Asset Pricing notes - John Cochrane

[CH1]

- Investor's Utility Function

$$u(c_t, c_{t+1}) = u(c_t) + \beta E[u(c_{t+1})]$$

β → subjective discount rate → captures investor's impatience

→ often use power utility : $u(c_t) = \frac{1}{1-\gamma} c^{1-\gamma}$ (γ → gamma)

when $\gamma \to 1$: $u(c_t) = \ln c_t$

- Investor's Objective

$$\max_{\xi} \; u(c_t) + \beta E[u(c_{t+1})]$$

ξ → x_i [ZZAI] [KSAI]

s.t. $c_t = e_t - p_t \xi$

p_t → price at t

ξ → the amount of the asset he choose to buy

$c_{t+1} = e_{t+1} + x_{t+1} \xi$

x_{t+1} → payoff at $t+1$

e : original consumption level

- FOC.

$$p_t u'(c_t) = E_t[\beta u'(c_{t+1}) x_{t+1}] \tag{1.1}$$

$$p_t = E_t\left[\frac{\beta u'(c_{t+1})}{u'(c_t)} x_{t+1}\right] \tag{1.2}$$

- Stochastic discount factor

define $m_t = \beta \frac{u'(c_{t+1})}{u'(c_t)}$ as SDF (1.3)

→ is stochastic because it's not known at t

→ other names : marginal rate of substitution
pricing kernel
change of measure
state-price density

plug (1.3) into (1.2)

$$p_t = E_t[m_{t+1} x_{t+1}] \tag{1.4}$$

$$m_{t+1} = f(\text{data}, \text{parameter})$$

- Price, Payoff and Notation

	price $= p_x$	Payoff $= x_{t+1}$
stock	p_t	$p_{t+1} + d_{t+1}$
Return	1	R_{t+1}
Price-dividend ratio	p_t/d_t	$[(p_{t+1}/d_{t+1})+1]\, d_{t+1}/d_t$
Risk-free return	1	R_f
Excess return	0	$R^a_{t+1} - R^b_{t+1}$
Zero-coupon bond	p_t	1
Option	C	$\max(S_T - K, 0)$

- Riske-free Interest Rate

from above table, $x_{t+1} = p_{t+1} + d_{t+1}$

devide p_t on the both side, we can therefore get,

$$\frac{x_{t+1}}{p_t} = \frac{p_{t+1}+d_{t+1}}{p_t} \equiv R_{t+1}$$

we can think return as a payoff with price one.

↳ applying $p=1 \Rightarrow x=R$; recall $p=E(mx)$

we get $1 = E(mR)$

↳ applying risk-free interest rate

impatience $= \beta$

consumption growth $= \frac{C_{t+1}}{C_t}$

power parameter $= \gamma$

$$R^f = \frac{1}{E(m)}$$

$$E(m) = \beta \frac{u'(C_{t+1})}{u'(C_t)} = \beta \left(\frac{C_t}{C_{t+1}}\right)^{\gamma}$$

in power utility

$u(C_t) = \frac{C^{1-\gamma}}{1-\gamma}$

$u' = C^{-\gamma}$

° suppose c growth is lognormally distributed i.e. $\ln\left(\frac{C_{t+1}}{C_t}\right) \sim N(\mu, \sigma^2)$

using the fact that normal X means

$$E(e^X) = e^{E(X) + 0.5\sigma^2}$$

We have $R^f = \frac{1}{\beta}\left[E_t\left(e^{-\gamma \ln \frac{C_{t+1}}{C_t}}\right)\right]^{-1} = \left[e^{-\gamma E_t(\Delta \ln C_{t+1}) + 0.5\gamma^2\sigma^2(\Delta \ln C_{t+1})}\right]^{-1}$

$$r^f = \delta + \gamma E_t(\Delta \ln C_{t+1}) - 0.5\gamma^2\sigma^2(\Delta \ln C_{t+1})$$

where $r^f = \ln R^f$, $\beta = e^{-\delta}$, $\Delta \ln C_{t+1} = \ln C_{t+1} - \ln C_t$

c.f. textbook pp.11-12

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[CH1]

- Risk corrections

$$p = E(mx) = E(m)E(x) + cov(m,x)$$

applying $R^f = \frac{1}{E(m)}$

$$p = \frac{E(x)}{R^f} + cov(m,x)$$

applying $p=1$
x以R^i代入

$$E(R^i) - R^f = -R^f cov(m,R^i)$$

- Idiosyncratic Risk does not affect price

$$x = proj(x|m) + \varepsilon$$

projection = projection means "linear regression without a constant"

$$proj(x|m) = \frac{E(mx)}{E(m^2)} m$$

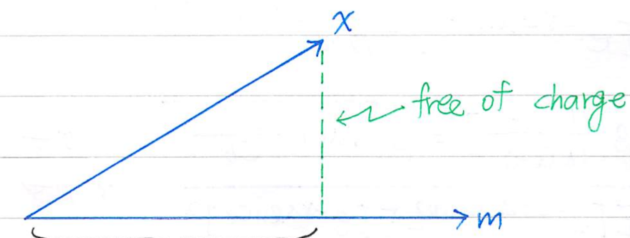
plug into

$$E(m\varepsilon) = 0$$

therefore, $p\left(\frac{E(mx)}{E(m^2)} m\right) = E\left(m^2 \frac{E(mx)}{E(m^2)}\right)$

$$= E(m^2)\frac{E(mx)}{E(m^2)} = E(mx) = p(x)$$

in chart



$$p(proj(x|m)) = p\left(\frac{E(mx)}{E(m^2)} m\right) = p(x)$$

- Expected Return-Beta Representation

think about "factor models"

$$E(R^i) = R^f + \beta_{i,m}\lambda_m$$

- FF (1993) 3 factor
- FF (2015) 5 factor
- Carhart (1997) 6 factor
- JW (1996) conditional CAPM
- AP (2005) liquidity-adjusted CAPM
- Belo (2010)、LWZ (2009) : Production-base CAPM

proof: $E(mR^i) = E(m)E(R^i) + cov(m,R^i) = 1$ (1.11)

2边 同乘以R^f

$$E(R^i) = R^f - R^f Cov(m,R^i)$$

$$= R^f + \underbrace{\frac{Cov(m,R^i)}{Var(m)}}_{\beta_{i,m}}\underbrace{\left(-\frac{Var(m)}{E(m)}\right)}_{\lambda_m}$$

$R_f = \frac{1}{E(m)}$ 代入

$\beta_{i,m}$: the quantity of risk in each asset and varies from asset to asset

λ_m: price of risk and is the same for all assets

[注意:有含R^i ⇒ $\beta_{i,m}$]

[CH1]

• Mean-Variance Frontier

Recall that: $\rho_{im} = \dfrac{cov(m, R^i)}{\sigma(R^i)\sigma(m)}$ plug into (1.11) (see previous page)

$$1 = E(mR^i) = E(m)E(R^i) + \rho_{i,m}\sigma(R^i)\sigma(m)$$

$$E(R^i) = R^f - \rho_{im}\frac{\sigma(m)}{E(m)}\sigma(R^i)$$

applying that $|\rho| \le 1$

$$|E(R^i) - R^f| \le \frac{\sigma(m)}{E(m)}\sigma(R^i)$$

$$\xrightarrow{\text{移项}} \left|\frac{E(R^i) - R^f}{\sigma(R^i)}\right| \le \frac{\sigma_m}{E(m)} \approx \gamma\sigma(\Delta \ln c)$$

↳ 這個格後証明

Sharp Ratio

風險趨避程度

消費成長率

• Equity Premium Puzzle

Assume: (1) Power utility: $u(C_t) = \frac{1}{1-\gamma} C_t^{1-\gamma}$

$u'(C_t) = C_t^{-\gamma}$

(2) $\ln \frac{C_{t+1}}{C_t} \sim N(\mu, \sigma^2)$

(a) $E(m) = E\left[\beta \frac{u'(C_{t+1})}{u'(C_t)}\right] = \beta E\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\right]$ [let $\Delta \ln C_t = \ln \frac{C_{t+1}}{C_t}$]

$= \beta E[e^{-\gamma \Delta \ln C_t}] = \beta e^{-\mu\gamma + \frac{1}{2}\sigma^2\gamma^2}$

$e^{-2\mu r} = e^{-\mu r} \cdot e^{-\mu r}$

(b) $\sigma(m) = \sigma\left[\beta \frac{u'(C_{t+1})}{u'(C_t)}\right] = \beta\sqrt{var\left(\frac{C_{t+1}}{C_t}\right)^{2}}$

$= \beta\sqrt{E[e^{-2\gamma\Delta \ln C_t}] - E[e^{-\gamma\Delta \ln C_t}]^2}$

$= \beta\sqrt{e^{-2\mu\gamma + 2\sigma^2\gamma^2} - e^{-2\mu\gamma + \sigma^2\gamma^2}}$

$\frac{分子}{分母} = \dfrac{e^{-\mu r}\sqrt{e^{2\sigma^2\gamma^2} - e^{\sigma^2\gamma^2}}}{e^{-\mu\gamma} \cdot e^{\sigma^2\gamma^2/2}}$

$= \sqrt{\dfrac{e^{\sigma^2\gamma^2} \cdot e^{\sigma^2\gamma^2} - e^{\sigma^2\gamma^2}}{e^{\sigma^2\gamma^2}}} = \sqrt{e^{\sigma^2\gamma^2} - 1}$

By (a) and (b): $\dfrac{\sqrt{e^{-2\mu\gamma + 2\sigma^2\gamma^2} - e^{-2\mu\gamma + \sigma^2\gamma^2}}}{e^{-\mu\gamma + \frac{1}{2}\sigma^2\gamma^2}} = \sqrt{e^{\sigma^2\gamma^2} - 1} \approx \gamma\sigma(\Delta \ln C)$

过去50年US资料顯示: real average stock return = 9% with S.D. = 16% and real return on treasury bills = 1%; the mean and S.D. of consumption growth = 1%

therefore, Sharp Ratio = 0.5 if $\rho_{m,R^{mv}} = 1$, then risk aversion = 50

but, actually $\rho_{m,R^{mv}} = 0.2$, then 〃 = 250 !!

⇒ equity premium puzzle why? (Cochrane): (1) investor可能真的很風險趨避

(2) equity premium其實沒那麼高, 过去太lucky

(3) model specification error (utility)

(4) other factors: macro variables, distored belief ... etc

如何修正

(i) habit formation

(ii) EZ utility ...

Asset Pricing notes - John Cochrane

[CH4]

- SDF and the Law of one price

▣ Portfolio Formation and Linearity

- if $x_1, x_2 \in X$, then for any real numbers, a and b, the linear combination is also a payoff.

$$ax_1 + bx_2 \in X$$

- It is a portfolio of 2 assets that is a linear combination of them. The pricing of a portfolio is also linear.

$$p(ax_1 + bx_2) = ap(x_1) + bp(x_2)$$

- It means that the price of a portfolio must have the same price of the linear combination of its components' prices.

- SDF ⟹ Law of One Price [algebraic proof]

[Theorem] The existence of a discount factor m implies the law of one price

[proof]

$$x \equiv ax_1 + bx_2$$

$$E(mx) = E(m(ax_1 + bx_2))$$

$$= aE(mx_1) + bE(mx_2)$$

applying $p(x) = E(mx)$, we can get

$$p(x) = ap(x_1) + bp(x_2)$$

[Q.E.D.]

▪ if two portfolios have the same payoff (in every state of nature), then they must have the same price.

- Law of One price ⟹ SDF [algebraic proof]

[proof] suppose the space X is spanned by N basic payoffs, then the payoff space X can be written as a vector. So does the price.

$$X \equiv \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} \qquad p \equiv \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_N \end{pmatrix}$$

Now, consider an N by N matrix,

$$xx' = \begin{pmatrix} x_1^2 & x_1x_2 & \cdots & x_1x_N \\ x_2x_1 & x_2^2 & \cdots & x_2x_N \\ \vdots & \vdots & \ddots & \vdots \\ x_Nx_1 & x_Nx_2 & \cdots & x_N^2 \end{pmatrix}$$

since it's a symmetric matrix, the inverse matrix $(x'x)^{-1}$ exists.

we want a discount factor that is in the payoff space, as the theorem requires, Thus, it must be the form,

$$\underset{(SDF)}{x^*} = c'x$$

constructing c so that x^* prices the basis assets, and we want

$$p = E(x^* x) = E(xx'c)$$

therefore, $c = E(xx')^{-1}p$ plug into x^*

$$x^* = p'E(xx')^{-1}x$$

$$\begin{aligned} E(x^*(x'c)) &= E[p'E(xx')^{-1}xx'c] \\ &= p'E(xx')^{-1}E(xx')c \\ &= p'c \\ &= c'p \\ &= p(c(c'x)) \end{aligned}$$

therefore, x^* is the SDF $\in X$ ※ [Q.E.D.]

- What is Arbitrage.
  - one cannot get for free a portfolio that might pay off positively.
  - For all non-negative (positive) payoffs $x \in X$, the price $p(x)$ is positive.

- $m > 0 \Longrightarrow$ No Arbitrage

  we have $m > 0$. $x \geq 0$ and $x > 0$ for some states, Therefore, in some states with positive probability, $mx > 0$. Thus, $p > 0$ because $p = E(mx)$ ※ [Q.E.D.]

- No Arbitrage $\Longrightarrow m > 0$

  想法 $\Longrightarrow$ A counter example for no arbitrage: the payoff is strictly

  [proof] we want to show that $m = x^* > 0$, we can prove it by contradiction. suppose $x^* < 0$ for some states of nature. We form a pay off that is 1 in those state and 0 for all others.

  This payoff is positive, but price is as below,

  $$p = \sum_s \pi(s) x^*(s) \cdot 1 = E(x^* \cdot 1) \leq 0$$

  which contradiction with no arbitrage. #

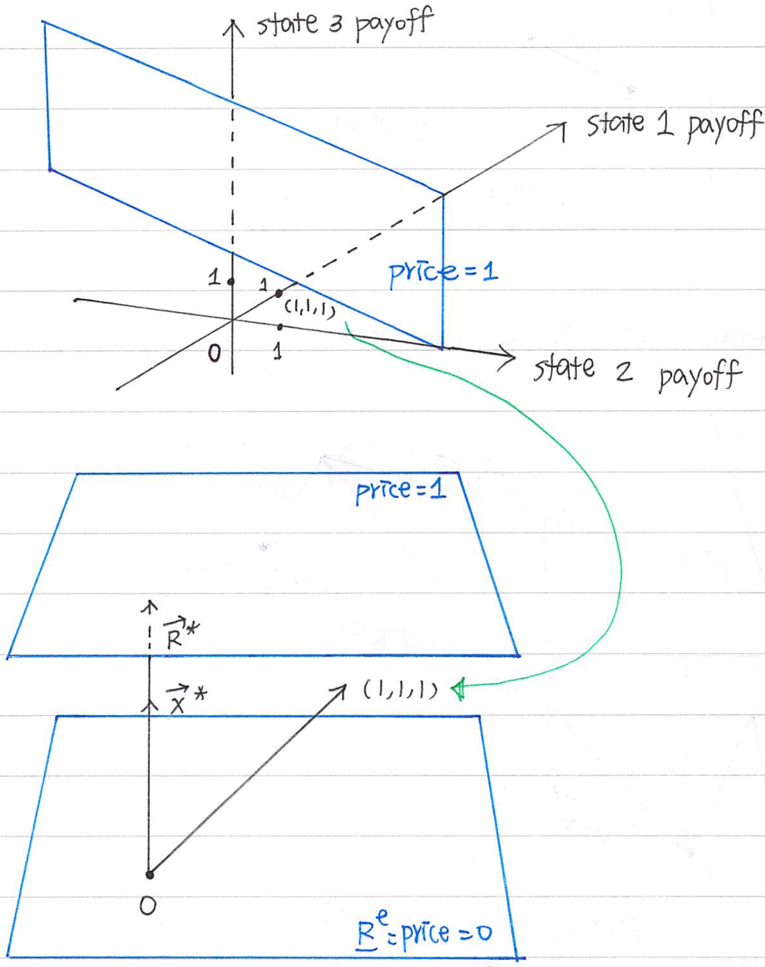
  [Q.E.D.]

Asset Pricing notes - John Cochrane

- $R^i = R^* + w^i R^{e*} + n^i$

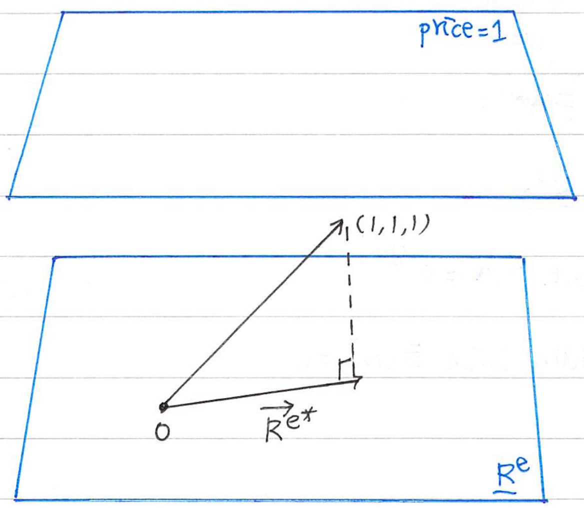
1. we consider a K states world, where $K \geq 3$

   The isoprice levels become flat planes.



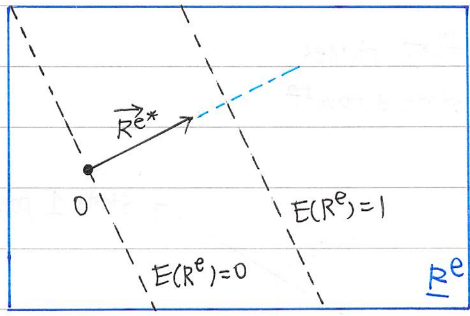
1.1. Since $\vec{x}^* \perp$ price, and $\vec{R}^* = \frac{\vec{x}^*}{E(x^{*2})}$ just streches $\vec{x}^*$

2. By $R^{e*} = \text{proj}(1 \mid \underline{R}^e)$, we have

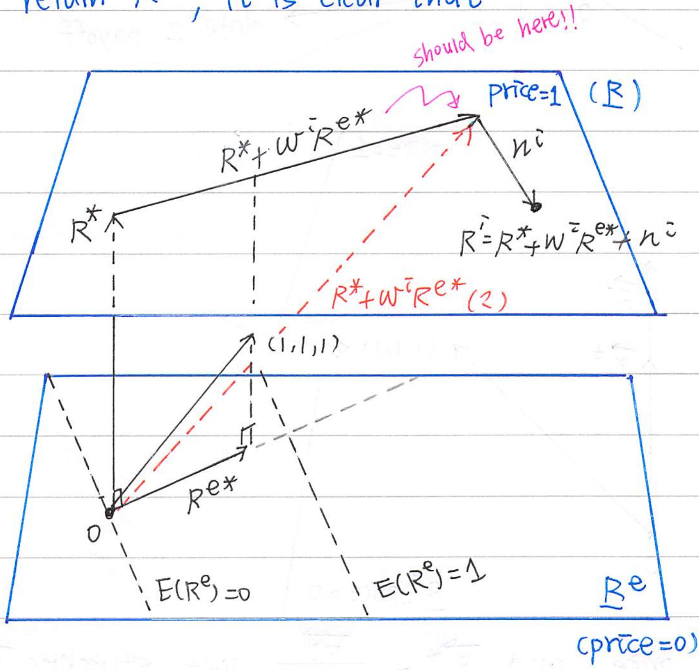


(turn next page!)

3. Furthermore, by $E(R^*) = E(R^{e*} R^e) = \vec{R}^{e*} \cdot \vec{R}^e$ (as figure 3.2) [CH5]



4. Finally, for any return R^i, it is clear that



• 解題時，先寫上：

the definition：

(1) $R^* = \dfrac{x^*}{p(x^*)} = \dfrac{x^*}{E(x^{*2})}$

(2) $R^{e*} \equiv proj(1 | \underline{R}^e)$

(3) $\underline{R}^e = \{x \in \underline{X}, s.t.\ p(x) = 0\}$

(4) n^i = excess return with $E(n^i) = 0$

(5) w^i = scalar

Asset Pricing notes : John Cochrane

[CH9]

- What is Factoring Pricing?
  - consumption-based models do not work well in practice. Linear factor models are the most popular models in discrete-time empirical work
  - To determine a particular list of factors that can proxy marginal consumption growth. They should be linear.

$$\beta \frac{u'(C_{t+1})}{u'(C_t)} \approx a + b' f_{t+1}$$

- CAPM (Capital Asset Pricing Model)
  - SDF for pricing : $m_{t+1} = a + b R^{W}_{t+1}$
  - The beta representation is as follows

$$E(R^i) = r + \beta_{i,R^W}[E(R^W) - r]$$

  - How to get m?
    - two-period quadratic utility
    - exponential utility, with normal returns.
    - Infinite horizon, quadratic utility, with i.i.d. returns
    - log utility

(1) investors are born with wealth W_t in the 1st period and earn no labor income. They can invest in N asset whose returns are R^i.

(2) They close their position by selling their assets in the next period, t+1, and consume the wealth W_{t+1}

- Two-Period Quadratic Utility
  - investors have quadratic preferences and only live two period

Assume $u(C_t, C_{t+1}) = -\frac{1}{2}(C^* - C_t)^2 - \frac{1}{2}\beta E[(C^* - C_t)^2]$

Budget constraint :

(1) $C_{t+1} = W_{t+1}$ ← 只活2期,第二期通通消費

(2) $W_{t+1} = R^W_{t+1}(W_t - C_t)$ ← 前期投资

(3) $R^W_{t+1} = \sum_{i=1}^{N} w_i R^i_{t+1}$, $\sum_{i=1}^{N} w_i = 1$

$$m_{t+1} = \beta \frac{u'(C_{t+1})}{u'(C_t)} = \beta \cdot \frac{(C^* - C_{t+1})}{(C^* - C_t)}$$

$$= \beta \cdot \frac{C^* - R^W_{t+1}(W_t - C_t)}{C^* - C_t}$$

$$= \underbrace{\boxed{\frac{\beta C^*}{C^* - C_t}}}_{a_t} - \underbrace{\boxed{\frac{\beta(W_t - C_t)}{C^* - C_t}}}_{b_t} R^W_{t+1}$$

Two-period investors with no labor income and quadratic utility imply the CAPM

$$m_{t+1} = a_t - b_t R^W_{t+1}$$

- Exponential Utility, Normal returns

Assume $u(C) = -e^{-\alpha c}$, where $c \sim N(E(C), \sigma^2(c))$

α is known as the coefficient of absolute risk aversion

Budget constraint: $\begin{cases} (1)\ C_t = y^f R^f + y' R \\ (2)\ W = y^f + y' 1 \end{cases}$ initial wealth $= W$ $\rightarrow$ risk-free $: R^f$, $\searrow$ risky $: R$

y^f and y' denote the amount of different asset

$$E[u(C)] = E[-e^{-\alpha c}] = -e^{-\alpha E(C) + \frac{\alpha^2}{2}\sigma^2(c)}$$
$$= -e^{-\alpha[y^f R^f + y' E(R)] + \frac{\alpha^2}{2} y' \Sigma y}$$

$\max E[u(C)]$

F.O.C. $y = \Sigma^{-1} \dfrac{E(R) - R^f}{\alpha} \Rightarrow E(R) - R^f = \alpha \Sigma y$
$= \alpha \operatorname{cov}(R, R^W)$

therefore, $E[R^W] - R^f = \alpha \cdot \operatorname{cov}(R, R^W)$

$\Rightarrow E(R^W) - R^f = \alpha \sigma^2(R^W)$ ※

- Quadratic Value Function

特点[1]: infinite horizon, quadratic utility and i.i.d. returns

特点[2]: replace the 2nd-period quadratic utility function with a quadratic value function, assuming i.i.d. returns.

$$U = u(C_t) + \beta E_t[\underbrace{V(W_{t+1})}_{\text{value function}}]$$

$\max U$

F.O.C. $P_t u'(C_t) = \beta E_t[V'(W_{t+1}) x_{t+1}]$

$$m_{t+1} = \beta \frac{V'(W_{t+1})}{u'(C_t)}$$

Now, suppose value function were quadratic.

$$V(W_{t+1}) = -\frac{\eta}{2}(W_{t+1} - W^*)^2$$

therefor $m_{t+1} = \beta \dfrac{V'(W_{t+1})}{u'(C_t)} = -\beta\eta \dfrac{W_{t+1} - W^*}{u'(C_t)}$

再一次地应用 $W_{t+1} = R^W_{t+1}(W_t - C_t)$

$$= -\beta\eta \frac{R^W_{t+1}(W_t - C_t) - W^*}{u'(C_t)}$$

$$= \underbrace{\boxed{\frac{\beta\eta W^*}{u'(C_t)}}}_{a_t} + \underbrace{\boxed{-\frac{\beta\eta(W_t - C_t)}{u'(C_t)}}}_{b_t} R^W_{t+1}$$

$\therefore m_{t+1} = a_t + b_t R^W_{t+1}$ ※

Asset Pricing notes - John Cochrane

[CH9]

- log utility

Assume $u(c) = \ln(c)$

$$P_t^w = E\left(\sum_{j=1}^{\infty} \beta^j \frac{u'(c_{t+j})}{u'(c_t)} c_{t+j}\right)$$

$$= E_t\left(\underbrace{\sum_{j=1}^{\infty} \beta^j}_{\text{等比级数}} \frac{c_t}{\cancel{c_{t+j}}} \cancel{c_{t+j}}\right) \quad \text{消去}$$

$$= \frac{\beta}{1-\beta} c_t$$

1. define the wealth portfolio as a claim to all future consumption.
2. The price of the wealth portfolio is proportional to consumption c_t.

Thus
$$R_{t+1}^w = \frac{P_{t+1}^w + c_{t+1}}{P_t^w} = \frac{\left(\frac{\beta}{1-\beta}+1\right) c_{t+1}}{\frac{\beta}{1-\beta} c_t}$$

$$= \frac{1}{\beta} \frac{c_{t+1}}{c_t}$$

$$= \frac{1}{\beta} \frac{u'(c_t)}{u'(c_{t+1})}$$

$$\therefore m_{t+1} = \beta \frac{u'(c_{t+1})}{u'(c_t)} = \frac{1}{R_{t+1}^w} \quad \text{※}$$