ is high, when consumption growth is high, higher γ makes interest rate is more sensitive to consumption growth.

2. New item σ^2 captures *precautionary savings*: when consumption is more volatile, people want to save more because they are worried about low consumption states. Thus, interest rate decreases.

3. For power utility function, curvature parameter γ controls: (i) *intertemporal substitution* (aversion to consumption varies over time), (ii) *risk aversion* (aversion to consumption varies across states of natures), and (iii) *precautionary savings* (which depend on third derivative of utility function ???).

Risk corrections Idea: Payoffs that are positively correlated with consumption growth have lower prices (or high returns) to compensate investors for risk:

$$p = \frac{E(x)}{R^f} + cov(m, x) = \frac{E(x)}{R^f} + \frac{cov\left[\beta u'(c_{t+1}), x_{t+1}\right]}{u'(c_t)} \tag{1.6}$$

$$E(R^i) - R^f = -R^f cov(m, R^i) = -\frac{cov\left[u'(c_{t+1}), R^i_{t+1}\right]}{E\left[u'(c_{t+1})\right]} \tag{1.7}$$

Marginal utility $u'(c)$ declines as c rises. Asset price is lowered if its payoff covaries positively with consumption (or negatively with marginal utility $u'(c)$). In contrast, asset price is higher if it covaries negatively with consumption.

Example: if asset pays you more money when consumption is low (when you are poor now or when consumption growth is high), it is a hedge for you such as insurance because it smooths your consumption. Thus, you are happy to pay more money to buy them (p will be higher) and return of this asset is lower ($E(R^i)$ is lower).

Idiosyncratic risk does not affect prices We can decompose payoff x into a part correlated with discount factor and an idiosyncratic part ε as: $x = proj(x|m) + \varepsilon$. where $proj(x|m) = \frac{E(mx)}{E(m^2)}m$ (fitted value of regress x on m).

Proof: that price of $proj(x|m)$ equals price of x

$p(proj(x|m)) = p\left(\frac{E(mx)}{E(m^2)}m\right) = E\left(m^2\frac{E(mx)}{E(m^2)}\right) = E(mx) = p(x)$

Expected return-beta We can write $p = E(mx)$ as: $E(R^i) = R^f + \beta_{i,m}\lambda_{i,m}$

where $\beta_{i,m}$is coefficient of regress return R^i on m, so beta varies for each asset; the $\lambda_{i,m}$ is risk factor (i.e., *price of risk*) so it is the same for all asset.

In consumption-based model, $m = \beta(c_{t+1}/c_t)^{-\gamma}$ so we can rewrite:

$E(R^i) = R^f + \beta_{i,\triangle c}\lambda_{i,\triangle c}$ with $\lambda_{i,\triangle c} = \gamma var(\triangle c)$.

Easily to see, the price of risk depends on the volatility of the discount factor (is consumption growth here) and risk aversion. The more risk averse investors and a riskier environment, the higher expected return premium one must pay to persuade investors to hold risky (high beta) assets.

Mean-Variance frontier Some intuitions are:

1. All asset returns must lie in the mean-variance region, where the boundary lines are *mean-variance frontier*.

2. In the frontiers, returns are perfectly correlated with the discount factor or $|\rho_{m,R^i}| = 1$.

   (a) In the upper frontier, returns are perfectly negatively correlated with discount factor ($\triangle c$) or positively correlated with consumption (C_t). They are maximally risky (pay less income when consumption is lower) so it has higher expected returns.

   (b) In the lower frontier, returns are perfectly positively correlated with discount factor ($\triangle c$) or negatively correlated with consumption (C_t). It provides best insurance againts consumption fluctuation so it has lower returns.

3. Any two frontier returns are perfectly correlated, so we can *span* or *synthesize* any frontier return from two such returns. For example, if we have a frontier return is R^m then for any frontier return R^{mv}, we can write: $R^{mv} = R^f + a(R^m - R^f)$ for some number a (yeah, it looks like a CAPM model).

4. Any mean-variance frontier return carries all pricing information. Given a mean-variance frontier return and risk-free rate, we can find a discount factor m to price all assets and vice versa.

5. Given a discount factor, we can construct a single-beta representation, so expected returns can be described in a single-beta representation using any mean-variance efficient return (except for risk free rate):

$$E(R^i) = R^f + \beta_{i,mv}[E(R^{mv}) - R^f]$$

The beta model can be applied to any asset, including the R^{mv} with beta equals to one. We can identify the risk premium factor as $\lambda = E[R^{mv} - R^f]$.

Equity premium Equity premium is represented by the Sharpe ratio limited by the volatility of the discount factor:

$$\left|\frac{E(R^i) - R^f}{\sigma(R^i)}\right| \leq \frac{\sigma(m)}{E(m)} \approx \gamma\sigma(\triangle c)$$

The maximal risk-return trade-off is steeper if (i) investors are more risk-averse or (ii) more risk (consumption is more volatile). Both situations make investors reluctant to invest in risky assets.

In 50 years of post-war in the US, the returns of equity is 9% on average and standard deviation is 16%, while the real return of risk-free (T-bill rate) is 1% so Sharpe ratio is around 0.5. Aggregate non-durable and service consumption growth has mean and standard deviation of 1%. We can only reconcile this fact if investors have risk-aversion coefficient of 50! But the Sharpe ratio equals 0.5 is in frontier, but in real data, the correlation between aggregate consumption and market return is only 0.2 rather than perfect correlation as in the frontier. If we add this fact, we need a risk aversion coefficient of 250.

Chapter 2

Discount factor

2.1 First theorem: Law of one price and Existence of Discount factor

Law of one price indicates that if two portfolios have the same payoffs in every states of nature, then they must have the same price. But this section does not assume the complete market but just consider a set of payoffs x or a *payoff space* $\underline{X}$. Thus, we can think of $\underline{X}$ as a proper subset of complete markets R^s, meaning that the market is *incomplete*.

- The payoff space includes primitive assets and its combination or portfolios of primitive assets.
- The return space is a subset of the payoff space.

Law of one price (linearity)

$$p(ax_1 + bx_2) = ap(x_1) + bp(x_2) \tag{2.1}$$

Investors cannot make instantaneous profits by repacking portfolios. If there is a violation of law of one price, then traders will quickly eliminate them so they cannot survive in equilibrium.

Theorem 1 Given free portfolio formation and the law of one price, there exists a unique payoff $x^* \in \underline{X}$ such that $p(x) = E(x^* x)$ for all $x \in \underline{X}$.

Remember that here the discount factor is also a payoff in the payoff space. See appendix of this chapter for the proof.

Intuitions

1. If the market is incomplete, there are an infinite number of discount factor to price asset. Any discount factor m (that satisfies $p = E(mx)$) can be represented as $m = x^* + \varepsilon$ with $E(\varepsilon x) = 0$. In contrast, if the market is complete, then x^* is the only possible discount factor.
2. Algebraically, $p = E(mx) = E\left[(proj(m|\underline{X}) + \varepsilon)\, x\right] = E\left[proj(m|\underline{X})x\right] = E(x^* x)$. Thus, x^* is the projection of any SDF m into the payoff space.

2.2 Second theorem: No arbitrage and Positive Discount Factor

Definition of arbitrage: positive payoff ($x > 0$) implies positive price ($p(x) > 0$). Implication of this definition is you cannot get a free portfolio with positive payoff but never cost you anything.

Theorem 2 $p = E(mx)$ and $m(s) > 0$ imply no arbitrage.

Prove it easily: we have $m > 0$, $x \geq 0$ and some states that $x > 0$. In some states with positive probability $mx > 0$ and in other states $mx = 0$. Therefore, $E(mx) > 0$.

Theorem 3 In complete markets, no arbitrage implies that there exists a unique $m > 0$ such that $p = E(mx)$.

See proof in textbook.

Theorem 4 No arbitrage implies the existence of a strictly positive discount factor, $m > 0$, $p = E(mx) \forall x \in \underline{X}$.

Chapter 3

Mean-Variance Frontier and Beta Representations

3.1 Expected return-Beta Representations

The expected return-Beta expression of a factor pricing model is (time-series coefficient):

$$E(R^i) = \gamma + \beta_{i,a}\lambda_a + \beta_{i,b}\lambda_b + ... \tag{3.1}$$

where $i = 1, 2, ..., N$ asset, λ_a could be understood as the price of risk factor a (it is the slope in the figure of $E(R^i)$ and $\beta_{i,a}$ space.

When the factors are excess returns then $\lambda_a = E(f^a)$ (because each factor has beta of one on itself: $E(f^a) = \beta_{i,a}\lambda_a = 1 \times \lambda_a = \lambda_a$). If the test assets are also excess returns ($R^{ei} = R^i - R^j$, where R^j is often risk-free), then the intercept should be zero. Some famous factors are: market return (in CAPM) or consumption growth (CCAPM). Assets with high beta are more risky (i.e., higher exposure to risk factors), so have higher expected returns.

The main idea of this model is if assets' payoff is high in good times and low in bad times (as measured by the factors), the price is low (higher returns). For example, beta in market model (or CAPM) means that if a stock pays more payoff in times of high market returns and pays low in poor market returns (such as crisis), this stock is risky so its price should be low and its expected return should be high enough for investors to hold.

3.2 Mean-Variance Frontier

Mean-variance frontier is the boundary that minimize return variance for a given mean return. In the figure, the mean-variance of all risky assets is the hyperbola, while that of risky assets and a risk-free asset is the larger wedge-shaped region. The risk-free rate is below the minimum variance of the risky frontier, to ensure that investors do not short the risky assets (if risk-free rate is higher).

When does the mean-variance frontier exists?

Theorem: So long as the variance-covariance matrix of return ($\Sigma = E\left[(R-E)(R-E)'\right]$) is nonsingular (i.e., could inverse), there is a mean-variance frontier.

Problem: A vector of asset return R, mean return vector $E = E(R)$, and variance-covariance matrix $\Sigma = E\left[(R-E)(R-E)'\right]$. The weights of portfolio are w, so portfolio return is $w'R$ and weights sum is $w'1 = 1$. The problem is to choose a portfolio to minimize variance for a given mean.

$$\underset{\{w\}}{min}\, w'\Sigma w \; s.t. \, w'E = \mu; \; w'1 = 1 \tag{3.2}$$

Solution Let $A = E'\Sigma^{-1}E$; $B = E'\Sigma^{-1}1$; and $C = 1'\Sigma^{-1}1$, then
$var(R^{mvp}) = \frac{C\mu^2 - 2B\mu + A}{AC - B^2}$ and $w = \Sigma^{-1}\frac{E(C\mu - B) + 1(A - B\mu)}{(AC - B^2)}$

The proofs of this solution could be found in text book and in another appendix that we prepared in Capital Market class.

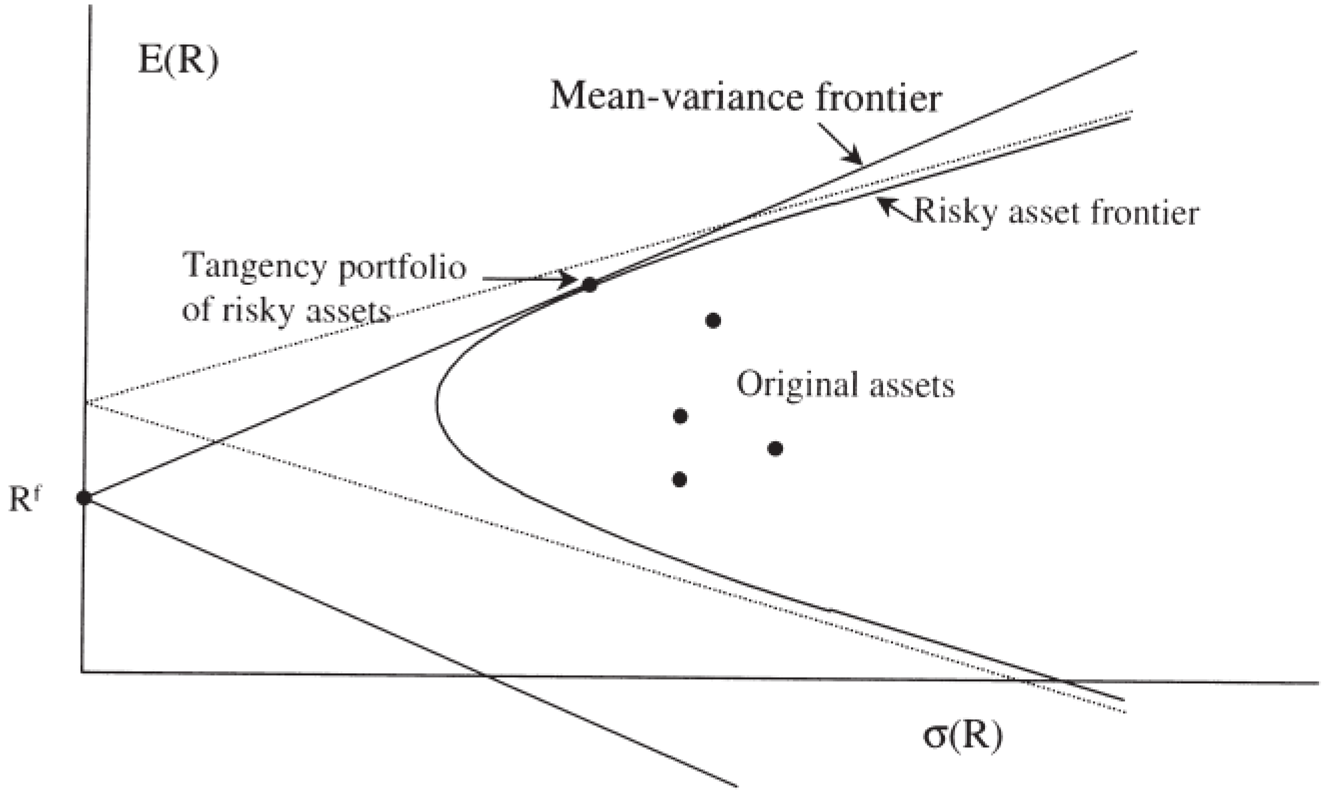

Figure 3.1: Mean-Variance frontier

Minimum variance portfolio (*mvp*) This portfolio is interesting. Its mean portfolio return is $\mu^{mvp} = B/C$. The weight is $w^{mvp} = \Sigma^{-1}1/(1'\Sigma^{-1}1)$.

The whole frontier is spanned by any two frontier returns. For example, if we have two distinct frontier returns μ_1 and μ_2 then the weight on a third portfolio with mean $\mu_3 = \lambda\mu_1 + (1-\lambda)\mu_2$ are given by $w_3 = \lambda w_1 + (1-\lambda)w_2$.

3.3 An orthogonal characterization of Mean-Variance frontier

The above approach is classic and cumbersome. This section follows Hansen and Richard (1987) to derive a more attractive frontier. The idea is we first describle any return by a 3D orthogonal decomposition. Then we derive the mean-variance frontier without any algebra.

Definitions of R^* and R^{e*}

First, R^* is return of any asset with payoff x^* that can act as a discount factor (e.g., consumption growth or market return). Price of x^* is $p(x^*) = E(x^*x^*)$. Then, $R^* = \frac{x^*}{p(x^*)} = \frac{x^*}{E(x^{*2})}$.

And $R^{e*} = proj(1|\underline{R}^e)$ where $\underline{R}^e$ is space of excess returns.

The decomposition of R^i is illustrated as follows:

Theorem: Every return R^i could be expressed as $R^i = R^* + w^i R^{e*} + n^i$

Theorem: R^{mv} is on the mean-variance frontier if and only if $R^{mv} = R^* + w^i R^{e*}$ (or $n^i = 0$)

In a 2D figure, we can re-draw the orthogonal decomposition into the mean-standard deviation space as Figure 3.4. R^* is the return closest to the origin.

We find that:

$$
\begin{aligned}
E(R^i) &= E(R^*) + w^i E(R^{e*}), \\
\sigma^2(R^i) &= \sigma^2(R^* + w^i R^{e*}) + \sigma^2(n^i)
\end{aligned}
$$

Return with $n^i = 0$ minimize variance for each mean. We can understand the part n as the idiosyncratic return that just moves and asset off the frontier (but does not priced as higher return, although more risky).

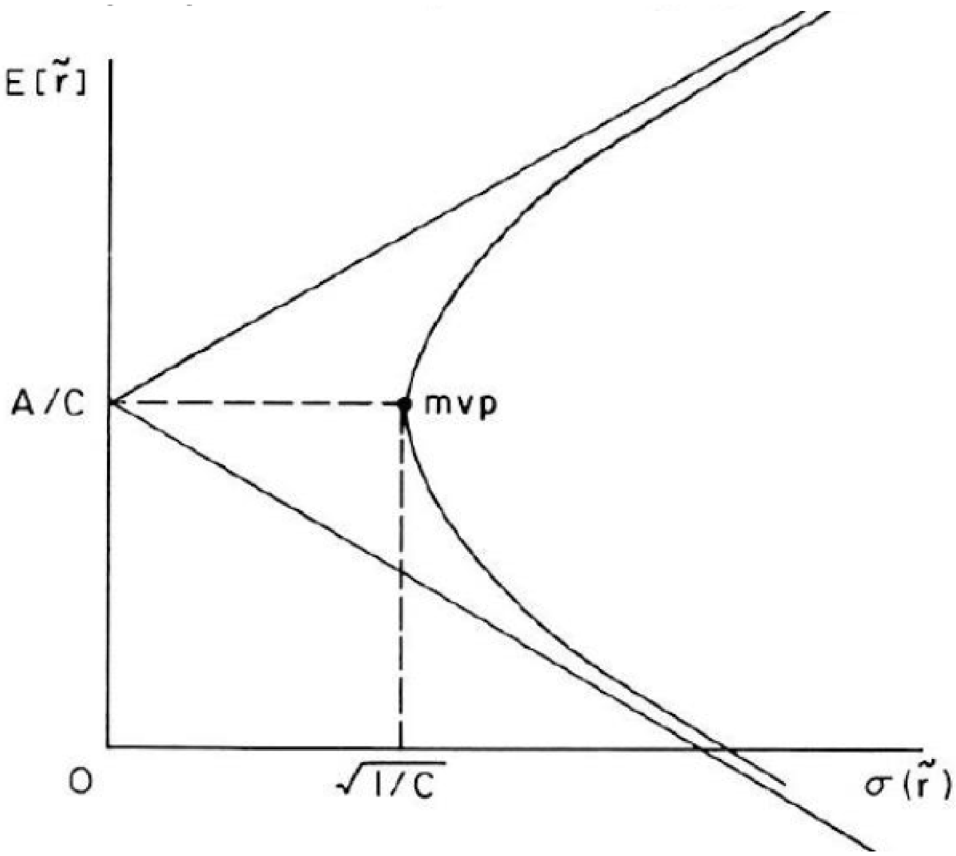

Figure 3.2: Minimum variance portfolio

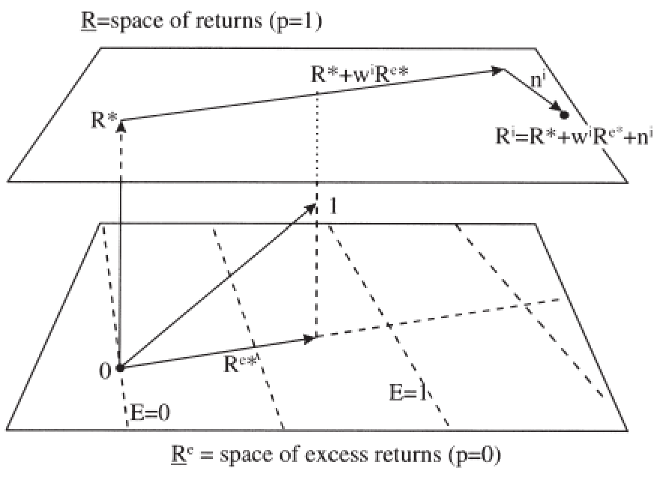

Figure 3.3: Orthogonal decomposition and mean-variance frontier

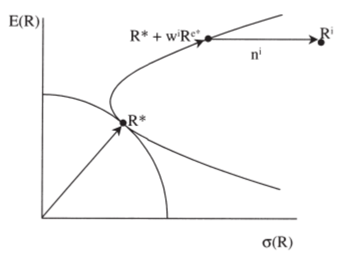

Figure 3.4: Orthogonal decomposition in mean-SD space

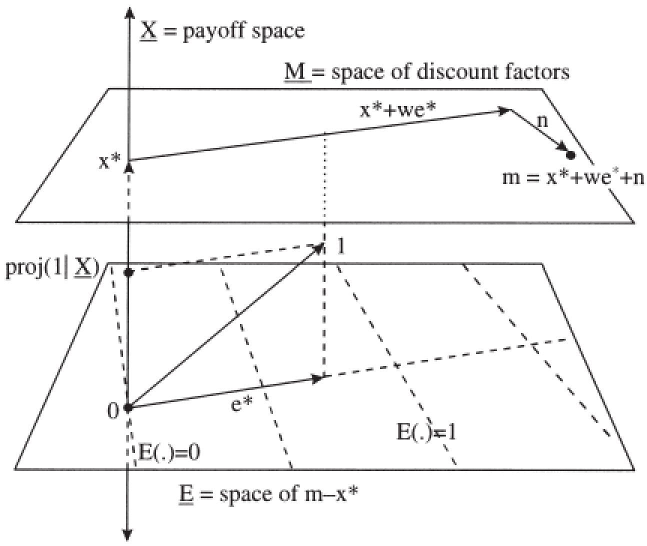

Figure 3.5: Decomposition of discount factor *m*

3.4 Hansen-Jagannathan bounds

The Hansen-Jagannathan bound:

$$\frac{\sigma(m)}{E(m)} \geq \frac{|E(R^e)|}{\sigma(R^e)}$$

The righ-hand side is the slope of tangency line to the frontier. When $E(m)$ increases, the slope is lower and the bound decreases (which attains its lowest at *mvp*).

This gives a duality problem:

$$\min_{\{m\}} \frac{\sigma(m)}{E(m)} = \max_{\{R^e\}} \frac{E(R^e)}{\sigma(R^e)}$$

For discount factors, analogously as mean-variance frontier of assets, we can derive:

$$\begin{aligned} m &= x^* + we^* \\ e^* &= 1 - proj(1|\underline{X}) = proj(1|\underline{E}) = 1 - E(x)'E(xx')^{-1}x \\ \underline{E} &= \{m - x^*) \end{aligned}$$

Chapter 4

Discount factors, Betas, and Mean-Variance Frontiers

Bibliography