LIN MOD M2 : 3

MD.

3A Find $p(\beta_i \mid y, \lambda_i, \tau, \sigma^2)$. First find joint distribution over vector forms of $\underline{\beta}$, $\underline{y}$, and $\underline{\lambda}$, then get full conditional of $\underline{\beta}$ by finding the kernel w.r.t. $\underline{\beta}$, and finally isolate just the β_i element.

Joint "J" $= P(\underline{y}, \underline{\beta}, \underline{\lambda}, \tau, \sigma^2) = N(\underline{y} \mid \underline{\beta}, \sigma^2 I) \cdot N(\underline{\beta} \mid \underline{0}, \underline{\lambda}^2 I)$

$\cdot\, C^+(\underline{\lambda} \mid \phi, \tau) \; C^+(\tau \mid \phi, \sigma) \cdot P(\sigma^2)$

Get kernel w.r.t. $\underline{\beta}$:

$$P(\underline{\beta} \mid \underline{y}, \underline{\lambda}, \tau, \sigma^2) \propto J \propto \exp\left\{-\frac{1}{2}\left[(\underline{y}-\underline{\beta})'(\sigma^2 I)^{-1}(\underline{y}-\underline{\beta})\right]\right\}$$

$$\times \exp\left\{-\frac{1}{2}\left[\underline{\beta}'(\underline{\lambda}^2 I)^{-1}\underline{\beta}\right]\right\}$$

$$= \exp\left\{-\frac{1}{2}\left[\underbrace{(\underline{y}-\underline{\beta})'}_{1\times n}\underbrace{(\sigma^2 I)^{-1}}_{n\times n}\underbrace{(\underline{y}-\underline{\beta})}_{n\times 1} + \underbrace{\underline{\beta}'}_{1\times n}\underbrace{(\underline{\lambda}^2 I)^{-1}}_{n\times n}\underbrace{\underline{\beta}}_{n\times 1}\right]\right\}$$

Taking just β_i, $\propto \exp\left\{-\frac{1}{2}\left[\frac{(y_i-\beta_i)^2}{\sigma^{2}} + \frac{\beta_i^2}{\lambda_i^2}\right]\right\}$

$$= \exp\left\{-\frac{1}{2}\left[\frac{y_i^2}{\sigma^2} - \frac{2y_i\beta_i}{\sigma^2} + \frac{\beta_i^2}{\sigma^2} + \frac{\beta_i^2}{\lambda_i^2}\right]\right\}$$

$$= \exp\left\{-\frac{1}{2}\left[\left(\frac{1}{\sigma^2}+\frac{1}{\lambda_i^2}\right)\beta_i^2 - 2\left(\frac{y_i}{\sigma^2}\right)\beta_i + C_1\right]\right\}$$

By completing the square, $\sim N\left(\left(\frac{1}{\sigma^2}+\frac{1}{\lambda_i^2}\right)^{-1}\left(\frac{y_i}{\sigma^2}\right), \left(\frac{1}{\sigma^2}+\frac{1}{\lambda_i^2}\right)^{-1}\right)$

$$= P(\beta_i \mid y, \lambda_i, \tau, \sigma^2)$$

$\boxed{3B}$ Assume $\sigma^2 = \tau = 1$, let $m_i = E[\beta_i \mid y, \lambda]$. From 3A, found

$$m_i = \left(\frac{1}{\sigma^2} + \frac{1}{\lambda_i^2}\right)^{-1}\left(\frac{y_i}{\sigma^2}\right) = \left(1 + \frac{1}{\lambda_i^2}\right)^{-1} \cdot y_i = \left(\frac{1}{1 + \frac{1}{\lambda_i^2}}\right) \cdot y_i$$

$$= \left(\frac{\lambda_i^2}{\lambda_i^2 + 1}\right) \cdot y_i = \left(1 - \frac{1}{1+\lambda_i^2}\right) \cdot y_i = (1 - K_i) \cdot y_i, \quad \text{for } K_i = \frac{1}{1+\lambda_i^2}.$$

Find $P(K_i)$ by change of variable and plot it.

$$K_i = \frac{1}{1+\lambda_i^2} \Rightarrow \lambda_i = \pm\left(\frac{1}{K_i} - 1\right)^{\frac{1}{2}} \Rightarrow \frac{\partial \lambda_i}{\partial K_i} = \frac{1}{2}\left(\frac{1}{K_i} - 1\right)^{-\frac{1}{2}} (-1)(K_i^{-2})$$

$$= \frac{-1}{2K_i^2\left(\frac{1}{K_i} - 1\right)^{\frac{1}{2}}}$$

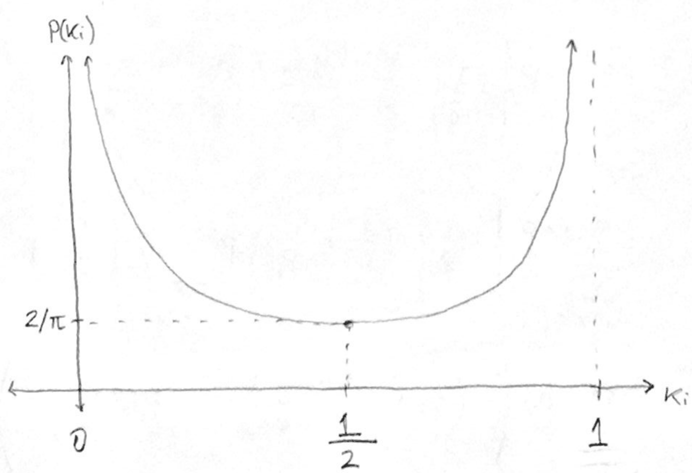
$$P_{K_i}(K_i) = P_{\lambda_i}\left(\lambda_i = \left(\tfrac{1}{K_i} - 1\right)^{\frac{1}{2}}\right) \cdot \left|\frac{\partial \lambda_i}{\partial K_i}\right|$$

$$= \frac{\cancel{2}}{\pi\left(1 + \left[\left(\frac{1}{K_i} - 1\right)^{\frac{1}{2}}\right]^2\right)} \cdot \frac{1}{\cancel{2}K_i^2\left(\frac{1}{K_i} - 1\right)^{\frac{1}{2}}}$$

$$= \frac{1}{\pi\left(\cancel{1} + \frac{1}{K_i} - \cancel{1}\right) K_i^2 \left(\frac{1}{K_i} - 1\right)^{\frac{1}{2}}} = \frac{1}{\pi K_i \left(\frac{1}{K_i} - 1\right)^{\frac{1}{2}}} \cdot \frac{(K_i)^{\frac{1}{2}}}{(K_i)^{\frac{1}{2}}} = \frac{1}{\pi K_i^{+\frac{1}{2}} (1 - K_i)^{\frac{1}{2}}}$$

$$= \frac{1}{\pi} K_i^{-\frac{1}{2}} (1 - K_i)^{-\frac{1}{2}} = \text{Beta}\left(\tfrac{1}{2}, \tfrac{1}{2}\right) \blacksquare$$

Plot.



$$P_{K_i}\left(\tfrac{1}{2}\right) = \frac{1}{\pi}\left(\tfrac{1}{2}\right)^{-\frac{1}{2}}\left(1 - \tfrac{1}{2}\right)^{-\frac{1}{2}}$$

$$= \frac{2}{\pi}$$