MD

Lin Mod M1 : 4A

Estimate β_1, β_2, and σ^2 for model $y_i = \beta_1 x_{i1} + \beta_2 x_{i2} + e_i$, $e_i \sim N(0, \sigma^2)$, i.i.d.

Given x_{i1}, x_{i2}, y_i.

$$X = \begin{bmatrix} 10 & 15 \\ 9 & 14 \\ 9 & 13 \\ 11 & 15 \\ 11 & 14 \\ 10 & 14 \\ 10 & 16 \\ 12 & 13 \end{bmatrix} \quad Y = \begin{bmatrix} 82 \\ 79 \\ 74 \\ 83 \\ 80 \\ 81 \\ 84 \\ 81 \end{bmatrix}$$

$$\hat{\beta} = \begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{pmatrix} = (X'X)^{-1}X'Y$$

$$\approx \boxed{\begin{pmatrix} 2.65 \\ 3.74 \end{pmatrix}}$$

$$\hat{\sigma}_2 = MSE = \frac{Y'(I-M)Y}{n-r} \quad \text{where } M = X(X'X)^{-1}X', \quad n = 8, \quad r = 2$$

$$= \frac{Y'Y - Y'MY}{n-r} = \frac{Y'Y - Y'(X(X'X)^{-1}X')Y}{n-r} \approx \boxed{4.70}$$

Lin Mod M1 : 4B

Give 95% C.I. for β_1 and $\beta_1 + \beta_2$

$$\frac{\lambda'\hat{\beta} - \lambda'\beta}{(MSE\, \lambda'(X'X)^{-}\lambda)^{\frac{1}{2}}} \sim t(df\,E)$$

For β_1, use $\lambda_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. For $\beta_1+\beta_2$, use $\lambda_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Generally, the C.I. is defined as: $\lambda'\beta = \lambda'\hat{\beta} \pm (MSE\, \lambda'(X'X)^{-}\lambda)^{\frac{1}{2}} \cdot$ "two-sided, 95%, df=6, t-score"

Here, for β_1: $\beta_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}'\hat{\beta} \pm \left(MSE \begin{pmatrix} 1 \\ 0 \end{pmatrix}'(X'X)^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)^{\frac{1}{2}} \cdot 2.45 \approx \boxed{[1.12, 4.17]}$

For $\beta_1+\beta_2$: $\beta_1+\beta_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}'\hat{\beta} \pm \left(MSE \begin{pmatrix} 1 \\ 1 \end{pmatrix}'(X'X)^{-1}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)^{\frac{1}{2}} \cdot 2.45 \approx \boxed{[5.93, 6.84]}$

LinMod M1 : 4C

Perform $\alpha = 0.01$ test for $H_0 : \beta_2 = 3$

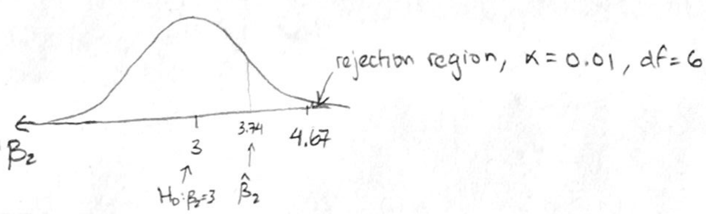
Use same approach as in (4B) : $$\frac{\lambda'\hat{\beta} - \lambda'\beta}{\left(MSE\, \lambda'(X'X)^{-}\lambda\right)^{\frac{1}{2}}} \sim t(df\,E)$$

where here, $\lambda = \begin{pmatrix}0\\1\end{pmatrix}$, $\lambda'\hat{\beta}$ is judged against $\lambda'\beta = 3$ and the resulting t-score is compared to the critical t-score for two-sided, 99%, df = 6 score. ($t_{critical} \approx 3.71$)

$$\frac{\begin{pmatrix}0\\1\end{pmatrix}'\hat{\beta} - 3}{\left[4.70 \begin{pmatrix}0\\1\end{pmatrix}'(X'X)^{-1}\begin{pmatrix}0\\1\end{pmatrix}\right]^{\frac{1}{2}}} \approx 1.64$$

$1.64 < 3.71$

Not extreme enough to reject $H_0 : \beta_2 = 3$ at $\alpha = 0.01$.



Lin Mod M1 : 4D

Find p-value for test $H_0 : \beta_1 - \beta_2 = \emptyset$

Here, use above t-distribution formula with $\lambda = \begin{pmatrix}1\\-1\end{pmatrix}$

$$\frac{\begin{pmatrix}1\\-1\end{pmatrix}'\hat{\beta} - \overbrace{\begin{pmatrix}1\\-1\end{pmatrix}\beta}^{\emptyset}}{\left[mse \cdot \begin{pmatrix}1\\-1\end{pmatrix}'(X'X)^{-1}\begin{pmatrix}1\\-1\end{pmatrix}\right]^{\frac{1}{2}}} \approx -1.02$$

In R: pt(q=-1.02, df=6) gives probability that a value is that far below the null or farther: $P_{low} \approx 0.17$. Since the null is equality, the alternative is non-equality, i.e. a two-tailed test. The p-value is therefore $2 \times P_{low} \approx 0.34$.