Lin Mod M1 : 3A

MD

Show $E\left[(X_0\hat{\gamma} - X\beta)'(X_0\hat{\gamma} - X\beta)\right] = \sigma^2 \, tr(M_0) + \beta' X'(I - M_0) X\beta$,

where M_0 is p.p.m. onto $C(X_0)$ and $\hat{\gamma}$ is L.S. estimator in simple model.

Here: $Y = X\beta + \varepsilon$, $\varepsilon \sim N(0, \sigma^2 I_n) \Rightarrow Y \sim N(X\beta, \sigma^2 I_n)$, $E[Y] = X\beta$

$cov(Y) = \sigma^2 I_n$

Since $\hat{\gamma}$ is LS, $X_0\hat{\gamma} = M_0 Y$. $E[X_0\hat{\gamma}] = E[M_0 Y] = M_0 \cdot E[Y] = M_0 X\beta$

The left side, above, can be written with terms swapped: $E\left[(X\beta - X_0\hat{\gamma})'(X\beta - X_0\hat{\gamma})\right] = ⊛$

Now call $z = X\beta - X_0\hat{\gamma}$, so that $⊛ = E[z'z]$.

Using the variance identity: $E[z'z] = cov(z) + E[z]^2$.

First, simplify $z = X\beta - X_0\hat{\gamma} = X\beta - M_0 Y$

Then find $E[z] = E[X\beta - M_0 Y] = X\beta - M_0 \cdot E[Y] = X\beta - M_0 X\beta = (I - M_0) X\beta$

and $cov(z) = cov(X\beta - M_0 Y) = M_0 \cdot cov(Y) \cdot M_0' = M_0 \cdot \sigma^2 I_n \cdot M_0' = \sigma^2 I_n \cdot M_0 M_0' = \sigma^2 I_n \cdot M_0$

idempotent

Therefore $E[z'z] = cov(z) + E[z]^2$

$= \sigma^2 I_n \cdot M_0 + \left[(I - M_0) X\beta\right]'\left[(I - M_0) X\beta\right]$

$= \sigma^2 \cdot M_0 + \beta' X'(I - M_0)'(I - M_0) X\beta$

$= \sigma^2 \cdot tr(M_0) + \beta' X'(I - M_0) X\beta$ ← idempotent

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