MCMC M2 : 1

Joint Posterior

MAURICE DIESENDRUCK

$$\theta = (k, \beta_k, \sigma^2), \quad P(\theta|y) \propto P(y, k, \beta_k, \sigma^2) = P(y|k, \beta_k, \sigma^2)\, P\left(\tfrac{1}{\sigma^2}\right) P(\beta_k|k)\, P(k) = *$$

$$* = N(y|X_k\beta_k, \sigma^2)\, Ga\left(\tfrac{1}{\sigma^2}\Big|1,1\right) N(\beta_k|\phi, 10\cdot I_{2k+1})\, Poi^+(k|\lambda)$$

$$\propto \left[\prod_{i=1}^{n} \left(\tfrac{1}{\sigma^2}\right)^{\frac{1}{2}} \exp\left\{-\tfrac{1}{2}\left(\tfrac{1}{\sigma^2}\right)(y_i - [X_k\beta_k]_i)^2\right\}\right]\left[\left(\tfrac{1}{\sigma^2}\right)^{1-1}\exp\left\{-\left(\tfrac{1}{\sigma^2}\right)\right\}\right]$$

$$\times \left[\prod_{j=1}^{2k+1} \exp\left\{-\frac{1}{2}\frac{\beta_{kj}^2}{10}\right\}\right]\left[\exp\{-\lambda\}\frac{\lambda^k}{k!}\right]$$

$$= (\sigma^2)^{-\frac{n}{2}} \exp\left\{-\tfrac{1}{2}(\sigma^2)^{-1}(y - X_k\beta_k)'(y - X_k\beta_k)\right\} \cdot \exp\left\{-(\sigma^2)^{-1}\right\} \exp\left\{-\frac{1}{2}\frac{\beta_k'\beta_k}{10}\right\} \exp\{-\lambda\}\frac{\lambda^k}{k!}$$

MCMC M2 : 2

Complete Conditionals

MD

$$P(\beta_k \mid k, \sigma^2, y) \propto \exp\left\{-\frac{\tau}{2}(y - X_k\beta_k)'(y - X_k\beta_k)\right\} \exp\left\{-\frac{1}{2}\frac{\beta_k'\beta_k}{10}\right\}$$

$$= \exp\left\{-\frac{1}{2}\left[\tau\left(y'y - y'X_k\beta_k - \beta_k'X_k'y + \beta_k'X_k'X_k\beta_k\right) + \frac{1}{10}\left(\beta_k'\beta_k\right)\right]\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\left[\beta_k'\left(\frac{X_k'X_k}{\sigma^2} + \frac{I_{2k+1}}{10}\right)\beta_k - \beta_k'\left(\frac{X_k'y}{\sigma^2}\right) + \ldots\right]\right\}$$

$$\sim N\left(\left(\frac{X_k'X_k}{\sigma^2} + \frac{I_{2k+1}}{10}\right)^{-1}\frac{X_k'y}{\sigma^2},\ \left(\frac{X_k'X_k}{\sigma^2} + \frac{I_{2k+1}}{10}\right)^{-1}\right)$$

Used "pattern matching" technique: $(y-m)'\Sigma^{-1}(y-m)$
$= y'\Sigma^{-1}y - y'\Sigma^{-1}m + \ldots$

$$P(\tau \mid k, \beta_k, y) \propto \tau^{\frac{n}{2}} \exp\left\{-\frac{\tau}{2}(y - X_k\beta_k)'(y - X_k\beta_k)\right\} \cdot \exp\{-\tau\}$$

$$= \tau^{\frac{n}{2}} \exp\left\{-\frac{\tau}{2}(y - X_k\beta_k)'(y - X_k\beta_k) - \tau\right\}$$

$$= \tau^{\frac{n}{2}} \exp\left\{-\left[\frac{(y - X_k\beta_k)'(y - X_k\beta_k)}{2} + 1\right]\tau\right\}$$

$$\sim Ga\left(\frac{n}{2} + 1,\ \frac{(y - X_k\beta_k)'(y - X_k\beta_k)}{2} + 1\right)$$

MCMC M2 : 3

Birth Acceptance Probability

MD

$$\alpha_{Birth}(K, K+1) = \min\left\{1, \rho(K, K+1)\right\} \quad \text{where}$$

$$\rho(K,K+1) = \frac{\text{target}_{K+1}}{\text{target}_K} \cdot \frac{\text{transition}_{K+1\to K}}{\text{transition}_{K\to K+1}} \cdot \frac{1}{\text{auxiliary}(\mu)} \cdot |\text{Jacobian}|$$

$$= \underbrace{\frac{P(K+1, \beta_{K+1}, \sigma^2 \mid y)}{P(K, \beta_K, \sigma^2 \mid y)}}_{(A)} \cdot \underbrace{\frac{P(K \mid K+1)}{P(K+1 \mid K)}}_{(B)} \cdot \underbrace{\frac{1}{P(\mu)}}_{(C)} \cdot \underbrace{\frac{\partial T}{\partial x_K \partial \mu}}_{(D)}$$

(A) is defined as the joint posterior probability, calculated in part (1).

(B) is the transition ratio, where probability of $P(K|K+1) = P(K+1|K) = \frac{1}{2}$,

unless $K=1 \Rightarrow P(K+1|K) = 1$, or $K = K_{MAX} \Rightarrow P(K|K+1) = 1$.

(C) is the density of the randomly sampled μ, in the birth move.

(D) is the Jacobian of the transformation $T(x_K, \mu) = x_{K+1}$

MCMC M2 : 4

Death Acceptance Probability

$$\alpha_{Death}(K+1, K) = \min\left\{1, \frac{1}{\rho(K, K+1)}\right\}$$

As defined in (3).

MCMC M2 : 5 Plots for RJ MCMC

MB

See attachment

MCMC M2 : 6 Marginal of Y

Simple form of candidate's formula :

$$P(\beta|y) = \frac{P(y|\beta)P(\beta)}{P(y)} \Rightarrow P(y) = \frac{P(y|\beta)P(\beta)}{P(\beta|y)}$$

In this case :

$$P(\bar{\beta}_K, K \mid y) = \frac{P(y|\bar{\beta}_K, K)\, P(\bar{\beta}_K|K)\, P(K)}{P(y|K)}$$

$$\Rightarrow P(y|K) = \frac{P(y \mid \bar{\beta}_K, K)\, P(\bar{\beta}_K \mid K)\, P(K)}{P(\bar{\beta}_K, K \mid y)}$$

For a known K: Can find $P(K) \sim Pois^{+}(\lambda)$, and the first two numerator terms yield

$$P(\bar{\beta}_K | y, K) \sim N\left(\underbrace{\left(2X_K'X_K + \frac{I_{2K+1}}{10}\right)^{-1} \cdot 2X_K'y}_{\bar{\beta}_K},\ \underbrace{\left(2X_K'X_K + \frac{I_{2K+1}}{10}\right)^{-1}}_{S_K} \right)$$

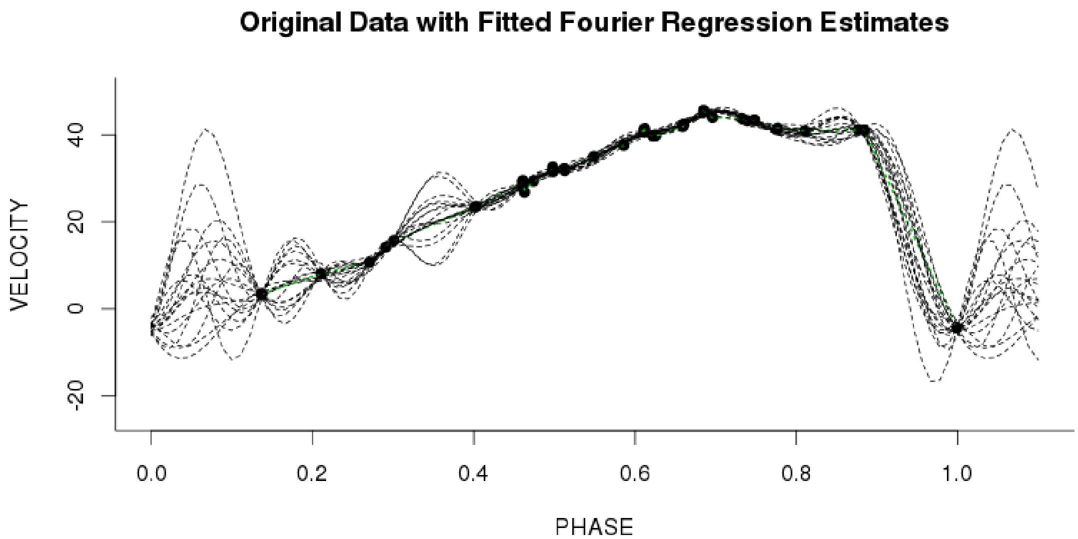
Therefore $P(y|K) \propto N(\bar{\beta}_K, S_K)$

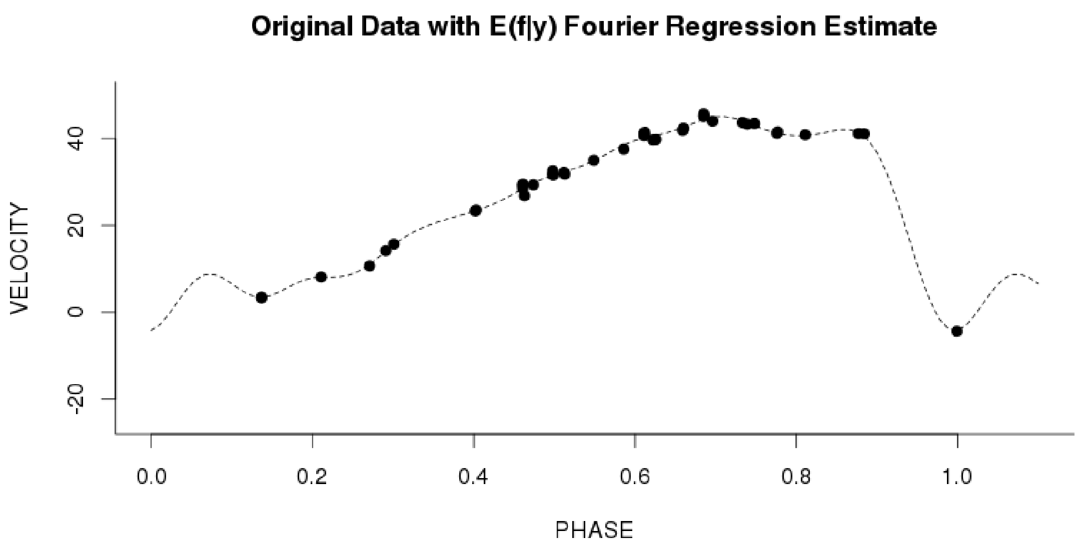
$$\propto |S_K|^{-\frac{1}{2}} P(y|\bar{\beta}_K, K) P(\bar{\beta}_K)$$

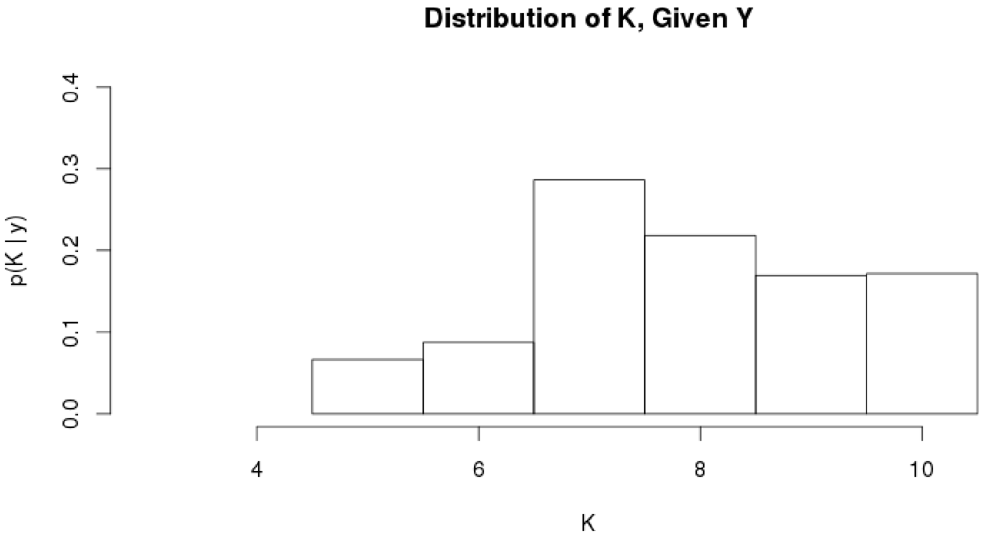
MCMC Midterm 2: Attachments

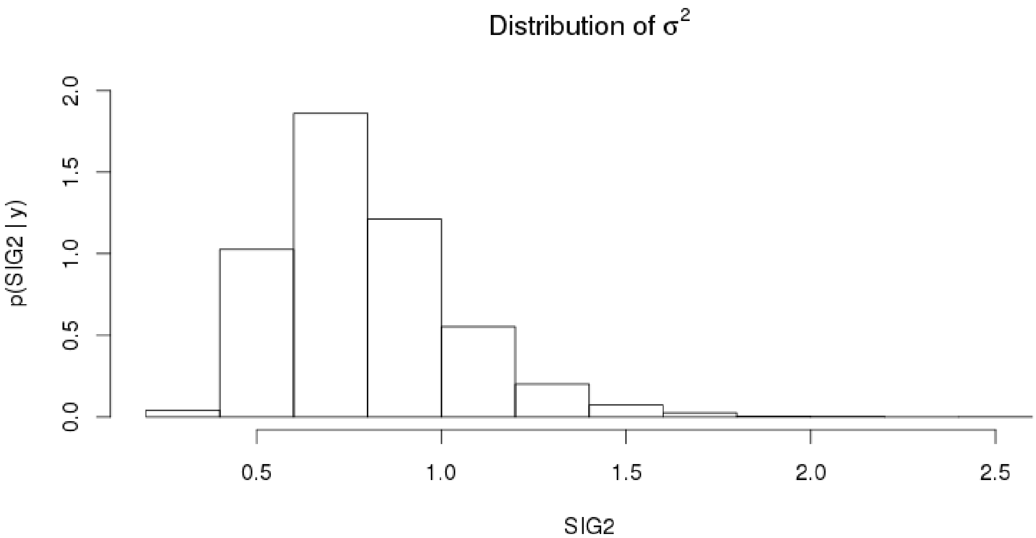
Maurice Diesendruck

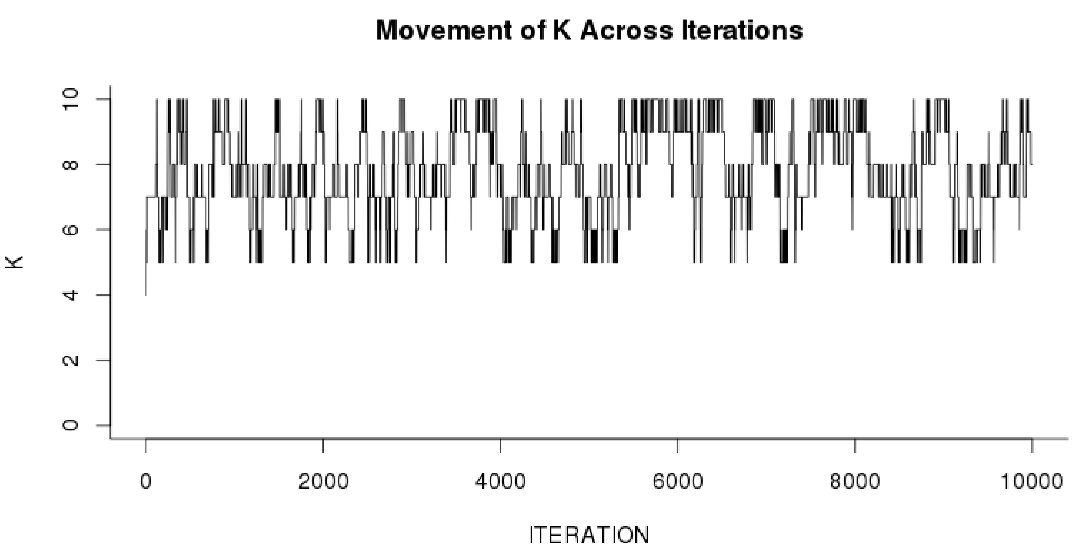
April 10, 2015

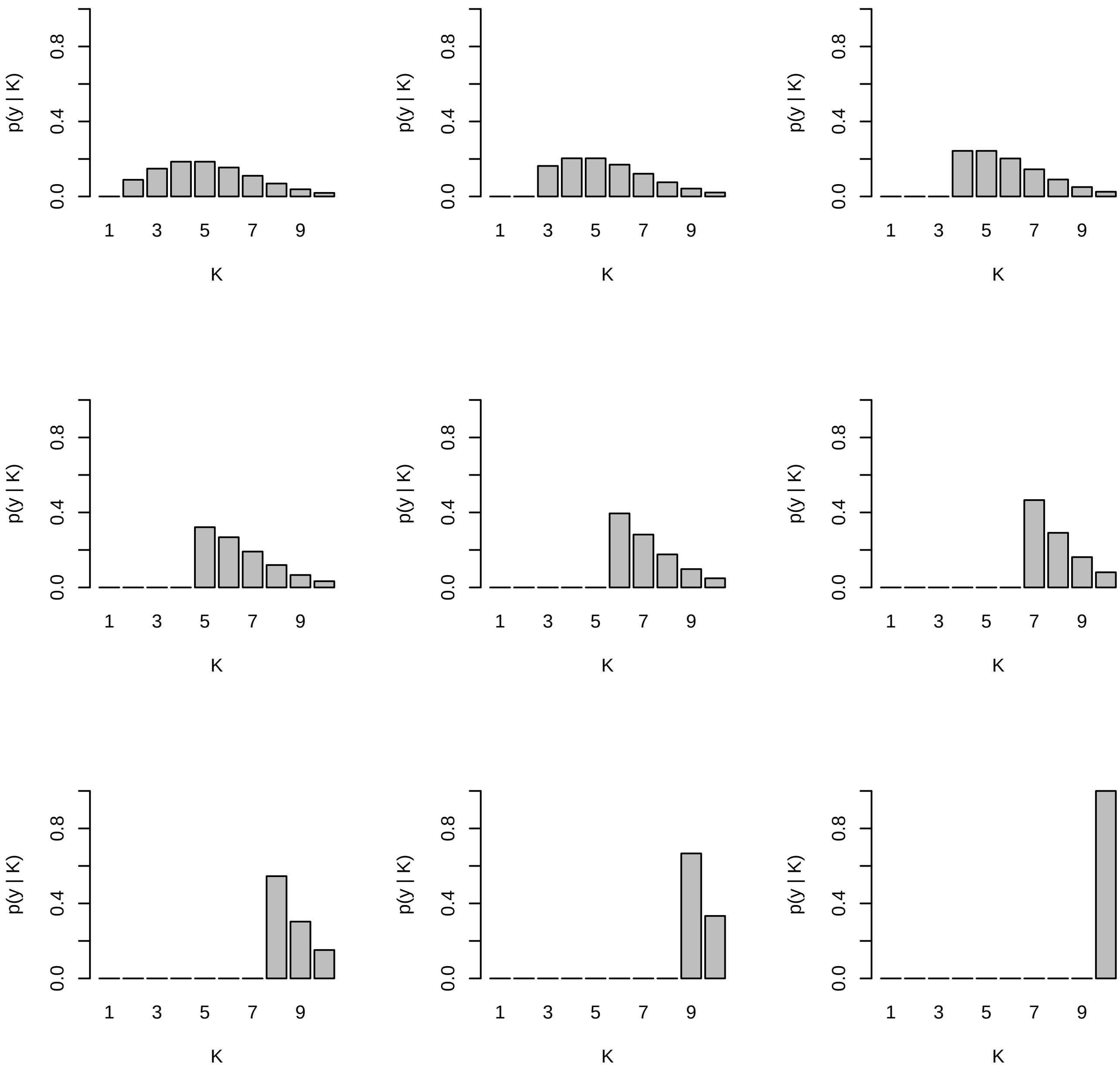


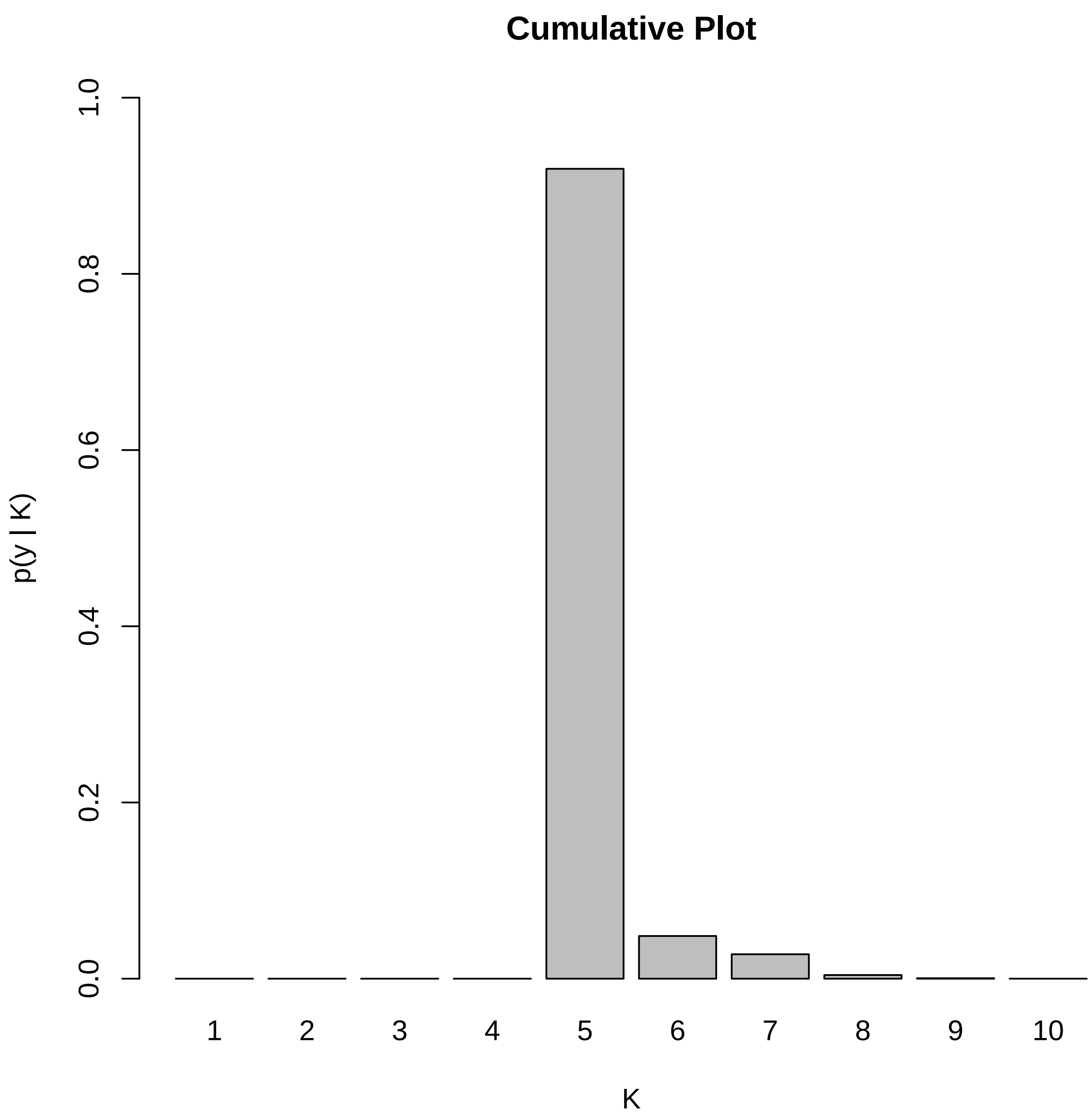



Distribution of K, Given Y
p(K | y)
0.0
0.1
0.2
0.3
0.4
4
6
8
10
K


Distribution of σ²
p(SIG2 | y)
0.0
0.5
1.0
1.5
2.0
0.5
1.0
1.5
2.0
2.5
SIG2


Movement of K Across Iterations
K
0
2
4
6
8
10
0
2000
4000
6000
8000
10000
ITERATION


p(y | K)
0.0
0.4
0.8
1
3
5
7
9
K



Relevant Code Snippets

0.1 Log Joint Posterior for Ratio

```
ljointpost=function(K, b, sig2){
  #print("calculating posterior with param:");
  n = length(y);
  p <- length(b)
  yhat <- X[,1:p] %*% b
```

```
  # Posterior. Cancel out terms without K, yhat, b.
  lpo  <- (-1/2)*(1/sig2)*sum((y-yhat)^2) - 1/2*sum(b^2)/10 + K*log(lambda) -
          log(factorial(K))

  return(lpo);
}
```

0.2 Gibbs Conditional Posterior Distributions

```
sample.b <- function(K,sig2) { # generate b ~ p(b | K, sig2, y)
  idx <- 1:(2*K+1)    # select columns (elements) for K harmonics
  Xk <- X[,idx]       # Subset of design matrix, with 2K+1 columns.
  A0k <- A0[idx,idx] # Precision matrix of beta prior.
  b0k <- b0[idx]      # Mean vector of zeros from beta prior.

  # Full conditionals from Question 2
  V <- solve(t(Xk)%*%Xk/sig2+A0k)
  mm <- V%*%(t(Xk)%*%y/sig2)
  L <- t( chol(V))              # LL' = V
  b <- mm + L %*% rnorm(2*K+1) # b ~ N(m,V)

  return (b)
}

sample.sig2 <- function(K,b) { # generate 1/sig2 ~ p(1/sig2 | K,b,y)
  p <- length(b)
  idx <- 1:(2*K+1)    # select columns (elements) for K harmonics
  Xk <- X[,idx]       # Subset of design matrix, with 2K+1.

  a1 <- (n/2)+1
  b1 <- t(y-Xk%*%b)%*%(y-Xk%*%b)/2 + 1

  sig2inv <- rgamma(1,shape=a1,rate=b1)
  sig2 <- 1/sig2inv
  return (sig2)
}
```

0.3 Auxiliary Variable Transformation and Reverse Transformation

```
qu <- function(K, b, sig2) {
  ## find m,L for mapping T: (b,u) -> b1,  below
  ## bnew = m + L*u,
  ## Use a regression of residuals on (K+1)-st harmonic
```

```
  ##     to determine m and L
  idx <- 1:(2*K+1)    # select columns (elements) for K harmonics
  Xk <- X[,idx]       # Subset of design matrix, with columns for setting of K.
  eps <- y-Xk%*%b

  regression <- lm(eps ~ sin((K+1)*2*pi*x) + cos((K+1)*2*pi*x))
  mk <- regression$coefficients[2:3]
  Vk <- vcov(regression)[2:3,2:3]
  Lk <- t(chol(Vk))
  return (list(m=mk, V=Vk, L=Lk))
}

Tinv <- function(K1,b1,sig2) {
  ## proposed (shorter) par vector
  ## bnew = m + Lu or u = L^-1 (bnew-m)
  K <- K1-1
  p <- 2*K+1
  b <- b1[1:p]
  bnew <- b1[c(p+1,p+2)]

  ## back out auxiliary u, and logJ
  fit <- qu(K, b, sig2)
  u <- solve(fit$L)%*%(bnew-fit$m)
  logJ <- sum(log(diag(fit$L)))

  return (list(b=b, u=u, logJ=logJ))
}
```

0.4 Acceptance Probability for Birth Move

```
rho <- function(K, b1, b, u, logJ, sig2) {
  ##  acceptance ratio for birth move,
  ##  moving from b -> (b,u)
  # current parameter: b
  # proposal:          b1
  K1 <- K+1

  lqu <- sum(dnorm(u, m=0, sd=1, log=TRUE)) # TODO: Shouldn't this be dnorm?

  ljointpostratio <- ljointpost(K1, b1, sig2) - ljointpost(K, b, sig2)

  # Priors and likelihood
  rho <- exp(ljointpostratio)

  # Transition probabilities
```

```
  if (K==1) {
    rho <- rho*2
  } else if (K==Kmx-1) {
    rho <- rho/2
  } else {
    # Scaling rule smoothely penalizes larger values of K.
    birth.weight <- qbeta((Kmx-K+0.01)/Kmx, 1, 5)
    rho <- rho*birth.weight
  }
  rho <- rho/exp(lqu)         # Auxiliary
  rho <- rho*exp(logJ)        # Jacobian

  return (rho)
}
```