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# LinMod M1 : 1A

Generally, for $M$ p.p.m onto $C(X)$, $M = X(X'X)^{-}X'$

$M_0$: orthogonal projection onto $\mathbf{1} = \begin{bmatrix}1\\1\\1\end{bmatrix}$ $\Rightarrow M_0 = \mathbf{1}(\mathbf{1}'\mathbf{1})^{-}\mathbf{1}' = \begin{bmatrix}1\\1\\1\end{bmatrix}(3)^{-}\begin{bmatrix}1 & 1 & 1\end{bmatrix}$

$$= \begin{bmatrix}1\\1\\1\end{bmatrix}\begin{bmatrix}1 & 1 & 1\end{bmatrix}\cdot\frac{1}{3} = \begin{bmatrix}1/3 & 1/3 & 1/3\\1/3 & 1/3 & 1/3\\1/3 & 1/3 & 1/3\end{bmatrix}$$

$M_1$: orthog proj onto $X_1 = \begin{bmatrix}1\\0\\-1\end{bmatrix}$ $\Rightarrow M_1 = X_1(X_1'X_1)^{-}X_1' = \begin{bmatrix}1\\0\\-1\end{bmatrix}\left(\begin{bmatrix}1 & 0 & -1\end{bmatrix}\begin{bmatrix}1\\0\\-1\end{bmatrix}\right)^{-}\begin{bmatrix}1 & 0 & -1\end{bmatrix}$

$$= \begin{bmatrix}1\\0\\-1\end{bmatrix}(2)^{-}\begin{bmatrix}1 & 0 & -1\end{bmatrix} = \begin{bmatrix}1\\0\\-1\end{bmatrix}\begin{bmatrix}1 & 0 & -1\end{bmatrix}\cdot\frac{1}{2} = \begin{bmatrix}\frac{1}{2} & 0 & -\frac{1}{2}\\0 & 0 & 0\\-\frac{1}{2} & 0 & \frac{1}{2}\end{bmatrix}$$

$M_2$: orthog. proj. onto $X_2 = \begin{bmatrix}1/3\\-2/3\\1/3\end{bmatrix}$ $\Rightarrow M_2 = X_2(X_2'X_2)^{-}X_2' = \begin{bmatrix}1/3\\-2/3\\1/3\end{bmatrix}\left(\begin{bmatrix}\frac{1}{3} & -\frac{2}{3} & \frac{1}{3}\end{bmatrix}\begin{bmatrix}1/3\\-2/3\\1/3\end{bmatrix}\right)^{-}\begin{bmatrix}\frac{1}{3} & -\frac{2}{3} & \frac{1}{3}\end{bmatrix}$

$$= \begin{bmatrix}1/3\\-2/3\\1/3\end{bmatrix}\left(\frac{6}{9}\right)^{-}\begin{bmatrix}\frac{1}{3} & -\frac{2}{3} & \frac{1}{3}\end{bmatrix} = \begin{bmatrix}\frac{1}{3}\\-\frac{2}{3}\\1/3\end{bmatrix}\begin{bmatrix}\frac{1}{3} & -\frac{2}{3} & \frac{1}{3}\end{bmatrix}\cdot\frac{9}{6} = \begin{bmatrix}\frac{1}{6} & -\frac{1}{3} & \frac{1}{6}\\-\frac{1}{3} & \frac{2}{3} & -\frac{1}{3}\\\frac{1}{6} & -\frac{1}{3} & \frac{1}{6}\end{bmatrix}$$

$$M_0 + M_1 + M_2 = \begin{bmatrix}\frac{1}{3} & \frac{1}{3} & \frac{1}{3}\\\frac{1}{3} & \frac{1}{3} & \frac{1}{3}\\\frac{1}{3} & \frac{1}{3} & \frac{1}{3}\end{bmatrix} + \begin{bmatrix}\frac{1}{2} & 0 & -\frac{1}{2}\\0 & 0 & 0\\-\frac{1}{2} & 0 & \frac{1}{2}\end{bmatrix} + \begin{bmatrix}\frac{1}{6} & -\frac{1}{3} & \frac{1}{6}\\-\frac{1}{3} & \frac{2}{3} & -\frac{1}{3}\\\frac{1}{6} & -\frac{1}{3} & \frac{1}{6}\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} = I_3 \quad \blacksquare$$

Lin Mod M1 : 1B

MB

Given $Y = \mathbf{1} \cdot \beta_0 + X_1 \cdot \beta_1 + X_2 \cdot \beta_2 + \varepsilon$ with $\varepsilon \sim N(\underline{0}, \sigma^2 I_3)$, if only $\mathbf{1}$ and $X_1$ are considered, are the OLS estimates of intercept ($\beta_0$) and slope ($\beta_1$) unbiased estimates of $\beta_0$ and $\beta_1$ in the true model?

Yes.

The OLS estimator for the simpler model, $Y = \mathbf{1} \cdot \beta_0 + X_1 \beta_1 + \varepsilon$ is $\hat{\beta} = \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = (X'X)^{-1} X'Y$

Here, $X = \begin{bmatrix} | & | \\ \mathbf{1} & X_1 \\ | & | \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & -1 \end{bmatrix}$

$$\hat{\beta} = \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \left( \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & -1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix}$$

$$= \left( \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 2 & 2 & 2 \\ 3 & 0 & -3 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 2Y_1 + 2Y_2 + 2Y_3 \\ 3Y_1 - 3Y_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{3}(Y_1 + Y_2 + Y_3) \\ \frac{1}{2}(Y_1 - Y_3) \end{bmatrix}$$

The OLS estimator for the true model, $Y = \mathbf{1} \cdot \beta_0 + X_1 \cdot \beta_1 + X_2 \cdot \beta_2 + \varepsilon$ is $\hat{\beta} = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix} = (X'X)^{-1} X'Y$

Here, $X = \begin{bmatrix} | & | & | \\ \mathbf{1} & X_1 & X_2 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1/3 \\ 1 & 0 & -2/3 \\ 1 & -1 & 1/3 \end{bmatrix}$

$$\hat{\beta} = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix} = \left( \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1/3 & -2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1/3 \\ 1 & 0 & 2/3 \\ 1 & -1 & 1/3 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix}$$

$$= \left( \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \frac{2}{3} \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3}(Y_1 + Y_2 + Y_3) \\ \frac{1}{2}(Y_1 - Y_3) \\ \frac{1}{2}(Y_1 - 2Y_2 + Y_3) \end{bmatrix}$$

In both cases, $\beta_0 = \frac{1}{3}(Y_1 + Y_2 + Y_3)$ and $\beta_1 = \frac{1}{2}(Y_1 - Y_3)$. ∎

# Lin Mod M1 : 1C

MD

Let $\hat{\varepsilon}_s$ denote residuals from fitting simpler model, $Y = 1\cdot\beta_0 + X_1\beta_1 + \varepsilon_s$.

$\hat{\varepsilon}_s = Y - (1\hat{\beta}_0 + X_1\hat{\beta}_1)$. Find $E[\hat{\varepsilon}_t]$ where $\hat{\varepsilon}_t = Y - (1\cdot\hat{\beta}_0 + X_1\hat{\beta}_1 + X_2\hat{\beta}_2)$

$$E[\hat{\varepsilon}_t] = E\left[Y - 1\cdot\hat{\beta}_0 - X_1\hat{\beta}_1 - X_2\hat{\beta}_2\right]$$

$$\hat{\varepsilon}_s = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} - \left(\tfrac{1}{3}(Y_1+Y_2+Y_3)\right)\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \left(\tfrac{1}{2}(Y_1-Y_3)\right)\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \underset{(3\times 1)}{\begin{bmatrix} Y_1 - \frac{1}{3}(Y_1+Y_2+Y_3) - \frac{1}{2}(Y_1-Y_3) \\ Y_2 - \frac{1}{3}(Y_1+Y_2+Y_3) \\ Y_3 - \frac{1}{3}(Y_1+Y_2+Y_3) + \frac{1}{2}(Y_1-Y_3) \end{bmatrix}} = \underset{(3\times 1)}{\begin{bmatrix} \frac{1}{6}Y_1 - \frac{1}{3}Y_2 + \frac{1}{6}Y_3 \\ -\frac{1}{3}Y_1 + \frac{2}{3}Y_2 - \frac{1}{3}Y_3 \\ \frac{1}{6}Y_1 - \frac{1}{3}Y_2 + \frac{1}{6}Y_3 \end{bmatrix}}$$

Note: $\hat{\varepsilon}_s$ is the residual after projecting $Y$ onto $1$ and $X_1$.
$\hat{\varepsilon}_t$ is the residual after projecting $Y$ onto $1$, $X_1$, and $X_2$.

Therefore, by subtracting out $\hat{\varepsilon}_s$'s projection onto $X_2$, what remains is the residual after projecting onto $1$, $X_1$, and $X_2$. So $\hat{\varepsilon}_s - M_2\hat{\varepsilon}_s = \hat{\varepsilon}_t$.

$$M_2\hat{\varepsilon}_s = \underset{(3\times 3)}{\begin{bmatrix} \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \end{bmatrix}} \underset{(3\times 1)}{\begin{bmatrix} \frac{1}{6}Y_1 - \frac{1}{3}Y_2 + \frac{1}{6}Y_3 \\ -\frac{1}{3}Y_1 + \frac{2}{3}Y_2 - \frac{1}{3}Y_3 \\ \frac{1}{6}Y_1 - \frac{1}{3}Y_2 + \frac{1}{6}Y_3 \end{bmatrix}} = \underbrace{\begin{bmatrix} \frac{1}{6}(\frac{1}{6}Y_1 - \frac{1}{3}Y_2 + \frac{1}{6}Y_3) - \frac{1}{3}(-\frac{1}{3}Y_1 + \frac{2}{3}Y_2 - \frac{1}{3}Y_3) + \frac{1}{6}(\frac{1}{6}Y_1 - \frac{1}{3}Y_2 + \frac{1}{6}Y_3) \\ -\frac{1}{3}(\quad\downarrow\quad) + \frac{2}{3}(\quad\downarrow\quad) - \frac{1}{3}(\quad\downarrow\quad) \\ \frac{1}{6}(\quad\downarrow\quad) - \frac{1}{3}(\quad\downarrow\quad) + \frac{1}{6}(\quad\downarrow\quad) \end{bmatrix}}_{(3\times 1)}$$

$$\hat{\varepsilon}_t = \hat{\varepsilon}_s - M_2\hat{\varepsilon}_s = \underset{(3\times 1)}{\begin{bmatrix} \frac{1}{6}Y_1 - \frac{1}{3}Y_2 + \frac{1}{6}Y_3 \\ -\frac{1}{3}Y_1 + \frac{2}{3}Y_2 - \frac{1}{3}Y_3 \\ \frac{1}{6}Y_1 - \frac{1}{3}Y_2 + \frac{1}{6}Y_3 \end{bmatrix}} - \underset{(3\times 1)}{\nabla}$$

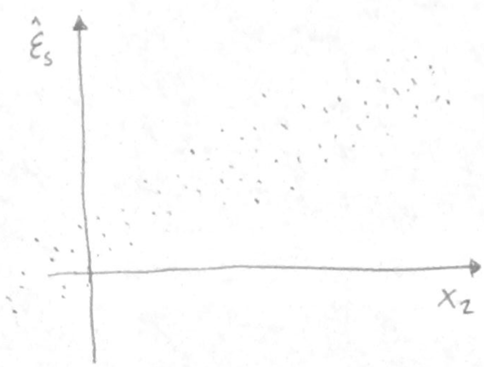
$$= \begin{bmatrix} \frac{5}{6}(\frac{1}{6}Y_1 - \frac{1}{3}Y_2 + \frac{1}{6}Y_3) - \frac{1}{3}(-\frac{1}{3}Y_1 + \frac{2}{3}Y_2 - \frac{1}{3}Y_3) + \frac{1}{6}(\frac{1}{6}Y_1 - \frac{1}{3}Y_2 + \frac{1}{6}Y_3) \\ -\frac{1}{3}(\quad\downarrow\quad) + \frac{1}{3}(\quad\downarrow\quad) - \frac{1}{3}(\quad\downarrow\quad) \\ \frac{1}{6}(\quad\downarrow\quad) - \frac{1}{3}(\quad\downarrow\quad) + \frac{5}{6}(\quad\downarrow\quad) \end{bmatrix} = \begin{bmatrix} \frac{5}{18}Y_1 - \frac{5}{9}Y_2 + \frac{5}{18}Y_3 \\ -\frac{2}{9}Y_1 + \frac{4}{9}Y_2 - \frac{2}{9}Y_3 \\ \frac{5}{18}Y_1 - \frac{5}{9}Y_2 + \frac{5}{18}Y_3 \end{bmatrix}$$

$$\hat{\varepsilon}_t = \begin{bmatrix} \frac{5}{18}(Y_1 - 2Y_2 + Y_3) \\ -\frac{2}{9}(Y_1 - 2Y_2 + Y_3) \\ \frac{5}{18}(Y_1 - 2Y_2 + Y_3) \end{bmatrix} = \begin{bmatrix} \frac{5}{18} \\ -\frac{2}{9} \\ \frac{5}{18} \end{bmatrix} (Y_1 - 2Y_2 + Y_3)$$

$(3 \times 1)$

## Lin Mod M1 : 1D

Given $\hat{\varepsilon}_s$ as residuals on the simple model, $Y = 1 \cdot \beta_0 + X_1\beta_1 + \varepsilon_1$; if the data comes from the true model, $Y = 1\beta_0 + X_1\beta_1 + X_2\beta_2 + \varepsilon_t$, then plotting $\hat{\varepsilon}_s$ versus $X_2$ will show some correlation, since the information that "should" be expressed by $X_2$ is forced into the simple model's error term $\varepsilon_s$.




$\hat{\varepsilon}_s \not\perp X_2$ if the data comes from the true model.