StatMod2 - Gibbs - Exercises 4

Maurice Diesendruck

March 31, 2015

1 Model

$$
\begin{aligned}
\boldsymbol{y} \sim &\ N(X\boldsymbol{\beta}, \sigma^2 \boldsymbol{I_n}) \text{ for i.i.d. } y_1, \cdots, y_n \\
\boldsymbol{\beta}|\tau^2 \sim &\ N(\mathbf{0}, \tau^2 \boldsymbol{I_p}) \text{ for i.i.d. } \beta_1, \cdots, \beta_p \\
\sigma^2 \sim &\ IG(a/2, b/2) \equiv \Big(\frac{1}{\sigma^2}\Big) \sim \ Ga(a/2, b/2) \ \dagger \\
\tau^2 \sim &\ IG(c/2, d/2) \equiv \Big(\frac{1}{\sigma^2}\Big) \sim \ Ga(a/2, b/2) \ \dagger
\end{aligned}
$$

† For notational convenience, the following distributions use $Ga(a, b)$. In R, a and b are initialized as $1/2$.

2 Joint Posterior

Let $\boldsymbol{\theta} = (\boldsymbol{\beta}, \sigma^2, \tau^2)$.

$$
\begin{aligned}
P(\boldsymbol{\theta}|y) \propto &\ P(y|\boldsymbol{\theta})P(\boldsymbol{\theta}) \\
&= N(y|X\boldsymbol{\beta}, \sigma^2) N(\boldsymbol{\beta}|\mathbf{0}, \tau^2 \boldsymbol{I_p}) Ga(\frac{1}{\sigma^2}|a, b) Ga(\frac{1}{\tau^2}|c, d) \\
&\propto \ \Big(\frac{1}{\sigma^2}\Big)^{n/2} exp\Big(-\frac{1}{2}\Big(\frac{(\boldsymbol{y} - X\boldsymbol{\beta})'(\boldsymbol{y} - X\boldsymbol{\beta})}{\sigma^2}\Big)\Big) \\
&\quad \text{x } \Big(\frac{1}{\tau^2}\Big)^{p/2} exp\Big(-\frac{1}{2}\frac{\boldsymbol{\beta}'\boldsymbol{\beta}}{\tau^2}\Big)\Big) \\
&\quad \text{x } \Big(\frac{1}{\sigma^2}\Big)^{a-1} exp^{-b\Big(\frac{1}{\sigma^2}\Big)} \\
&\quad \text{x } \Big(\frac{1}{\tau^2}\Big)^{c-1} exp^{-d\Big(\frac{1}{\tau^2}\Big)}
\end{aligned}
$$

3 Full Conditionals

$$
\begin{aligned}
P(\boldsymbol{\beta}|\boldsymbol{y},\sigma^2,\tau^2) &\propto \; exp\Big(-\frac{1}{2}\Big(\frac{(\boldsymbol{y}-X\boldsymbol{\beta})'(\boldsymbol{y}-X\boldsymbol{\beta})}{\sigma^2}+\frac{\boldsymbol{\beta}'\boldsymbol{\beta}}{\tau^2}\Big)\Big) \\
&= exp\Big(-\frac{1}{2}\Big(\frac{\boldsymbol{y}'\boldsymbol{y}}{\sigma^2}-\frac{\boldsymbol{y}'X\boldsymbol{\beta}}{\sigma^2}-\frac{\boldsymbol{\beta}'X'\boldsymbol{y}}{\sigma^2}+\beta'(\frac{X'X}{\sigma^2}+\frac{I_p}{\tau^2})\beta\Big)\Big) \\
&\sim \; N\left(\Big(\frac{X'X}{\sigma^2}+\frac{I_p}{\tau^2}\Big)^{-1}\frac{X'Y}{\sigma^2},\Big(\frac{X'X}{\sigma^2}+\frac{I_p}{\tau^2}\Big)^{-1}\right)
\end{aligned}
$$

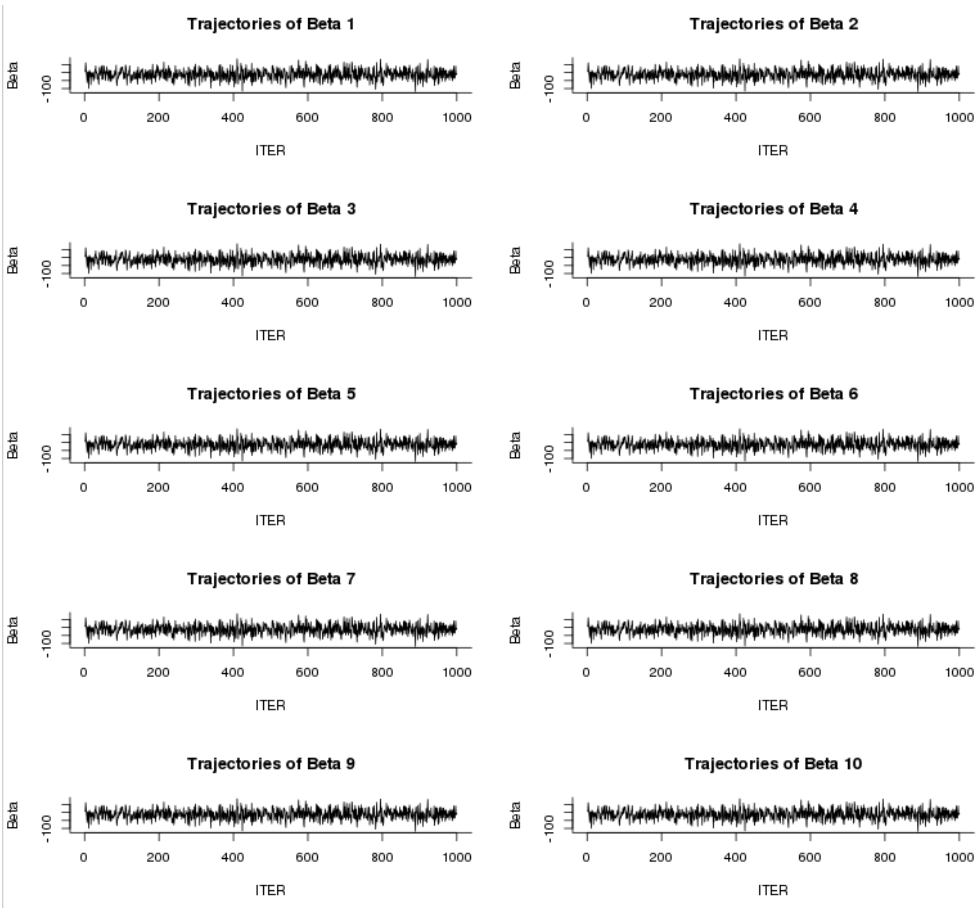
$$
\begin{aligned}
P(\sigma^2|\boldsymbol{y},\boldsymbol{\beta},\tau^2) &\propto \; \Big(\frac{1}{\sigma^2}\Big)^{n/2}exp\Big(-\frac{1}{2}\Big(\frac{(\boldsymbol{y}-X\boldsymbol{\beta})'(\boldsymbol{y}-X\boldsymbol{\beta})}{\sigma^2}\Big)\Big) \\
&\qquad \text{x } \Big(\frac{1}{\sigma^2}\Big)^{a-1}exp^{-b\Big(\frac{1}{\sigma^2}\Big)} \\
&= \Big(\frac{1}{\sigma^2}\Big)^{n/2+a-1}exp\Big(-\Big(\frac{(\boldsymbol{y}-X\boldsymbol{\beta})'(\boldsymbol{y}-X\boldsymbol{\beta})}{2}+b\Big)\Big(\frac{1}{\sigma^2}\Big)\Big) \\
&\sim \; Ga\Big(n/2+a,\Big(\frac{(\boldsymbol{y}-X\boldsymbol{\beta})'(\boldsymbol{y}-X\boldsymbol{\beta})}{2}+b\Big)\Big)
\end{aligned}
$$

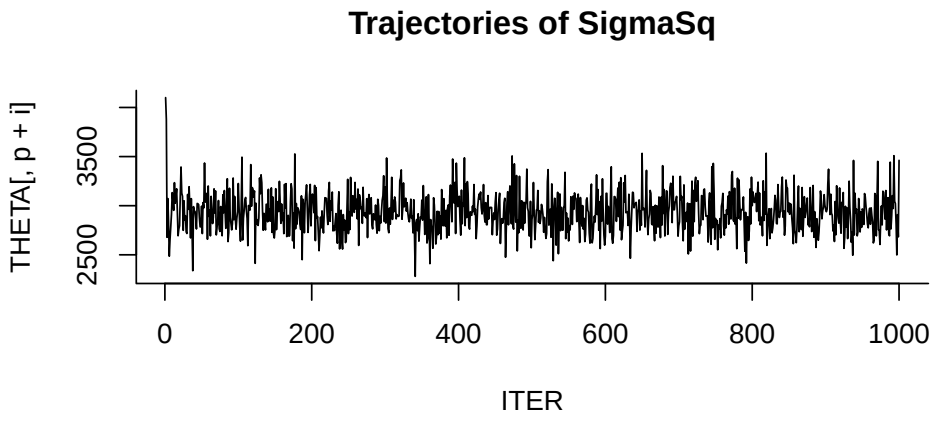
$$
\begin{aligned}
P(\tau^2|\boldsymbol{y},\boldsymbol{\beta},\sigma^2) &\propto \; \Big(\frac{1}{\tau^2}\Big)^{p/2}exp\Big(-\frac{1}{2}\Big(\frac{\boldsymbol{\beta}'\boldsymbol{\beta}}{\tau^2}\Big)\Big) \\
&\qquad \text{x } \Big(\frac{1}{\tau^2}\Big)^{c-1}exp^{-d\Big(\frac{1}{\tau^2}\Big)} \\
&= \Big(\frac{1}{\tau^2}\Big)^{p/2+c-1}exp\Big(-\Big(\frac{\boldsymbol{\beta}'\boldsymbol{\beta}}{2}+d\Big)\Big(\frac{1}{\tau^2}\Big)\Big) \\
&\sim \; Ga\Big(p/2+c,\Big(\frac{\boldsymbol{\beta}'\boldsymbol{\beta}}{2}+d\Big)\Big)
\end{aligned}
$$

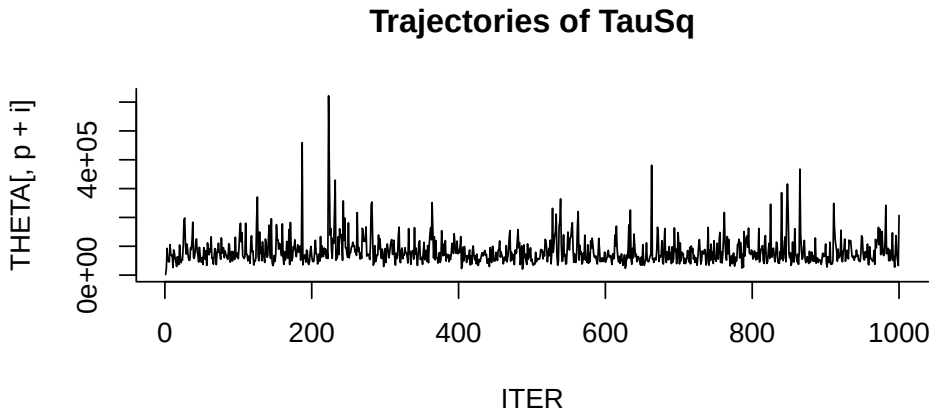
4 Results

These are the means of $\beta_1,\cdots,\beta_p$. Each also mixed appropriately.

```
[1]   -2.051309 -221.537769  510.610845  310.677515 -180.066386   -7.965675
[7] -159.422452  111.922047  507.486380   77.530884
```


Trajectories of Beta 1
Trajectories of Beta 2
Trajectories of Beta 3
Trajectories of Beta 4
Trajectories of Beta 5
Trajectories of Beta 6
Trajectories of Beta 7
Trajectories of Beta 8
Trajectories of Beta 9
Trajectories of Beta 10
Beta
-100
0
200
400
600
800
1000
ITER


Trajectories of SigmaSq
THETA[, p + i]
2500
3500
0
200
400
600
800
1000
ITER


Trajectories of TauSq
THETA[, p + i]
0e+00
4e+05
0
200
400
600
800
1000
ITER