LinMod M1: 6B

Complete Linear Model to a Bayesian Inference Model

MD

Bayes: $P(Y|\beta) = N(X\beta, \sigma^2 I_p)$, $p(\beta) = N(m_0, \Sigma_0)$

Note: σ^2 is fixed.

Goal: Find m_0, Σ_0 such that $E[\beta|Y] = \tilde{\beta}$

① $Y = X\beta + \varepsilon$, $\varepsilon \sim N(0, \sigma^2 I_p)$ and want to add a prior on β. Try $m_0 = 0$, as an "uninformed" guess of the value (rather than expecting $\beta > 0$ or $\beta < 0$). So $\beta \sim N(0, \Sigma_0)$.

② Just as ① alone implies $Y \sim N(X\beta, \sigma^2 I_p)$; $0 = I\beta + \tilde{\varepsilon}$, $\tilde{\varepsilon} \sim N(\phi, \Sigma_0)$ implies $\beta \sim N(\phi, \Sigma_0)$.

$$\begin{bmatrix} Y \\ 0 \end{bmatrix} = \begin{bmatrix} X \\ I \end{bmatrix} \beta + \begin{bmatrix} \varepsilon \\ \tilde{\varepsilon} \end{bmatrix}, \text{ where } \begin{bmatrix} \varepsilon \\ \tilde{\varepsilon} \end{bmatrix} \sim N\left(\begin{bmatrix} \phi_{p\times 1} \\ \phi_{n-p\times 1} \end{bmatrix}, \sigma^2 \begin{bmatrix} I_p & 0 \\ 0 & \Sigma_0 \end{bmatrix} \right)$$

Above is L.S. model whose estimates: $P(\beta|Y) \propto P(Y|\beta) \cdot P(\beta)$ characterize the complete Bayesian model.

Setting $m_0 = 0$ and keeping Σ_0 such that variance is large puts low weight on the prior and more weight on the L.S. estimate $\tilde{\beta}$: e.g. $a I_{n-p}$, $a >> 1$.

The prior would thus be $\beta \sim N(\phi, (a I_{n-p})\sigma^2)$ for large a, and $E[\beta|Y] = \tilde{\beta}$.

$$\boxed{\begin{aligned} & m_0 = 0 \\ & \Sigma_0 = a I_{n-p} \sigma^2, \quad a >> 1 \end{aligned}}$$