Gaussian processes HW

Given $\begin{bmatrix} y \\ \theta \end{bmatrix}$ is a joint distr. and that

① $y|\theta \sim N(R\theta, \Sigma)$ and ② $\theta \sim N(m, V)$

Show: $\begin{bmatrix} y \\ \theta \end{bmatrix}$ is MVN.

$P(y|\theta)P(\theta) \equiv \text{joint} \triangleq \begin{bmatrix} y \\ \theta \end{bmatrix}$, To be MVN, a linear combo must be ~~MVN~~ UVN.

Consider vector $\begin{bmatrix} y|\theta \\ \theta \end{bmatrix}$ and $a \neq 0$, such that $a\begin{bmatrix} y|\theta \\ \theta \end{bmatrix}$ is a linear combo of MVN's, so is UVN. Therefore the joint $\begin{bmatrix} y|\theta \\ \theta \end{bmatrix} = \begin{bmatrix} y \\ \theta \end{bmatrix}$ is MVN.

What are the moments of $\begin{bmatrix} y \\ \theta \end{bmatrix}$?

So $\begin{bmatrix} y \\ \theta \end{bmatrix} \sim N\left(\begin{bmatrix} Rm \\ m \end{bmatrix}, \begin{bmatrix} \Sigma + RVR' & RV \\ (RV)' & V \end{bmatrix} \right)$

$E[Y] = E\big[E[Y|\theta]\big] = E[R\theta] = RE[\theta] = Rm$, and $E[\theta] = m$

$\overset{\text{var}}{\cancel{\text{cov}}}(Y) = E[\text{var}(Y|\theta)] + \text{Var}(E[Y|\theta])$, and $\text{var}(\theta) = V$,

$= E[\Sigma] + \text{Var}(R\theta)$

$= \Sigma + R\,\text{var}(\theta)R'$

$= \Sigma + RVR'$

and $\text{cov}(Y,\theta) = E\big[(Y - E[Y])(\theta - E(\theta))'\big]$

$= E\big[(Y - R\theta)(\theta - m)'\big]$

$= E\big[Y\theta' - Ym' - R\theta\theta' + R\theta m'\big]$ where $Y = R\theta + \varepsilon$

$= E[Y\theta' - Ym'] - \cancel{Rmm'} + \cancel{Rmm'}$

$= E\big[(R\theta + \varepsilon)\theta' - (R\theta + \varepsilon)m'\big]$

$= E\big[R\theta\theta' + \varepsilon\theta' - R\theta m' - \varepsilon m'\big]$

$= E[R\theta\theta'] + \underbrace{E[\varepsilon\theta']}_{0} - E[R\theta m'] - \underbrace{E[\varepsilon m']}_{0}$

$\cancel{Rmm'} = E[R\theta\theta'] - E[R\theta m']$

$\text{cov}(Y,\theta) = R\,E[\theta\theta' - mm'] = R[E(\theta\theta') - mm']$

$= \boxed{R \cdot V}$

$\text{var}(\theta) = E[(\theta - m)(\theta - m)']$

$= E[\theta\theta'] - mm'$

Gaussian Processes

Given GP prior $f \sim GP(0, C)$, and X's, and $y_i \sim N(f_i, \sigma^2)$

Find posterior $P(f \mid y, x)$

$$P(f \mid y, x) \propto P(y \mid f)\, P(f)$$

$$\propto \exp\left\{ \frac{(f-Y)'(f-Y)}{-2\sigma^2} \right\} \exp\left\{ -\frac{1}{2} f' C^{-1} f \right\}$$

$$= \exp\left\{ -\frac{1}{2} \left[\frac{f'f}{\sigma^2} - \frac{f'Y}{\sigma^2} - \frac{Y'f}{\sigma^2} + \frac{Y'Y}{\sigma^2} + f' C^{-1} f \right] \right\}$$

$$\propto \exp\left\{ -\frac{1}{2} \left[f'\left(\frac{I}{\sigma^2} + C^{-1} \right) f - \frac{2Y'f}{\sigma^2} + \dots \right] \right\}$$

$$\sim N\left(\left(\frac{I}{\sigma^2} + C^{-1} \right)^{-1} \left(\frac{Y}{\sigma^2} \right), \left(\frac{I}{\sigma^2} + C^{-1} \right)^{-1} \right)$$