MD

LINMOD M3 : 3

For model with intercept, expect y and $\hat{y}$ to share the same mean, since

$$y_i = \hat{y}_i + e_i \Rightarrow \sum_{i=1}^{n} y_i = \sum_{i=1}^{n} \hat{y}_i + \sum_{i=1}^{n} e_i \text{, where } e_i \sim N(0,\sigma^2) \text{, so } \sum_{i=1}^{n} e_i = \emptyset$$

$$\sum_{i=1}^{n} y_i = \sum_{i=1}^{n} \hat{y}_i \Rightarrow E[y] = \frac{\sum_{i=1}^{n} y_i}{n} = \bar{y}\mathbb{1}_n = \frac{\sum_{i=1}^{n} \hat{y}_i}{n} = E[\hat{y}]$$

Note: $\bar{\mathbf{y}} = \bar{y} \cdot \mathbb{1}_n$

Correlation of y and $\hat{y}$ $\equiv$ cosine of angle between their mean-centered vectors $y-\bar{y}$ and $\hat{y}-\bar{y}$,

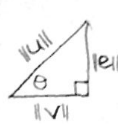
because $$\text{corr}(y,\hat{y}) = \frac{\sum_{i=1}^{n}(y_i-\bar{y})(\hat{y}_i-\bar{y})}{\sqrt{\sum_{i=1}^{n}(y_i-\bar{y})^2}\sqrt{\sum_{i=1}^{n}(\hat{y}_i-\bar{y})^2}}$$ and letting $u = y - \bar{y}$, $v = \hat{y} - \bar{y}$

$$= \frac{u_1v_1 + \ldots + u_nv_n}{\sqrt{u_1^2+\ldots+u_n^2}\sqrt{v_1^2+\ldots+v_n^2}} = \frac{u'v}{\sqrt{u'u}\sqrt{v'v}} = \frac{u \cdot v}{\|u\|\,\|v\|} = \cos\theta$$

where θ is the angle between u and v.

Geometric argument, using Pythagorean, u, and v, finds $\cos\theta$.

First, note:
1. Model has intercept, so $\mathbb{1}_n \in C(X)$ and $\bar{y}\mathbb{1}_n \equiv \bar{\mathbf{y}} \in C(X)$
   Since $\hat{y}$ is a projection onto $C(X)$, $\hat{y} \in C(X) \Rightarrow v = \hat{y} - \bar{y} \in C(X)$
2. Given orthogonal decomposition of $y = \hat{y} + e$, where $\hat{y} \in C(X)$ and $e \in C(X)^\perp \Rightarrow e = y - \hat{y} \in C(X)^\perp$.
3. Recognize that $u = y - \bar{y}$, "triangulates" v and e, as the hypoteneuse:



since $e \perp v$.

Pythagorean Theorem states $\|u\|^2 = \|v\|^2 + \|e\|^2 \Rightarrow \|y-\bar{y}\|^2 = \|\hat{y}-\bar{y}\|^2 + \|y-\hat{y}\|^2$

$$\Rightarrow \underbrace{\sum_{i=1}^{n}(y_i-\bar{y})^2}_{SS_{total}} = \underbrace{\sum_{i=1}^{n}(\hat{y}_i-\bar{y})^2}_{SS_{regr}} + \underbrace{\sum_{i=1}^{n}(y_i-\hat{y}_i)^2}_{SS_{error}}$$, which is equivalent to

$SS_{total} = SS_{regr} + SS_{error}$

Finally, $\cos\theta = \frac{\|v\|}{\|u\|} \Rightarrow (\cos\theta)^2 = [\text{corr}(y,\hat{y})]^2 = \frac{\|v\|^2}{\|u\|^2} = \frac{SS_{reg}}{SS_{total}} = R^2$ ∎