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The Solow–Swan model with a bounded population growth rate

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Abstract

The paper analyzes the dynamic of the Solow–Swan growth model when the labor growth rate is non-constant but variable and bounded over time. Per capita capital is seen to stabilize to the non-trivial steady state of the Solow–Swan model with a particular constant labor growth rate. The solution of the model is proved to be asymptotically stable. In case of a Cobb-Douglas production function and a generalized logistic population growth law, the solution is shown to have a closed-form expression via Hypergeometric functions.

Keywords: Solow–Swan model; Bounded population growth rate

1. Introduction

The neoclassical growth model theory originated with the work of Solow (1956), Swan (1956), who independently proposed similar one-sector models. Solow developed an analysis of growth which is in several ways closely related to the one provided by the neoclassical model. Swan gave a rather similar analysis but in a fashion that was less mathematically explicit. The Solow–Swan model is designed to show how growth in the capital stock, growth in the labor force, advances in technology interact and how they affect a nation's total output. The model has a supply of goods based on a production function with constant returns to scale. Labor grows at a constant rate, the level of technology is constant over time, the saving rate is constant and capital depreciates at a positive constant rate, that is, at each point in time, a constant fraction of the capital stocks wears out and, hence, can no longer be used for production. At any moment the capital stock is a key determinant of the economy's output, but the capital stock can change and this can

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lead to economic growth. Since Solow and Swan, growth theory has evolved into a voluminous literature and different generations of models have been considered (see, for example, Barro and Sala-i-Martin, 1995).

In this paper we modify the Solow–Swan model by considering a population growth rate $n(t)$ that is non-constant over time. We assume that $n(t)$ is bounded and convergent to a positive number n_∞ as t tends to infinity. The corresponding model, described by a non-linear first order differential equation, is investigated by making use of results referred to as Comparison theorems (see, for example, Birkhoff and Rota, 1978). In the long run the per capita capital of the model converges to the non-trivial steady state of the Solow–Swan model with the constant labor growth rate equal to n_∞. The solution of the model is asymptotically stable, and it can be written in closed-form using Hypergeometric functions, when the production function is Cobb-Douglas and the population growth is similar to the logistic growth law.

2. The model

The economy consists of a single good that can be used either for consumption or investment. This good is produced by labor L and capital K in a process described by a neoclassical production function $Y(t) = F(K(t), L(t))$. This means that F has constant returns to scale, $F(\lambda K, \lambda L) = \lambda F(K, L)$, for all $\lambda > 0$, positive and diminishing marginal returns, $F_K > 0, F_{KK} < 0, F_L > 0, F_{LL} < 0$ (where subscripts denote partial derivatives), and satisfies the Inada conditions $\lim_{K\to 0} F_K = \lim_{L\to 0} F_L = +\infty, \lim_{K\to +\infty} F_K = \lim_{L\to +\infty} F_L = 0$. The condition of constant returns to scale implies that output can be written as $Y = F(K, L) = LF(K/L, 1) = Lf(k)$, where $k \equiv K/L$ is the capital-labor ratio, $y \equiv Y/L$ is per capita output, and the function $f(k)$ is defined to equal $F(k, 1)$. The production function expressed in intensive form is $y = f(k)$. Moreover, $f(0) = 0$, $f(\infty) = +\infty$, $f'(0) = +\infty$, $f'(\infty) = 0$, $f'(k) > 0$, $f''(k) < 0$, for all $k > 0$.

The net increase in the stock of physical capital equals gross investment less depreciation, $\dot{K} = I - \delta K = sF(K, L) - \delta K$, where $\dot{K}$ denotes differentiation with respect to time, s is the fraction of output that is saved, δ is the depreciation rate. The change in capital stock over time is given by $\dot{K}/L = sf(k) - \delta k$. Using the condition $\dot{k} = (\mathrm{d}/\mathrm{d}t)(K/L) = \dot{K}/L - (\dot{L}/L)k$, and substituting into the previous equation, we get $\dot{k} = sf(k) - (\delta + \dot{L}/L)k$.

If $\dot{L}/L$ is constant, say n, then we have the well-known Solow–Swan model. Its fundamental dynamic equation is $\dot{k} = sf(k) - (\delta + n)k$.

Now, suppose that today's population $L(0)$ is given, $L(0) = 1$, and that all population growth rates $n(t) = \dot{L}/L$ are controllable subject only to be between prescribed upper and lower limits, $0 \le n(t) \le n^*$, for all t. Moreover, let assume that there exists $\lim_{t\to +\infty} n(t) = n_\infty$. This modified version of the Solow–Swan model has the dynamic described by

$$\dot{k} = sf(k) - (\delta + n(t))k.$$

This first-order non-linear differential equation governs the trajectory of k from any arbitrary starting value $k(0) = k_0$ and so characterizes the system.

3. Some facts on the dynamic of the model

Let consider the following Cauchy problem

$$\begin{cases} \dot{k} = sf(k) - (\delta + n(t))k, \\ k(0) = k_0. \end{cases} \tag{1}$$

This problem has a unique solution $k(t)$ defined on $[0, \infty)$ (see Birkhoff and Rota, 1978).

An immediate consequence is the following result.

Proposition 1. *Let $k_i(t), i = 1, 2$, be the solution of $\dot{k} = sf(k) - (\delta + n(t))k, k_i(0) = k_{i,0}$. If $k_{1,0} < k_{2,0}$, then $k_1(t) < k_2(t)$, for all t.*

Proof. This follows from the uniqueness of the solution of the Cauchy problem. Geometrically, the uniqueness means that the graphs of two distinct solutions of the same differential equation cannot intersect. □

The differential equation in (1) cannot be solved in terms of elementary functions. A common technique in this case is to compare the unknown solutions of the given equation with the known solutions of another, i.e. to use the so-called Comparison theorems. We will use the following one (see Birkhoff and Rota, 1978):

If $k_i(t), i = 1, 2$, is the solution of the Cauchy problem $\dot{k} = \varphi_i(t, k), k(0) = k_0$, and $\varphi_1(t, k) \leq \varphi_2(t, k)$, for all (t, k), then $k_1(t) \leq k_2(t)$, for all t.

Lemma 2. *Let $k_i(t), i = 1, 2$, be the solution of $\dot{k} = sf(k) - (\delta + n_i(t))k, k(0) = k_0$. If $n_1(t) \leq n_2(t)$, for all t, then $k_1(t) \geq k_2(t)$, for all t.*

Proof. It follows from the previous Comparison theorem. □

Proposition 3. *Let $k_1(t)$ be the solution of $\dot{k} = sf(k) - (\delta + n^*)k, k(0) = k_0$, and $k_2(t)$ be the solution of $\dot{k} = sf(k) - \delta k, k(0) = k_0$. If $k(t)$ solves problem* (1), *then $k_1(t) \leq k(t) \leq k_2(t)$, for all t.*

Proof. Immediate from the previous Comparison theorem. □

This result relates $k(t)$ to $k_1(t)$ and $k_2(t)$. It is therefore important to understand the dynamic behavior of $k_1(t)$ and $k_2(t)$.

Proposition 4. *Let $k(t)$ be the solution of $\dot{k} = sf(k) - (\delta + M)k, k(0) = k_0$, with M a positive constant. As $t \to +\infty$, $k(t)$ converges to the unique non-trivial steady state k^* of the system.*

Proof. Steady states are characterized by $\dot{k} = 0$. The curvature of production function f along with the Inada conditions insure that there is a unique non-trivial steady state k^*. This value corresponds to the unique point of intersection of the curves $z = sf(k)/k$ and $z = \delta + M$ in the (k, z)-plane. If $k < k^*$, the growth rate of k is positive and k increases monotonically toward k^*. If $k > k^*$, the growth rate is negative and k declines monotonically toward k^*. Therefore $k(t)$ increases to k^* if $k_0 < k^*$ and decreases to k^* if $k_0 > k^*$. □

Remark 5. k^* is the unique solution of the equation $sf(k) = (\delta + M)k$.

Remark 6. The map $k \mapsto sf(k)/k$ is monotone decreasing.

Theorem 7. *Let $k_1(t), k_2(t)$ be as in Proposition 3. Let k_1^*, k_2^* be the unique non-trivial steady state of the corresponding Cauchy problem.*

(i) $k_1^* < k_2^*$ if $n^* > 0$.
(ii) $\lim_{t\to+\infty} k(t) = k_\infty^*$, *the steady state of the Solow–Swan model with $n = n_\infty$.*
(iii) *Let $n(t)$ be a monotone decreasing function $(n_\infty = 0, n^* = n(0))$.*

(a) *If* $k_0 \le k_1^*$, $\dot{k}(t) \ge 0$, *for all* t.
(b) *If* $k_1^* < k_0 \le k_2^*$, *there exists* $\tau > 0$ *such that* $\dot{k}(t) \le 0$ *for* $t \in (0, \tau]$ *and* $\dot{k}(t) \ge 0$, *for* $t \in [\tau, \infty)$.
(c) *If* $k_2^* < k_0$, $\dot{k}(t) \le 0$, *for all* t.

Proof.

(i) If $k_1^* \ge k_2^*$, we should have $\delta = sf(k_2^*)/k_2^* \ge sf(k_1^*)/k_1^* = \delta + n^*$, a contradiction.
(ii) Let $\lim_{t\to+\infty} k(t) = k^*$ and $\lim_{t\to+\infty} \dot{k}(t) = M$. By Lagrange's theorem, $k(t+1) - k(t) = \dot{k}(\xi_t)$, for some $\xi_t \in (t, t+1)$. As $t \to +\infty$, the right hand side of this identity gives $\lim_{t\to+\infty} \dot{k}(\xi_t) = M$ (since $\lim_{t\to+\infty} \xi_t = +\infty$), while the left hand side, $\lim_{t\to+\infty}[k(t+1) - k(t)] = 0$ (since $k^* < \infty$ by Proposition 3). So $M = 0$, i.e. $sf(k^*) - (\delta + n_\infty)k^* = 0$. Since this equation has a unique solution k_∞^*, $k^* = k_\infty^*$.
(iii) (a) There exists a right neighborhood $(0, \eta)$ of $t = 0$ where $\dot{k}(t) > 0$. In fact, if $k_0 < k_1^*$, from Remarks 5 and 6, $sf(k_0)/k_0 > sf(k_1^*)/k_1^* = \delta + n(0)$, i.e. $\dot{k}(0) > 0$, and so, by continuity, the statement. If $k_0 = k_1^*$, then $\dot{k}(0) = 0$. From $\ddot{k}(t) = \{sf'(k(t)) - (\delta + n(t))\}\dot{k}(t) - \dot{n}(t)k(t)$, we have $\ddot{k}(0) = -\dot{n}(0)k_0 > 0$. The statement follows by continuity. We are now able to conclude that $\dot{k}(t) \ge 0$, for all t. In fact, if there were $t_0 > 0$ such that $\dot{k}(t_0) < 0$, then there would be $\tau \in (\eta, t_0)$ such that $\dot{k}(\tau) = 0$. Since $\ddot{k}(\tau) = -\dot{n}(\tau)k(\tau) > 0$, this would give a contradiction.
(b) Let $k_1^* < k_0 \le k_2^*$. Remarks 5 and 6 imply $\dot{k}(0) < 0$. If $\dot{k}(t) < 0$, for all t, then $sf(k(t))/k(t) < \delta + n(t)$, for all t. As $t \to +\infty$, we should have $\delta + n(0) = sf(k_2^*)/k_2^* < \delta$, a contradiction. Therefore, there exists $t_1 > 0$ such that $\dot{k}(t_1) > 0$. Set $\tau = \inf\{t > 0 : \dot{k}(t) > 0\}$. Proceeding as in (iii)-(a), we see there is no $t > \tau$ such that $\dot{k}(t) < 0$.
(c) Let $k_2^* < k_0$. Remarks 5 and 6 give $\delta = sf(k_2^*)/k_2^* > sf(k(t))/k(t) > sf(k_0)/k_0$, i.e. $\delta > \dot{k}(t)/k(t) + \delta + n(t) > \dot{k}(0)/k_0 + \delta + n(0)$. Therefore $\dot{k}(t) < -n(t) < 0$, for all t.

Remark 8. The term $sf(k(t))/k(t)$ makes sense if $k(t) > 0$. The Inada condition implies that $k(t) > 0$ a.e.

4. Asymptotic stability of the solution

The solution $k(t, k_0)$ of the Cauchy problem $\dot{k} = sf(k) - (\delta + n(t))k$, $k(0) = k_0$, is called Lyapunov stable, if for any real number $\varepsilon > 0$ and any t there exists a positive real number $\eta(\varepsilon, t)$ (i.e. η is dependent on ε and t) such that $|\hat{k}_0 - k_0| < \eta$ implies $|k(t, \hat{k}_0) - k(t, k_0)| < \varepsilon$, for all t. In essence, this says that if k_0 is sufficiently close to $\hat{k}_0$, then $k(t, \hat{k}_0)$ remains close to $k(t, k_0)$, for all t.

If $k(t, k_0)$ is Lyapunov stable and there exists a sufficiently positive small number $\mu > 0$ such that $|\hat{k}_0 - k_0| < \mu$ implies $\lim_{t\to+\infty}(k(t, \hat{k}_0) - k(t, k_0)) = 0$, then $k(t, k_0)$ will be called asymptotically stable. In other words, among stable solutions we may encounter some with the property that all the solutions close to the given one at the initial moment of time become infinitely close to the given one.

Theorem 9. *Let* $k_0 > 0$. *Then* $k(t, k_0)$ *is asymptotically stable.*

Proof. Let $k(t) = k(t, k_0)$, $\hat{k}(t) = k(t, \hat{k}_0)$. By Theorem 7, $\lim_{t\to+\infty}(\hat{k}(t) - k(t)) = 0$, i.e. for every $\varepsilon > 0$ there is $T > 0$ such that $|\hat{k}(t) - k(t)| < \varepsilon$, for $t > T$. Therefore, for every $\varepsilon > 0$

there is $\eta_1 > 0$ such that $|\hat{k}_0 - k_0| < \eta_1$ implies $|\hat{k}(t) - k(t)| < \varepsilon$, for $t > T$. The statement will follow as soon as we show this is also valid when $t \leq T$.

Let $k_0 < \hat{k}_0$. By Proposition 1, $k(t) < \hat{k}(t)$, for all t. Writing $k(t)$, $\hat{k}(t)$ as

$$k(t) = k_0 + \int_0^t [sf(k(t)) - (\delta + n(t))k(t)]\mathrm{d}t,$$
$$\hat{k}(t) = \hat{k}_0 + \int_0^t [sf(\hat{k}(t)) - (\delta + n(t))\hat{k}(t)]\mathrm{d}t,$$

we get

$$\hat{k}(t) - k(t) \leq (\hat{k}_0 - k_0) + \int_0^t \{s[f(\hat{k}(t)) - f(k(t))] + (\delta + n(t))(\hat{k}(t) - k(t))\}\mathrm{d}t.$$

By the Mean Value theorem, $f(\hat{k}(t)) - f(k(t)) = f'(\xi)(\hat{k}(t) - k(t))$, for some $\xi \in (k(t), \hat{k}(t))$. The dynamic of $k_1(t)$, $k_2(t)$ (see Proposition 2) as well as the fact f' is decreasing imply $f'(\xi) < M$, for some constant M. In fact, if $k_0 < k_1^*$, then $f'(\xi) < f'(k_0)$. If $k_1^* < k_0 < k_2^*$ or $k_2^* < k_0$, then $f'(\xi) < f'(k_1^*)$.

When $t \leq T$, the above and Gronwall's inequality (see Birkhoff and Rota, 1978) give

$$\hat{k}(t) - k(t) < (\hat{k}_0 - k_0) + \int_0^t [sM + \delta + n(t)](\hat{k}(t) - k(t))\mathrm{d}t,$$

$$\leq (\hat{k}_0 - k_0)\mathrm{e}^{\int_0^t [sM+\delta+n(t)]\mathrm{d}t} \qquad \leq (\hat{k}_0 - k_0)\mathrm{e}^{(sM+\delta+n^*)T}.$$

Choosing $\eta = \varepsilon/\mathrm{e}^{(sM+\delta+n^*)T}$, we have that for any $\varepsilon > 0$ there is $\eta_2 > 0$ such that $|\hat{k}_0 - k_0| < \eta_2$ implies $|\hat{k}(t) - k(t)| < \varepsilon$ if $t \leq T$. Therefore, setting $\eta = \min\{\eta_1, \eta_2\}$, we conclude that for any $\varepsilon > 0$ there is $\eta > 0$ such that $|\hat{k}_0 - k_0| < \eta$ implies $|\hat{k}(t) - k(t)| < \varepsilon$, for all t. $\square$

5. The model solution for the Cobb-Douglas production function

Let consider a neoclassical production function having the Cobb-Douglas form, $Y(t) = K(t)^{\alpha}L(t)^{1-\alpha}$, $0 < \alpha < 1$. We can now work out an explicit solution to the differential equation of the corresponding model.

Theorem 10. *Let $k(t)$ be the solution of* (1) *when $y = k^{\alpha}$, $0 < \alpha < 1$. Then*

$$k(t) = \mathrm{e}^{-\delta t}L(t)^{-1}\left(k_0^{1-\alpha} + (1-\alpha)s\int_0^t \mathrm{e}^{(1-\alpha)\delta u}L(u)^{1-\alpha}\mathrm{d}u\right)^{1/(1-\alpha)}. \tag{2}$$

Proof. The equation $\dot{k} = sk^{\alpha} - (\delta + n(t))k$ is a non-linear differential equation of Bernoulli type. By taking the substitution $z = k^{1-\alpha}$, we get a linear differential equation in z, $\dot{z} = (1-\alpha)s - (1-\alpha)(\delta + \dot{L}(t)/L(t))z$, which is solved by

$$z(t) = \mathrm{e}^{-\int_0^t (1-\alpha)(\delta+(\dot{L}(u)/L(u)))\mathrm{d}u}\left(z_0 + \int_0^t (1-\alpha)s\,\mathrm{e}^{\int_0^u (1-\alpha)(\delta+(\dot{L}(v)/L(v)))\mathrm{d}v}\mathrm{d}u\right).$$

Since $\int_0^t (\delta + \dot{L}(u)/L(u))\mathrm{d}u = \delta t + \ln L(t)$, it follows

$$z(t) = (\mathrm{e}^{\delta t}L(t))^{-(1-\alpha)}\left(z_0 + (1-\alpha)s\int_0^t \mathrm{e}^{(1-\alpha)\delta u}L(u)^{1-\alpha}\mathrm{d}u\right).$$

The statement follows by rewriting this in terms of k. $\square$

Remark 11. Hopital's rule applied to (2) gives $\lim_{t\to+\infty} k(t) = [s/(\delta + n_{\infty})]^{1/(1-\alpha)}$. This result agrees with what was theoretically proved in Theorem 7.

6. The solution expressed through Hypergeometric functions

Let consider the case of a Cobb-Douglas production function and of a population growth law of the form

$$\dot{L}(t) = aL(t) - bL(t)^{2-\beta}, \quad 0 < b < a,\ 0 \leq \beta < 1. \tag{3}$$

Eq. (3) is a Bernoulli differential equation solved by

$$L(t) = \left(1 - \frac{b}{a}\right)^{-1/(1-\beta)} \mathrm{e}^{at} \left(1 - \frac{b}{b-a}\mathrm{e}^{(1-\beta)at}\right)^{-1/(1-\beta)}. \tag{4}$$

Remark 12. This growth law is a generalization of the familiar logistic model, $\beta = 0$ in (3). The logistic law is an example where $n(t)$ is a monotone function decreasing to zero. Its solution can be obtained from (4) for $\beta = 0$, $L(t) = a\mathrm{e}^{at} / \left(a - b + b\mathrm{e}^{at}\right)$.

$L(t)$ is a bounded increasing function converging to $(a/b)^{1/(1-\beta)}$, $L(0) = 1$. Moreover, its growth rate $n(t)$ decreases monotonically to 0. So $n^* = n(0)$ and $n_\infty = 0$.

We are now going to show that the solution $k(t)$ of (2) can be written in closed-form through the Hypergeometric function ${}_2F_1$ (see Appendix A).

Theorem 13. *Let* $\gamma_1 = (1-\alpha)(\delta + a)$, $\gamma_2 = (1-\beta)a$, $\gamma_3 = (1-\alpha)/(1-\beta)$, $B = b/(b-a)$. *Let* ${}_2F_1$ *be the Hypergeometric function. Then*

$$k(t) = \mathrm{e}^{-\delta t} L(t)^{-1} \left\{ k_0^{1-\alpha} + \frac{(1-\alpha)s(1-B)^{(1-\alpha)/(1-\beta)}}{\gamma_1} \left[\mathrm{e}^{\gamma_1 t}\, {}_2F_1\left(\frac{\gamma_1}{\gamma_2}, \gamma_3, \frac{\gamma_1}{\gamma_2} + 1; B\mathrm{e}^{\gamma_2 t}\right) \right.\right.$$
$$\left.\left. - {}_2F_1\left(\frac{\gamma_1}{\gamma_2}, \gamma_3, \frac{\gamma_1}{\gamma_2} + 1; B\right)\right]\right\}^{1/(1-\alpha)}.$$

Proof. Using (4), the definitions of $\gamma_1, \gamma_2, \gamma_3, B$, and operating the change of variable $x = \mathrm{e}^{\gamma_2 u}$, we get

$$\int_0^t \mathrm{e}^{(1-\alpha)\delta u} L(u)^{1-\alpha} \mathrm{d}u \tag{5}$$
$$= \frac{(1-B)^{(1-\alpha)/(1-\beta)}}{\gamma_1\gamma_2} \int_1^{\mathrm{e}^{\gamma_2 t}} x^{\frac{\gamma_1}{\gamma_2}-1} (1-Bx)^{-\gamma_3} \mathrm{d}x,$$
$$= \frac{(1-B)^{(1-\alpha)/(1-\beta)}}{\gamma_1\gamma_2} \left(\int_0^{\mathrm{e}^{\gamma_2 t}} x^{(\gamma_1/\gamma_2)-1} (1-Bx)^{-\gamma_3} \mathrm{d}x - \int_0^1 x^{(\gamma_1/\gamma_2)-1} (1-Bx)^{-\gamma_3} \mathrm{d}x \right).$$

The purpose is now to write in a different way the above two integrals. For the second integral of the above expression we have

$$\int_0^1 x^{(\gamma_1/\gamma_2)-1} (1-Bx)^{-\gamma_3} \mathrm{d}x = \int_0^1 x^{(\gamma_1/\gamma_2)-1} (1-x)^0 (1-Bx)^{-\gamma_3} \mathrm{d}x,$$
$$= \frac{\gamma_2}{\gamma_1}\ {}_2F_1\left(\frac{\gamma_1}{\gamma_2}, \gamma_3, \frac{\gamma_1}{\gamma_2} + 1; B\right).$$

We have used the integral representation of an Hypergeometric function ${}_2F_1$ and the two properties of the Γ-function, $\Gamma(1) = 1$ and $\Gamma(v+1) = v\Gamma(v)$, for all $v > 0$. For the first integral, by changing variable $r = e^{-\gamma_2 t}x$, we get

$$\int_0^{e^{\gamma_2 t}} x^{(\gamma_1/\gamma_2)-1}(1-Bx)^{-\gamma_3}\mathrm{d}x = e^{\gamma_1 t}\int_0^1 r^{(\gamma_1/\gamma_2)-1}(1-Be^{\gamma_2 t}r)^{-\gamma_3}\mathrm{d}r,$$

$$= \frac{\gamma_2}{\gamma_1}e^{\gamma_1 t}\ {}_2F_1\left(\frac{\gamma_1}{\gamma_2}, \gamma_3, \frac{\gamma_1}{\gamma_2}+1; Be^{\gamma_2 t}\right).$$

The statement now follows by substituting these in (5) and then in (2). □

Remark 14. Let $c \geq 1$. Since $\int_0^c x^{(\gamma_1/\gamma_2)-1}(1-Bx)^{-\gamma_3}\mathrm{d}x \sim \int_0^c x^{(\gamma_1/\gamma_2)-1}\mathrm{d}x$, the fact $\gamma_1/\gamma_2 > 0$ implies that the integrals used in the previous proof are convergent.

7. Conclusion

The economic meaning of the results of this paper is as follows. A country with more initial per capita capital gets more per capita capital when all the other conditions are the same. Independently from its initial value, the per capita capital of a country with labor growth rate $n(t)$ will tend to stabilize to the non-trivial steady state of the country whose dynamic is described by the Solow–Swan model with labor growth rate equal to n_∞. Moreover, small variations of the initial per capita capital do not change very much the economic growth process, i.e. the solution of the model is asymptotically stable. Countries with same initial per capita capital but with higher labor force growth rate will have less per capita capital and so less per capita consumption. In the long run, they will stabilize to the same per capita capital if their labor growth rate limits are equal. From the above, we understand the importance for a country to have an efficient population control policy.

Appendix A

Let recall some facts about Hypergeometric functions (see, for example, Whittaker and Watson, 1927). The power series

$${}_2F_1(c_1, c_2, c_3; z) = \frac{\Gamma(c_3)}{\Gamma(c_1)\Gamma(c_2)}\sum_{m=1}^{\infty}\frac{\Gamma(c_1+n)\Gamma(c_2+m)}{\Gamma(c_3+m)}\frac{z^m}{m!}$$

in the complex variable z is called the Hypergeometric series or Gauss's series. It is convergent for any c_1, c_2 and c_3 if $|z| < 1$, and it is convergent for $\mathrm{Re}(c_1+c_2-c_3) < 0$ if $|z| = 1$. The Hypergeometric functions are obtained as analytic continuations of the functions determined by Hypergeometric series that are single-valued analytic functions defined on the domain obtained from the complex plane by deleting a line connecting the branch points $z = 1$ and ∞. A Hypergeometric function is a solution of the differential equation

$$z(1-z)\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} + [c_3 - (c_1+c_2+1)z]\frac{\mathrm{d}w}{\mathrm{d}z} - c_1 c_2 w = 0,$$

which is called the Hypergeometric differential equation or Gaussian differential equation. For $\mathrm{Re}(c_1) > 0$ and $\mathrm{Re}(c_3 - c_1) > 0$, there is the following integral representation

$${}_2F_1(c_1, c_2, c_3; z) = \frac{\Gamma(c_3)}{\Gamma(c_1)\Gamma(c_3-c_1)}\int_0^1 t^{c_1-1}(1-t)^{c_3-c_1-1}(1-zt)^{-c_2}\mathrm{d}t.$$

References

Barro, R.J., Sala-i-Martin, X., 1995. Economic growth. McGraw-Hill, New York.
Birkhoff, G., Rota, G., 1978. Ordinary differential equations, third ed. John Wiley and Sons, New York.
Solow, R.M., 1956. A contribution to the theory of economic growth. Quarterly Journal of Economics 70 (1), 65–94.
Swan, T.W., 1956. Economic growth and capital accumulation. Economic Record 32 (63), 334–361.
Whittaker, E.T., Watson, G.N., 1927. A course of modern analysis. Cambridge University Press, London, New York.