LINMOD HW 5 $\boxed{1}$ MD

$$y_i = b_0 + b_1 x_{i1} + b_2 x_{i2} + \varepsilon_i \quad , \quad \varepsilon_i \sim N(\emptyset, \sigma^2)$$

Find $SSR(X_1, X_2 | J) = R(\beta_1, \beta_2 | \beta_0)$. Are b_0, b_1, b_2 estimable?

$$\begin{array}{cc} & \begin{array}{ccc} J & X_1 & X_2 \end{array} \\ X = & \begin{bmatrix} 1 & 4 & 2 \\ 1 & -1 & -3 \\ 1 & 2 & 0 \\ 1 & 0 & -2 \\ 1 & -2 & -4 \\ 1 & 3 & 1 \end{bmatrix} \end{array} = [\, J \;\; X_1 \;\; X_2 \,] \;, \quad X_0 = [\, X_1 \;\; X_2 \,] \;, \quad X_J = \begin{bmatrix} J \end{bmatrix}$$

$$Y = \begin{bmatrix} -2 \\ 7 \\ 2 \\ 5 \\ 8 \\ -1 \end{bmatrix} = [\, J \;\; X_1 \;\; X_2 \,] \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix} + e = J\beta_0 + X_1\beta_1 + X_2\beta_2 + \epsilon$$

$$Y'MY = Y'M_JY + \underbrace{\underset{\substack{\| \\ M(I-M_J)}}{Y'(M-M_J)Y}}_{R(\beta_1, \beta_2 | \beta_0)}$$

M is ppm onto $C(X)$

M_J is ppm onto $C(X_J)$

$M - M_J$ is ppm onto $C(X_0)$

$$M = X(X'X)^- X'$$

$$M_J = J(J'J)^- J' = \frac{1}{n} \cdot JJ'$$

$$R(\beta_1, \beta_2 | \beta_0) = Y'MY - Y'M_JY \approx 145.9 - 60.2 = \boxed{\begin{array}{l} SSR(X_1, X_2 | J) = \\ 85.7 \end{array}}$$

$\beta_0, \beta_1, \beta_2$ are not estimable, because X is not full rank $X_2 = X_1 - 2 \cdot J$,

so $X'X$ is singular, and $X'X\beta = X'Y$ cannot be solved uniquely for β.

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What happens to the test of $\Lambda' b = d$ if $\Lambda' b$ has no estimable part?

$\Lambda' b$ not estimable $\Rightarrow$ no unique solution to $b = (X'X)^{-} X'Y$

$\Rightarrow$ for $b_1 \neq b_2$, $(Y - Xb_1)'(Y - Xb_1) = (Y - Xb_2)'(Y - Xb_2)$ ⊛

$Xb_1 = Xb_2$ $\qquad SSE(b_1) = SSE(b_2)$

$\Lambda' \beta = \Lambda' b = d \Rightarrow d \in C(\Lambda')$ $\Rightarrow Y = XU\gamma + Xb + \varepsilon$

$\Rightarrow \beta - b \in C(\Lambda)^{\perp} = CU$ $\qquad Y - Xb = XU\gamma + \varepsilon = X_0 \gamma + \varepsilon$

$= U\gamma$ $\qquad Y - Xb - X_0\gamma + \Lambda' b - d = \varepsilon$

$X\beta - Xb = XU\gamma$

$Y = X\beta + kXb^{\,0} + \varepsilon$

$= X(\beta + kb) + \varepsilon$

$Y \sim N(X(\beta + kb), \sigma^2 I)$

Test / Likelihood Ratio numerator is the same for any b that solves $\Lambda' b = d$ because of ⊛ above, so the test does not change.

Corollary 3.3.8: $\Lambda' \beta$ has no estimable part iff the constraint <u>does not affect the model</u>, it defines a reparameterization, a side condition.

$$C(\Lambda) \cap C(X') = \{\emptyset\} \quad \text{iff} \quad C(XU) = C(X)$$

Can obtain numerator sum of squares for testing $\Lambda' \beta = 0$ by finding $X_0 = XU$ directly and using it to get $M - M_0$, or by finding Λ_0 with $\Lambda_0' = P_0' X$ and using M_{MP_0}