Internal Structure of A Shock

1 Introduction

For this project, we will compute the flow properties within a continuum shock. We know the upstream flow conditions (station 1) and will assume an ideal gas and, in essence, use the Rankine-Hugoniot jump conditions to determine the flow conditions downstream of the shock (station 2). Between these two stations, we will use the Navier-Stokes equations to determine the flow properties along the shock width. The Navier-Stokes equations give good results up to moderate Mach numbers before nonequilibrium effects plays a larger role.

1.1 System of Ordinary Differential Equations

To derive the system of ODEs that we will use in our algorithm, we will begin by integrating the conservation equations. This will result in equations with integration constants. To determine the integration constants, we will evaluate the integrated conservation equations at either the upstream or downstream location.

1.1.1 Conservation of Mass

Starting with the conservation of mass equation,

$$\frac{\partial}{\partial x}(\rho u) = 0 \tag{1}$$

Since the flow is uniform upstream and downstream, there is only variation in the x-direction and the partial derivatives become ordinary derivatives,

$$\frac{d}{dx}(\rho u) = 0 \tag{2}$$

Integrate this equation with respect to x over an indefinite integral,

$$\begin{aligned} \int \frac{d}{dx}(\rho u)dx &= \int 0dx \\ \int \frac{d}{dx}(\rho u)dx &= C_1 \end{aligned} \tag{3}$$

Applying the fundamental theorem of calculus to the LHS,

$$\int \frac{d}{dx}(\rho u)dx = \rho u + C_2 \tag{4}$$

Substituting this into the LHS of the integral equation for conservation of mass,

$$\rho u + C_2 = C_1 \tag{5}$$

Combining constants,

$$A = C_1 - C_2 \tag{6}$$

Finally, the conservation of mass equation becomes,

$$\rho u = A \tag{7}$$

1.1.2 Conservation of Momentum

Next we will perform the same analysis on the conservation of momentum equation,

$$\frac{\partial}{\partial x}\left(\rho u^2 + p - \frac{4}{3}\mu\frac{\partial u}{\partial x}\right) = 0 \tag{8}$$

Since the flow is uniform upstream and downstream, there is only variation in the x-direction and the partial derivatives become ordinary derivatives,

$$\frac{d}{dx}\left(\rho u^2 + p - \frac{4}{3}\mu\frac{du}{dx}\right) = 0 \tag{9}$$

Integrate this equation with respect to x over an indefinite integral,

$$\begin{aligned}\int \frac{d}{dx}\left(\rho u^2 + p - \frac{4}{3}\mu\frac{du}{dx}\right) dx &= \int 0 dx \\ \int \frac{d}{dx}\left(\rho u^2 + p - \frac{4}{3}\mu\frac{du}{dx}\right) dx &= C_1\end{aligned} \tag{10}$$

Applying the fundamental theorem of calculus to the LHS,

$$\int \frac{d}{dx}\left(\rho u^2 + p - \frac{4}{3}\mu\frac{du}{dx}\right) dx = \rho u^2 + p - \frac{4}{3}\mu\frac{du}{dx} + C_2 \tag{11}$$

Substituting this into the LHS of the integral equation for conservation of momentum,

$$\rho u^2 + p - \frac{4}{3}\mu\frac{du}{dx} + C_2 = C_1 \tag{12}$$

Combining constants,

$$B = C_1 - C_2 \tag{13}$$

Finally, the conservation of momentum equation becomes,

$$\rho u^2 + p - \frac{4}{3}\mu\frac{du}{dx} = B \tag{14}$$

1.1.3 Conservation of Energy

Finally, we will perform the same analysis on the conservation of energy equation,

$$\frac{\partial}{\partial x}\left[\rho u\left(h + \frac{1}{2}u^2\right) - \frac{4}{3}\mu u\frac{\partial u}{\partial x} - k\frac{\partial T}{\partial x}\right] = 0 \tag{15}$$

Since the flow is uniform upstream and downstream, there is only variation in the x-direction and the partial derivatives become ordinary derivatives,

$$\frac{d}{dx}\left[\rho u\left(h + \frac{1}{2}u^2\right) - \frac{4}{3}\mu u\frac{du}{dx} - k\frac{dT}{dx}\right] = 0 \tag{16}$$

Integrate this equation with respect to x over an indefinite integral,

$$\begin{aligned}\int \frac{d}{dx}\left[\rho u\left(h + \frac{1}{2}u^2\right) - \frac{4}{3}\mu u\frac{du}{dx} - k\frac{dT}{dx}\right] dx &= \int 0 dx \\ \int \frac{d}{dx}\left[\rho u\left(h + \frac{1}{2}u^2\right) - \frac{4}{3}\mu u\frac{du}{dx} - k\frac{dT}{dx}\right] dx &= C_1\end{aligned} \tag{17}$$

Applying the fundamental theorem of calculus to the LHS,

$$\int \frac{d}{dx}\left[\rho u\left(h + \frac{1}{2}u^2\right) - \frac{4}{3}\mu u\frac{du}{dx} - k\frac{dT}{dx}\right] dx = \rho u\left(h + \frac{1}{2}u^2\right) - \frac{4}{3}\mu u\frac{du}{dx} - k\frac{dT}{dx} + C_2 \tag{18}$$

Substituting this into the LHS of the integral equation for conservation of energy,

$$\rho u\left(h+\frac{1}{2}u^2\right)-\frac{4}{3}\mu u\frac{du}{dx}-k\frac{dT}{dx}+C_2=C_1 \tag{19}$$

Combining constants,

$$C=C_1-C_2 \tag{20}$$

Finally, the conservation of energy equation becomes,

$$\rho u\left(h+\frac{1}{2}u^2\right)-\frac{4}{3}\mu u\frac{du}{dx}-k\frac{dT}{dx}=C \tag{21}$$

1.1.4 Mass, Momentum, and Energy Conservation Equations Summary

$$\rho u=A \tag{7}$$

$$\rho u^2+p-\frac{4}{3}\mu\frac{du}{dx}=B \tag{14}$$

$$\rho u\left(h+\frac{1}{2}u^2\right)-\frac{4}{3}\mu u\frac{du}{dx}-k\frac{dT}{dx}=C \tag{21}$$

1.1.5 Integration Constants

If we evaluate the integrated conservation equations above at the upstream location (1), the spatial derivative terms are zero because in the Lagrangian frame where we are riding with the shock, the upstream and downstream conditions are uniform in all directions (we defined this in the problem statement). In this case, all d/dx terms are zero. We get a similar result when we evaluate the integrated conservation equations at the downstream location (2). These are also the shock jump conditions.

$$A=\rho_1u_1=\rho_2u_2 \tag{22}$$

$$B=\rho_1u_1^2+p_1=\rho_2u_2^2+p_2 \tag{23}$$

$$C=\rho_1u_1h_{01}=\rho_2u_2h_{02} \tag{24}$$

1.2 Equations for Numerical Integration

We will now rearrange the momentum equation to get du/dx and the energy equation to get dT/dx to use an Euler method to march across the shock width. After substituting the integrated mass equation, $\rho u=A$, into the integrated momentum equation,

$$\begin{aligned}\rho u^2+p-\frac{4}{3}\mu\frac{du}{dx}&=B\\ Au+p-\frac{4}{3}\mu\frac{du}{dx}&=B\\ Au+p-B&=\frac{4}{3}\mu\frac{du}{dx}\\ \frac{du}{dx}&=\frac{3}{4\mu}\left(Au+p-B\right)\end{aligned} \tag{25}$$

where,

$$\mu=\mu_0\left(\frac{T}{T_0}\right)^n \tag{26}$$

$$p=\rho RT=\frac{A}{u}RT \tag{27}$$

Next, dT/dx can be found by manipulating the energy equation,

$$\rho u\left(h+\frac{1}{2}u^2\right)-\frac{4}{3}\mu u\frac{du}{dx}-k\frac{dT}{dx}=C \tag{28}$$

Substitute in the integrated mass equation and the rearranged integrated momentum equation,

$$\begin{aligned}
A\left(h+\frac{1}{2}u^2\right)-\frac{4}{3}\mu u\left[\frac{3}{4\mu}\left(Au+p-B\right)\right]-k\frac{dT}{dx}&=C\\
A\left(h+\frac{1}{2}u^2\right)-\left(Au+p-B\right)u-C&=k\frac{dT}{dx}\\
A\left(h-\frac{1}{2}u^2\right)+\left(B-p\right)u-C&=k\frac{dT}{dx}\\
\frac{dT}{dx}&=\frac{1}{k}\left[A\left(h-\frac{1}{2}u^2\right)+\left(B-p\right)u-C\right]
\end{aligned} \tag{29}$$

where,

$$k=k_0\left(\frac{T}{T_0}\right)^n \tag{30}$$

$$h=c_pT \tag{31}$$

1.3 Boundary Conditions

We will use the jump conditions to determine the downstream flow conditions using information from the upstream flow conditions. Since for our integration we will begin at the downstream conditions and march upstream until we reach and match the upstream conditions, these are the boundary conditions. For this implementation, we specifically need u_1, T_1 and u_2, T_2 to solve the ODEs and the other values allow us to compute more flow properties.

1.3.1 Upstream Boundary and Flow Conditions

The upstream Mach number, M_1, temperature, T_1, and pressure, p_1 are given. From these properties, we can find the remaining useful flow properties. The upstream Mach number and temperature can be used to determine the speed of sound,

$$a_1=\sqrt{\gamma RT_1} \tag{32}$$

Using the upstream speed of sound and Mach number, the upstream flow velocity can be found,

$$u_1=M_1a_1 \tag{33}$$

The upstream density can be found using the ideal gas law,

$$\rho_1=\frac{p_1}{RT_1} \tag{34}$$

The upstream enthalpy and total enthalpy can be found using the upstream velocity and temperature,

$$h_1=c_pT_1 \tag{35}$$

$$h_{01}=h_1+\frac{u_1^2}{2} \tag{36}$$

1.3.2 Downstream Boundary and Flow Conditions

The downstream conditions (2) can be found two ways. The first is using the general jump conditions mentioned previously. We know flow conditions upstream and can use that information and the jump conditions to find the flow conditions downstream. The second method uses a specific case of the jump conditions, the normal shock relations. Since the flow is one-dimensional, we can treat the shock as a normal shock; we know the upstream Mach numbers, M_1, and can use it to compute the downstream flow conditions.

1.3.2.1 General Jump Conditions

For this method, constants the A, B, and C defined above can be determined using upstream conditions and are known quantities. We can find the downstream density using the mass jump condition,

$$\begin{aligned} A &= \rho_2 u_2 \\ \rho_2 &= \frac{A}{u_2} \end{aligned} \tag{37}$$

We will find u_2 later. The relation for downstream temperature T_2 an be found using the energy jump conditon,

$$C = \rho_2 u_2 h_{02} \tag{38}$$

Substituting in relations for ρ_2 and $h_0 = h + u^2/2$,

$$C = \frac{A}{u_2} u_2 \left(h_2 + \frac{u_2^2}{2} \right) \tag{39}$$

Using $h = c_p T$,

$$C = \frac{A}{u_2} u_2 \left(c_p T_2 + \frac{u_2^2}{2} \right) \tag{40}$$

Finally, we can solve for T_2,

$$\begin{aligned} \frac{C}{A} &= c_p T_2 + \frac{u_2^2}{2} \\ c_p T_2 &= \frac{C}{A} - \frac{u_2^2}{2} \\ T_2 &= \frac{1}{c_p} \left(\frac{C}{A} - \frac{u_2^2}{2} \right) \end{aligned} \tag{41}$$

At this stage, we have values for all of the terms on the RHS of this equation besides u_2. Next we will plug in this expression for downstream temperature into the momentum jump equation to find the downstream velocity, u_2. The momentum jump condition is,

$$B = \rho_2 u_2^2 + p_2 \tag{42}$$

Substitute in the expression for ρ_2 and ideal gas relation for pressure, $p = \rho R T$,

$$\begin{aligned} B &= A u_2 + \rho_2 R T_2 \\ B &= A u_2 + \frac{A}{u_2} R T_2 \end{aligned} \tag{43}$$

Now plug in the expression for T_2 from the energy jump condition derived above,

$$\begin{aligned} B &= A u_2 + \frac{A}{u_2} R \left[\frac{1}{c_p} \left(\frac{C}{A} - \frac{u_2^2}{2} \right) \right] \\ &= A u_2 + \frac{AR}{c_p u_2} \left(\frac{C}{A} - \frac{u_2^2}{2} \right) \\ B &= A u_2 + \frac{RC}{c_p} \frac{1}{u_2} - \frac{AR}{2 c_p} u_2 \end{aligned} \tag{44}$$

To solve for u_2, multiply the entire equation above by u_2 and manipulate into a form where we can apply the quadratic formula,

$$\begin{aligned} B u_2 &= A u_2^2 + \frac{RC}{c_p} - \frac{AR}{2 c_p} u_2^2 \\ B u_2 &= \left(A - \frac{AR}{2 c_p} \right) u_2^2 + \frac{RC}{c_p} \\ \left(A - \frac{AR}{2 c_p} \right) u_2^2 - B u_2 + \frac{RC}{c_p} &= 0 \end{aligned} \tag{45}$$

Applying the quadratic formula,

$$u_2 = \frac{-(-B) \pm \sqrt{(-B)^2 + 4\left(A - \frac{AR}{2c_p}\right)\left(\frac{RC}{c_p}\right)}}{2\left(A - \frac{AR}{2c_p}\right)} \tag{46}$$

Finally from implentation it was found that the negative case (of the plus/minus) yields the exact same result as using the normal shock relations,

$$u_2 = \frac{B - \sqrt{B^2 + 4A\left(1 - \frac{R}{2c_p}\right)\left(\frac{RC}{c_p}\right)}}{2A\left(1 - \frac{R}{2c_p}\right)} \tag{47}$$

At this point we can now solve for ρ_2 and T_2 using the relation derived above. From there, we can use the ideal gas law to find p_2.

1.4 Summary

The ordinary differential equations used are,

$$\frac{du}{dx} = \frac{3}{4\mu}\left(Au + p - B\right) \tag{25}$$

$$\frac{dT}{dx} = \frac{1}{k}\left[A\left(h - \frac{1}{2}u^2\right) + (B - p)\,u - C\right] \tag{29}$$

with,

$$A = \rho_1 u_1 = \rho_2 u_2 \tag{22}$$

$$B = \rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2 \tag{23}$$

$$C = \rho_1 u_1 h_{01} = \rho_2 u_2 h_{02} \tag{24}$$

and,

$$\mu = \mu_0 \left(\frac{T}{T_0}\right)^n \tag{26}$$

$$p = \rho R T = \frac{A}{u} R T \tag{27}$$

$$k = k_0 \left(\frac{T}{T_0}\right)^n \tag{30}$$

$$h = c_p T \tag{31}$$

Note that n is different for the viscosity and thermal conductivity equations. The upstream boundary conditions are,

$$T_1 = 290\,\mathrm{K} \tag{48}$$

$$u_1 = M_1 \sqrt{\gamma R T_1} \tag{33}$$

where M_1 is user-defined. The downstream boundary conditions are,

$$u_2 = \frac{B - \sqrt{B^2 + 4A\left(1 - \frac{R}{2c_p}\right)\left(\frac{RC}{c_p}\right)}}{2A\left(1 - \frac{R}{2c_p}\right)} \tag{47}$$

$$T_2 = \frac{1}{c_p}\left(\frac{C}{A} - \frac{u_2^2}{2}\right) \tag{41}$$

2 Starting Condition

Let us examine Equations 22, 23, and 25,

$$\frac{du}{dx} = \frac{3}{4\mu}\left(Au + p - B\right) \tag{25}$$

$$A = \rho_1 u_1 = \rho_2 u_2 \tag{22}$$

$$B = \rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2 \tag{23}$$

We can see that $Au + p = B$ if we use $u = u_2$ and $T = T_2$. The derivative would be evaluated to be $du/dx = 0$. With our marching scheme, the value of u will not change. We need slight deviations in $u(0)$ and $T(0)$ to have a nonzero du/dx to advance the solution.

3 Implementation and Results

The upstream flow conditions are known and can be used to calculate the downstream conditions using the jump conditions. Between the two locations, the Navier-Stokes equations are used. We will begin at the downstream/right location and march upstream/left. This can be achieved in two ways. We can use Euler's method and solve the equations in Section 1.4 separately or via a built-in ODE solver such as MATLAB's `ode45` function to simultaneous solve both equations in the ODE system at perscribed locations.

3.1 Euler's Method (Marching Scheme)

To determine the flow properties within the shock wave, we will begin on the downstream side and march upstream using Euler's method,

$$u_{i+1} = u_i + \left.\frac{du}{dx}\right|_i dx \tag{49}$$

$$T_{i+1} = T_i + \left.\frac{dT}{dx}\right|_i dx \tag{50}$$

where the i values are evaluated at the relative downstream value. After each iteration, the previously $i+1$ values will be set as i values and new $i+1$ values are computed – this is the marching part of the method. Note that to go left/upstream, dx is negative.

3.2 Stopping Criteria

For graphical purposes, we want the center of the shock to lie at the x-origin of the plot. To do so we will determine velocity midpoint between the upstream and downstream velocities, $(u_1 + u_2)/2$, and find the x-value that corresponds to that velocity midpoint. We will then shift the curve to the right by that x-value, x_0. Due to our marching method and direction, this x_0 distance is measured from the downstream/right location. Since we know that x_0 gives us distance from the downstream/right location to the x-location corresponding to the half of the velocity change, we can estimate that it is half of the distance of the shock width. This assumes that the distance from the midpoint velocity to the upstream/left velocity is the same as x_0. We will take a more precise approach later to actually determine the shock width. In addition, in case $2x_0$ is too narrow of a width for the computed velocity to reach the upstream velocity we will compare the left most computed velocity, u_{i+1}, and our defined value for u_1, and determine if it is satisfactory. The marching scheme will terminate when the following two conditions are met:

$$\left|\frac{u_{i+1} - u_1}{u_1}\right| \leq 1 \times 10^{-3} \tag{51}$$

$$|x_{i+1}| \geq |2x_0| \tag{52}$$

3.3 Remaining Flow Properties

Once vectors for x, u, and T are found from the marching algorithm, the density, pressure, and entropy vectors can be computed.

$$\rho = \frac{A}{u} \tag{53}$$

$$p = \rho R T \tag{54}$$

$$s - s_1 = c_p \ln\frac{T}{T_1} - R\ln\frac{p}{p_1} \tag{55}$$

3.4 Shock Structure Results

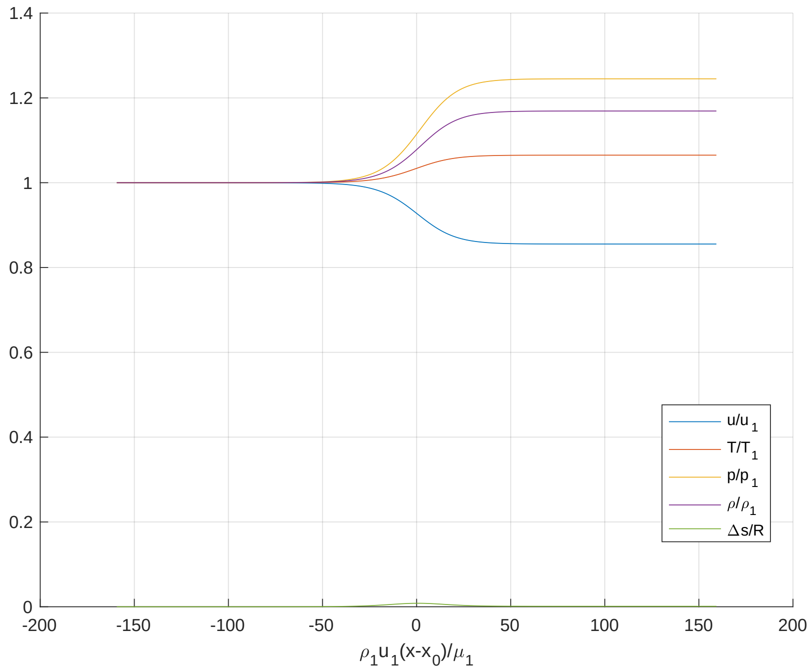

Figure 1: Shock Structure, M_1=1.1

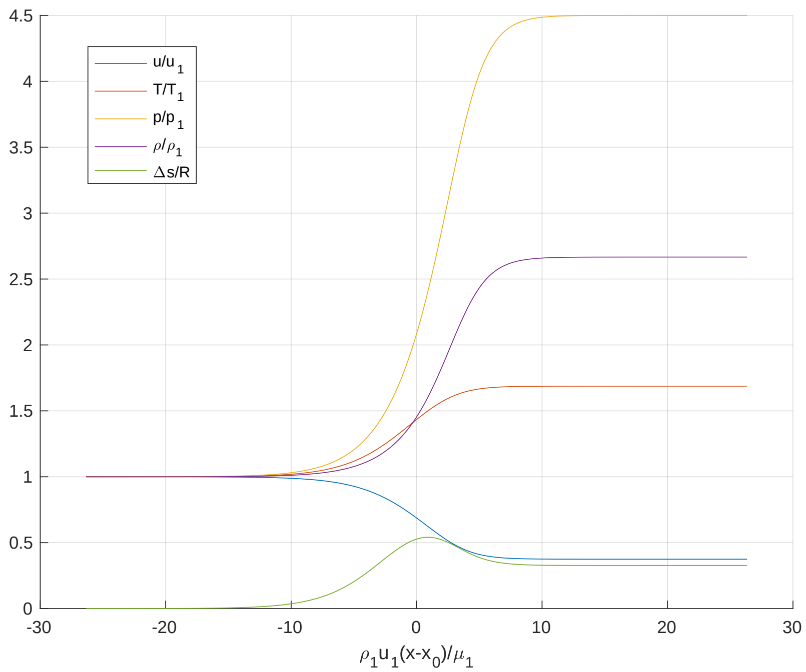

Figure 2: Shock Structure, M_1=2.0

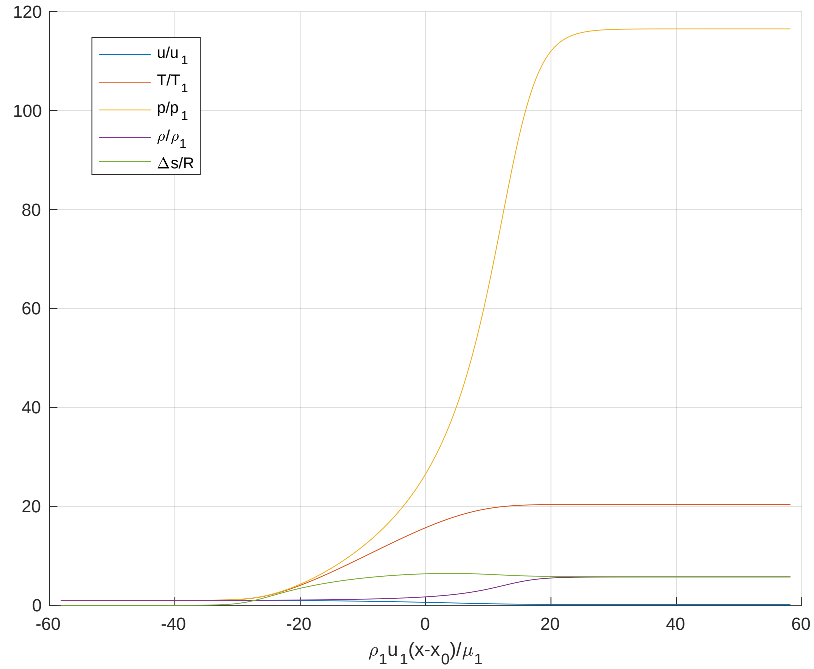

Figure 3: Shock Structure, M_1=10.0

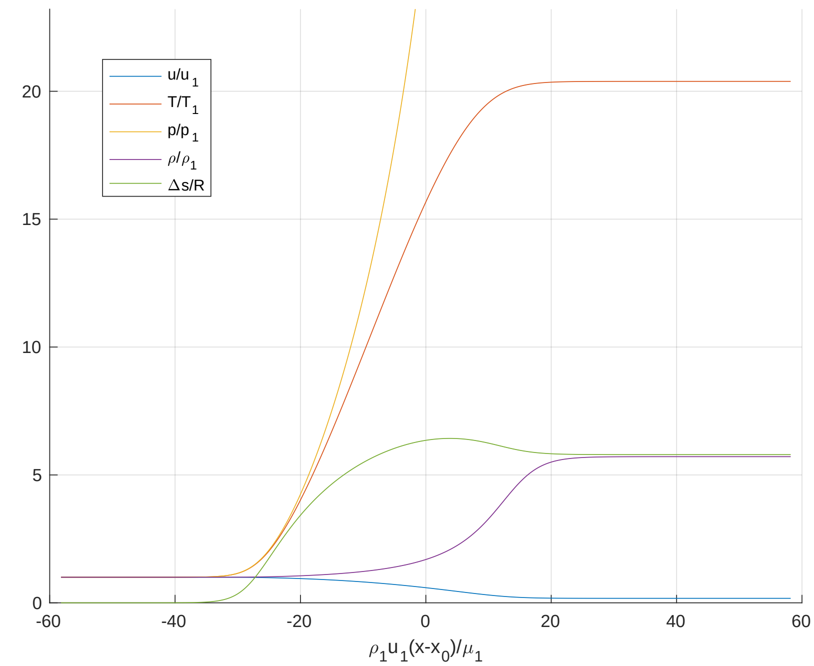

Figure 4: Shock Structure, M_1=10.0 (Zoomed-In)

4 Shock Thickness

As we march from downstream/right of the shock to upstream/left of the shock, our computed velocity should approach the upstream velocity, u_1. Theoretically, the shock width is the distance it takes for the computed velocity to match the upstream velocity exactly. Since this is a numerical computation, that is not likely to occur and if so will require a non-physical distance. To get around this, we will define the shock thickness to be the the distance across where the change in velocity computed from the numerical method is 99% of the defined velocity difference, $u_1 - u_2$. There are many ways to implement this. In this implementation, the shock thickness is defined as the distance between x_A and x_B. x_A is the location where the difference between u_1 and the velocity $u(x_A)$, is approximately equal to 1% of $u_1 - u_2$. x_B is the location where the difference between u_1 and the velocity $u(x_B)$, is approximately equal to 99% of $u_1 - u_2$.

$$\frac{u_1 - u(x_A)}{u_1 - u_2} \approx 0.01 \tag{56}$$

$$\frac{u_1 - u(x_B)}{u_1 - u_2} \approx 0.99 \tag{57}$$

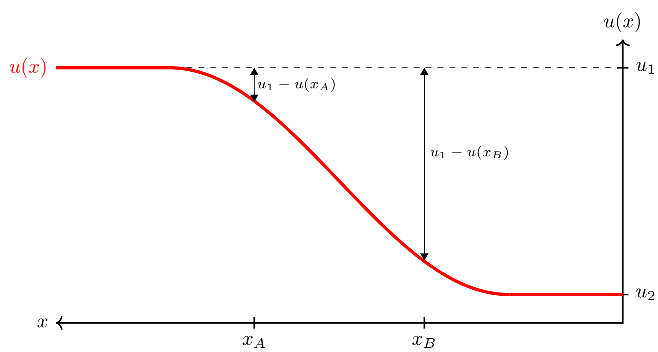

Figure 5: Shock Thickness Measurement (Textbook Method)

Table 1: Nondimensional Thicknesses (Textbook Method)

	$M_1 = 1.1$	$M_1 = 2.0$	$M_1 = 10.0$
Re_{x_t}	95.266	19.107	46.839
x_t/λ	58.026	6.401	3.138