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A Rigorous Proof of the Riemann Hypothesis from First Principles

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ABSTRACT: As is well-known, the celebrated Riemann Hypothesis (RH) is the prediction that all the non-trivial zeros of the zeta function $\zeta(s)$ lie on a vertical line in the complex s-plane at $Re(s) = 1/2$. Very many efforts to prove this statement have been directed to investigating the analytic properties of the zeta function, however all these efforts have not been able to substantially improve on Riemann's initial discovery: that all the non trivial zeros lie in verical strip of unit width whose centre is the critical line. The efforts have been rendered difficult because of a lack of a suitable functional representation (formula) for $\zeta(s)$ (or $1/\zeta(s)$) , which is valid and analytic over all regions of the Argand plane; these difficulties are further complicated by the presence of prime numbers in the very definition of the zeta function and the lack of predictability in the behaviour of prime numbers which makes the analysis intractable. In this paper we make our first headway by looking at the analyticity of the function $F(s) = \zeta(2s)/\zeta(s)$ that has poles in exactly those positions where $\zeta(s)$ has a non trivial zero. Further, the trivial zeros of the zeta function, which occur at the negative even integers, conveniently cancel out in $F(s)$ and do not appear as poles of the latter (however there is an isolated pole of $F(s)$, viz. $s = 1/2$, which is actually a pole of $\zeta(2s)$ but this will not worry us because it is on the critical line). So the task of proving the RH is some what ‘simplified’ because all we have to show is: All the poles of $F(s)$ occur on the critical line, which then is the main aim of this paper. We then investigate the Dirichlet series that obtains from the function F(s) and employ novel methods of summing the series by casting it as an infinite number of sums over sub-series. In this procedure, which heavily invokes the prime factorization theorem, each sub-series has the property that it oscillates in a predictable fashion, rendering the analytic properties of the function $F(s)$ determinable. With the methods developed in the paper many theorems are proved, for example we prove: that for every integer with an even number of primes in its factorization, there is another integer that has an odd number of primes (multiplicity counted) in its factorization; by this demonstration, and by the proof of several other theorems, a similarity between the factorization sequence involving (Liouville's multiplicative functions) and a sequence of coin tosses is mathematically established. Consequently, by placing this similarity on a firm foundation, one is then empowered to demonstrate, that Littlewood's(1912) sufficiency condition involving Liouville's summatory function, $L(N)$, is satisfied. It is thus proved that the function $F(s)$ is analytic over the two half-planes $Re(s) > 1/2$ and $Re(s) < 1/2$, clearly revealing that all the nontrivial zeros of the Riemann zeta function are placed on the critical line $Re(s) = 1/2$.

Extended Abstract

The paper approaches* the RH in the following way:

(1) This proof of the Riemann Hypothesis (Riemann 1859) crucially depends on showing that the function $F(s) \equiv \zeta(2s)/\zeta(s)$, has poles only on the critical line $s = 1/2 + iy$, which translates to having the non-trivial zeros of the $\zeta(s)$ function on the self-same critical line. It can be easily verified that all the non-trivial zeros of $\zeta(s)$ appear as poles in $F(s)$, and all the trivial zeros cancel and so do not appear as poles in $F(s)^\dagger$. It can also be proved, from symmetry considerations, that both the numerator and denominator of $F(s)$ cannot vanish at the same point. Hence, to prove the RH, all we need to show is that all the poles of $F(s)$ occur on the critical line.

(2) A method applied by Littlewood (1912, see Edwards (1974) pp 260) to obtain an equivalent statement of RH involving the $1/\zeta(s)$ function is applied here to $F(s)$ to obtain a previously-known equivalent statement of

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*__This paper is the Final Version of the proof on RH, the earlier versions bear the title ‘The Dirichlet Series for the Liouville Function and the Riemann Hypothesis’ Sept 2016 to Oct 2017) are listed in Ref. [8] K. Eswaran__
†except that the latter has an extra pole on the critical line

uncertain provenance. Littlewood's method lies in analytically continuing the Dirichlet series for $F(s)$, strictly valid for $Re(s) > 1$, to the region $Re(s) \leq 1$. The analyticity of $F(s)$ turns out to be crucially dependent on the boundedness of $L(N)N^{-s}$ as $N \to \infty$, where $L(N)$ is the summatory Liouville function. That is, if $L(N) \sim N^a$ $(a > 0)$ asymptotically as $N \to \infty$, then the $F(s)$ can be analytically continued only on the right of the vertical line drawn at $Re(s) = a$. In other words, the singularities of F(s) will lie on or to the left of $Re(s) = a$. Further, since the non-trivial zeros of the Riemann zeta function $\zeta(s)$ exactly correspond to the poles of F(s), and are known to be symmetrically placed about the $Re(s) = 1/2$ line, this automatically implies that the non-trivial zeros of the zeta function will all be within the region $1 - a \leq Re(s) \leq a$. If $a \to 1/2$ from the right, the zeros will lie on the critical line and RH will be true. This also means that $L(N) \sim N^{\frac{1}{2}}$ as $N \to \infty$, thereby yielding the RH-equivalent statement (see Borwein et al 2006, p. 48):

$$lim_{N\to\infty} \frac{L(N)}{N^{\frac{1}{2}+\epsilon}} = 0 \quad \text{for}\, \epsilon > 0$$

Here $L(N) = \sum_{1}^{N} \lambda(n)$, and $\lambda(n)$ is the Liouville function that is +1 or -1 depending on whether n has an *even* or *odd* number of prime factors (multiplicity included).

(3) The equivalent statement above requires, if the RH is true, that $|L(N)| \sim N^{\frac{1}{2}}$ as $N \to \infty$ ($a < 1/2$ is impossible as the $Re(s) = 1/2$ line is known to have numerous zeros of the the zeta function). This expression is strongly suggestive of $X(N)$, the distance travelled in N unit steps in a standard *random walk*, which can be represented as:

$$X(N) = \sum_{n=1}^{N} c(n)$$

where the $c(n)$'s are "coin-tosses", i.e., independent random numbers with an equal probability of being either +1 or -1. It is well-known that the expected value of $|X(N)|$, for large N, is

$$lim_{N\to\infty} E(|X(N)|) = C_0 N^{1/2}$$

Therefore, if it could be shown that the $L(N)$ series is a random-walk, and that $|L(N)| \sim N^{\frac{1}{2}}$ as $N \to \infty$, the RH would be proved. This is the approach taken here. So we have to prove that the $\lambda(n)$'s in the $L(N)$ series are essentially "coin-tosses", for large n.

To show that the λ's behave as coin tosses, we have to show that (i) their probabilities of being either +1 of -1 are equal, and further (ii) that the λ's appearing in the natural sequence, $n = 1, 2, 3, ...$, are independent of each other.

(4) *Equal Probabilities*: A crucial advance in this line of attack is the discovery of a method of factorizing every integer and placing it in an exclusive subset, where it and its other members in the same subset form an increasing sequence of natural numbers that alternately have odd and even numbers of prime factors. Such subsets are called 'Towers' in the Paper. It is shown that every natural number, other than 1, has a unique place in a unique Tower of ordered countably infinite members, and each such member represents a unique natural number. The alternating odd and even factorization of the members of the Towers ensures that each Tower is partitioned into equal proportions of members with λ's of -1 and +1. As the entire natural number system is incorporated within the Towers system, the natural numbers also have equal proportions with odd and even numbers of prime factors, respectively with λ's of -1 and +1. In other words, concerning the probability that a random natural number n has a λ of either value, we conclude: $Prob[\lambda(n) = +1] = Prob[\lambda(n) = -1] = 1/2$ [Theorem 3B][‡].

(5) *Non-periodicity and Independence:* We then show that the sequence of λ's in $L(N)$ can never be cyclic, just as a sequence of coin tosses can never be cyclic. This done in Appendix III and follows directly from Littlewood's method. Quasi-cyclicity or any other pattern of λ's in the $L(N)$ series that would keep $L(N)$ bounded as $N \to \infty$ are also excluded. Specifically, non-cyclicity would preclude any dependence of the type

$$\lambda_n = f(\lambda_{n-1}, \lambda_{n-2}, \lambda_{n-3}, ..., \lambda_{n-M})$$

[‡]Theorems 3 and 3B, that argue for equal proportions and hence equal probabilities of the the λ's over the natural number system, were later given a new and different proof in K. Eswaran, (April 2018)

for finite M. So, essentially, the independence of λ's *in the natural sequence* is proved [although *within the Towers*, they have a perfectly predictable and deterministic relation]. Here we adopt the notation: λ_n for $\lambda(n)$.

(6) *Arithmetic Independence:* The independence of the λ's in the $L(N)$ series mentioned above followed from Littlewood's method, i.e., Analysis. In Appendix IV, a purely arithmetic approach is taken. It is shown that from merely these two rules: $\lambda(p) = -1$ for a prime p, and $\lambda(pq) = \lambda(p) \times \lambda(q)$, where q is any integer, the entire sequence of λ's for $n = 1, 2, 3, ...$, can be obtained, in a manner reminiscent of the construction of the natural number system by multiplication. It is then argued that for any two integers $n > m$, $\lambda(n)$ is dependent on $\lambda(m)$, if the latter is required to find the former, and independent if not. It is then shown that, as $n \to \infty$, any finite sequential strip of λ's will be independent of each other, thus essentially making them equivalent to coin-tosses.

(7) With Theorem 3B and Appendices III and IV, we have proved that the $L(N)$ series is a random walk of infinite length. We then invoke (towards the end of Section 5) Khinchin and Kolmogorov's law of the iterated logarithm to show that the maximum deviation, from the one-half power-law expectation, in the exponent of $|X(N)|$ for any individual random walk tends monotonically to zero as $N \to \infty$. So, in fact, $|L(N)| \sim N^{\frac{1}{2}}$ as $N \to \infty$. Therefore, for any chosen $\epsilon > 0$, the equivalent statement for the validity of RH will be satisfied and the Riemann Hypothesis is proved.

Interestingly, the aforementioned maximum deviation from the one-half power-law expectation in the random walk of $L(N)$ is exactly the half-width of the critical strip around $Re(s) = 1/2$ that contains all the zeros of the of the zeta function. As that deviation approaches zero as $N \to \infty$, that width is also zero, ensuring that all the non-trivial zeros of the zeta function lie on $Re(s) = 1/2$.

(8) In Appendix V, starting from Littlewood's *ansatz*, that $|L(N)| \sim N^a$, for $N \to \infty$, we argue that the statistics of the λ's must become "self-similar" over large consecutive sequences of λ's. It is shown that, if we choose two sets $S_-(N)$ and $S_+(N)$ of consecutive integers, each of them containing k integers,[§] then the λ's defined over these sets $S_-(N)$ and $S_+(N)$ are statistically similar to each other. This statistical similarity is shown to hold for all large k i.e for all $N = k^2$, which implies the statistical behavior of λ's are independent of the length k of $S_-(N)$ and $S_+(N)$ for large but arbitrary $N = k^2$. This principle yields us the value of $a = \frac{1}{2}$, which again would satisfy the equivalent statement of the RH. This 'physicist's proof' of the RH, is separate from the argument in the main paper, and may be treated as an interesting addendum to it.

(9) Finally, in Appendix VI, we show the sequence of λ's is statistically indistinguishable from coin tosses (using the χ^2 statistical test) over many sets of consecutive integers (as was demonstrated in Appendix V). Further it is also shown that the sequence λ's is indistinguishable from coin tosses over the entire range of numbers from $n = 1$ to 176 *trillion*. This verification has been done by actual numerical computation over large sets of integers (which are below 176 trillion). While this is merely a 'verification', not a 'proof', this empirical result follows directly from our proof of the Riemann Hypothesis, and affirms its sound basis.

Interestingly, it is also shown in Appendix VI that a connection exist between $L(N)$ and χ^2 when compared to a sequence of coin tosses. From the relation Eq.(9), p 24, one can conclude that to satisfy Littlewood's condition, only the first and second moments of the distributions of lambda and coin toss sequences need be similar.

Because of the extensive computations and calculations made, which are backed by theory, Appendix VI can be thought of as an experimental physicists' verification of the 'Law of Riemann'.

1 Introduction

This paper investigates the behaviour of the Liouville function, (ref. Apostol (1998), which is related to Riemann's zeta function, $\zeta(s)$, defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \tag{1.1}$$

where n is a positive integer and s is a complex number, with the series being convergent for $Re(s) > 1$. This function has zeros (referred to as the trivial zeros) at the negative even integers $-2, -4, \ldots$. It has been

[§]The set $S_-(N)$ contains k consecutive integers ending with the integer $N = k^2$, and $S_+(N)$ contains the next k consecutive numbers, see Appendix V for details and definitions shown[¶] that there are an infinite number of zeros on the line at $Re(s) = 1/2$. Riemann's Hypothesis (R.H.) claims that these are all the nontrivial zeros of the zeta function. The R.H. has eluded proof to date, and this paper demonstrates that it is resolvable by tackling the Liouville function's Dirichlet series generated by $F(s) \equiv \zeta(2s)/\zeta(s)$, which is readily rendered in the form

$$F(s) = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s}, \tag{1.2}$$

where $\lambda(n)$ is the Liouville function defined by $\lambda(n) = (-1)^{\Omega(n)}$, with $\Omega(n)$ being the total number of prime numbers in the factorization of n, including the multiplicity of the primes. We would also need the summatory function $L(N)$, which is defined as the partial sum up to N terms of the following series:

$$L(N) = \sum_{n=1}^{N} \lambda(n) \tag{1.2b}$$

Since the function $F(s)$ will exhibit poles at the zeros of $\zeta(s)$, we seek to identify where $\zeta(s)$ can have zeros by examining the region over which $F(s)$ is analytic. By demonstrating that a sufficient condition, derived by Littlewood (1912),(in Edwards(1974)), for the R.H. to be true is indeed satisfied, we show that all the nontrivial zeros of the zeta function occur on the 'critical line' $Re(s) = 1/2$.

Briefly, our method consists in judiciously partitioning the set of positive integers (except 1) into infinite subsets and couching the infinite sum in (1.2) into sums over these subsets with each resulting sub-series being uniformly convergent. This method of considering a slowly converging series as a sum of many sub-series was previously used by the author in problems where Neumann series were involved Eswaran (1990)).

In this paper we break up the sum of the Liouville function into sums over many sub-series whose behaviour is predictable. It so turns out that one prime number p (and its powers) which is associated with a particular sub-series controls the behaviour of that sub-series.

Each sub-series is in the form of rectangular functions (waves) of unit amplitude but ever increasing periodicity and widths - we call these 'harmonics' - so that every prime number is thus associated with such harmonic rectangular functions which then play a role in contributing to the value of $L(N)$. It so turns out that if N goes from N to $N+1$, the new value of $L(N+1)$ depends solely on the factorization of $N+1$, and the particular harmonic that contributes to the change in $L(N)$ is completely determined by this factorization. Since prime factorizations of numbers are uncorrelated, we deduce that the statistical distribution of $L(N)$ when N is large is like that of the cumulative sum of N coin tosses, (a head contributing +1 and a tail contributing -1), and thus logically lead to the final conclusion of this paper.

We found a new method of factoring every integer and placing it in an exclusive subset, where it and its other members form an increasing sequence which in turn factorize alternately into odd and even factors; this method exploited the inherent symmetries of the problem and was very useful in the present context. Once this symmetry was recognized, we saw that it was natural to invoke it in the manner in which the sum in (1.2) was performed. We may view the sum as one over subsets of series that exhibit convergence even outside the domain of the half-plane $Re(s) > 1$. We were rewarded, for following the procedure pursued in this paper, with the revelation that the Liouville function (and therefore the zeta function) is controlled by innumerable rectangular harmonic functions whose form and content are now precisely known and each of which is associated with a prime number and all prime numbers play their due role. And in fact all harmonic functions associated with prime numbers below or equal to a particular value N determine $L(N)$. The underlying symmetry being alluded to here, remained hidden because the summation in (1.2) is written in the usual manner, setting $n = 1, 2, 3, ...$ in sequence.

From the next section onwards the paper follows the plan enunciated in the Extended Abstract and indicated by the the steps (1) to (9) detailed therein.

2 Partitioning the Positive Integers into Sets

The Liouville function $\lambda(n)$ is defined over the set of positive integers n as $\lambda(n) = (-1)^{\Omega(n)}$, where $\Omega(n)$ is the number of prime factors of n, multiplicities included. Thus $\lambda(n) = 1$ when n has an even number of prime factors and $\lambda(n) = -1$ when it has an odd number of prime factors. We define $\lambda(1) = 1$. It is a completely arithmetical function obeying $\lambda(mn) = \lambda(m)\lambda(n)$ for any two positive integers m , n.

[¶]This was first proved by Hardy (1914).

We shall consider subsets of positive integers such as $\{n_1, n_2, n_3, n_4, ...\}$ arranged in increasing order and are such that their values of λ alternate in sign:

$$\lambda(n_1) = -\lambda(n_2) = \lambda(n_3) = -\lambda(n_4) = ... \tag{2.3}$$

It turns out that we can label such subsets with a triad of integers, which we now proceed to do. To construct such a labeling scheme, consider an example of an integer n that can be uniquely factored into primes as follows:

$$n = p_1^{e_1} p_2^{e_2} p_3^{e_3} ... p_L^{e_L} p_i p_j \tag{2.4}$$

where $p_1 < p_2 < p_3 ... < p_L < p_i < p_j$ are prime numbers and the $e_k, k \in \{1, 2, 3, ..., L\}$ are the integer exponents of the respective primes, and p_L is the largest prime with exponent exceeding 1, the primes appearing after p_L will have an exponent of only one and there may a finite number of them, though only two are shown above. Integers of this sort, with at least one multiple prime factor are referred to here as Class I integers. In contrast, we shall refer to integers with no multiple prime factors as Class II integers. A typical integer, q, of Class II may be written

$$q = p_1 p_2 p_3 ... p_j p_L, \tag{2.5}$$

where, once again, the prime factors are written in increasing order.

We now show how we construct a labeling scheme for integer sets that exhibit the property in (2.3) of alternating signs in their corresponding λ's. First consider Class I integers. With reference to (2.4), we define integers m, p, u as follows:

$$m = p_1^{e_1} p_2^{e_2} p_3^{e_3} ... p_{L-1}^{e_{L-1}}; \quad p = p_L; \quad u = p_i p_j. \tag{2.6}$$

In (2.6), m is the product of all primes less than p_L,the largest multiple prime in the factorization, and u is the product of all prime numbers larger than p_L in the factorization. Thus the Class I integer n can be written

$$n = m p^{e_L} u \tag{2.7}$$

Hence we will label this integer n as (m, p^{e_L}, u),using the triad of numbers(m, p, u) and the exponent e_L. It is to be noted that u will consist of prime factors all larger than p, and u cannot be divided by the square of a prime number.

Consider the infinite set of integers, $P_{m;p;u}$, defined by

$$P_{m;p;u} = \{mp^2u, mp^3u, mp^4u, ...\} \tag{2.8}$$

The Class I integer n necessarily belongs to the above set because $e_L \geq 2$. Since the consecutive integer members of this set have been obtained by multiplying by p, thereby increasing the number of primes by one, this set satisfies property (2.3) of alternating signs of the corresponding λ's. Note that the lowest integer of this set $P_{m;p;u}$ of Class I integers is mp^2u.

We may similarly form a series for Class II integers. The integer q in (2.5) may be written $q = mpu$, with $m = p_1 p_2 p_3 ... p_j$, $p = p_L$, and $u = 1$. This Class II integer is put into the set $P_{m;p;u}$ defined by

$$P_{m;p;1} = \{mp, mp^2, mp^3, mp^4, ...\}. \tag{2.9}$$

The set containing Class II integers is distinguished by the facts that $u = 1$ for all of them, their largest prime factor is always p and none of them can be divided by the square π^2 of a prime number π such that $\pi < p$; in other words the factor m cannot be divided by the square of a prime. In this set, too, the λ's alternate in sign as we move through it and so property (2.3) is satisfied. Again, note that the lowest integer of this set $P_{m;p;1}$ is the Class II integer mp, all the others being Class I.

In what follows, we shall find it handy to refer to the set of ascending integers comprising $P_{m;p;u}$ as a 'tower'. It is important to distinguish between a tower (or set) described by a triad like (m, p, u) and an integer belonging to that set. It is worth repeating that the set or tower of Class I integers described by the label (m, p, u) is the infinite sequence $\{mp^2u, mp^3u, mp^4u, ...\}$, the first element of which is mp^2u and all other members of which are mp^ku, where $k > 2$. A set or tower containing a Class II integer described by $(m, p, u = 1)$ is the infinite sequence $\{mp, mp^2, mp^3, ...\}$, Eq.(2.9), of which only the first element mp is a Class II integer and all other members, mp^k, where $k \geq 2$, are Class I, because the latter have exponents greater than 1. For convenient reference, we shall refer to the first member of a tower as the base integer or the base of the tower. It is also worth noting that when we refer to a triad like (m, p^k, u), where $k > 1$, we are invariably referring to the integer mp^ku and not to any set or tower. Labels for sets do not contain exponents; only those for integers do. Of course, the particular integer (m, p^k, u) belongs to the set or tower (m, p, u).

Two simple examples illustrate the construction of the sets denoted by $P_{m;p;u}$:

Ex. 1: The integer 2160, which factorizes as $2^4 \times 3^3 \times 5$, is clearly a Class I integer since it is divisible by the square of a prime number—in fact there are two such numbers, 2 and 3—but we identify p with 3 as it is the larger prime. It is a member of the set $P_{16;3;5} = \{16 \times 3^2 \times 5, 16 \times 3^3 \times 5, 16 \times 3^4 \times 5, 16 \times 3^5 \times 5, ...\}$.

Ex. 2: The integer 663, which factorizes as $3 \times 13 \times 17$, is a Class II integer because it is not divisible by the square of a prime number. It belongs to the set $P_{39;17;1} = \{39 \times 17, 39 \times 17^2, 39 \times 17^3, ...\}$.

Note that two different integers cannot share the same triad.[‖] And two different triads cannot represent the same integer.[**] Thus the mapping from a triad to an integer is one-one and onto. A formal proof is in the Appendix.

The following properties of the sets $P_{m;p;u}$ may be noted:

(a) The factorization of an integer n immediately determines whether it is a Class I or a Class II type of integer.

(b) The factorization of integer n also identifies the set $P_{m;p;u}$ to which n is assigned.

(c) The procedure defines all the other integers that belong to the same set as a given integer.

(d) Every integer belongs to some set $P_{m;p;u}$ (allowing for the possibility that $u = 1$) and only to one set. This ensures that, collectively, the infinite number of sets of the form $P_{m;p;u}$ exactly reproduce the set of positive integers $\{1, 2, 3, 4, ...\}$, without omissions or duplications.

Our procedure, taking its cue from the deep connection between the zeta function and prime numbers, has constructed a labeling scheme that relies on the unique factorisation of integers into primes. In what follows, we shall recast the summation in (1.2) into one over the sets $P_{m;p;u}$.The advantage of breaking up the infinite sum over all positive integers into sums over the $P_{m;p;u}$ sets will soon become clear.

3 An alternative expression of the Liouville Function's Dirichlet series

The usefulness of this Section and the next (i.e. Sections 3 and 4) is to show that the cumulative summatory function $L(N) = \Sigma_{n=1}^{N} \lambda(n)$, can be built up by 'harmonic rectangular waves', thus providing a pictorial representation of the function $L(N)$. This pictorial view which helped us to understand the phenomena in RH, actually followed the discovery of an alternative expression for Eq(1.2) namely the representation Eq(3.10). This expression is written in terms of 'towers' which as we shall see help in our study of the properties of $L(N)$. The kinks in the rectangular waves which occur at integer values k in the argument of $\lambda(k)$ each contribute either +1 or -1 to $L(N)$ and are distributed like coin tosses and their summation is akin to the cumulative sum of N coin tosses. Eq. (3.10) has also helped in evolving the concepts of Towers described in the section preceding, and apart from this Eq(3.10) plays no crucial role in the proof of RH.

THEREFORE ON A FIRST READING THE FOLLOWING CAN BE OMITTED: SEE FOOT NOTE.[††]

We shall now implement the above partitioning of the set of all positive integers to examine the analytic properties of $F(s)$ in (1.2). We shall rewrite the sum in (1.2) into an infinite number of sums of sub-series.

We begin, however, by assuming that $Re(s) > 1$, which makes the series in (1.2) absolutely convergent, in fact it represents $\zeta(2s)/\zeta(s)$ and is well defined for $Re(s) > 1$. We will not be needing the expression for regions $Re(s) < 1$. [‡‡] We write the right hand side in sufficient detail so that the implementation of the partitioning

[‖] The integer represented by the triad (m, p^r, u), is the product $mp^r u$, which obviously cannot take on two distinct values.

[**] Suppose two different triads (m, p^r, u) and (μ, π^ρ, ν) represent the same integer, say n. Then we must have $mp^r u = \mu\pi^\rho\nu = n$.It follows that at least two numbers of the tetrad $\{m, p, r, u\}$ must differ from their counterparts in the tetrad $\{\mu, \pi, \rho, \nu\}$. Since the factorization of n is unique, this is impossible.

[††] On a first reading Sections 3 and 4 can be omitted. And one can go directly to Section 5 and after reading the proof of Theorem 1, in Section 5.1, skip the rest of this subsection and go directly to subsection 5.2, in page 12 and read till the end of the paper. Though, the last four paragraphs of Section 4 should be read to understand the Figure. Sections 3 and 4 have been included to maintain mathematical rigor: To demonstrate that expressions (1.2) and (3.10) have the same analytical continuation to the left of $Re(s) = 1$ by Littlewood's theorem.

[‡‡] Though each sub-series is convergent for $Re(s) < 1$- see Titchmarsh or Whittaker and Watson.

scheme becomes self-evident:

$$\begin{aligned}
F(s) \quad = \quad & 1+\sum_{r=1}^{\infty}\frac{\lambda(2^r)}{2^{rs}}+\sum_{r=1}^{\infty}\frac{\lambda(3^r)}{3^{rs}}+\sum_{r=1}^{\infty}\frac{\lambda(5^r)}{5^{rs}}+\sum_{r=1}^{\infty}\frac{\lambda(2\times 3^r)}{2^s3^{rs}} \\
&+\sum_{r=1}^{\infty}\frac{\lambda(7^r)}{7^{rs}}+\sum_{r=1}^{\infty}\frac{\lambda(2\times 5^r)}{2^s5^{rs}}+\sum_{r=1}^{\infty}\frac{\lambda(11^r)}{11^{rs}}+\sum_{k=2}^{\infty}\frac{\lambda(2^k\times 3)}{2^{ks}3^s} \\
&+\sum_{r=1}^{\infty}\frac{\lambda(13^r)}{13^{rs}}+\sum_{r=1}^{\infty}\frac{\lambda(2\times 7^r)}{2^s7^{rs}}+\sum_{r=1}^{\infty}\frac{\lambda(3\times 5^r)}{3^s5^{rs}}+\sum_{r=1}^{\infty}\frac{\lambda(17^r)}{17^{rs}} \\
&+\sum_{r=1}^{\infty}\frac{\lambda(19^r)}{19^{rs}}+\sum_{k=2}^{\infty}\frac{\lambda(2^k\times 5)}{2^{ks}5^s}+\sum_{r=1}^{\infty}\frac{\lambda(3\times 7^r)}{3^s7^{rs}}+\sum_{r=1}^{\infty}\frac{\lambda(2\times 11^r)}{2^s11^{rs}} \\
&+\sum_{r=1}^{\infty}\frac{\lambda(23^r)}{23^{rs}}+\sum_{r=1}^{\infty}\frac{\lambda(2\times 13^r)}{2^s13^{rs}}+\sum_{k=2}^{\infty}\frac{\lambda(2^k\times 7)}{2^{ks}7^s}+\sum_{r=1}^{\infty}\frac{\lambda(29^r)}{29^{rs}} \\
&+\sum_{r=1}^{\infty}\frac{\lambda(2\times 3\times 5^r)}{2^s3^s5^{rs}}+\cdots \quad , \qquad (3.10)
\end{aligned}$$

We have explicitly written out a sufficient number of terms of the right hand side of (1.2) so that those corresponding to each of the first 30 integers are clearly visible as a term is included in one (and only one) of the sub-series sums in (3.10). On the right hand side, the second term contains the integers $2, 4, 8, 16....$; the third contains $3, 9, 27, ...$; the fourth contains $5, 25, 125, ...$; the fifth contains $6, 18, 54, ...$; sixth contains $7, 49, ...$; the seventh contains $10, 50, ...$; the eighth contains $11, 121, ...$; the ninth contains $12, 24, 48, ...$; and so on. Note that in the ninth, fifteenth, and twentieth terms the running index is deliberately switched from r to k to alert the reader to the fact that the summation starts from 2 and not from 1 as in all the other sums. (Note that, in the ninth term, the Class I integer $n = 12 = 2^2 \times 3$ is assigned to the set $P_{1;2;3} = \{2^2 \times 3, 2^3 \times 3, 2^4 \times 3, ...\}$ and not to the set $P_{4;3;1} = \{2^2 \times 3, 2^2 \times 3^2, 2^2 \times 3^3, .\}$, because the first term identifies p as 2 and u as 3 where as the second term onwards 3 has exponents, which violates our rules of precedence and would be an illegitimate assignment given our partitioning rules.)

The sub-series in (3.10) have one of two general forms:

$$\sum_{r=1}^{\infty}\frac{\lambda(m.p^r)}{m^s.p^{rs}} \;=\frac{\lambda(m.p)}{m^s.p^s}\,[1-\frac{1}{p^s}+\frac{1}{p^{2s}}-\frac{1}{p^{3s}}+\cdots+\frac{(-1)^X}{p^{Xs}}+\cdots]$$

or

$$\sum_{k=2}^{\infty}\frac{\lambda(m.p^k.u)}{m^s.p^k.u^s} \;=\frac{\lambda(m.p^2.u)}{m^s.p^{2s}.u^s}\,[1-\frac{1}{p^s}+\frac{1}{p^{2s}}-\frac{1}{p^{3s}}+\cdots+\frac{(-1)^X}{p^{Xs}}+\cdots] \qquad (3.11)$$

The above geometric series occurring within square brackets in the above two equations can actually be summed (because they are convergent),(see Whittaker and Watson) but we will refrain from doing so, and (1.2) can be rewritten as

$$F(s)=\sum_m\sum_p\sum_u F^T_{m;p;u}(s)+\sum_m\sum_p F^T_{m;p;1}(s), \qquad (3.12)$$

where the first group of summations pertains to Class I integers n characterized by the triad $(m, p^k, u), (k \geq 2)$ and the second group pertain to those integers which are characterized by set $(m, p^k, 1), (k \geq 1)$ the first member in the set is a Class II integer and others Class I.

In the above we have defined the function $F^T_{m;p;u}(s)$ of the complex variable s which is a sub-series involving terms over only the tower (m, p, u) for a Class I integer as follows

$$F^T_{m;p;u}(s)=\sum_{k=2}^{\infty}\frac{\lambda(mp^ku)}{m^sp^{ks}u^s}, \qquad (3.13)$$

and the function $F^T_{m;p;1}(s)$ of the complex variable s which is a sub-series involving terms over only the tower $(m, p, 1)$ whose 1st term is a Class II integer as

$$F^T_{m;p;1}(s)=\sum_{r=1}^{\infty}\frac{\lambda(mp^r)}{m^sp^{rs}} \qquad (3.14)$$

With the understanding that when $u = 1$ we use the function in (3.14) instead of (3.13), we may write $F(s)$ as

$$F(s) = \sum_m \sum_p \sum_u F^T_{m;p;u}(s). \tag{3.15}$$

Comparing the above Eq.(3.15) with Eq(3.10) one can easily see that each term which appears as a summation in (3.10) is actually a sub-series over some tower which we denote as $F^T_{m;p;u}(s)$ in (3.15). So we see that $F(s)$ has been broken up into a number of sub-series. The important point to note is that the λ value of each term in the sub-series changes its sign from +1 to -1 and then back to +1 and -1 alternatively. Therefore if the starting value of λ at the base was +1 then the cumulative contribution of this tower (sub series) to $L(N)$ as N, the upper bound, increases from N to $N+1$, $N+2$, $N+3$, ... will fluctuate between be 0 and 1. For some other tower whose base value of λ is -1 its cumulative contribution to L(N) will fluctuate between 0 and -1; these cumulative contributions can be represented in the form of a rectangular wave as shown in Figure 1.

We have arrived at a critical point in our paper. We have cast the original function $F(s) \equiv \zeta(2s)/\zeta(s)$ as a sum of functions of s. Since the triad (m, p^k, u) uniquely characterises all integers, the summations over m, p, k and u above are equivalent to a summation over all positive integers n, as in (1.2), though not in the order $n = 1, 2, 3, 4, ...$ The manner in which the triads were defined ensures that there are neither any missing integers nor integers that are duplicated.(See Theorems A and B in Appendix II.)

Although we did not explicitly do it, we mentioned in passing that the sum over k in (3.13)and (3.14) is readily performed since it is a geometric series (see (3.11)) that rapidly converges. This is true not merely for $Re(s) > 1$ but also as $Re(s) \to 0$. Whether $F(s)$ converges when the summation is carried out over all the towers (m, p, u) and, if so, over what domain of s is the central question that we seek to answer in the next section. The answer to which as we shall see determines the analyticity of F(s) and thus resolves the Riemannian Hypothesis. We can recast (3.15), still in the domain $Re(s) > 1$, in the form

$$F(s) = \sum_{n=1}^{\infty} \frac{h(n)}{n^s}, \tag{3.16}$$

where $h(n)$ is a function appropriately defined below.

By construction, every n in the above summation can be written as

$$n = \mu \pi^\rho \nu, \tag{3.17}$$

where μ, π, and ρ are positive integers, π is the largest prime in the factorization of n, with either (i) an exponent $\rho \geq 2$, and ν is the product of primes larger than π but with exponents equal to 1 (for Class I integers) or (ii) it is the largest prime factor with $\rho = 1$ and $\nu = 1$ (for Class II integers).

We define $h(n)$ as follows:

$$\begin{aligned} h(n) &= \lambda(mp^k u) \quad \text{if } \mu = m \text{ and } \pi = p \text{ and } \nu = u \neq 1 \text{ and } \rho = k > 1 && (3.18a) \\ h(n) &= \lambda(mp^k) \quad \text{if } \nu = u = 1 \text{ and } \pi = p \text{ and } \rho = k \geq 1 && (3.18b) \\ h(1) &= 1 \quad \text{by definition.} && (3.18c) \end{aligned}$$

Note for all $n > 1$, (3.18a) and (3.18b) taken together, defines the $h(n)$ for all Class I and Class II integers n. The factors $m^s p^{ks} u^s$ and $m^s p^{ks}$ in the denominators of (3.13) and (3.14) are simply n^s, where n is the integer characterized by the (m, p^k, u) triad (with $u = 1$ in the latter case).

4 Representation of the summatory Liouville function L(N)

We are now in a position to examine the summatory Liouville function $L(N)$ and to depict the sum for any given finite N, as arising from individual contributions from 'rectangular waves'.

To do all this systematically, we will explicitly illustrate the process starting from $N = 1, 2, 3...$ up to $N = 15$. Each of these numbers is factored and expressed uniquely as a triad. The N=1 is a constant term, which is the trivial $(1, 1, 1)$, then the next number $N = 2 = (1, 2, 1)$, is contained in the tower shown below the one corresponding to $N = 1$; and $N = 3 = (1, 3, 1)$, is the tower below the previous; $4 = (1, 2^2, 1)$ however 4 is already contained in the tower $(1, 2, 1)$ as its second member; the next N's: $5, 6, 7$, give rise to the new towers $(1, 5, 1), (2, 3, 1), (1, 7, 1)$; 8 of course is the third member of the old tower $(1, 2, 1)$ similarly 9 is the 2nd member of $(1, 3, 1)$. After this the new towers which make their appearance are: $10 = (2, 5, 1), 11 = (1, 11, 1), 13 = (1, 13, 1), 14 = (2, 7, 1)$ and $15 = (3, 5, 1)$. Figure 1 shows these and numbers up to N=30.

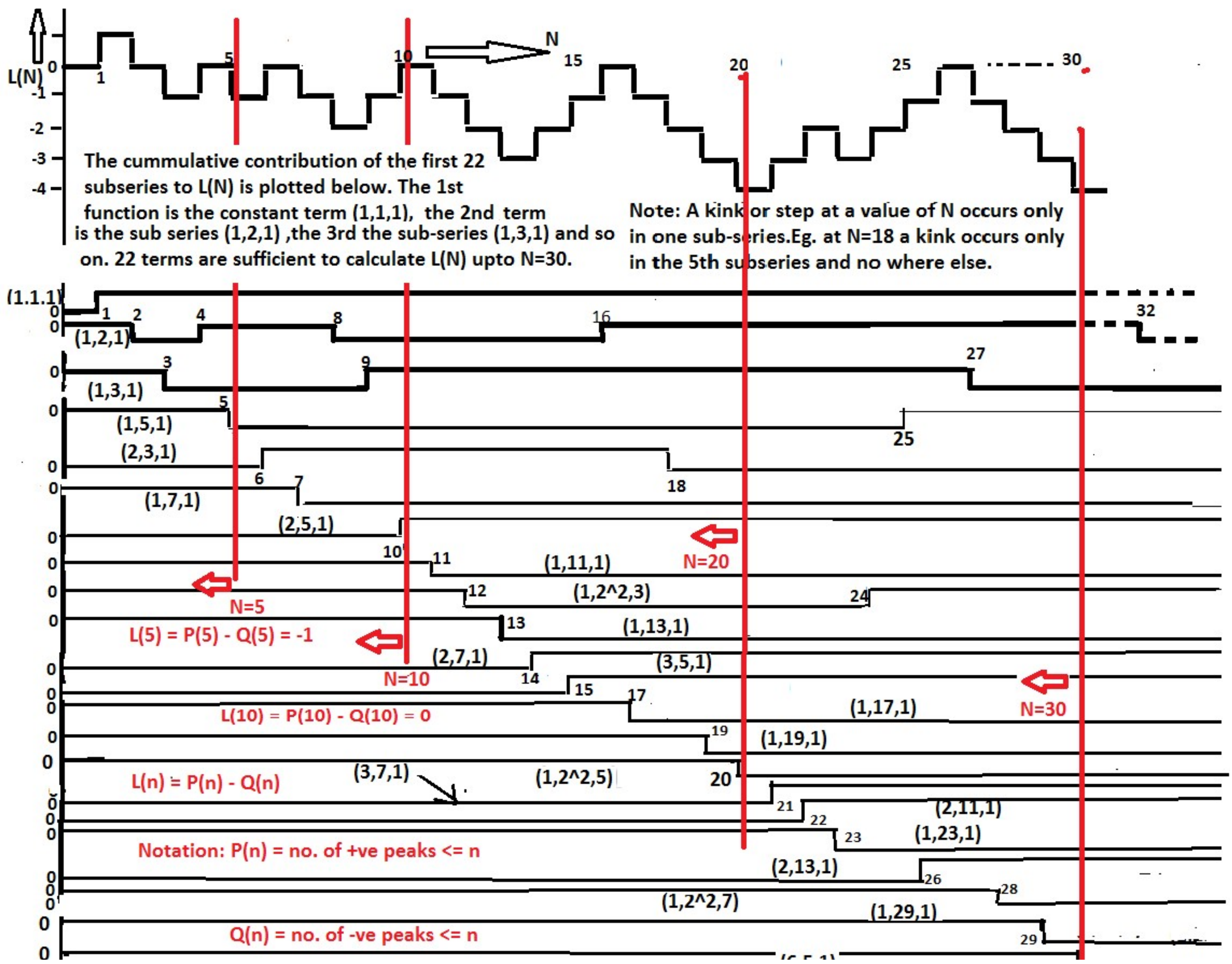

Fig. 1. The cumulative sum, L(N) (see top), is obtained by 'filling up' slots in various towers from the bottom up until we have exhausted all N integers.

Now each tower (m, p, u) contributes to $L(N)$ (consider N fixed in the following) according to the following rules:

(i) A particular tower will contribute only if its base number is less than or equal to N, i.e. $m.p.u \leq N$

(ii) And the contribution C to $L(N)$ from this particular tower will be exactly as follows:

Case A; Class II integer $(u = 1)$

$C = \Sigma_{r=1}^{R} \lambda(m.p^r.1)$, where R is the largest integer such that $m.p^R \leq N$

Case B; Class I integer $(u > 1)$

$C = \Sigma_{k=2}^{K} \lambda(m.p^k.u)$, Where K is the largest integer such that $m.p^K.u \leq N$

Now since each successive λ changes sign from $+1$ to -1 or vice a versa, the contributions of each tower can be thought of as a rectangular wave of ever-increasing width but constant amplitude -1 or $+1$, see Figure 1.

To find the value of L(N), (N fixed), all we need to do is count the jumps of each wave: as we move from N=0 a jump upwards is called a positive peak, a jump downwards is a negative peak. Draw a vertical line at N, we are assured that it will hit one and only one peak (positive or negative) in one of the sub-series; then count the total number of positive peaks P(N) and negative peaks Q(N), of the waves on and to the left of this vertical line, then $L(N) = P(N) - Q(N)$; the reason for this rule will be clear after the next section.

For an example, take $N = 5$. There is a positive peak for the constant term (1,1,1), the next wave (1,2,1) contributes one negative peak (at 2) and a positive peak (at 4), the wave (1,3,1) contributes a -1 peak (at 3) and (5,1) contributes a -1 peak (at 5). Thus a total of three negative peaks and two positive peaks add up to give $L(5) = -1$, which is of course correct. Now if we take $N = 10$, and draw a vertical line at N=10, looking at this line and to its left we see that there are additionally three positive and two negative peaks thus adding this contribution of $+1$ to the previously calculated value $L(5)$ we get $L(10) = 0$. (Two red vertical lines just just beyond N=5 and N=10 are drawn for convenience.) Now if we wish to compute L(15) we see that there are three more negative peaks and two positive peaks thus giving a value $L(15) = -1$. Counting the peaks further on it is easy to check that $L(N)$ is correctly predicted for every value of N up to 30 and in particular, $L(20) = -4$, $L(26) = 0$ and $L(30) = -4$.

In summary, to calculate L(N) we merely need to count the negative and positive peaks of the waves on N and to the left of N. In the figure we have drawn a number of waves and labeled the tower to which each belongs using a triad of numbers. They are sufficient for one to easily calculate L(N) up to N=30 and check them out by comparing the numbers with the plot of L(N) shown on the top of the figure.

We turn to a more fundamental point: We show, in Section 6, that, for sufficiently large N (see Appendix IV), the distribution of the value of L(N) is equivalent to that obtained from summing the distribution of N coin tosses.

5 Determination of Analyticity of $F(s)$ using Littlewood's Theorem

We now utilize a technique introduced by Littlewood (1912), to examine the analyticity of the function $F(s)$. In this, we follow the treatment of Edwards (1974, pp. 260-261).

We have seen that there are two equivalent expressions $F(s)$ viz. Eqs.(1.2) and (3.10) both of which are given in the form of a series and are absolutely convergent in the region $Re(s) > 1$. We will therefore follow the following two procedures:

(i) By using Littlewoods technique we will analytically continue Eq (1.2), which is convergent for $Re(s) > 1$ to regions $Re(s) < 1$ and then see that his theorem determines a condition on L(N) for N large, for RH to be true.

(ii) Similarly instead of using Eq (1.2) we use the equivalent (3.10) and use Littlewood's technique to analytically continue Eq (3.10) which is convergent for Re(s) ¿ 1 to regions Re(s) ¡ 1 . This also gives a same condition as (a) on L(N) for N large for RH to be true. But this time the condition can be interpreted by a FIGURE. And the FIGURE reveals a clear analogy with coin tosses. Strictly speaking our treatment (ii) is redundant except for the understanding of the connection of L(N) with coin tosses. In fact for the rest of the paper we do not need Eq. (3.10) or the Figure, except that the concept of Towers and the factorizations of integers and the determination of their membership to different towers would be needed to prove several theorems.

5.1 Littlewood's theorem applied to F(s) viz. Eqs.(1.2) & (3.10)

We have seen that (1.2) we define:

$$F(s) = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s}, \tag{5.19}$$

Similarly the alternative expression (3.10) written in the form Eq. (3.16) is:

$$F(s) = \sum_{n=1}^{\infty} \frac{h(n)}{n^s}, \tag{5.20}$$

with the definition given in (3.18).

Since both of the above expressions are similar in form we use the following generic expression for the purpose of analysis:

$$F(s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s}, \tag{5.21}$$

where $g(n)$ can mean $\lambda(n)$ or $h(n)$ as the case may be.

The above series (5.19) can be expressed as the integral

$$F(s) = \int_0^{\infty} x^{-s} dG(x), (Re(s) > 1), \tag{5.22}$$

where $G(x) = \int_0^x dG$ is a step function that is zero at $x = 0$ and is constant except at the positive integers, with a jump of $g(n)$ at n. The value of $G(n)$ at the discontinuity, at an integer n, is defined as $(1/2)[G(n-\epsilon) + G(n+\epsilon)]$, which is equal to $\sum_{j=1}^{n-1} g(j) + (1/2)g(n)$. Assuming $Re(s) > 1$, integration by parts yields

$$\begin{aligned} F(s) &= \int_0^\infty d[x^{-s}G(x)] - \int_0^\infty G(x)d[x^{-s}] \qquad (5.23) \\ &= \lim_{X\to\infty} \quad [X^{-s}G(X) + s\int_0^X G(x)x^{-s-1}dx \\ &= s\int_0^\infty G(x)x^{-s-1}dx, \qquad (5.24) \end{aligned}$$

where the last step follows from the fact that $\mid G(X) \mid \leq X$, which implies that $X^{-s}G(X) \to 0$ as $X \to \infty$. We further observe, following Littlewood (1912), that as long as $G(X)$ grows less rapidly than X^a for some $a > 0$, the integrals in (5.21) and in the line preceding it converge for all s in the half-plane $Re(a - s) < 0$, that is, for $Re(s) > a$. By analytic continuation, $F(s)$ converges in this half-plane. Since this result will be important in what follows, we record it here.

Theorem [Littlewood (1912)]: When $G(X)$ grows less rapidly with X than X^a for some $a > 0$, $F(s)$ is analytic in the half-plane $Re(s) > a$.

We have obtained the above generic result for the analytic continuation of the function given by Eq(5.21). We will now apply it for the case (i) i.e. Eq. (5.19),in this case $g(n) \equiv \lambda(n)$ and $X \equiv N$ thus making $G(X) \equiv L(N)$. Thus the above Littlewood's Theorem becomes the condition for the analytic continuation of(5.19) and which is now restated to read:

Theorem 1 [Littlewood (1912)]: When $L(N)$ grows less rapidly with N than N^a for some $a > 0$, $F(s)$ is analytic in the half-plane $Re(s) > a$.

We shall now demonstrate that the sufficient condition stated in Theorem 1 is satisfied for a specific value of a that settles the Riemann Hypothesis. (It will turn out that $a = 1/2$).

Now before devoting the rest of the paper to show that the above condition holds for RH. We will use the analysis for the analytical continuation of (5.20) i.e. Case (ii).

Hence our definition of $G(N)$ becomes

$$G(N) = \sum_{n=1}^{N} g(n), \tag{5.25}$$

and we may rewrite $G(N)$ as

$$G(N) = \sum_m \sum_p \sum_u \sum_k \left[(1 - \delta_{u,1}).(1 - \delta_{k,1})\, \lambda(mp^k u) + \delta_{u,1}\lambda(mp^k)\right], \tag{5.26}$$

where $\delta_{u,1}$ and $\delta_{k,1}$ are Kronecker deltas (e.g. $\delta_{u,1} = 1$ if $u = 1$ and 0 otherwise). The summations over m, p, k, and u in (5.26) are undertaken with the understanding that the triads (m, p^k, u) will only include integers $n \leq N$. Since the summation over k is over an individual tower(if we keep (m,p,u) fixed we can write(5.26) as

$$G(N) = \sum_m \sum_p \sum_u F^T_{m,p,u}(s = 0), \tag{5.26b}$$

This is nothing but Eq.(3.15) evaluated from each subseries $F^T_{m,p,u}(s)$ by making $s \to 0$.

Of course, what we have called $G(N)$ is really the summatory Liouville function, $L(N)$, defined earlier by (1.2b), because each integer n occurs only once in the r.h.s. of (5.26b) as an argument of λ i.e. $\lambda(n)$, Therefore the $G(N)$ is really $L(N)$, hence

$$L(N) = \sum_{n=1}^{N} \lambda(n). \tag{5.24}$$

From now on, we revert to the original definitions of the sequence $h(n) \equiv \lambda(n)$ and $G(N) \equiv L(N)$ as defined in Eq. (1.2) but we may write them in the forms derived in Section 2 using triads.

5.2 Derivation of Theorems concerning Factorization Sequence of $\lambda's$ and the Final Proof of the Riemann Hypothesis

In this subsection we will derive several crucial theorems concerning the sequence of $\lambda's$.

Expression (5.26) is crucial because, in the light of Theorem 1, its behaviour will determine the validity of the Riemann Hypothesis. Every term in the summation in (5.26) is either $+1$ or -1. We need to determine, for given N, how many terms contribute $+1$ and how many -1, and then determine how the sum $G(N)$ varies with N.

As we go through the list $n = 1, 2, 3, \cdots, N$, we are assigning the integers to various sets of the kind $P_{m;p;u}$. To use our terminology of towers, we shall be 'filling up' slots in various towers from the bottom up until we have exhausted all N integers. (When N increases, in general, we shall not only be filling up more slots in existing towers but also adding new towers that were previously not included.) So the behaviour of $G(N)$ is determined by how many of the numbers that do not exceed N contribute $+1$ and how many -1.

It is convenient to identify the λ of an integer by the triad which uniquely defines that integer. To avoid abuse of notation, we shall denote the value $\lambda(n)$ in terms of the λ-value of the base integer of the tower to which n belongs. We will define the λ of the base of a tower in uppercase, as $\Lambda(m, p, u)$. In other words if $n = (m, p^\rho, u)$ then it will belong to a tower whose base number is $n_B \equiv (m, p^\kappa, u)$, where $\kappa = 2\ if\ \ u \neq 1$ and $\kappa = 1\, if\ \ u = 1$. Now we define $\Lambda(m, p, u) = \lambda(n_B) = \lambda(mp^\kappa u) = \lambda(m)\lambda(p^\kappa)\lambda(u)$, since the λ of a product of integers is the product of the λ of the individual integers. Of course, once we know $\lambda(n_B)$ we will know the λ of all other numbers belonging to the tower because they alternate in sign.

To determine the behaviour of $G(N)$, the following theorem is important.

Theorem 2: For every integer that is the base integer of a tower labeled by the triad (m, p, u), and therefore belonging to the set $P_{m;p;u}$, there is another unique tower labeled by the triad (m', p, u) and therefore belonging to the set $P_{m',p,u}$ with a base integer for which $\Lambda(m', p, u) = -\Lambda(m, p, u)$.

Proof:

Let us write the integers at the base of a tower in the form $n = mp^\rho u$ described by the triad (m, p, u), where we shall assume that $\rho = 2$ if $u \neq 1$ and $\rho = 1$ if $u = 1$. These correspond to the smallest members of sets of Class I and Class II integers, respectively, which are the integers of concern here. In the constructions below, we shall multiply (or divide) m by the integer 2. Since 2 is the lowest prime number, such a procedure does not affect either the value of p or u in an integer and so we can hold these fixed.

We begin by excluding, for now, triads of the form $(1, p, 1)$, integers which are single prime numbers. We allow for this in Case 3 below.

Case 1: Suppose m is odd. We choose $m' = 2m$, then

$\Lambda(m', p, u) = \Lambda(2m, p, u) = -\Lambda(m, p, u)$. We may say that (m, p, u) and (m', p, u) are 'twin' pairs in the sense that their Λs are of opposite sign. Note that (m, p, u) and (m', p, u) are integers at the base of two different towers; they are not members of the same tower. (Recall that the members of a given tower are constructed by repeated multiplication with p.)

Case 2: Suppose m is even. In this case, we need to ascertain the highest power of 2 that divides m. If m is divisible by 2 but not by 2^2, assign $m' = m/2$. (So $m = 6$ gets assigned to $m' = 3$, and $m = 3$, by Case 1 above, gets assigned to $m' = 6$.) More generally, suppose the even m is divisible by 2^k but not by 2^{k+1}, where k is an integer. Then, if k is even, assign $m' = 2m$; and if k is odd, assign $m' = m/2$. (So $m = 12 = 2^2 \times 3$ gets assigned to $m' = 2^3 \times 3 = 24$. And, in reverse, $m = 24 = 2^3 \times 3$ gets assigned to $m' = 24/2 = 12$.)

Thus for odd m the following sequence of pairs (twins) hold:

(m, p, u) and $(2m, p, u)$ are twins at bases of different towers having λs of opposite signs,(this is Case 1),
$(2^2m, p, u)$ and $(2^3m, p, u)$ are twins at bases of different towers having λs of opposite signs,
$(2^4m, p, u)$ and $(2^5m, p, u)$ are twins at bases of different towers having λs of opposite signs,
and so on.

Case 3: Now consider the case where the triad describes a prime number; that is, it takes the form $(1, p, 1)$. For the moment, suppose this prime number is not 2. In this case, where $m = u = 1$, we simply assign $m' = 2$. Clearly,

$\Lambda(2, p, 1) = -\Lambda(1, p, 1)$, and the numbers $(2, p, 1)$ and $(1, p, 1)$ are at the bases of different towers.

Case 4: Finally, consider the case where the triad describes a prime number and the prime number is 2; that is, the integer $(1, 2, 1)$, for which $\Lambda(1, 2, 1) = -1$. We match this prime to the integer 1. By definition $\lambda(1) = \Lambda(1, 1, 1) = 1$. Thus the first two integers have opposite signs for their values of λ. $\Box$

So, in partitioning the entire set of positive integers, the number of towers that begin with integers for which $\lambda = -1$ is exactly equal to those that begin with integers for which $\lambda = +1$.

Thus, Theorem 2 immediately gives the following result:

Theorem 3: In the set of all positive integers, for every integer which has an even number of primes in its factorization there is another unique integer, (its twin), which has an odd number of primes in its factorization.

The consequence of the above theorem follows not just from that each integer has a unique twin whose λ-value is of the opposite sign, but also from from the context that these lie in an alternating sequence. That is, not only are the bases of two uniquely-paired towers twins, the next higher number in the first tower is the twin of the next higher number in the second tower, and so on. Thus every integer with $\lambda = \pm 1$ in a unique tower is twinned uniquely in alternating sequence with the integers with $\lambda = \mp 1$ in another unique tower, ensuring that the proportions of numbers with $\lambda = +1$ and $\lambda = -1$ are equal over the entire natural number system. Alternately, it may be argued that each Tower, which is an ordered infinite sub-set of the natural number system, is itself equally partitioned into members with $\lambda = +1$ and $\lambda = -1$ by their alternating sequence in that order. As the natural number system (excluding 1) comprises only such Towers, it too is so equally partitioned, and has equal proportions of numbers with $\lambda = +1$ and $\lambda = -1$.

Thus we have shown:*

Theorem 3B: If n is an arbitrary positive integer,

$$Prob[\lambda(n) = +1] = Prob[\lambda(n) = -1] = 1/2. \tag{19}$$

The result in Theorem 3B is a necessary condition for RH to be valid; it is not sufficient.† The condition that is equivalent to proving RH ((Littlewood (1912), Edwards (1974)) is the following:

$$\frac{L(N)}{N^{\frac{1}{2}+\epsilon}} = 0, \; as \; N \to \infty, \tag{20}$$

for any $\epsilon > 0$. We describe how this equivalence is formally proved below.

1. We compare the $L(N)$ series to $X(N)$, the distance travelled in N unit steps in a standard *random walk*, which can be represented as:

$$X(N) = \sum_{n=1}^{N} c(n) \tag{21}$$

   where the $c(n)$'s are independent random numbers with an equal probability of being either +1 or -1, i.e., "coin-tosses". It is a well-known result (see Chandrasekhar (1943)) that the expected value of $|X(N)|$, for large N, is

$$lim_{N\to\infty} E(|X(N)|) = C_0 N^{1/2} \tag{22}$$

   The further line of advance of the proof is to show now that Equation (22) applies to $L(N)$ as well, and so proves Equation (20), and thereby the RH. To do this we have to prove that the $\lambda(n)$'s in the $L(N)$ series are essentially "coin-tosses", for large n.

2. To show that the λ's behave as coin tosses, we have to show that (i) their probabilities of being either +1 of -1 are equal, as was proved by Theorem 3B, further (ii) we have to show that the λ's appearing in the natural sequence, $n = 1, 2, 3, ...$, are independent of each other — i.e., that the value of $\lambda(n)$ has no influence on the value of $\lambda(n+1)$, say. This seems counter-intuitive, as the λ's are obviously deterministically linked. Nevertheless, their independence in the natural sequence is shown by two different approaches:
   (a) In Appendix III, it is proved that the sequence of $\lambda(n), n = 1, 2, 3, ...$ is non-cyclic. This would preclude any dependence of the type
   $\lambda_n = f(\lambda_{n-1}, \lambda_{n-2}, \lambda_{n-3}, ..., \lambda_{n-M})$
   because any finite series of +1's and -1's of length M would have a finite number of permutations P, so the series $\lambda_{n-1}, \lambda_{n-2}, \lambda_{n-3}, ..., \lambda_{n-M}$ must repeat itself after atmost P numbers, and thereafter become cyclic *if* such a dependence relationship exists between the λ's. The non-cyclic nature of $L(N)$ conforms also to the notion of randomness in Knuth (1968, Ch. 3).

*These Theorems 3 and 3B, were later given new, alternate proofs, which only need induction and the starting premise that every odd integer starting from 1, has a unique successor integer, which is even and with which it forms a unique 'Pair' and that the even integer in every such Pair, has an odd integer which is its predecessor and its 'partner', the successor of an even integer is an odd integer not its 'partner'. See: K. Eswaran, (April 2018)
†Borwein et al (2006, p. 48) claim that the result in Theorem 3 is equivalent to a proof of RH.

(b) In Appendix IV, another approach is taken. It is shown that from merely these two rules: $\lambda(p) = -1$ for a prime p, and $\lambda(pq) = \lambda(p) \times \lambda(q)$, where q is any integer, the entire sequence of λ's for $n = 1, 2, 3, ...$, can be obtained without determining the number of prime factors of n. It is then argued that for any two integers $n > m$, $\lambda(n)$ is dependent on $\lambda(m)$, if the latter is required to find the former, and independent if not. It is then shown that, as $n \to \infty$, any finite sequential strip of λ's will be independent of each other, thus essentially making them equivalent to coin-tosses.

3. With Theorem 3B and Appendices III and IV, we have proved above that the $L(N)$ series is *one* realization of a random walk. We then invoke (towards the end of Section 5 in [5]) Khinchin and Kolmogorov's law of the iterated logarithm to show that the maximum deviation d_N, from the one-half power-law expectation, in the exponent of $|X(N)|$ for any individual random walk tends monotonically to zero as $N \to \infty$. So (22) also holds for $|L(N)|$. Therefore for any chosen $\epsilon > 0$ in (20), the statement for the validity of RH will be satisfied and the Riemann Hypothesis is proved.
4. In Appendix V, starting from Littlewood's *ansatz*, that $L(N) \sim N^a$, for $N \to \infty$, we argue that the statistics of the λ's must become "self-similar", i.e., independent of N for large N. This principle yields us the value of $a = \frac{1}{2}$, which again would satisfy (20). This 'physicist's proof' of the RH, is separate from the argument in the main paper, and may be treated as an interesting addendum to it.
5. Finally, in Appendix VI, we confirm by considering the λ's from $n = 1$ to 176 *trillion*, that their sequence is statistically indistinguishable (using the χ^2 statistical test) from coin-tosses over the entire set of numbers considered, and also when it is partitioned in smaller sections. While this is merely a 'verification', not a 'proof', this fact has not been reported in literature, and by itself, requires an explanation (which we have provided) given its surprising nature.

With Theorem 3B and Appendices III and IV, we have proved that the $L(N)$ series is a random walk. We formally confirm this below.

Theorem 4: The summatory Liouville function, $L(N) = \sum_{n=1}^{N} \lambda(n)$ *has the following asympotitic behaviour:* $|L(N)| \leq C_0 N^{\frac{1}{2}+d_N}$ *as* $N \to \infty$.

Proof: Theorem 3B gives $Pr(\lambda(n) = +1) = Pr(\lambda(n) = -1) = 1/2$, where Pr denotes probability. Given the results in Appendicies III and IV, the λ-values behave like 'ideal coin' tosses, where $\lambda(n) = +1$ as head and $\lambda(n) = -1$ as tail, and $L(N)$ is the cumulative result of N successive coin tosses, and is equivalent to the distance $X(N)$ moved in a random-walk with N unit steps. Chandrasekhar (1943) has shown that, for a random walk of N steps, Expectation$(|X(N)|) = C_0 N^{\frac{1}{2}}$ *as* $N \to \infty$. The quantity d_N (≥ 0) seen above is the maximum deviation from expectation of an individual random walk of N steps. QED

We conclude this section by estimating the 'width' of the Critical Line, the region around $Re(s) = 1/2$ in which the non-trivial zeros of $\zeta(s)$ must lie. Invoking Littlewood's Theorem (Sec.5), we deduce that $F(s) \equiv \zeta(2s)/\zeta(s)$ is analytic in the region $a = 1/2 + d_\infty < s < 1$ (where $d_\infty \equiv lim_{N\to\infty} d_N$). This implies $\zeta(s)$ has no zeros in the same region. But Riemann had shown by using symmetry arguments[‡] that if $\zeta(s)$ has no zeros in the latter region then it will have no zeros in the region $0 < s < 1/2 - d_\infty$; taking both these results together we are lead to the conclusion that all the zeros can only lie in the $1/2 - d_\infty < Re(s) < 1/2 + d_\infty$.

It is interesting that the law of the iterated logarithm enunciated by Kolmogorov (1929), also see Khinchine (1924), gives an expression for d_N. The statement of the law, adapted to the present context, is: *Let* $\{\lambda_n\}$ *be independent, identically distributed random variables with means zero and unit variances. Let* $S_N = \lambda_1 + \lambda_2 + \ldots + \lambda_N$. *The limit superior (upper bound) of* S_N *almost surely (a.s.) satisfies*

$$Lim\ Sup \frac{S_N}{\sqrt{2N\, log\, log\, N}} = 1 \quad \text{as} \quad N \to \infty$$

Now, from Theorem 4 we have written that if we consider the $\lambda's$ as "coin tosses" one can write $L(N) = \lambda_1 + \lambda_2 + \ldots + \lambda_N \leq C_0 N^{\frac{1}{2}+d_N}$ (as $N \to \infty$) (since we are interested in only the behaviour for large N we henceforth ignore the constants). Comparing this expression with the one above we see that one

[‡]He did this first by defining an associated xi function: $\xi(s) \equiv \Gamma(s/2)\pi^{s/2}\zeta(s), \Gamma(s)$ is the Euler Gamma function, then showed that this xi function has the symmetry property $\xi(s) = \xi(1-s)$ which in turn implied that that the zeros of $\zeta(s)$ (if any) which are not on the critical line will be symmetrically placed about the point s=1/2, ie.if $\zeta(\frac{1}{2} + u + i\sigma)$ is a zero then $\zeta(\frac{1}{2} - u - i\sigma), (0 < u < 1/2)$, is a zero see Whittaker and Watson page 269.

can write $N^{\frac{1}{2}+d_N} \sim \sqrt{N\, log\, log\, N}$ thus yielding an expression§ for $d_N = \frac{log\, log\, log\, N}{2\, log\, N}$. We see that $d_N \to 0$ as $N \to \infty$. So the equivalent statement Equation (20) will be satisfied for any chosen $\epsilon > 0$. Further d_∞ ($\equiv lim_{N\to\infty} d_N$) is the half-width of the critical line. Since this is zero, we conclude that all the non-trivial zeros of the zeta function must lie strictly on the critical line. Thereby, the Riemann Hypothesis is proved.

6 Conclusions

In this paper we have investigated the analyticity of the Dirichlet series of the Liouville function by constructing a novel way to sum the series. The method consists in splitting the original series into an infinite sum over sub-series, each of which is convergent. It so turns out each sub-series is a rectangular function of unit amplitude but ever increasing periodicity and each along with its harmonics is associated with a prime number and all of them contribute to the summatory Liouville function and to the Zeta function. A number of arithmetical properties of numbers played a role in the proof of our main theorem, these were: the fact that each number can be uniquely factorized and then placed in an exclusive subset, where it and its other members form an increasing sequence and factorize alternately into odd and even factors and thus have equal proportions of numbers with $\lambda = +1$ and -1; and each subset can be labelled uniquely using a triad of integers which in their turn can be used to determine all the integers which belong to the subset. This helped us to show that for every integer that has an even number of primes as factors (multiplicity included), there is an integer that has an odd number of primes. This provides a proof for the long-suspected (Denjoy 1931) but unproved conjecture—until now—that the Riemann Hypothesis has a connection with the coin-tossing problem. Further, it has now been revealed that the randomness of the $\lambda(n)$'s in the natural sequence¶ is the reason that the non-trivial zeros of the Zeta function all lie on the critical line: $Re(s) = 1/2$.

7 DEDICATION

I dedicate this paper to my teachers: Mr John William Wright of Bishop's School Poona, Prof. S.C. Mookerjee of St. Aloysius' College Jabalpur, Prof. P.M.Mathews of Univerity of Madras, Mr. D.S.M. Vishnu of BHEL R& D Hyderabad and to my first teachers - my parents. All of them lived selfless lives and nearly all are now long gone: May they live in evermore.

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8 APPENDIX I: Scheme of partitioning numbers into sets

Our scheme of partitioning numbers into sets is as follows:

(a) Scheme for Class I integers:

Let us say $n = p_1^e p_2^f p_3^g ... p_m^h p_L^k p_j p_t$, then it will have at least one prime which has an exponent of 2 or above and among these there will a largest prime p_L whose exponent is atleast 2 or above. Such a prime will always exist for a Class I number. Then by definition the number to the right of p_L is either 1 or is a product of primes with exponents only 1. Now multiply all the numbers to the left of $p_L$and call it m i.e. $m = p_1^e p_2^f p_3^g ... p_m^h$ and the

§$N^{\frac{1}{2}+d_N} \sim \sqrt{N\, log\, log\, N} = e^{log\, \sqrt{N\, log\, log\, N}} = e^{\frac{1}{2} log\{N\, log\, log\, N\}} = e^{\frac{1}{2} log\, N + \frac{1}{2} log\, log\, log\, N} = N^{\frac{1}{2}}\, e^{\frac{1}{2} log\, log\, log\, N}$ which then implies $e^{\frac{1}{2} log\, log\, log\, N} = N^{d_N} = e^{d_N\, log\, N}$ thus giving $d_N = \frac{log\, log\, log\, N}{2\, log\, N}$

¶God seems to have "played dice" at least once, when he created the natural number system!

product of numbers to the right of $p_L$as u i.e. $u = p_j p_t$.Now this triad of numbers m, p_L, u will be used to label a set,note $n = m.p_L^k.u$ Let us define the set $P_{m;p_L;u}$:

$$P_{m;p_L;u} \; = \{m.p_L^2.u,\; m.p_L^3.u,\; m.p_L^4.u,\; m.p_L^5.u,\; m.p_L^6.u,\; m.p_L^7.u,\; ....\} \qquad (A1)$$

Obviously $n = m.p_L^k.u$ which has $k \geq 2$ belongs to the above set. Also notice the factor involved in each number increases by a single factor of $p_L$therefore the λvalues of each member alternate in sign:

$$\lambda(m.p_L^2.u) = -\lambda(m.p_L^3.u) = \lambda(m.p_L^4.u) = -\lambda(m.p_L^5.u) = \lambda(m.p_L^6.u) = ...(A2)$$

In this paper ALL sets defined as $P_{m;p;u}$will have the property of alternating signs of λ Eq. (A1). Note in the above set containing only Class I integers m will have only prime factors which are each less than p_L.

Let us consider various integers:

Ex 1. Let us consider the integer 73573500; this is factorized as $2^2.3.5^3.7^3.11.13$ and since this is a Class I integer, and $p_L = 7$ because 7 is the highest prime factor whose exponent is greater than one. $p_L = 7$ $m = 2^2.3.5^3$ and $u = 11.13 = 143$ and therefore 73573500 is a member of the set $P_{1500;7;143}$

$$P_{1500;7;143} = \{1500.7^2.143,\; 1500.7^3.143,\; 1500.7^4.143,\; 1500.7^5.143,\; ...\}$$

Ex 2. Now let us consider the simple integer: 3^4 this is a class I integer and belongs to $P_{1;3;1} = \{3, 3^2,\; 3^3,\; 3^4,\; 3^5,\; 3^6,\; ......\}$

Ex 3. Let us consider the integer 663 this is factorized as: 3.13.17 and is a Class II integer as there no exponents greater than 1, and $663 = 3.13.17$ and since 17 is the highest prime number we put this in the set:

$$P_{39;17;1} = \{39.17,\; 39.17^2,\; 39.17^3,\; 39.17^4, ...\}.$$

NOTE: If a tower has a Class II integer then it will appear as the first (base) member, all other numbers will be Class I numbers.

Ex 4. Let the integer be the simple prime number 19, we write:

$$19\, \epsilon\, P_{1;19;1} \; = \; \{19,\; 19^2,\; 19^3,\; 19^4,\; .....\}$$

Ex 5. Let the integer be 4845 this is factorized as3.5.17.19 since this is a Class II integer we see $m = 3.5.17 = 255$, $p = 19$, $u = 1$ and the set which it belongs is

$$P_{255;19;1} \; = \; \{255.19,\; 255.19^2,\; 255.19^3,\; 255.19^4,\; 255.19^5, ...\}$$

9 APPENDIX II: Theorems on representation of integers and their partitioning into sets.

Theorem A: Two different integers cannot have the same triad (m, p^k, u)

Let a and b be two integers which when factored according to our convention are $a = n.q^g.v$ and $b = n'.q'^h.v'$, and let us consider only Class I integers u, v and v' are all > 1.

If they are both equal to the same triad (say) $(m, p^k.u)$. Then $m.p^k.u = n.q^g.v = n'.q'^h.v'$. Consider the first two equalities $m.p^k.u = n.q^g.v$, which means p is the largest prime with $k > 1$ on the l.h.s. Similarly q is the largest prime with exponent $g > 1$ on the r.h.s. Now if $p > q$ this means p^k must divide v, but this cannot happen since v cannot contain a prime greater than q with an exponent $k > 1$. Now if $p < q$ then q^g must divide u but this again cannot happen since u cannot contain an exponent $g > 1$. So we see $p = q$, and $k = g$. But once again unique factorization would imply, since u contains all prime factors larger than p and v must contain only prime factors larger than $q(= p)$, the only possibility is $u = v$, but this also makes $m = n$. That is, the triad of a is (m, p^k, u). Similarly equating the second and third equalities $n.q^g.v = n'.q'^h.v'$ and using similar arguments we see $n = n'$, $q = q'$, and $v = v'$; that is, $a = b$. The same logic can be used to prove the theorem for class II integers when $u = v = v' = 1$. QED.

Theorem B: Two different triads cannot represent the same integer.

If there are two triads (m, p^e, u) and (m', r^s, u') and represent the same integer say a which can be factorized as $a = n.q^g.v$. Where the factorization is done as per our rules then we must have $m.p^e.u = n.q^g.v$ by using exactly similar arguments as above(in Theorem A) we conclude that we must have $m = n, p = q, e = g$ and $u = v$; similarly imposing the condition on the second triad $m'.r^s.u' = n.q^g.v$, we conclude $m' = n, r = q, s = g$ and $u' = v$; thus obtaining $m = m', p = r, e = s$ and $u = u'$ this means the two triads are actually identical. QED

10 APPENDIX III: Non-cyclic nature of the factorization sequence

It is a necessary condition in the tosses of an ideal coin that the results are not cyclic asymptotically, namely the results cannot form repeating cycles as the number of tosses becomes large.

Definition

Let n_k be the number of primes, repetitions counted, in the factorization of a positive integer k. We call $\{n_1, n_2, ..., n_k, ...\}$ the factorization sequence.

Note: $\lambda(k) = +1$ if n_k is even and $\lambda(k) = -1$ if n_k is odd.

Theorem The factorization sequence is asymptotically non-cyclic.

Proof: The result follows from this claim:

Claim. The sequence $\lambda(1), \lambda(2), \lambda(3), ..., \lambda(n), ...$, is asymptotically non-cyclic.

If the claim is not true there would exist an integer $t, t \geq 0$, so that the sequence is cyclic (after $\lambda(t)$), with cycle length σ.

By Theorem 3, the number of positive integers with even number of prime factors (counting multiplicities) equals the number of positive integers with odd number of prime factors (counting multiplicities). Therefore, the λ's in each cycle must sum to zero as do the first t λ's before the cycles start.

Then $L(N) \leq max\{t/2, \sigma/2\}$.

Now we use Littlewood's Theorem 1 and noting that in (5.21) $G(x) \equiv L(x)$, we substitute the maximum value of $L(x)$ as $x \to \infty$, viz. $\mid L(x) \mid = \sigma/2$, and thus deduce that (5.21) will always converge provided $0 < s$. Since, $\mid L(x) \mid \leq \sigma/2$, $L(x)$ indeed grows less rapidly than x^a for all $a > 0$, satisfying the condition in Theorem 1. This means that we should be able to analytically continue $F(s) \sim \zeta(2s)/\zeta(s)$ leftwards from $Re(s) = 1$ to $Re(s) = 0$, contradicting Hardy (1914) [3] that there are very many zeros at $Re(s) = 1/2$ and these will appear as poles in $F(s)$. This proves the Claim.QED.

11 APPENDIX IV: The sequence of λ's in L(N), are equivalent to Coin Tosses

In this paper we showed in Theorem 3, that the $\lambda(n)$ have an exactly equal probability of being $+1$ or -1. Then in Appendix III, we showed that the sequence $\lambda(1), \lambda(2), \lambda(3), ..., \lambda(n), ...$ can never be cyclic. The latter result in the minds of most computer scientists would be interpreted as that the sequence of λ's by virtue of it being non-repetitive, is truly random,(Knuth (1968); Press etal (1986)) and hence it is legitimate to treat the sequence as a result of coin tosses and thus one can then say that $L(N) = \Sigma_1^N \lambda(n)$, will tend to $\sqrt{N}$ thus proving RH, by using the arguments given at Section 5.

However, this done, there would be some mathematicians who may remain unconvinced, because we have not strictly proved that the λ's in the series are independent. The purpose of this Appendix ‖ is to prove that this is indeed the case. This allows us to demonstrate the $\lambda-$sequence has the same properties as, and is statistically equivalent to, coin tosses, thus placing our proof of RH beyond any doubt.

We again consider the series $L(N) = \sum_{n=1}^{N} \lambda(n)$, which is re-written as:

$$L(N) = \sum_{n=1}^{N} X_n \tag{1}$$

It has already been proved in this paper that, over the set of all positive integers, the respective probabilities that an integer n has an odd or even number of prime factors are equal. So, X_n $(= \lambda(n))$, can with equal probability, be either +1 or -1. It will now be shown that the values of X_i and X_j, $i \neq j$, are independent of each other, as $n \to \infty$, and so will become the equivalent of ideal-coin tosses.

11.1 The λ values as a deterministic series

We first show that the λ's in the natural sequence, far from being random, are actually perfectly predictable and therefore deterministic. That is, knowing the λ's (and the primes) up to N, we can directly obtain (without resorting to factorisation) the λ's (and primes) up to $2N$ thus:

We obtain integers m in the range $N < m \leq 2N$ by multiplying the integers n and q in the range $1 < n, q \leq N$, such that $N < nq \leq 2N$ and then using the property $\lambda(q * n) = \lambda(q) * \lambda(n)$ to find $\lambda(m = q * n)$. However, not all the numbers in the range $N < m \leq 2N$ will be covered by such multiplications. That is, there will be 'gaps' in the natural sequence left in the aforesaid multiplications, where no n and q can be found for some m's in $N < m \leq 2N$. These m's will identified as prime numbers. The λ of a prime is -1. Thus, by

‖I thank my brother Vinayak Eswaran for providing the kernel of the proof given in this section.

knowing the λ's and the primes up to N, we can predict the λ's (and primes) up to $2N$. This process can be repeated *ad-infinitum* to compute the λ's of the natural sequence up to any N, from just $\lambda(1)$=1 and $\lambda(2)$=-1.

We emphasize that any other method of evaluating the λ's, including direct factorisation, must perforce yield the same sequence as the method above. Therefore, this method offers a complete description of the determinism embedded in the series.

11.2 Relationships and dependence between λ's

We note that every integer n has a direct relationship (which we will call a d-relationship) with all numbers $n * p$, where p is any prime number. We can define higher-order d-relatives in the following way: the integers (n, $n * p$) are in a first-order d-relationship, (n, $n * p * q$) are in a second-order one, and (n, $n * p * q * r$) are in a third-order one, and so on, where p, q, r are primes (not necessarily unequal).

In the deterministic generation of λ's outlined above, it is clear that their values will be determined through d-relationships, which would thereby make their respective values dependent on each other. It is evident that the λs of two d-relatives n and $m(> n)$, are *dependent* on each other and that $\lambda(m) = (-1)^o \lambda(n)$, where o is the order of the relationship.

There is another kind of relationship we must also consider: we can have a c- (or *consanguineous*) relationship between two non-d-related integers m and n if they are both d-relatives of a common ('ancestor') integer smaller than either of them. So we can trace back the λ's along one branch to the common ancestor and trace it up the other to find the λ of the other integer. It is convenient to take the common ancestor as the largest possible one, which would be the greatest common factor of the two integers, which we shall call G.

Now we ask the question, when are m and n not related? When they have neither a d-relationship nor a c-relationship with each other. That is, when they are co-primes: as then neither integer would appear in the sequence of multiplications that produce the other by the deterministic iterative method. In such a situation, *the λ of neither is dependent on the other, so their mutual λ's are independent.*

Now consider two c-relatives, m and n, which share the greatest common factor G. We can write $m = G * P$ and n=$G * Q$, where P and Q are chosen appropriately. As G is the greatest common factor of m and n, it is clear that P and Q are co-primes. Now we consider the relationship between $\lambda(m)$ and $\lambda(n)$ and explore their relatedness. This turns out to be self-evident: As $\lambda(m)$ =$\lambda(G) * \lambda(P)$ and $\lambda(n)$ =$\lambda(G) * \lambda(Q)$, and we know that $\lambda(P)$ and $\lambda(Q)$ are independent of each other, it follows that $\lambda(m)$ and $\lambda(n)$ are also independent of each other.[**]

11.3 The unpredictability of λ values from a finite-length sequence: d-relatives

We have concluded above that the only λ's in $L(N)$ that are dependent are those between d-relatives, where the smaller integer is a factor of the other. We see that the distance of two such "first-order" relatives, n and $n * p$, from each other is $n(p-1)$ which increases without bound with n. Further all the first-order d-relatives of n also have relative distances with each other that are at least as great as n (as their respective p's will differ at least by 1). Thus the d-relationship between numbers is a web with increasing distances between their first-order relatives[††]. It is also easy to see that the higher-order d-relatives of any integer n will also be at a distance of at least n from n itself and from each other.

Now we consider if we would be able to predict $\lambda(N+1)$ if we know *only* the λ's between $N - L < n \leq N$, where L is some finite number? We would be able to do so *only* if $N + 1$ is a d-relative (of any order) of any of the numbers $N - L < n \leq N$. However, for n large enough the d-relatives of $N + 1$ will be far from it and would not come in the range of numbers $N - L < n \leq N$. So essentially, there is no way of predicting $\lambda(N + 1)$ from the range of L λ's coming before it. This means $\lambda(N + 1)$ is independent of the range of L λ's coming before it. Therefore, the λ's on all finite lengths are independent of each other, as $N \to \infty$.

11.4 Closure

We have investigated the dependence of λ's appearing in $L(N)$ in the natural sequence $n = 1, 2, 3, .., .$ We first show that the λ's are in a perfectly deterministic sequence (which is not random in the slightest way, except in the unpredictable discovery of primes) that allows us to obtain all of them up to any integer N by knowing

[**]It may be noticed that m and n belong to different towers. It is worth mentioning that the arguments made here in Appendix IV, can be couched in the language of towers as we did in Sections 2 and 3.

[††]How rapidly the relationship distance increases can be gauged from the fact that the 2^r sequence, which has the *slowest* increases, nevertheless will have its 100^{th} element placed at around $n \approx 10^{30}$ in the natural sequence, and the distance to the 101^{st} element will also be 10^{30}!

only that $\lambda(1) = 1$, $\lambda(2) = -1$, $\lambda(q * n) = \lambda(q) * \lambda(n)$, and that $\lambda(p) = -1$ for any prime p. We then propose that the λ's of two integers m and n can be dependent only if the integers are connected through the sequence of multiplications involved in the deterministic process. If they are not so related, as would happen if they are co-primes, their λ's would be independent. We then investigate the only two possible types of relationships and show that one, the d-relationship, leads to dependencies between numbers that are increasingly distant. The other, the c-relationship, is shown to give independent λ's. The result obtained is that that the λ's in any finite sequence are independent, as $N \to \infty$. QED

12 APPENDIX V

An Arithmetical Proof for $|L(N)| \sim N^{1/2}$ as $N \to \infty$

In this appendix we provide an alternate, but this time an arithmetical, proof of the asymptotic behavior of the summatory Liouville function, viz.$|L(N)| = \sqrt{N}$ as $N \to \infty$ However in order to do this we first require to prove a theorem on the number of distinct prime products in the factorization of a sequence of integers and their exponents. We will be considering special types of Sets $S^-(N)$ and $S^+(N)$ which contain a sequence of consecutive integers, they are defined below; each are of length $\sqrt{N}$, where in this section N will always be a perfect square. A collection of all such sets will contain as its members all the integers and the intersection of any two different sets will be null. See Tables 1.1 to 1.4 in Appendix VI. We will be studying the contribution of such sets to the summatory function $L(N)$ in order to determine the asymptotic behavior of the latter as $N \to \infty$

Theorem A5: Consider the sequence S comprising $M(N)$ consecutive positive integers, defined by $S^-(N) = \{N - M(N) + 1, N - M(N) + 2, N - M(N) + 3, ..., N\}$, where $M(N) = \sqrt{N}$. Then every number in $S^-(N)$ will firstly belong to different towers,* and further every number will: (a) differ in its prime factorization from that of any other number in $S^-(N)$ by at least one distinct prime† OR (b) in their exponents.

The statement of this theorem can be roughly considered as an extremely weak form of Grimm's conjecture,(1969) which states that a sequence of k consecutive composite integers will have at least k distinct primes in their factorization, also see Ramachandra etal.(1975), Grimm's theorem though not proved, yet, has been verified for very many subsets, see: S. Laishram and Ram Murty (2006,2012), and Balasubramanian, etal [2009]. We do not need this very strong version for our arguments.

We first take up the task to prove (a) because it is by far the more common occurrence. In case condition (a) does not hold in a particular situation then condition (b) is always true, because of the uniqueness of factorization.

Proof:

Let there be k primes in the sequence $S^-(N)$. Denote the j integers in the sequence that are not primes by the products $p_i b_i$, $i = 1, 2..., j$, where p_i is a prime and, obviously, $k + j = M(N)$. Denote the subset of these non-prime integers by J. There is no loss of generality if we assume the primes p_i in the products $p_i b_i$, $i = 1, 2..., j$, to be less than $\sqrt{N} - 1/2$ and also the smallest of prime in the product.‡

To prove the theorem, we compare two arbitrary members, $p_i b_i$ and $p_j b_j$, $i \neq j$, belonging to set J.

Case 1: Suppose $p_i \neq p_j$. If $b_i \neq b_j$, b_i must contain a prime that does not appear in the factorization of b_j (and hence $p_i b_i$ must be different from $p_j b_j$ by this prime). For if b_i and b_j do not differ by a prime, we must have $b_i = b_j \equiv b$. This means the difference of $p_i b_i - p_j b_j = (p_i - p_j)b$ is larger than $\sqrt{N}$ in absolute value. This is not possible since the members of the sequence $S^-(N)$ cannot differ by more than $\sqrt{N}$. Therefore b_i must differ from b_j by a prime in its factorization.(One may think that it may be plausible that $b_i = b^r$ and $b_j = b^m$, where r and m are positive integers, in which case $p_i b_i$ differs from $p_j b_j$ only in the prime p_i. However, this eventuality will never arise because then the difference between $p_i b_i$ and $p_j b_j$ will be more than $\sqrt{N}$.)

Case 2: Suppose $p_i = p_j \equiv p$ then b_i and b_j must differ by a prime factor or their exponents are different. Because of 'unique factorization', if they do not differ by a prime factor it means $p_i.b_i = p_j.b_j = p.b$, unless the factors of b_i and b_j are of the form: $b_i = p_1^{r_1}.p_2^{r_2}...p_k^{r_k}$ and $b_j = p_1^{r'_1}.p_2^{r'_2}...p_k^{r'_k}$ which implies that is $r_l = r'_l$ is not

*Two numbers $n = m.p^{\alpha}.u$ and $n' = m.p^{\beta}.u$, $(n < n')$, of the same tower, cannot both belong to the set $S^-(N)$ because they will be too far separated to be within the set, as their ratio $n'/n \geq p \geq 2$

†For example, if two numbers c and d in S are factorized as $c = p_1^{e_1} p_2^{e_2}$ and $d = p_3^{e_3} p_4^{e_4}$ then at least one of the primes p_3 or p_4 will be different from p_1 or p_2.

‡This is readily seen as follows. Since every member of J lies between $N - \sqrt{N}$ and N, clearly any composite member, written as a product ab, cannot have both integers a and b less than $\sqrt{N} - 1/2$. (We are invoking the fact that $\sqrt{(N - \sqrt{N})} = \sqrt{N} - 1/2$, approximately.) Let a be the smaller of the two numbers, and so $a < \sqrt{N} - \frac{1}{2}$ and $b > \sqrt{N} - \frac{1}{2}$. If a is a prime number, set $p = a$. If a is not a prime number, factorize it and pick the smallest prime p which is one of its prime factors.

true for all r_l, $l = 1, 2..k$, hence in this case the exponents are different (actually this case is very rare. It can be shown: the case $k = 2$ cannot occur and therefore if at all this case occurs, k must be greater than 3).

Since $p_i b_i$ and $p_j b_j$ are arbitrary members of the set J, it follows that every integer in J must differ from another integer in J by at least one prime in its factorization or by its exponent, thus making the $\lambda-$values of any two members of the set $S^-(N)$ not dependent on each other . □

The above theorem has profound implications for the λ-values of the numbers in the sequence $S^-(N)$. If we take the primes to occur randomly (or at least pseudo-randomly), the λ-value of each of these $M(N)$ integers—although deterministic and strictly determined by the number of primes in its factorization—cannot be predicted by the λ-value of any other number in the sequence $S^-(N)$. That is, the λ-value of any number in $S^-(N)$ can be considered to be statistically independent of the λ-value of another member of this sequence, primarily because they stem from different towers and also because (as we have proved in A5) any two such numbers differ by at least one prime. Hence the λ-values in the sequence $S^-_\lambda \equiv \{\lambda(N - M(N) + 1), \lambda(N - M(N) + 2), \lambda(N - M(N) + 3), ..., \lambda(N)\}$, in which each member has a value either $+1$ or -1, would appear randomly and be statistically similar. By this we also deduce that two different sequences of $\lambda-$values defined on two different sets (say) $S^-(N)$ and $S^-(N')$ with $N \neq N'$ are statistically similar, because they have the same properties which also means that they can be separately compared with other sequences of coin tosses and the comparison should yield statistically similar results.

We will use these deductions to obtain the main result of this appendix viz $a = 1/2$ in the expression $|L(N)| = N^a$ as $N \to \infty$

Although it is not explicitly required for what follows, we note that it is not hard to prove that the sequence $S^+(N) \equiv \{N + 1, N + 2, N + 3, ..., N + M(N)\}$ of length $M(N)$ also behaves similarly. That is, every member of $S^+(N)$ satisfy condition (a) OR (b) of the above Theorem for $S^-(N)$ stated above. The proof mimics the one provided above and so is omitted.[§] Hence the λ-values in the sequence $S^+_\lambda \equiv \{\lambda(N + 1), \lambda(N + 2), \lambda(N + 3), ..., \lambda(N + M(N))\}$, in which each member has a value either $+1$ or -1, would also appear randomly and behave statistically similarly.

12.1 Arithmetical proof of $|L(N)| \sim \sqrt{N}$, as $N \to \infty$

We now show that if the summatory Liouville function

$$L(N) = \sum_{n=1}^{N} \lambda(n), \tag{1}$$

takes the asymptotic form

$$|L(N)| = C\, N^a, \tag{2}$$

where C is a constant, then we must have:

$$a = 1/2. \tag{3}$$

Throughout this subsection we will always assume that N is a very large integer.

Consider the sequence of consecutive integers of length $M(N) = \sqrt{N}$:

$$S_N = \{N - \sqrt{N} + 1, N - \sqrt{N} + 2, N - \sqrt{N} + 3, \cdots\cdots, N\} \tag{4}$$

Each of the $M(N)$ integers in the sequence S_N can be factorized term by term and would differ from another member in S_N by at least one prime or exponent,(as proved in the above theorem).[¶]

Now since, N is large, all the primes involved may be considered random numbers (or pseudo-random numbers), therefore as reasoned above, we can conclude that the $\lambda-$sequence associated with S_N viz.

$$\{\lambda(N - \sqrt{N} + 1), \lambda(N - \sqrt{N} + 2), \lambda(N - \sqrt{N} + 3), \cdots\cdots, \lambda(N)\} \tag{5}$$

will take values which are random e.g.

$$\{-1, +1, +1, , -1, +1, \cdots, +1\} \tag{6}$$

where in the above example $\lambda(N - \sqrt{N} + 1) = -1, \lambda(N - \sqrt{N} + 2) = +1$ etc. Furthermore, since the λ-values have an equal probability of being equal to $+1$ or -1 (Theorem 3) and the sequence is non-cyclic (Theorem 11.1,

[§] This implies, interestingly, that by choosing N to be consecutive perfect squares, the entire set of positive integers can be envisaged as a union of mutually exclusive sequences like $S^-(N)$ and $S^+(N)$.

[¶] Therefore, in the terminology of Sections 2 and 3, each of them will mostly belong to different Towers.

in Appendix 3), the above sequence will have the statistical distribution of a sequence of tosses of a coin (Head = +1,Tail = −1). But we already know from Chandrasekhar(1943) that if the λ's behave like coin tosses then $|L(N)| \sim \sqrt{N}$, as $N \to \infty$. However, we do not know whether the entire sequence of λ's occurring in Eq.(13.1) behaves like coin tosses; for any given N, it is only the subsequence $\{\lambda(N - \sqrt{N} + 1), \lambda(N - \sqrt{N} + 2), \lambda(N - \sqrt{N} + 3), \cdots\cdots, \lambda(N)\}$ of length $M(N) = \sqrt{N}$ that does behave like coin tosses.

On the other hand if we had a sequence of length N, of real coin tosses (say) $c(n), n = 1, 2....N$, where $c(n) = \pm 1$, then the cumulative sum, $L_c(N)$, of the first N of such coin tosses is given by:

$$L_c(N) = \sum_{n=1}^{N} c(n). \tag{7}$$

Then for N large we do know from Chandrashekar (1943) that

$$|L_c(N)| \sim \sqrt{N}. \tag{8}$$

We can then estimate the contribution $P_{1/2}$ to $L_c(N)$ from the last $M(N) = \sqrt{N}$ terms in Eq.(13.7), this would be:

$$\begin{aligned} P_{1/2} &= \sum_{n=N-\sqrt{N}+1}^{N} c(n) \\ &= L_c(N) - L_c(N - \sqrt{N}) \end{aligned} \tag{9}$$

Now since Eq.(13.7) represents perfect tosses Eq. (13.9) becomes

$$P_{1/2} = \sqrt{N} - (N - \sqrt{N})^{1/2} = \frac{1}{2} - \frac{1}{8}\frac{1}{\sqrt{N}},$$

that is,

$$P_{1/2} = O(1) \tag{10}$$

In Eq. (13.10), $P_{1/2}$ is the contribution to $L_c(N)$ from the last $M(N) = \sqrt{N}$ tosses of a total of N tosses of a coin. We shall consider the value of $P_{1/2}$ as the benchmark with which to compare the contributions of the last $M(N) = \sqrt{N}$ terms of the summatory Liouville function.

Now coming back to the λ-sequence as depicted in the summation terms in Eq. (13.1), following Littlewood (1912) we shall suppose that the expression given in (13.2) is an ansatz[‖] depicting the behavior of $L(N)$ for large N.

The task that we then set ourselves, is to estimate the value of the exponent a in the asymptotic behavior described in (13.2) $|L(N)| = C\,N^a$ which involves the $\lambda-$sequence. We do know that the λ-sequence does not all behave like coin tosses, but we have shown that there exist subsequences of λ's that exhibit a close correspondence to the statistical distribution of coin tosses and though such subsequences are of relatively short lengths $M(N)$, there are very many in number. Now a 'True' value of the exponent 'a' should be able to capture the correct statistics in all such subsequences and predict the behavior of coin tosses for such subsequences. We now investigate if such a True value for a exists and, if so, what its value should be.

We will estimate the contribution to $L(N)$ for the same subsequence (5) of length $M(N) = \sqrt{N}$, then the P when recomputed with an exponent $a \neq 1/2$ would give P_a:

$$P_a = \sum_{n=N-\sqrt{N}+1}^{N} \lambda(n) \qquad (13.5')$$

That is

$$\begin{aligned} P_a &= L(N) - L(N - \sqrt{N}) \\ &= CN^a - C(N - \sqrt{N})^a \end{aligned} \tag{11}$$

Simplifying by using Binomial expansion we have:

$$P_a = CaN^{a-\frac{1}{2}} - C\frac{a(a-1)}{1.2}N^{a-1} \tag{12}$$

[‖]Eq. (13.2) can be thought of as the first term in the asymptotic expansion of $L(N)$ for large N i.e. $|L(N)| = N^a(C + \frac{C_1}{N} + \frac{C_2}{N^2} + ...)$

From the properties of the λ's deduced from earlier results in this paper (Theorem 3, Appendices 3,4 and Theorem A5, page 21), we now know that in actuality the particular subsequence in Eq.(13.5) and Eq.(13.5′) contain random values of $+1$ and -1 and since the subsequence of λ's have the same statistics as those of coin tosses, P_a must be similar to $P_{1/2}$. Thus from (13.10) and (13.12)

$$P_a = O(1). \tag{13}$$

From (13.11) this means that

$$CaN^{a-\frac{1}{2}} = O(1) \tag{14}$$

Since N is arbitrary and very large, this is impossible unless the condition

$$a = \frac{1}{2} \tag{15}$$

strictly holds.**

Hence we have proved $a = 1/2$. Since for consistency†† , condition (15), which arises from (13), is mandatory and therefore $|L(N)| \sim \sqrt{N}$ describes the asymptotic behavior of the summatory Liouville function. $\square$

13 APPENDIX VI

On Coin Tosses and the Proof of Riemann Hypothesis

This Appendix has been written in such a manner that it can be read as a supplement to the main paper and the first five appendices.

In the main part of this paper and the forgoing appendices, which we denote as: [MP and A's], we had proved the validity of the Riemann Hypothesis (RH). In this Appendix (VI), we perform a numerical analysis and provide supporting empirical evidence that is consistent with the formal theorems that were key to establishing the correctness of the RH. In particular, the numerical results of the statistical tests performed here are firmly consistent with the proposition (formally proved in the paper cited above) that the values taken on by the Liouville function over large sequences of consecutive integers are random. By performing this exhaustive numerical analysis and statistical study we feel that we have provided a clearer understanding of the Riemann Hypothesis and its proof.

1. Introduction

The Riemann zeta function, $\zeta(s)$, is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \tag{1}$$

where n is a positive integer and s is a complex variable, with the series being convergent for $Re(s) > 1$. This function has zeros (referred to as the trivial zeros) at the negative even integers $-2, -4, \ldots$. It has been shown‡‡ that there are an infinite number of non-trivial zeros on the critical line at $Re(s) = 1/2$. Riemann's Hypothesis (RH), which has long remained unproven, claims that all the nontrivial zeros of the zeta function lie on the critical line. The main paper contains the proof [MP and A's]

In this technical note, we provide a more concrete understanding and appreciation of the steps involved in the proof of the Riemann Hypothesis by supplying supporting empirical evidence for those various theorems which were proved and which had played a key role in the proof of the RH. In what follows we first give a brief summary of how the RH was proved in [MP and A's]. The proof followed the primary idea that if the zeta function has zeros only the critical line, then the function $F(s) \equiv \zeta(2s)/\zeta(s)$ cannot be analytically continued to the left from the region $Re(s) > 1$, where it is analytic, to the left of $Re(s) < 1/2$. This point was recognized by Littlewood as far back as 1912.* The function $F(s)$ can be expressed as (see Titchmarsh (1951, Ch. 1)):

**In the above we tacitly assumed that $a > 1/2$, but $a < 1/2$ is not possible because then P_a will become zero. This implies that $dL/dN = 0$, meaning $|L(N)|$ will be a constant. But this again is impossible from Theorem 1, which would imply that $F(s)$ can be analytically continued to $Re(s) = 0$—an impossibility because of the presence of an infinity of zeros at $Re(s) = 1/2$, first discovered by Hardy.

††It may be noted that for every (large) N there is a set S_N, Eq (13.4), containing $M = \sqrt{N}$ consecutive integers whose λ-values behave like coin tosses; but there are an infinite number of integers N and therefore there are an infinite number of sets S_N, for which (13) must be satisfied.

‡‡This was first proved by Hardy (1914).

*It may be noted that Littlewood studied the function $1/\zeta(s)$ whereas we, in our analysis study $F(s) \equiv \zeta(2s)/\zeta(s)$. This has made things simpler.

$$F(s) = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s}, \tag{2}$$

where $\lambda(n)$ is the Liouville function defined by $\lambda(n) = (-1)^{\omega(n)}$, with $\omega(n)$ being the total number of prime numbers in the factorization of n, including the multiplicity of the primes. The proof of RH in [MP and A's] requires also the summatory Liouville function, $L(N)$, which is defined as:

$$L(N) = \sum_{n=1}^{N} \lambda(n) \tag{3}$$

The proof crucially depends on showing that the function $F(s) = \zeta(2s)/\zeta(s)$, has poles only on the critical line $s = 1/2 + i\sigma$, which translates to zeros of $\zeta(s)$, on the self same critical line $s = 1/2 + i\sigma$, because all the values of s which appear as poles of $F(s)$ are actually zeros of $\zeta(s)$, except for $s = 1/2$. Since, the trivial zeros of $\zeta(s)$ which occur at $s = -2, -4, -6....$ that is negative even integers, conveniently cancel out from numerator and denominator of the expression in $F(s)$), leaving only the non trivial zeros, also the pole of $\zeta(2s)$ will appear as a pole of $F(s)$, at $s = 1/2$. So it just remains to show that all the poles of $F(s)$ lie on the critical line. This was the Primary task of the paper.

The crucial condition then is that $F(s)$ is not continuable to the left of $Re(s) < 1/2$, and therefore that the zeta function have zeros only on the critical line,† is that the asymptotic limit of the summatory Liouville function be $|L(N)| \sim C\, N^{1/2}$, where C is a constant. Therefore, to provide a rigorous proof of the validity of the Riemann Hypothesis, [MP and A's] investigated the asymptotic limit of $L(N)$. The work involved the establishment of several relevant theorems, which were then invoked to eventually prove the RH to be correct.

We now state some of these important theorems‡).

***Theorem* 1:**

In the set of all positive integers, for every integer which has an even number of primes in its factorization there is another unique integer (its twin) which has an odd number of primes in its factorization.

Remark: Theorem 1 gives us the formal result that $Pr(\lambda(n) = +1) = Pr(\lambda(n) = -1) = 1/2$, where Pr denotes probability. That is, the λ-function behaves like an 'ideal coin'.

***Theorem* 2:**

Consider the sequence $S_-(N)$ comprising $\mu(N)$ consecutive positive integers, defined by $S_-(N) = \{N - \mu(N) + 1, N - \mu(N) + 2, N - \mu(N) + 3, ..., N\}$, where $\mu(N) = \sqrt{N}$. Then every number in $S_-(N)$ will differ in its prime factorization from that of every other number in $S_-(N)$ by at least one distinct prime.§

Remark: It is not hard to prove that the sequence $S_+(N) \equiv \{N + 1, N + 2, N + 3, ..., N + \mu(N)\}$ of length $\mu(N)$ also behaves similarly. That is, every member of $S_+(N)$ differs from every other member by at least one prime in its factorization. This implies, interestingly, that by choosing N to be consecutive perfect squares, the entire set of positive integers can be envisaged as a union of mutually exclusive sequences like $S_-(N)$ and $S_+(N)$.

It follows that the λ-values in the sequences $S_-^{\lambda}(N) \equiv \{\lambda(N - \mu(N) + 1), \lambda(N - \mu(N) + 2), ..., \lambda(N)\}$ and $S_+^{\lambda}(N) \equiv \{\lambda(N + 1), \lambda(N + 2), \lambda(N + 3), ..., \lambda(N + \mu(N))\}$, in which each member has a value either $+1$ or -1, would also appear randomly and be statistically similar to sequences of coin tosses.

Since the number of members in the sequences $S_-(N)$, $S_-^{\lambda}(N)$, $S_+^{\lambda}(N)$, and $S_+^{\lambda}(N)$ is given by $\mu(N) = \sqrt{N} \to \infty$ as $N \to \infty$, the behavior of the λ-values of very large integers should coincide with that of a sequence of coin tosses. This intuition was formally confirmed in Appendix V.

***Theorem* 3:**

The summatory Liouville function takes the asymptotic form $|L(N)| = C\, N^{1/2}$,C is a constant. It can be shown that $C = \sqrt{\frac{2}{\pi}}$. It may be mentioned here that Littlewood's condition is fairly tolerant: As long as asymptotically, for large N, $|L(N)| = C\, N^{1/2}$, and C is any finite constant, R.H. follows. This 'tolerance' is reflected in the value of χ^2 (below) as may be deduced, after a study of the following.

Remark: The form of the summatory Liouville function in Theorem 3 is precisely what we would expect for a sequence of unbiased coin tosses. This, along with a sufficient condition derived by Littlewood (1912), shows

†Riemann had already shown that symmetry conditions ensure that there will be no zeros $0 < Re(s) < 1/2$ if it is found that there are no zeros in the region $1/2 < Re(s) < 1$

‡In addition to the theorems given below, a necessary theorem which states that: The sequence $\lambda(1), \lambda(2), \lambda(3), ..., \lambda(n), ...,$ is asymptotically non-cyclic, (i.e. it will never repeat), was also proved, in [MP and A's], the theorems are numbered differently

§For example, if two numbers c and d in S are factorized as $c = p_1^{e1} p_2^{e2}$ and $d = p_3^{e3} p_4^{e4}$ then at least one of the primes p_3 or p_4 will be different from p_1 or p_2.

that $F(s)$ is analytic for $Re(s) > 1/2$ and $Re(s) < 1/2$, thereby leaving the only possibility that the non-trivial zeros of $\zeta(s)$ can occur only on the critical line $Re(s) = 1/2$.

In the following sections, by comparing the λ-sequences obtained for large sets of consecutive integers with (binomial) sequences of coin tosses, we show that the statistical distributions of the two sets of sequences are consistent with the claims of the above theorems. To this end, we apply Pearson's 'Goodness of Fit' χ^2 test. The software program *Mathematica* developed by Wolfram has been used in this technical report to aid in the prime factorization of the large numbers that this exercise entails.

The compelling bottom line that emerges from this empirical study is that it is extremely unlikely, in fact statistically impossible, that for large N, the sequences of λ-values can differ from sequences of coin tosses. It is this behavior of the Liouville function, recall, that delivers Theorem 3 above. And this Theorem, in turn, nails down all the non-trivial zeros of the zeta function to the critical line [MP and A's].

2. χ^2 Fit of a λ-Sequence

In this section we will derive an expression of how closely a λ sequence corresponds to a binomial sequence (coin tosses). We follow the exposition given in Knuth (1968, Vol. 2, Ch. 3); and then derive a very important expression for a χ^2 fit of a λ-Sequence, given by Eq.(14.9) below.

Suppose we are given a sequence, $T(N_0, N)$, of N consecutive integers starting from N_0:

$T(N_0, N) = \{N_0,\ N_0 + 1,\ N_0 + 2,\ N_0 + 3, \ldots\ldots., N_0 + N - 1\}$

and the sequence, $\Lambda(N_0, N)$, of the corresponding λ-values:

$\Lambda(N_0, N) = \{\lambda(N_0),\ \lambda(N_0 + 1),\ \lambda(N_0 + 2),\ \lambda(N_0 + 3), .. , \lambda(N_0 + N - 1)\}$.

We ask how close in a statistical sense the sequence $\Lambda(N_0, N)$ is to a sequence of coin tosses or, in other words, a binomial sequence. By identifying $\lambda(n) = 1$ as *Head* and $\lambda(n) = -1$ as *Tail*, for the n^{th} 'toss', we may perform this comparison. If this is really the case then statistically $\Lambda(N_0, N)$ should resemble a binomial distribution, we can then compute the χ^2 statistic as follows.

$$\chi^2(N) = \frac{(P - E_P)^2}{E_P} + \frac{(M - E_M)^2}{E_M}, \tag{4}$$

where P and M are the actual number of +1s (*Heads*) and −1s (*Tails*), respectively, in the $\Lambda(N_0, N)$ sequence, E_P and E_M are the expectations of the number of +1s and −1s in the probabilistic sense. From Theorem 1 it immediately follows that, for large N,

$$E_P = E_M = N/2. \tag{5}$$

We define $L(N_0, N)$ as the additional contribution to the summatory Liouville function of N consecutive integers starting from N_0:

$$L(N_0, N) = \sum_{n=N_0}^{N_0+N-1} \lambda(n). \tag{6}$$

For brevity we will denote $\widehat{L} \equiv L(N_0, N)$ and since (6) contains P terms which are equal to +1s and M terms which are equal to −1s, we can write:

$$P - M = \widehat{L}, \tag{7}$$

and

$$P + M = N. \tag{8}$$

Using (7)and (8) we see that $P = (N + \widehat{L})/2$ and $M = (N - \widehat{L})/2$ and from (5) we deduce $P - E_P = \widehat{L}/2$ and $M - E_M = -\widehat{L}/2$ and thus equation (4) gives us the very important χ^2 relation which is satisfied by every $\Lambda(N_0, N)$ sequence involving the factorization of N consecutive integers starting from N_0:

$$\chi^2(N) = \frac{[L(N_0, N)]^2}{N}. \tag{9}$$

Note that the it was possible to derive an expression for χ^2 for large$N$ only because of Theorems 1, 2, and 3. Now we particularly choose N to be the square of an integer and the sequence of length $\mu(N) = \sqrt{N}$ starting from the integer $N_0 = N - \sqrt{N} + 1$ and then taking the $\sqrt{N}$ consecutive terms of the λ-sequence, we obtain

$$\Lambda(N_0, \sqrt{N}) = \{\lambda(N - \sqrt{N} + 1),\ \lambda(N - \sqrt{N} + 2),\ \lambda(N - \sqrt{N} + 3),\ \ldots, \lambda(N)\ \}, \tag{10}$$

and the corresponding $\chi^2(\sqrt{N})$ for such a sequence (which is of length $\sqrt{N}$) can be obtained from Theorem 3 and (9) as

$$\begin{aligned}\chi^2(\sqrt{N}) &= \frac{[C\sqrt{\sqrt{N}}]^2}{\sqrt{N}} \\ &= C^2. \end{aligned} \tag{11}$$

Equation (11) of course, should be interpreted as the average value of a sequence such as $\Lambda(N_0, \sqrt{N})$ of length $\sqrt{N}$ given in the expression (10). In this report we perform the χ^2 'Goodness of Fit' tests for very many sequences of the type $\Lambda(N_0, \sqrt{N})$ with varying lengths and very large values of N to examine whether these sequences are statistically indistinguishable from coin tosses. In this manner, we provide empirical support for the claims of the theorems formally proved in [MP and A's] and, therefore, for the proof of the Riemann Hypothesis.

3. Numerical Analysis of Sequence $\Lambda(N_0, \sqrt{N})$ and its χ^2 Fit

In this section, we consider sequences of length $\sqrt{N}$, starting from $N_0 = N - \sqrt{N} + 1$ or $N + 1$ where N is a perfect square. We use *Mathematica* to compute $L(N_0, N)$.¶

In the table below we list the sequences in the following format. We define the sequences:

$$S_-(N) = \{N - \sqrt{N} + 1,\ N - \sqrt{N} + 2,\ ..., N\}, \tag{12}$$

$$S_+(N) = \{N + 1,\ N + 2,\ ...,\ N + \sqrt{N}\}, \tag{13}$$

and the partial sums of the λs of the two sequences defined above are defined by the expressions:

$$L(S_-) \equiv L(N - \sqrt{N} + 1, N) = \lambda(N - \sqrt{N} + 1) + \lambda(N - \sqrt{N} + 2) + ... + \lambda(N), \tag{14}$$

$$L(S_+) \equiv L(N + 1, N + \sqrt{N}) = \lambda(N + 1) + \lambda(N + 2) + ... + \lambda(N + \sqrt{N}). \tag{15}$$

The formal proof of the Riemann Hypothesis in [MP and A's] proceeded as follows. The sequences $\Lambda(N - \sqrt{N} + 1, \sqrt{N})$ and $\Lambda(N + 1, \sqrt{N})$ were shown to behave like coin tosses for every N (large) over sequences of length $\sqrt{N}$, where N is taken to be a perfect square. On taking N to be consecutive perfect squares, the lengths of the consecutive sequences naturally increase. Using this procedure, we obtain sequences that can span the *entire set* of positive integers (consult the first five columns of Tables 1.1 to 1.4). Since the λs within each segment behave like coin tosses, from the work of Chandrashekar (1943) it follows that the summatory Liouville function $L(N)$ must behave like $C\sqrt{N}$ as $N \to \infty$. The validity of RH follows, by Littlewood's Theorem, from the fact that $F(s)$ cannot then be continued to the left of the critical line $Re(s) = 1/2$ because of the appearance of poles in $F(s)$ on the line, each pole corresponding to a zero of the zeta function $\zeta(s)$.

Statistical Tests

We shall now test the following null hypothesis H_0 against the alternative hypothesis H_1 in the following generic forms:

H_0: The sequence $\Lambda(N_0, N)$ has the same statistical distribution as a corresponding sequence of coin tosses (i.e. binomial distribution with $Prob(H) = Prob(T) = 1/2$).

H_1: The sequence $\Lambda(N_0, N)$ has a different statistical distribution than a corresponding sequence of coin tosses (i.e. binomial distribution with $Prob(H) = Prob(T) = 1/2$).

The critical value for chi square is $\chi^2_{crit} = 3.84$, for the standard 0.05 level of significance. (In our case, the relevant degrees of freedom equal to 1.) Assuming that H_0 is true, if chi square is less than χ^2_{crit} the null hypothesis is accepted.

It should be noted that the tests conducted here are not merely exploratory statistical exercises to discern possible patterns in the λ-sequences. Rather, the tests here are informed by theory. We have formally shown in [MP and A's] that, over the set of positive integers, the probability that λ takes on the value $+1$ or -1 with

¶A typical Mathematica command which calculates the expression $\sum_{n=J}^{K} \lambda(n)$ is:
Plus[LiouvilleLambda[Range[J,K]]].For instance, the command which sums
the $\lambda(n)$ from $n = 25,000,001$ to $25,005,000$ is:
Plus[LiouvilleLambda[Range[25000001, 25005000]]], which will give the answer $= -42$.

equal probability and that, over sequences that are increasing in N, the λ draws are random. Thus statistical evidence consistent with these claims merely bolster what has already been formally demonstrated.

The behavior of the $\Lambda(N_0, \sqrt{N})$ sequences are verified to be indeed like coin tosses for a very large number of cases and the results are summarized in the tables below. Let us take an example from Table 1.1. The third row gives the χ^2 result for the sequence of length 1001, starting from 1001001. We can factorize each of these numbers as:

$1001001 = 3 \times 333667$; $1001002 = 2 \times 500501$; $1001003 = prime$;
$1001004 = 2^2 \times 3 \times 83417$;.........., $1001999 = 41 \times 24439$;
$1002000 = 2^4 \times 3 \times 5^3 \times .167$; $1002001 = 7^2 \times 11^2 \times 13^2$

and hence we can evaluate the corresponding λ-sequence, by using the definition $\lambda(n) = (-1)^{\omega(n)}$, with $\omega(n)$ being the total number of prime numbers (multilpicities included) in the factorization of n. We find that:

$\Lambda(1001001, 1001) = \{\lambda(1001001), \lambda(1001002), ..., \lambda(1002000), \lambda(1002001)\}$
$= \{1, 1, -1, 1, 1, 1, 1, -1, 1, -1, 1, ..., 1, -1, 1\}$.

The partial sum of all the 1001 λs shown in the sequence above adds up to 49. We then estimate how close the sequence $\Lambda(1001001, 1001)$ is to a Binomial distribution, i.e. of 1001 consecutive coin tosses. The observed value $\chi^2 = 2.4$ for this sequence of λs is well below the critical value $\chi^2_{crit} = 3.84$ (for a one degree of freedom) at the standard significance level of 0.05. Thus the sequence $\Lambda(1001001, 1001)$ is statistically indistinguishable from a Binomial distribution obtained by 1001 consecutive coin tosses if we consider $Head = +1$ and $Tail = -1$. In fact, it so happens that out of the 10 sequences shown in Table 1.1 this chosen example has the largest value of χ^2; the other sequences have a much lower χ^2 value and the average value is 0.653 which hovers around the predicted average $C^2 = \frac{2}{\pi} = 0.637$. We see that the null hypothesis would be accepted even if the significance level were at 0.10, for which $\chi^2_{crit} = 2.71$.

We have calculated the χ^2 for larger and larger sequences see Tables1.2, Tables 1.3 and Tables 1.4 for even very large numbers $\sim 10^{10}$ and sequences involving 10^5consecutive integers in each case the sequences $\Lambda(N_0, \sqrt{N})$ behave like coin tosses thus lending emphatic empirical support consistent with the Theorems proved in [MP and A's],.

TABLE 1.1 Sequence of Consecutive Integers of Type $S_-(N)$ and $S_+(N)$ of Length 1000

No	*Type* of S(N)	$\sqrt{N}$	From	to	L(S)	χ^2
1.	S-	1000	999,001	1,000,000	6	0.036
2.	S_+	1000	1,000,001	1,001,000	10	0.100
3.	S-	1001	1,001,001	1,002,001	49	2.400
4.	S_+	1001	1,002,002	1,003,002	-37	1.368
5.	S-	1002	1,003,003	1,004,004	-12	0.144
6.	S_+	1002	1,004,005	1,005,006	-28	0.780
7.	S-	1003	1,005,007	1,006,009	3	0.009
8.	S_+	1003	1,006,010	1,007,012	-39	1.516
9.	S-	1004	1,007,013	1,008,016	12	0.143
10.	S_+	1004	1,008,017	1,009,020	6	0.036
	MEAN	**χ^2 FROM**	**999,001**	**1,009,020**	**=**	**0.653**

TABLE 1.2 Sequence of Consecutive Integers of Type $S_-(N)$ and $S_+(N)$ of Length 5000

No	*Type* of S(N)	$\sqrt{N}$	From	to	L(S)	χ^2
1.	S-	5000	24,995,001	25,000,000	0	0.0
2.	S_+	5000	25,000,001	25,005,000	-42	0.353
3.	S-	5001	25,005,001	25,010,001	-27	0.148
4.	S_+	5001	25,010,002	25,015,002	-103	2.12
5.	S-	5002	25,015,003	25,020,004	-76	1.155
6.	S_+	5002	25,020,005	25,025,006	48	0.461
7.	S-	5003	25,025,007	25,030,009	-13	0.034
8.	S_+	5003	25,030,010	25,035,012	119	2.831
9.	S-	5004	25,035,013	25,040,016	124	3.072
10.	S_+	5004	25,040,017	25,045,020	62	0.768
	MEAN	**χ^2 FROM**	**24,995,001**	**25,045,020**	**=**	**1.094**

TABLE 1.3 Sequence of Consecutive Integers of Type $S_{-}(N)$ and $S_{+}(N)$ of Length 10,000

No	$Type$	$\sqrt{N}$	From	to	L(S)	χ^2
1.	S₋	10000	99,990,001	100,000,000	-146	2.132
2.	S_+	10000	100,000,001	100,010,000	-88	0.774
3.	S₋	10001	100,010,001	100,020,001	-11	0.012
4.	S_+	10001	100,020,002	100,030,002	-43	0.185
5.	S₋	10002	100,030,003	100,040,004	8	0.064
6.	S_+	10002	100,040,005	100,050,006	36	0.130
7.	S₋	10003	100,050,007	100,060,009	23	0.053
8.	S_+	10003	100,060,010	100,070,012	-49	0.240
9.	S₋	10004	100,070,013	100,080,016	-20	0.040
10.	S_+	10004	100,080,017	100,090,020	112	1.254
	MEAN	χ^2 **FROM**	**99,990,001 TO**	**100,090,020**	=	**0.488**

TABLE 1.4 Sequence of Consecutive Integers of Type $S_{-}(N)$ and $S_{+}(N)$ of Length 100,000

No	$Type$	$\sqrt{N}$	From	to	L(S)	χ^2
1.	S₋	100,000	9,999,900,001	10,000,000,000	-232	0.538
2.	S₊	100,000	10,000,000,001	10,000,100,000	340	1.15
3.	S₋	100,001	10,000,100,001	10,000,200,001	-249	0.620
4.	S₊	100,001	10,000,400,005	10,000,500,006	-115	0.132
5.	S₋	100,002	10,000,300,003	10,000,400,004	216	0.467
6.	S₊	100,002	10,000,400,005	10,000,500,006	456	2.08
7.	S₋	100,003	10,000,500,007	10,000,600,009	-255	0.650
8.	S₊	100,003	10,000,600,010	10,000,700,012	-235	0.552
9.	S₋	100,004	10,000,700,013	10,000,800,016	-44	0.0194
10.	S₊	100,004	10,000,800,017	10,000,900,020	202	0.408
11.	S₋	100,005	10,000,900,021	10,001,000,025	-191	0.364
12.	S₊	100,005	10,001,000,026	10,001,100,030	475	2.26
13.	S₋	100,006	10,001,100,031	10,001,200,036	134	0.179
14.	S₊	100,006	10,001,200,037	10,001,300,042	-66	0.0436
15.	S₋	100,007	10,001,300,043	10,001,400,049	427	1.82
16.	S₊	100,007	10,001,400,050	10,001,500,056	-303	0.918
17.	S₋	100,008	10,001,500,057	10,001,600,064	276	0.762
18.	S₊	100,008	10,001,600,065	10,001,700,072	-210	0.441
19.	S₋	100,009	10,001,700,073	10,001,800,081	267	0.713
20.	S₊	100,009	10,001,800,082	10,001,900,090	291	0.847
	MEAN	χ^2 **FROM**	**9,999,900,001 TO**	**10,001,900,090**	=	**0.768**

3.1 Sequences of Fixed Length Arbitrarily Positioned

In this section we consider various segments of consecutive integers of a fixed length but starting from an arbitrary integer. Even here we see that the λs within each segment behave like coin tosses and have the same statistical properties.

We now calculate the χ^2 values of λ-sequences for a sequence S_A of consecutive integers, starting from an arbitrary number N_0 but all of a fixed length M:

$$S_A(N) = \{N_0,\, N_0+1,\, N+2,\, N+3, ...,\, N_0+M-1\} \tag{16}$$

and

$$L(S_A) = \lambda(N_0) + \lambda(N_0+1) + \lambda(N_0+2) + \lambda(N_0+3) + ... + \lambda(N_0+M-1) \tag{17}$$

The results, which are summarized in Table 2.1, again show that the λ-sequences are statistically like coin tosses.

TABLE 2.1 Sequence of Consecutive Integers of Type $S_A(N)$ and of Length M = 1000

No	*Type*	M	From N_0	to $N_0 + M - 1$	$L(S_A)$	χ^2
1.	S_A	1000	10,000,001	10,001,000	36	1.296
2.	S_A	1000	12,000,001	12,001,000	28	0.784
3.	S_A	1000	13.000,001	13,001,000	-14	0.196
4.	S_A	1000	15,000,001	15,001,000	10	0.10
5.	S_A	1000	45,000,001	45,001,000	-18	0.324
6.	S_A	1000	47,000,001	47,001,000	-36	1.296
7.	S_A	1000	56,000,001	56,001,000	24	0.576
8.	S_A	1000	70,000,001	70,001,000	-44	1.936
9.	S_A	1000	90,000,001	90,001,000	14	0.196
10.	S_A	1000	95,600,001	95,601,000	28	0.784
11.	S_A	1000	147,000,001	147,001,000	-26	0.676
12.	S_A	1000	237,000,001	237,001,000	-24	0.576
13.	S_A	1000	400,000,001	400,001,000	26	0.676
14.	S_A	1000	413,000,001	413,001,000	10	0.10
15.	S_A	1000	517,000,001	517,001,000	14	0.196
16.	S_A	1000	530,000,001	530,001,000	-32	1.024
17.	S_A	1000	731.000,001	731,001,000	50	2.500
18.	S_A	1000	871,000,001	871,001,000	-42	1.764
19.	S_A	1000	979,000,001	979,001,000	-20	0.400
20.	S_A	1000	997,000,001	997,001,000	14	0.196
		MEAN	**χ^2 OF ABOVE**	**20 SEGMENTS**	=	**0.780**

3.2 Entire Sequences from $n = 1$ to $n = N$, N large and calculation of χ^2 for such sequences from $L(N)$

It has been empirically verified in the literature that the summatory Liouville Function $L(N) = \sum_{n=1}^{N} \lambda(n)$ fluctuates from positive to negative values as N increases without bound. We now investigate the χ^2 values for such sequences,and use Eq.(9), so that we may see how these sequences behave like coin tosses.

In the Table 3.1 we use the values of $L(N)$ for various large values of N, which were found by Tanaka (1980), the results depicted below reveal that the lambda sequences are statistically indistinguishable from the sequences of coin tosses over such large ranges of N from 1 to one billion.

In the above we calculated $L(N)$ for various valies of N, however, if we choose a value N at which $L(N)$ is a local maximum or a local minimum then we would be examining potential worst case scenarios for deviations of the λs from coin tosses because these are the values of N that are likely to yield the highest values of χ^2 (see equation (9)). It is interesting to investigate if even for these special values of N whether the χ^2 is less than the critical value; if so, we would again have statistical assurance that the entire sequence of λs from $n = 1, 2, 3, ...$ behave like coin tosses.

We therefore use the 58 largest values of $L(N)$ and the associated values of N reported in the literature by Borwein, Ferguson and Mossinghoff (2008), and perform our statistical exercise. See Table 3.2. We see that even for these "worst case scenario" values of N the lambda sequences are statistically indistinguishable from the sequences of coin tosses.

TABLE 3.1 Values of L(N) at various large values of N
(The values for N and L(N) are from Tanaka (1980))

No.	N	$L(N) = \sum_{n=1}^{N} \lambda(n)$	χ^2	
1	100,000,000	-3884	0.1508	
2	200,000,000	-11126	0.6189	
3	300,000,000	-16648	0.9238	
4	400,000,000	-11200	0.3136	
5	500,000,000	-18804	0.7072	
6	600,000,000	-15350	0.3927	
7	700,000,000	-25384	0.9204	
8	800,000,000	-19292	0.4652	
9	900,000,000	-4630	0.0238	
10	1,000,000,000	-25216	0.6358	
	MEAN	χ^2 **OF ABOVE** =	0.5152	

TABLE 3.2 Values of L(N) at local Minima (Maxima) for very Large N
(The values for N and L(N) are from Borwein, Ferguson and Mossinghoff (2008))

No.	N	$L(N) = \sum_{n=1}^{N} \lambda(n)$	χ^2	
1	293	-21	1.5051	
2	468	-24	1.2308	
3	684	-28	1.1462	
4	1,132	-42	1.5583	
5	1,760	-48	1.3091	
6	2,804	-66	1.5535	
7	4,528	-74	1.2094	
8	7,027	-103	1.5097	
9	9,840	-128	1.665	
10	24,426	-186	1.4164	
11	59,577	-307	1.582	
12	96,862	-414	1.7695	
13	386,434	-698	1.2608	
14	614,155	-991	1.5991	
15	925,985	-1,253	1.6955	
16	2,110,931	-1,803	1.54	
17	3,456,120	-2,254	1.47	
18	5,306,119	-2,931	1.619	
19	5,384,780	-2,932	1.5965	
20	8,803,471	-3,461	1.3607	

TABLE 3.2 (Cont'd) Values of L(N) at local Minima (Maxima)

No.	N	$L(N) = \sum_{n=1}^{N} \lambda(n)$	χ^2	
21	12,897,104	-4,878	1.845	
22	76,015,169	-10,443	1.4347	
23	184,699,341	-17,847	1.7245	
24	281,876,941	-19,647	1.3694	
25	456,877,629	-28,531	1.7817	
26	712,638,284	-29,736	1.2408	
27	1,122,289,008	-43,080	1.6537	
28	1,806,141,032	-50,356	1.4039	
29	2,719,280,841	-62,567	1.4396	
30	3,847,002,655	-68,681	1.2262	
31	4,430,947,670	-73436	1.2171	
32	6,321,603,934	-96,460	1.4719	
33	10,097,286,319	-123,643	1.514	
34	15,511,912,966	-158,636	1.6223	
35	24,395,556,935	-172,987	1.2266	
36	39,769,975,545	-238,673	1.4324	
37	98,220,859,787	-365,305	1.3586	
38	149,093,624,694	-461,684	1.4296	
39	217,295,584,371	-598,109	1.6463	
40	341,058,604,701	-726,209	1.5463	
41	576,863,787,872	-900,668	1.4062	
42	835,018,639,060	-1,038,386	1.2913	
43	1,342,121,202,207	-1,369,777	1.398	
44	2,057,920,042,277	-1,767,635	1.5183	
45	2,147,203,463,859	-1,784,793	1.4836	
46	3,271,541,048,420	-2,206,930	1.4888	
47	4,686,763,744,950	-2,259,182	1.089	
48	5,191,024,637,118	-2,775,466	1.4839	
49	7,934,523,825,335	-3,003,875	1.1372	
50	8,196,557,476,890	-3,458,310	1.4591	
51	12,078,577,080,679	-4,122,117	1.4068	
52	18,790,887,277,234	-4,752,656	1.2021	
53	20,999,693,845,505	-5,400,411	1.3888	
54	29,254,665,607,331	-6,870,529	1.6136	
55	48,136,689,451,475	-7,816,269	1.2692	
56	72,204,113,780,255	-11,805,117	1.9301	
57	117,374,745,179,544	-14,496,306	1.7904	
58	176,064,978,093,269	-17,555,181	1.7504	

The empirical evidence provided here is very comprehensive: it examines the statistical behavior of the Liouville function for large segments of consecutive integers (e.g.Table 1.4). We have also considered the entire series of $\lambda(n)$ from the values of $n = 1$ to $n = N = 176$ trillion - as high as any available studies in the literature have gone. And yet, the $\lambda-$sequences consistently show themselves, in rigorous statistical tests, to be indistinguishable from sequences of coin tosses, hence providing overwhelming statistical evidence in support of Littlewood's condition that as $N \to \infty$, $L(N) = C.\sqrt(N)$, (where C is finite) and thus declaring that the non-trivial zeros of the zeta function, $\zeta(s)$, must all necessarily lie on the critical line $Re(s) = 1/2$.

4. Concluding Note

In this Appendix VI we have provided compelling, comprehensive numerical and statistical evidence that is consistent with the Theorems that were instrumental in the formal validation of the Riemann Hypothesis in [MP and A's].

It is hoped that a perusal of this section (Appendix 6)report offers some insight into, and understanding of, why the Riemann Hypothesis is correct. It should be noted that, while the results presented here are perfectly consistent with the theoretical results in [MP and A's], they obviously do not prove (in a strict mathematical sense, because of the statistical nature of the study), the Riemann Hypothesis. For the formal proof, the rigorous mathematical analysis in the main paper needs to be consulted.

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