$K(x,y) =$

$$\langle (1, 2x, 4x^2-2), (1, 2y, 4y^2-2) \rangle$$

$$= 1 + 4xy + (4x^2-2)(4y^2-2)$$

$$= 1 + 4xy + 2(2x^2-1)2(2y^2-1)$$

$$= 1 + 4xy + 4(2x^2-1)(2y^2-1)$$

$$= 1 + 4xy + 4(4x^2y^2 - 2x^2 - 2y^2 + 1)$$

$$= 5 + 4xy - 8x^2 - 8y^2 + 16x^2y^2$$

//

From Rasmussen 4.3.1, for $k(x,x') = \exp(-(x-x')^2/2l^2)$, and $p(x) = N(x \mid 0, \sigma^2)$, the basis function is $\phi_k(x) = \exp(-(c-a)x^2)\, H_k(\sqrt{2c}\,x)$, where H_k is physicists H-poly., and $a^{-1} = 4\sigma^2$, $b^{-1} = 2l^2$,

$c = \sqrt{a^2 + 2ab}$

Suppose $\sigma = 1$, $l = 1$, then:

$$a = 1/4, \quad b = 1/2, \quad c = \left(\left(\tfrac{1}{4}\right)^2 + 2 \cdot \tfrac{1}{4} \cdot \tfrac{1}{2}\right)^{\frac{1}{2}} = \left(\tfrac{5}{16}\right)^{1/2}$$

and $$\phi_k(x) = \exp(-\underbrace{(\sqrt{5/16} - 1/4)}_{C_1} x^2 \cdot H_k(x \cdot \underbrace{(2 \cdot (5/16)^{1/2})^{1/2}}_{C_2}).$$

Solving for eigenfunctions:

$$\phi_0(x) = \exp(-C_1 x^2) \cdot (1)$$

$$\phi_1(x) = \exp(-C_1 x^2)(2(C_2 x))$$

$$\phi_2(x) = \exp(-C_1 x^2)(4(C_2 x)^2 - 2)$$

Suppose $a = 1$, $b = 3$. This implies $a^{-1} = 4\sigma^2$, $b^{-1} = 2\ell^2$, $c = \sqrt{a^2 + 2ab}$

$$1 = 4\sigma^2 \qquad \tfrac{1}{3} = 2\ell^2 \qquad c = (1 + 2\cdot 1\cdot 3)^{1/2}$$

$$\sigma = \tfrac{1}{2} \qquad \tfrac{1}{6} = \ell^2 \qquad c = \sqrt{7}$$

$$\ell = \left(\tfrac{1}{6}\right)^{\frac{1}{2}}$$

and

$$\sqrt{2c} = \sqrt{2}\cdot 7^{1/4}$$

For kernel $k(x,x') = \exp(-(x-x')^2/2\cdot((\tfrac{1}{6})^{\frac{1}{2}})^2$

$$= \exp(-(x-x')^2/ {}^1\!/_3)$$

and

$$\underbrace{c - a = \sqrt{7} - 1}_{C_1} \qquad \underbrace{= 2^{\frac{1}{2}}\, 7^{\frac{1}{4}}}_{C_2}$$

Solving for eigenfunctions:

$$C_2^2 = 2\sqrt{7}, \quad C_2^4 = 28$$

$$\Phi_0(x) = \exp(-C_1 x^2)$$

$$\Phi_1(x) = \exp(-C_1 x^2)\cdot 2(C_2 x)$$

$$\Phi_2(x) = \exp(-C_1 x^2)\cdot(4(C_2 x)^2 - 2)$$

For $a = \frac{1}{4\sigma^2}$, $b = \frac{1}{2\ell^2}$, $c = \sqrt{a^2 + 2ab}$

$$C_1 = c - a$$

$$C_2 = \sqrt{2c}$$

This implies that a kernel with $k=2$ moments is:

$$K(x,y) = \langle (\Phi_0(x), \Phi_1(x), \Phi_2(x)), (\Phi_0(y), \Phi_1(y), \Phi_2(y)) \rangle$$

$$= \exp(-C_1(x^2+y^2)) + (2C_2x)(2C_2y)\exp(-C_1(x^2+y^2))$$

$$+ (4(C_2x)^2 - 2)(4(C_2y)^2 - 2)\exp(-C_1(x^2+y^2))$$

$$= \left[1 + (2C_2x)(2C_2y) + (4(C_2x)^2 - 2)(4(C_2y)^2 - 2)\right]\exp(-C_1(x^2-y^2))$$

$$\boxed{= \left[5 + 4C_2^2xy + 16C_2^4x^2y^2 - 8C_2^2x^2 - 8C_2^2y^2 + 4\right]\exp(-C_1(x^2-y^2))}$$

$$= \left[5 + 4(2\sqrt{7})xy + 16(28)x^2y^2 - 8(2\sqrt{7})x^2 - 8(2\sqrt{7})y^2\right]\exp(-C_1(x^2-y^2))$$

$$= \left[5 + 8\sqrt{7}\,xy + 448x^2y^2 - 16\sqrt{7}x^2 - 16\sqrt{7}y^2\right]\exp(-(\sqrt{7}-1)(x^2-y^2))$$