StatMod 2 - Multinomial-Dirichlet - Mixture of Normals

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remedial : $n_1 = 100 \quad y_i \sim N(55, 15)$

average : $n_2 = 400 \quad y_i \sim N(70, 10) \qquad n = \sum_{i=1}^{3} n_i$

honors : $n_3 = 150 \quad y_i \sim N(85, 5)$

Toy Example Model

$$(w_1, w_2, w_3) \sim Dir(n_1, n_2, n_3), \qquad \underline{a} = (n_1, n_2, n_3) = (100, 400, 150)$$

$$P\left(\gamma_i = k \,\middle|\, w, (n_1, n_2, n_3)\right)$$

$$D = \begin{bmatrix} \mathbb{1} N_1 \\ \mathbb{1} N_2 \\ \mathbb{1} N_3 \end{bmatrix}, \quad w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = D'\underline{w}$$

$$\text{Posterior} \propto \text{Joint} = \prod_{i=1}^{n} \Big[\mathbb{1}(\gamma_i = 1)\, N(y_i | 55, 15) \cdot w_1 + \mathbb{1}(\gamma_i = 2)\, N(y_i | 70, 10) \cdot w_2 + \mathbb{1}(\gamma_i = 3)\, N(y_i | 85, 5) \cdot w_3 \Big] = \prod_{i=1}^{n} D'\underline{w}$$

$$\propto \left[\prod_{i=1}^{n} \sum_{j=1}^{3} w_j\, N(y_i | \mu_j, \sigma_j^2) \right] \cdot \left[Dir(\underline{w} | 100, 400, 150) \right] = \left[\prod_{i=1}^{n} \sum_{j=1}^{3} w_j \left(\frac{1}{\sigma_j^2}\right)^{\frac{1}{2}} \exp\left\{-\frac{1}{2} \frac{(y_i - \mu_j)^2}{\sigma_j^2}\right\} \right] \cdot \left[\prod_{j=1}^{3} w_j^{a_j - 1} \right]$$

$$P(\gamma_i = k | \ldots) \propto \begin{cases} N(y_i | 55, 15)\, w_1 & \text{for } w = w_1 \\ N(y_i | 70, 10)\, w_2 & \text{for } w = w_2 \\ N(y_i | 85, 5) \cdot w_3 & \text{for } w = w_3 \end{cases}$$

$$P(\underline{w} | \ldots) \propto \prod_{j=1}^{3} w_j^{n_j} \prod_{j=1}^{3} w_j^{a_j - 1} \sim Dir(w | n_1 + a_1, n_2 + a_2, n_3 + a_3)$$

Joint: $P(\underline{y}, \underline{\mu}, \frac{1}{\underline{\sigma}^2}, \underline{\gamma}, \underline{w}) = \circledast$

$$\circledast = \prod_{i=1}^{n} N(y_i \mid \mu_{\gamma_i}, \sigma^2_{\gamma_i}) \cdot \prod_{j=1}^{J} \left[N(\mu_j \mid m, v) \cdot Ga\left(\frac{1}{\sigma_j^2} \middle| a, b\right)\right] \cdot Mn(\underline{\gamma} \mid N, \underline{w}) \cdot Dir(\underline{w} \mid \underline{\alpha})$$

Posterior: $P(\theta = \underline{\mu}, \frac{1}{\underline{\sigma}^2}, \underline{\gamma}, \underline{w} \mid \underline{y}) \propto P(\underline{y} \mid \theta)\, P(\theta) = \circledast$

Full conditionals $P(\mu_k \mid \ldots)$, $P(\sigma_k^2 \mid \ldots)$ + Extras, to complete Gibbs Sampler

$$P(\mu_k \mid \ldots) \propto \underbrace{N(\mu_k \mid m, v)}_{\text{Prior on } \mu_k} \cdot \underbrace{\prod_{i \in A_k} N(y_i \mid \mu_k, \sigma_k^2)}_{\text{Likelihood under } \mu_k, \sigma_k^2} \sim N(M, V)$$

where $V = \left(\frac{1}{v} + \frac{n_k}{\sigma_k^2}\right)^{-1}$

and $M = V \cdot \left(\frac{1}{v} \cdot m + \frac{n_k}{\sigma_k^2} \bar{y}_i\right)$

$$P(\sigma_k^2 \mid \ldots) \propto \underbrace{Ga\left(\frac{1}{\sigma_k^2} \middle| a, b\right)}_{\text{Prior on } \sigma_k^2} \cdot \underbrace{\prod_{i \in A_k} N(y_i \mid \mu_k, \sigma_k^2)}_{\text{Likelihood under } \mu_k, \sigma_k^2} \sim Ga\left(\frac{1}{\sigma_k^2} \middle| a + \frac{n_k}{2}, b + \frac{\sum_{i \in A_k} (y_i - \mu_k)^2}{2}\right)$$

$$P(\underline{w} \mid \ldots) \propto \prod_{i \in A_k} w_k \cdot \prod_{k=1}^{J} w_k^{\alpha - 1} = \prod_{k=1}^{J} w_k^{n_k + \alpha - 1} \sim Dir(\underline{w} \mid n_1 + \alpha_1, \ldots, n_J + \alpha_J)$$

$$P(\gamma_i = k \mid \ldots) \propto N(y_i \mid \mu_k, \sigma_k^2) \cdot w_k$$