MCMC HW 3

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(a)

$$Pr(y_i = 1|X_i) = \Phi(X_i'\beta). \tag{1}$$

$$y_i = I(z_i > 0) \tag{2}$$

$$z_i \sim\ N(X_i'\beta, 1) \tag{3}$$

Show that (1) $\equiv$ (2) and (3):
First, "$\Leftarrow$" (2) and (3) imply

$$y_i = \begin{cases} 1: & z_i > 0, \text{ or when } z_i = X_i'\beta + \epsilon_i > 0 \Rightarrow \epsilon_i > -X_i'\beta \\ 0: & \text{otherwise} \end{cases}$$

Where $z_i = X_i'\beta + \epsilon_i$, $\epsilon \sim\ N(0, 1)$.

So, $P(y_i = 1|X_i) = P(z_i > 0) = P(\epsilon_i > -X_i'\beta) = P(\epsilon_i < X_i'\beta) = \Phi(X_i'\beta) \equiv (1)$.

The inequality in the penultimate step is allowed to switch signs by the symmetry of the Normal distribution.

(b) Find conditional posterior $P(z_i|\beta, y)$ and $P(\beta|z, y)$, using prior on β, $P(\beta) = 1$.

The conditional posterior of z_i is identical to (3), except that the given value of y_i informs the value of z_i, by definition.

$$z_i|\beta, y \sim \begin{cases} N(X_i'\beta, 1), \text{ left truncated at 0, if } y_i = 1 \\ N(X_i'\beta, 1), \text{ right truncated at 0, if } y_i = 0 \end{cases}$$

In summary, if $y_i = 1$ then the distribution is the right side of the normal distribution centered at $X_i'\beta$, whereas if $y_i = 0$ then the distribution is the left side.

The conditional posterior of β, given a flat, uninformative prior, is simply the standard least squares result on the Normal z:

$$\beta|z, y \sim\ N(\hat{\beta}_z, 1 \cdot (X'X)^{-1}) \tag{4}$$

$$\text{Where } \hat{\beta}_z = (X'X)^{-1}X'z \tag{5}$$

(c) Gibbs sampler proposition:
Initialize value for β as MLE, and call it β_0. Then set $\beta_0 = \beta_k$.

1. Sample z_k from $P(z_k|\beta_k, y)$.
2. Sample β_{k+1} from $P(\beta_{k+1}|z_k, y)$.
3. Repeat until convergence and good mixing.

After discarding values from the beginning of the chain (the burn in period), the resulting values of β and z will be an estimate of the exact joint posterior distribution $P(\beta, z|y)$.

(d) Show the histogram of simulated β values as an estimate of $p(\beta_j|y)$.