MCMC M2 : 1

Joint Posterior

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$$\theta = (k, \beta_k, \sigma^2), \quad P(\theta|y) \propto P(y, k, \beta_k, \sigma^2) = P(y|k, \beta_k, \sigma^2)\, P(\tfrac{1}{\sigma^2})\, P(\beta_k|k)\, P(k) = *$$

$$* = N(y|X_k\beta_k, \sigma^2)\, Ga(\tfrac{1}{\sigma^2}|1,1)\, N(\beta_k|\phi, 10 \cdot I_{2k+1})\, Poi^+(k|\lambda)$$

$$\propto \left[\prod_{i=1}^{n} \left(\frac{1}{\sigma^2}\right)^{\frac{1}{2}} \exp\left\{-\frac{1}{2}\left(\frac{1}{\sigma^2}\right)(y_i - [X_k\beta_k]_i)^2\right\}\right]\left[\left(\frac{1}{\sigma^2}\right)^{1-1} \exp\left\{-\left(\frac{1}{\sigma^2}\right)\right\}\right]$$

$$\times \left[\prod_{j=1}^{2k+1} \exp\left\{-\frac{1}{2}\frac{\beta_{kj}^2}{10}\right\}\right]\left[\exp\{-\lambda\}\frac{\lambda^k}{k!}\right]$$

$$= (\sigma^2)^{-\frac{n}{2}} \exp\left\{-\frac{1}{2}(\sigma^2)^{-1}(y - X_k\beta_k)'(y - X_k\beta_k)\right\} \cdot \exp\left\{-(\sigma^2)^{-1}\right\} \exp\left\{-\frac{1}{2}\frac{\beta_k'\beta_k}{10}\right\} \exp\{-\lambda\}\frac{\lambda^k}{k!}$$

MCMC M2 : 2

MB

Complete Conditionals

$$P(\beta_K \mid K, \sigma^2, y) \propto \exp\left\{-\frac{\tau}{2}(y - X_K\beta_K)'(y - X_K\beta_K)\right\} \exp\left\{-\frac{1}{2}\frac{\beta_K'\beta_K}{10}\right\}$$

$$= \exp\left\{-\frac{1}{2}\left[\tau\left(y'y - y'X_K\beta_K - \beta_K'X_K'y + \beta_K'X_K'X_K\beta_K\right) + \frac{1}{10}\left(\beta_K'\beta_K\right)\right]\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\left[\beta_K'\left(\frac{X_K'X_K}{\sigma^2} + \frac{I_{2K+1}}{10}\right)\beta_K - \beta_K'\left(\frac{X_K'y}{\sigma^2}\right) + \dots\right]\right\}$$

$$\sim N\left(\left(\frac{X_K'X_K}{\sigma^2} + \frac{I_{2K+1}}{10}\right)^{-1}\frac{X_K'y}{\sigma^2},\ \left(\frac{X_K'X_K}{\sigma^2} + \frac{I_{2K+1}}{10}\right)^{-1}\right)$$

Used "pattern matching" technique: $(y-m)'\Sigma^{-1}(y-m) = y'\Sigma^{-1}y - y'\Sigma^{-1}m + \dots$

$$P(\tau \mid K, \beta_K, y) \propto \tau^{\frac{n}{2}} \exp\left\{-\frac{\tau}{2}(y - X_K\beta_K)'(y - X_K\beta_K)\right\} \cdot \exp\left\{-\tau\right\}$$

$$= \tau^{\frac{n}{2}} \exp\left\{-\frac{\tau}{2}(y - X_K\beta_K)'(y - X_K\beta_K) - \tau\right\}$$

$$= \tau^{\frac{n}{2}} \exp\left\{-\left[\frac{(y - X_K\beta_K)'(y - X_K\beta_K)}{2} + 1\right]\tau\right\}$$

$$\sim Ga\left(\frac{n}{2} + 1,\ \frac{(y - X_K\beta_K)'(y - X_K\beta_K)}{2} + 1\right)$$

MCMC M2 : 3

Birth Acceptance Probability

MD

$$\alpha_{Birth}(K, K+1) = \min\left\{1, \rho(K, K+1)\right\} \quad \text{where}$$

$$\rho(K, K+1) = \frac{\text{target}_{K+1}}{\text{target}_K} \cdot \frac{\text{transition}_{K+1\to K}}{\text{transition}_{K\to K+1}} \cdot \frac{1}{\text{auxiliary}(\mu)} \cdot |\text{Jacobian}|$$

$$= \underbrace{\frac{P(K+1, \beta_{K+1}, \sigma^2 \mid y)}{P(K, \beta_K, \sigma^2 \mid y)}}_{(A)} \cdot \underbrace{\frac{P(K \mid K+1)}{P(K+1 \mid K)}}_{(B)} \cdot \underbrace{\frac{1}{P(\mu)}}_{(C)} \cdot \underbrace{\frac{\partial T}{\partial x_K \partial \mu}}_{(D)}$$

(A) is defined as the joint posterior probability, calculated in part (1).

(B) is the transition ratio, where probability of $P(K|K+1) = P(K+1|K) = \frac{1}{2}$,

unless $K = 1 \Rightarrow P(K+1|K) = 1$, or $K = K_{MAX} \Rightarrow P(K|K+1) = 1$.

(C) is the density of the randomly sampled μ, in the birth move.

(D) is the Jacobian of the transformation $T(x_K, \mu) = x_{K+1}$

MCMC M2 : 4

Death Acceptance Probability

$$\alpha_{Death}(K+1, K) = \min\left\{1, \frac{1}{\rho(K, K+1)}\right\}$$

As defined in (3).

$\boxed{\text{MCMC M2: 5}}$ Plots for RJ MCMC

MB

See attachment

$\boxed{\text{MCMC M2: 6}}$ Marginal of Y

Simple form of candidate's formula:

$$P(\beta|y) = \frac{P(y|\beta)P(\beta)}{P(y)} \Rightarrow P(y) = \frac{P(y|\beta)P(\beta)}{P(\beta|y)}$$

In this case:

$$P(\bar{\beta}_k, k \mid y) = \frac{P(y|\bar{\beta}_k, k)\, P(\bar{\beta}_k|k)\, P(k)}{P(y|k)}$$

$$\Rightarrow P(y|k) = \frac{P(y|\bar{\beta}_k, k)\, P(\bar{\beta}_k|k)\, P(k)}{P(\bar{\beta}_k, k \mid y)}$$

For a known k: Can find $P(k) \sim \text{Pois}^+(\lambda)$, and the first two numerator terms yield

$$P(\bar{\beta}_k|y,k) \sim N\Bigg(\underbrace{\left(2X_k'X_k + \frac{I_{2k+1}}{10}\right)^{-1} \cdot 2X_k'y}_{\bar{\beta}_k},\ \underbrace{\left(2X_k'X_k + \frac{I_{2k+1}}{10}\right)^{-1}}_{S_k}\Bigg)$$

Therefore $P(y|k) \propto N(\bar{\beta}_k, S_k)$

$$\propto |S_k|^{-\frac{1}{2}} P(y|\bar{\beta}_k, k) P(\bar{\beta}_k)$$