StatMod 2 - Exe 5 - Multinom/Dir

Show Conjugacy

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Multinomial: $P(n_1,\ldots,n_K|\underline{w}) = \dfrac{N!}{n_1!\cdots n_K!} \prod_{k=1}^{K} w_k^{n_k} \quad , \quad \sum_{k=1}^{K} n_k = N$

Dirichlet: $P(\underline{w}|\underline{\alpha}) = \dfrac{\Gamma(\sum_k a_k)}{\prod_k \Gamma(a_k)} \prod_{k=1}^{K} w_k^{a_k-1}$

Joint Distribution: $P(\underline{n}|\underline{w})\cdot P(\underline{w}|\underline{\alpha}) = \underbrace{\dfrac{N!}{n_1!\cdots n_K!} \prod_{k=1}^{K} w_k^{n_k}}_{\text{Multinomial}} \cdot \underbrace{\dfrac{\Gamma(\sum_k a_k)}{\prod_k \Gamma(a_k)} \prod_{k=1}^{K} w_k^{a_k-1}}_{\text{Dirichlet}}$

$$= \frac{N!}{n_1!\cdots n_K!} \cdot \frac{\Gamma(\sum_{k=1}^{K} a_k)}{\prod_k \Gamma(a_k)} \cdot \prod_{k=1}^{K} w_k^{n_k+a_k-1}$$

Utilize: $N! = \left(\sum_{k=1}^{K} n_k\right)! = \Gamma\left(\left(\sum_{k=1}^{K} n_k\right)+1\right) \quad ; \quad n_1!\cdots n_K! = \prod_{k=1}^{K} n_k! = \prod_{k=1}^{K}\left(\Gamma(n_k+1)\right)$

$$= \frac{\Gamma\left(\left(\sum_{k=1}^{K} n_k\right)+1\right)\Gamma\left(\sum_{k=1}^{K} a_k\right)}{\prod_{k=1}^{K}\left(\Gamma(n_k+1)\right)\cdot \prod_{k=1}^{K}\Gamma(a_k)} \prod_{k=1}^{K} w_k^{n_k+a_k-1} = \frac{\Gamma\left(\sum_{k=1}^{K} n_k+a_k\right)}{\prod_{k=1}^{K}\Gamma(a_k+n_k)} \prod_{k=1}^{K} w_k^{n_k+a_k-1}$$

Posterior $P(\underline{w}|\underline{n},\underline{\alpha}) \propto Dir(n_1+a_1,\ldots,n_K+a_K)$

MD

Multinomial / Dirichlet

$\boxed{\text{Dirichlet is normalized Gammas}}$

Given $X_k \overset{iid}{\sim} Ga(a_k, 1)$, $k = 1, \dots, K$; $w_k = \dfrac{X_k}{\Sigma X_k}$, $W = \Sigma X_k$

Show $(\underline{w}) \sim Dir(\underline{a})$; $\underline{w} = w_1, \dots, w_K$, $\underline{a} = a_1, \dots, a_K$; also let $\underline{w}' = w_1, \dots, w_{K-1}$

Want to find joint density of $f(\underline{w}, W)$. First write pdfs of x_k.

$$f_{X_k}(x_k) = \frac{1}{\Gamma(a_k)} x_k^{a_k - 1} \exp\{-x_k\} \; ; \text{ by independence, } f_{\underline{X}}(\underline{x}) = \prod_{k=1}^{K} f_{X_k}(x_k)$$

Want transformation from $f_{\underline{X}}(\underline{x}) \rightarrow f_{\underline{w}', W}(\underline{w}', W)$.

$$f_{\underline{w}', W}(\underline{w}', W) = f_{\underline{X}}\left(x_1 = w_1 \cdot W, \dots, x_{K-1} = w_{K-1} \cdot W, \; x_K = \left(1 - \sum_{k=1}^{K-1} w_k\right) \cdot W\right) \cdot |J|$$

$\boxed{\text{since, } x_k = w_k \cdot W}$

$$|J| = \det \begin{array}{c} \\ x_1 \\ \\ \vdots \\ \\ x_{K-1} \\ x_K \end{array} \begin{array}{c} \begin{array}{cccc} w_1 \cdots & & w_{K-1} & W \end{array} \\ \begin{bmatrix} W & & \cdots 0 & w_1 \\ & W & \vdots & w_2 \\ & & \ddots & \vdots \\ \vdots & & & \\ 0 \cdots & & W & w_{K-1} \\ -W \; -W \cdots & & -W & \left(1 - \sum_{k=1}^{K-1} w_k\right) \end{bmatrix} \end{array} = \det \begin{bmatrix} W & 0 \cdots & \cdots 0 & w_1 \\ 0 & W & \vdots & \vdots \\ \vdots & & \ddots & \\ \vdots & & W & w_{K-1} \\ 0 & \cdots & 0 & 1 \end{bmatrix} = W^{K-1}$$

Substituting in:

$$f_{\underline{w}', W}(\underline{w}', W) = \prod_{k=1}^{K-1} \left[\frac{1}{\Gamma(a_k)} (w_k \cdot W)^{a_k - 1} \exp\{-w_k \cdot W\}\right] \cdot \frac{1}{\Gamma(a_K)} \left(\left(1 - \sum_{k=1}^{K-1} w_k\right) W\right)^{a_K - 1} \exp\left\{-\left(1 - \sum_{k=1}^{K-1} w_k\right) \cdot W\right\} \cdot W^{K-1}$$

$$= \frac{\left(\prod_{k=1}^{K-1} w_k^{a_k - 1}\right) \cdot \left(1 - \sum_{k=1}^{K-1} w_k\right)^{a_K - 1} \cdot W^{\left(\sum_{k=1}^{K} a_k\right) - 1} \cdot \exp\{-W\}}{\prod_{k=1}^{K} \Gamma(a_k)} \, , \quad \text{Let } C = \frac{\left(\prod_{1}^{K-1} w_k^{a_k - 1}\right)\left(1 - \sum_{1}^{K-1} a_k\right)^{a_K - 1}}{\prod_{1}^{K} \Gamma(a_k)}$$

Marginalizing out W:

$$f_{\underline{w}'}(\underline{w}') = \int_0^\infty f_{\underline{w}', W}(\underline{w}', W)\, dW = C \int_0^\infty W^{\left(\sum_1^K a_k\right) - 1} \cdot \exp\{-W\}\, dW = C \cdot \Gamma\left(\sum_{k=1}^{K} a_k\right)$$

$$= \frac{\Gamma(\Sigma a_k)}{\Pi\, \Gamma(a_k)} \cdot \left(\prod_1^{K-1} w_k^{a_k - 1}\right)\left(1 - \sum_1^{K-1} w_k\right)^{a_K - 1} \sim Dir(\underline{a})$$