Implementation of interior-point methods for positive semi-definite Toeplitz cone programming

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1 Introduction

Consider the cone of positive-semidefinite Toeplitz matrices

$$\mathcal{T}_n = \{x \in \mathbf{R}^n \mid T_n(x) \succeq 0\}$$

with $T_n(x)$ defined as

$$T_n(x)_{ij} = \begin{cases} x_{|i-j|+1} & 0 \leq |i-j| \leq n-1 \\ 0 & \text{otherwise} \end{cases}.$$

In words, $T_n(x)$ is the square Toeplitz matrix resulting from taking the ith entry of x as the ith sub- and super-diagonal. This cone appears in various applications in signal processing

Flesh out applications

. In this paper we investigate methods for cone programming over $\mathcal{T}$, i.e. we present methods for solving problems of the form

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \in \mathcal{T}_n. \end{array} \tag{1}$$

where $x \in \mathbf{R}^n$ is the optimization variable, $c \in \mathbf{R}^n$, $A \in \mathbf{R}^{m \times n}$, and $b \in \mathbf{R}^m$. The central difficulty for problems of this form is that $\mathcal{T}_n$ is not symmetric, so much of the standard primal-dual machinery becomes unweildly.

The remainder of this paper is organized as follows. We first explore the properties of $\mathcal{T}_n$, deriving its dual and showing that it is not symmetric. To motivate new algorithms we then investigate how un-symmetric cones present a problem for a standard primal-dual interior point method. We discuss a suitable barrier for $\mathcal{T}_n$ and its use in two different algorithms that make speed gains over the standard primal-dual interior-point method by forgoing pieces of the dual picture. Finally we present some numeric results from these algorithms. Note that nothing presented here is original, and in many cases follows our sources literally.

2 The Positive Semi-Definite Toeplitz Cone

To characterize the dual of $\mathcal{T}_n$ we show that $\mathcal{T}_n$ is the dual of the cone of finite autocorrelation sequences. A finite autocorrelation sequence is a vector $x \in \mathbf{R}^n$ with a $y \in \mathbf{R}^n$ such that

$$x_k = \sum_{i=1}^{n-k} y_i y_{i+k} \quad k = 1, \dots, n.$$

For any positive scalar α, αx can be represented as above by $\alpha^{1/2} y$, so the set of finite autocorrelation sequences forms a cone. The dual of $\mathcal{C}_n$ is

$$\mathcal{C}_n^* = \{z \in \mathbf{R}^n \mid z^T x \geq 0 \text{ for } x \in \mathcal{C}_n\}.$$

If $z^T x \geq 0$ for all $x \in \mathcal{C}_n$, then for all $y \in \mathbf{R}^n$

$$z^T x \geq 0 \iff \sum_{i=1}^{n} z_i \left(\sum_{i=1}^{n-k} y_i y_{i+k} \right).$$

probably don't want that iff symbol in there

The summation above can be expressed in terms of the Toeplitz form of z — the above is true if and only if $y^T T_n(z) y \geq 0$ for all $y \in \mathbf{R}^n$. Thus $\mathcal{C}_n^*$ is exactly the positive semi-definite Toeplitz cone, $\mathcal{T}_n$. Nonempty, closed, convex cones are reflexive, so $\mathcal{T}_n^* = \mathcal{C}_n$, and $\mathcal{T}_n$ is not symmetric.

Do we need more than this? Is it clear that these aren't the same thing or should we provide an example?

Do we need to show that it is nonempty, closed, and convex? Or is that too obvious?

Let $\psi_n(x)$ be a self-concordant ν logarithmically homogenous barrier for $\mathcal{T}_n$. Recall that ν logarithmically homogenous functions satisfy

$$\psi_n(tx) = -\nu \log(t) + \psi_n(x).$$

Similarly, let $\psi_n^*(z)$ be the conjugate barrier to ψ_n

$$\psi_n^*(x) = \max\{-s^T x - \psi_n(x) \mid x \in \mathcal{T}_n\}.$$

In TODO:ref(Nesterov 94 or 96) it is shown that ψ_n^* is a self-concordant and ν logarithmically homogenous barrier function for $\mathcal{C}_n$.

3 A Standard Primal-Dual Interior-Point Method

We turn now to consider primal-dual interior point methods and their unsuitability for solving un-symmetric cone problems. The dual problem of (1) is

$$\begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & c = A^T y + s \\ & s \in \mathcal{C}_n. \end{array} \tag{2}$$

where $y, s \in \mathbf{R}^n$ are the optimization variables. Any solution of (1) and (2) must satisfy the optimality conditions

$$Ax = bA^T y + s = c \tag{3}$$

4 Alternative Strategies

5 Numerical Results