Lin Mod M1 : 6A   Show $\tilde{\beta} = V\tilde{\gamma}$.   MD

Given model as in (5), let $\tilde{\beta} = (X'X + kI)^{-1}X'Y$, $\tilde{\gamma}_j = \frac{\lambda_j}{\lambda_j^2 + k} \cdot y_{*j}$ be canonical ridge regression estimates.

Recall: $X = ULV'$
$X' = VLU'$

Note: $U_r' \perp U$ by definition.

$V'V = U'U = I$ since cols of V, U form orthonormal basis

Let $\gamma = V'\beta$

Y_* is estimated by $\begin{bmatrix} L \\ \emptyset \end{bmatrix}\hat{\gamma}$

$L'L = L^2$, since L is diagonal.

$(L'L + kI)$ is square, and diag, so inverse can be expressed element-wise

$(L^2+kI)^{-1}$ and L^2 can commute, since each is diag. of same dimension

$X'XV = VL^2$ by definition

$X'X = VLU' \cdot ULV' = VL^2V'$

$(X'X)^{-1} = (VL^2V')^{-1}$

$V' = V^{-1}$, $(VV')^{-1} = VV'$

Setup: $Y = X\beta + \varepsilon$, $U_* = [U, U_r]$, $Y_* = U_*'Y = \begin{bmatrix} U' \\ U_r' \end{bmatrix} Y$, $\varepsilon_* = U_*'\varepsilon$

$Y = ULV'\beta + \varepsilon$

$Y_* = U_*'(ULV'\beta) + \varepsilon_*$

$= \begin{bmatrix} U' \\ U_r' \end{bmatrix} ULV'\beta + \varepsilon_* = \begin{bmatrix} U'UL \\ \emptyset \end{bmatrix} V'\beta + \varepsilon_*$

$Y_* = \begin{bmatrix} L \\ \emptyset \end{bmatrix}\gamma + \varepsilon_*$ This is the canonical regression model of Y_* with parameter γ, and design matrix $\begin{bmatrix} L \\ \emptyset \end{bmatrix}$.

Use the ridge regression estimate for the regression parameter γ:

$$\tilde{\gamma} = \left([L', \emptyset]\begin{bmatrix} L \\ \emptyset \end{bmatrix} + kI\right)^{-1}[L', \emptyset]\, Y_*$$

$$= (L'L + kI)^{-1}[L', \emptyset]\begin{bmatrix} L \\ \emptyset \end{bmatrix}\hat{\gamma}$$

$$= (L'L + kI)^{-1}(L'L)\hat{\gamma} = (L^2 + kI)^{-1}L^2\hat{\gamma}.$$

$$V\tilde{\gamma} = V\left[(L^2+kI)^{-1}L^2\hat{\gamma}\right]$$

$$= V(L^2+kI)^{-1}L^2 \cdot L^{-1}U'Y$$

$$= V(L^2+kI)^{-1}LU'Y$$

$$= V(L^2+kI)^{-1}(V'V)LU'Y$$

$$= V(L^2+kI)^{-1}V'X'Y$$

$$= (VL^2V' + kI)^{-1}X'Y$$

$$= (X'X + kI)^{-1}X'Y = \tilde{\beta} \quad \blacksquare$$

From above: $Y_* = \begin{bmatrix} L \\ \emptyset \end{bmatrix}\gamma + \varepsilon_*$

$\Rightarrow \hat{\gamma} = [L^{-1}, 0]\, Y_* = [L^{-1}, 0]\begin{bmatrix} U' \\ U_r' \end{bmatrix} Y$

$\hat{\gamma} = L^{-1}U'Y$

$\Rightarrow \hat{\gamma}_j = \frac{u_j'Y}{\lambda_j} = \frac{y_{*j}}{\lambda_j}$