MD

Let $\varepsilon_j = y_j - \hat{E}[y_j | x]$ for $j = 1, 2$. $\boxed{\text{Show } E[\varepsilon_j] = 0.}$

$$E[\varepsilon_j] = E\left[y_j - \hat{E}[y_j | x] \right] = E\left[E\left[y_j - \hat{E}[y_j|x] \,\middle|\, x \right] \right] = E[0] = 0 \quad \blacksquare$$

Uses law of iterated expectations, as in Thm 6.3.1.

Note: $\hat{E}[y_j|x] = \mu_{y_j} + (x-\mu_x)\beta_* = \mu_{y_j} + (x-\mu_x)V_{xx}^{-1}V_{xy_j}$, and therefore

$$\varepsilon_j = (y_j - \mu_{y_j}) - (x-\mu_x)V_{xx}^{-1}V_{xy_j}.$$

For $\rho_{y \cdot x} = \operatorname{corr}(\varepsilon_1, \varepsilon_2) = \dfrac{\operatorname{cov}(\varepsilon_1, \varepsilon_2)}{\cdots} = 0 \Rightarrow \operatorname{cov}(\varepsilon_1, \varepsilon_2) = 0$

$$\operatorname{cov}(\varepsilon_2, \varepsilon_1) = [\operatorname{cov}(\varepsilon_1, \varepsilon_2)]' = 0' = 0,$$

$\boxed{\text{Now show } \operatorname{cov}(\varepsilon_2, \varepsilon_1) = \operatorname{cov}(y_2, \varepsilon_1).}$

$$\operatorname{cov}(\varepsilon_2, \varepsilon_1) = \operatorname{cov}\left(y_2 - \hat{E}[y_2|x], \varepsilon_1 \right) = \operatorname{cov}(y_2, \varepsilon_1) - \underbrace{\operatorname{cov}\left(\hat{E}[y_2|x], \varepsilon_1 \right)}_{0} \quad \blacksquare$$

where the second term $= 0$, because predictions are orthogonal to errors, so their correlation (and thus covariance) is 0.

Together, $\rho_{yx} = 0 \Rightarrow \operatorname{cov}(\varepsilon_2, \varepsilon_1) = \operatorname{cov}(y_2, \varepsilon_1) = 0$ } No correlation between ε_1 and ε_2, an no correlation between y_2 and ε_1, or equivalently between y_1 and ε_2.

$\boxed{\text{Show } \operatorname{cov}(\varepsilon_j, x - \mu_x) = 0.}$

$$\begin{aligned}
\operatorname{cov}(\varepsilon_j, x-\mu_x) &= \operatorname{cov}\left((y_j - \mu_{y_j}) - (x-\mu_x)V_{xx}^{-1}V_{xy_j},\ x - \mu_x \right) \\
&= \operatorname{cov}\left(y_j - \mu_{y_j},\ x - \mu_x \right) - \operatorname{cov}\left((x-\mu_x)V_{xx}^{-1}V_{xy_j},\ x - \mu_x \right) \\
&= V_{y_j x} - \cancel{V_{xx}} \cdot \cancel{V_{xx}^{-1}} V_{xy_j} = 0 \quad \blacksquare
\end{aligned}$$

There is no correlation between the errors and the mean-centered data.