Lin Mod M1 : 2B

MB

$$\underline{Y} \mid \underline{\beta}, \sigma^2, \tfrac{1}{g} \sim N(X\beta, \sigma^2 I_n)$$

$$Y = X\beta + \varepsilon \quad , \quad \underline{\varepsilon} \sim N(\underline{0}, \sigma^2 I_n)$$

$$\underline{\beta} \mid g, \sigma^2 \sim N(\underline{0}, \sigma^2 g (X'X)^{-1})$$

$$\frac{1}{g} \sim Ga\left(\frac{1}{2}, \frac{n}{2}\right)$$

Find distribution of $\frac{1}{g} \,\Big|\, \underline{\beta}, \sigma^2, \underline{Y}$. Call $f = P\left(\frac{1}{g} \,\middle|\, \underline{\beta}, \sigma^2, \underline{Y}\right)$

$$f = \frac{P(\underline{Y}, \underline{\beta}, \sigma^2, \frac{1}{g})}{P(\underline{Y}, \underline{\beta}, \sigma^2)} = \frac{\text{joint}}{\text{marginal}} = \frac{P(\underline{Y} \mid \underline{\beta}, \sigma^2, \frac{1}{g}) P(\underline{\beta} \mid \sigma^2, \frac{1}{g}) P(\frac{1}{g})}{\text{marginal}}$$

$$\propto P(\underline{Y} \mid \underline{\beta}, \sigma^2, \tfrac{1}{g}) P(\underline{\beta} \mid \sigma^2, \tfrac{1}{g}) P(\tfrac{1}{g}) = N(\underline{Y} \mid X\beta, \sigma^2 I_n) \cdot N(\underline{\beta} \mid \underline{0}, \sigma^2 g (X'X)^{-1}) \cdot Ga\left(\tfrac{1}{g} \,\middle|\, \tfrac{1}{2}, \tfrac{n}{2}\right)$$

$$y_i \sim N((X\beta)_i, (\sigma^2 I_n)_{ii}) \qquad \beta_i \sim N(0, (\sigma^2 g (X'X)^{-1})_{ii})$$

$$y_i \sim N(m_i, v_i) \qquad \beta_i \sim N(0, w_i)$$

$$\propto \prod_{i=1}^{n} \left[(2\pi v_i)^{-\frac{1}{2}} \exp\left\{ -\frac{1}{2} \cdot \frac{(y_i - m_i)^2}{v_i} \right\} \right]$$

$$\times$$

$$\prod_{i=1}^{n} \left[(2\pi w_i)^{-\frac{1}{2}} \exp\left\{ -\frac{1}{2} \frac{\beta_i^2}{w_i} \right\} \right]$$

$$\times$$

$$\frac{\left(\frac{n}{2}\right)^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)} \left(\frac{1}{g}\right)^{-\frac{1}{2}} e^{-\frac{n}{2}\left(\frac{1}{g}\right)}$$

In this step, isolate only those terms with $\frac{1}{g}$. This includes the w_i term, as $w_i = [\sigma^2 g (X'X)^{-1}]_{ii}$. Here, $w_i^{-\frac{n}{2}}$ yields $\left(\frac{1}{g}\right)^{\frac{n}{2}}$.

$$\propto \prod_{i=1}^{n} \left[(2\pi v_i)^{-\frac{1}{2}} (2\pi w_i)^{-\frac{1}{2}} \right] \cdot \exp\left\{ -\frac{1}{2} \left[\sum_{i=1}^{n} \left(\frac{(y_i - m_i)^2}{v_i} + \frac{\beta_i^2}{w_i} \right) + \frac{n}{g} \right] \right\} \cdot \left(\frac{1}{g}\right)^{-\frac{1}{2}}$$

$$= C_1 \left(\frac{1}{g}\right)^{\frac{n}{2}} \cdot \exp\{\ldots\} \cdot \left(\frac{1}{g}\right)^{-\frac{1}{2}}$$

$$= C_1 \left(\frac{1}{g}\right)^{\frac{n}{2}} \cdot \exp\left\{-\frac{1}{2}\left(\frac{n}{g}\right) - \frac{1}{2}\left(\sum_{i=1}^{n}\left(\frac{(y_i-m_i)^2}{v_i} + \frac{\beta_i^2}{w_i}\right)\right)\right\} \cdot \left(\frac{1}{g}\right)^{-\frac{1}{2}}$$

Let $C_2 = \frac{(y_i-m_i)^2}{v_i}$

Let $w_i^{-1} = \frac{1}{g} \cdot C_3$

$$\propto \left(\frac{1}{g}\right)^{\frac{n-1}{2}} \exp\left\{-\frac{n}{2}\left(\frac{1}{g}\right) - \frac{1}{2}\left[\sum_{i=1}^{n}\left(C_2 + \beta_i^2 \cdot C_3 \cdot \left(\frac{1}{g}\right)\right)\right]\right\}$$

$$\propto \left(\frac{1}{g}\right)^{\frac{n-1}{g}} \exp\left\{-\frac{n}{2}\left(\frac{1}{g}\right) - \frac{1}{2}\sum_{i=1}^{n}\left(\beta_i^2 \cdot C_3\left(\frac{1}{g}\right)\right)\right\}$$

$$= \left(\frac{1}{g}\right)^{\frac{n-1}{2}} \cdot \exp\left\{-\frac{n}{2}\left(\frac{1}{g}\right) - \frac{1}{2} \cdot n\left(C_3\left(\frac{1}{g}\right)\right) \cdot \sum_{i=1}^{n} \beta_i^2\right\}$$

$$= \left(\frac{1}{g}\right)^{\frac{n-1}{2}} \exp\left\{-\left[\frac{n}{2} + \frac{n}{2} \cdot C_3 \cdot \sum_{i=1}^{n} \beta_i^2\right]\left(\frac{1}{g}\right)\right\}$$

$$\sim Ga\left(\frac{n+1}{2},\ \frac{n}{2}\left(1 + C_3 \cdot \sum_{i=1}^{n} \beta_i^2\right)\right) \quad \blacksquare$$

where $C_3 = \left[\sigma^2(X'X)^{-1}\right]_{ii}^{-1}$

Recall:

$$w_i = \left[\sigma^2 g (X'X)^{-1}\right]_{ii}$$

$$= g\left[\sigma^2 (X'X)^{-1}\right]_{ii}$$

$$w_i^{-1} = g^{-1}\left[\sigma^2 (X'X)^{-1}\right]_{ii}^{-1}$$

$$= \frac{1}{g} \cdot C_3$$