Lin Mod M1: 2A

MD

Suppose you have a prior $\beta | g, \sigma^2$ on a p-dimensional vector of regression coefficients: $\beta | g, \sigma^2 \sim N(\underline{0}, \sigma^2 g (X'X)^{-1})$, $\underset{n \times p}{X}$, $r(X) = p$, $g > 0$

Given $\frac{1}{g} \sim Ga(\frac{1}{2}, \frac{n}{2})$, find marginal $p(\beta | \sigma^2)$ (and be aware of relevent normalizing constants)

To solve, integrate out g from $\beta | g, \sigma^2$ to leave marginal $\beta | \sigma^2$.

① $\beta | g, \sigma^2 \sim N(\underline{0}, \sigma^2 g (X'X)^{-1})$

$$= \prod_{i=1}^{p} \left[(2\pi \sigma^2 g [(X'X)^{-1}]_{ii})^{-\frac{1}{2}} \exp\left\{ -\frac{1}{2} \frac{1}{\sigma^2 g [(X'X)^{-1}]_{ii}} \cdot (\beta_i - 0)^2 \right\} \right]$$

② $\frac{1}{g} \sim Ga(\frac{1}{2}, \frac{n}{2})$

$$= \frac{(\frac{n}{2})^{\frac{1}{2}}}{\Gamma(\frac{1}{2})} \left(\frac{1}{g}\right)^{\frac{1}{2}-1} e^{-\frac{n}{2}(\frac{1}{g})}$$

Combine (i.e. multiply) ① and ②, and integrate g (i.e. identify kernel) w.r.t $1/g$, regarding other variables as constant.

$$f = ① \times ② = \prod_{i=1}^{p} \left[(2\pi \sigma^2 g [(X'X)^{-1}]_{ii})^{-\frac{1}{2}} \exp\left\{ -\frac{1}{2} \frac{1}{\sigma^2 g [(X'X)^{-1}]_{ii}} \cdot \beta_i^2 \right\} \right] \cdot \frac{(\frac{n}{2})^{\frac{1}{2}}}{\Gamma(\frac{1}{2})} \left(\frac{1}{g}\right)^{-\frac{1}{2}} e^{-\frac{n}{2}(\frac{1}{g})}$$

Let $[(X'X)^{-1}]_{ii} = v_i$, $f = \prod_{i=1}^{p} \left[(2\pi \sigma^2 v_i)^{-\frac{1}{2}} \cdot \left(\frac{1}{g}\right)^{\frac{1}{2}} \cdot \exp\left\{ -\frac{\beta_i^2}{2\sigma^2 v_i} \cdot \left(\frac{1}{g}\right) \right\} \right] \cdot \frac{(\frac{n}{2})^{\frac{1}{2}}}{\Gamma(\frac{1}{2})} \left(\frac{1}{g}\right)^{-\frac{1}{2}} e^{-\frac{n}{2}(\frac{1}{g})}$

$$f = \left[\prod_{i=1}^{p} (2\pi \sigma^2 v_i)^{-\frac{1}{2}} \right] \cdot \left(\frac{1}{g}\right)^{\frac{p}{2}} \cdot \exp\left\{ \left[\sum_{i=1}^{p} \frac{-\beta_i^2}{2\sigma^2 v_i} \right] \cdot \frac{1}{g} \right\} \cdot \left[\frac{(\frac{n}{2})^{\frac{1}{2}}}{\Gamma(\frac{1}{2})} \right] \cdot \left(\frac{1}{g}\right)^{-\frac{1}{2}} e^{-\frac{n}{2}(\frac{1}{g})}$$

$$\propto \left(\frac{1}{g}\right)^{\frac{p-1}{2}} \exp\left\{ \left[\left(\sum_{i=1}^{p} \frac{-\beta_i^2}{2\sigma^2 v_i} \right) - \frac{n}{2} \right] \left(\frac{1}{g}\right) \right\} \sim Ga\left(\frac{p+1}{2}, \left(\sum_{i=1}^{p} \frac{\beta_i^2}{2\sigma^2 v_i} \right) + \frac{n}{2} \right)$$