SSIE 660: Applied Stochastic Processes
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Quiz 1 -Key

Name: Key

1. An urn contains five red, three orange, and two blue balls. Two balls are randomly selected.

   (a) What is the sample space of this experiment?

   The sample space is $S = \{(r,r),(r,o),(r,b),(o,o),(o,b),(b,b)\}$.

   (b) Let X represent the number of blue balls selected. What are the possible values of X?

   The possible values of X are 0, 1, and 2

   (c) Calculate $P[X = 0]$.
   $P[X = 0] =_8 C_2/_{10}C_2$.

2. Suppose X has a binomial distribution with parameters 6 and $1/2$. Show that $X = 3$ is the most likely outcome.
   $P\{X = 0\} = P\{X = 6\} = \frac{1}{2}^{6} = \frac{1}{64}$
   $P\{X = 1\} = P\{X = 5\} = 6\frac{1}{2}^{6} = \frac{6}{64}$
   $P\{X = 2\} = P\{X = 4\} =_6 C_2\frac{1}{2}^{6} = \frac{15}{64}$
   $P\{X = 3\} =_6 C_3 = \frac{1}{2}^{6} = \frac{20}{64}$

3. A coin is to be tossed until a head appears twice in a row. If the coin is fair, what is the probability that it will be tossed exactly four times?

   $P\{4 \text{ tosses}\} = P\{(t,t,h,h),(h,t,h,h)\} = 2 * (1/2)^4 = 1/8$

4. Three dice are thrown. Let E be the event that the same number appears on exactly two of the three dice, S be the event that all three numbers are the same, and D be the event that all three number are different. Calculate $P(E)$. (Hint:

$P(E) + P(S) + P(D) = 1$).

$P(S) = 6/216 = 1/36$, For $n(D) = 6 \cdot 5 \cdot 4 = 120, P(D) = 120/216 = 20/36$. Therefore, $P(E) = 1 - P(D) - P(S) = 5/12$

5. Suppose the distribution function of X is given by

$$F(b) = \begin{cases} 0, & b < 0 \\ 1/2, & 0 \leq b < 1 \\ 1, & 1 \leq b \leq \infty \end{cases}$$

Find the probability mass function of X.

$p(0) = 1/2, p(1) = 1/2$

6. A television store owner figures that 50 percent of the customers entering his store will purchase an low end television set, 20 percent will purchase a high end television set, and 30 percent will just be browsing. If five customers enter his store on a certain day, what is the probability that the owner sells 4 or more televisions?

E = buying a television. $P(E) = .7$

${}_4C_5(.7)^4(.3)^1 +_5 C_5(.7)^5(.3)^0$

7. In any binomial experiment in which n is large and p is small, $b(x; n, p) \approx p(x; \mu)$ where $b(x; n, p)$ is a binomial distribution with parameters n and p and $p(x; \mu)$ is a Poisson distribution with parameter μ. Compare the Poisson approximation with the correct binomial probability for the following case: $P[X = 0]$ when n = 10 and $p = 0.1$.

Binomial: $P[X = 0] =_{10} C_0(0.1)^0(0.9)^{10} \equiv 0.3487$,

Poisson: $\mu = np = 1.P[X = 0] = \dfrac{e^{-1}1^0}{0!} = \dfrac{e^{-1}}{1} = 0.3679$

8. Let the probability density of X be given by

$$f(x) = \begin{cases} c(4x - 2x^2), & 0 < x < 2 \\ 0, & \text{otherwise} \end{cases}$$

   (a) What is the value of c?

$$c\left[2x^2 - \frac{2}{3}x^3\right]_0^2 = 1$$

$$c\left(8-\frac{2}{3}\cdot(0.8)\right)=1$$

$$c=\frac{3}{8}$$

(b) $P[\frac{1}{2}<X<\frac{3}{2}]$?

$$\int_{\frac{1}{2}}^{\frac{3}{2}}\frac{3}{8}(4x-2x^2)\,\mathrm{d}x=\frac{3}{8}\left[2x^2-\frac{2}{3}x^3\right]_{\frac{1}{2}}^{\frac{3}{2}}=\left(\frac{9}{4}-\frac{5}{12}\right)\cdot\frac{3}{8}=\frac{11}{16}$$

9. The joint probability distribution of X and Y are given as follows:

X	1	1	-1	-1
Y	1	-1	1	-1
$P_{X,Y}(x,y)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{8}$	c

(a) What is the value of c?

$$\frac{1}{4}+\frac{1}{2}+\frac{1}{8}+c=1$$

$$c=\frac{1}{8}$$

(b) Find the pmf of X.

$$P(X=1)=\frac{3}{4}$$

$$P(X=-1)=\frac{1}{4}$$

(c) Find the pmf of Y.

$$P(Y=1)=\frac{3}{8}$$

$$P(Y=-1)=\frac{5}{8}$$

(d) Calculate $E[X]$.

$$E[X] = 1.\frac{3}{4} + (-1).(\frac{1}{4}) = \frac{1}{2}$$

(e) Calculate $E[Y]$.

$$E[Y] = 1.\frac{3}{8} + (-1).(\frac{5}{8}) = \frac{-1}{4}$$

(f) Calculate $E[XY]$.

$$\begin{aligned} E[XY] &= 1.\frac{1}{4} + (-1)(\frac{1}{2}) + (-1).\frac{1}{8} + (1)\frac{1}{8} \\ &= -\frac{1}{4} \end{aligned}$$

(g) Calculate $Var[X]$.

$$Var[X] = E[X^2] - (E(X))^2$$

$$1 - \left(\frac{1}{2}\right)^2 = \frac{3}{4}$$

10. Find $E[X]$ and $Var[X]$ if X is a Bernoulli random variable with parameter p.

$$E[X] = 0 \cdot (1-p) + 1 \cdot p = p$$

$$Var[X] = E[X^2] - (E[X])^2$$

$$E[X^2] = 0^2 \cdot (1-P) - 1^2 \cdot p = p$$

$$Var[X] = p - p^2 = p(1-p)$$

11. Assume that A and B are independent.

    a) Show that A and B^C are independent.

$$\begin{aligned} P(A) &= P(A \cap B) + P(A \cap B^C) \\ &= P(A|B)P(B) + P(A|B^C)P(B^C) \\ &= P(A)P(B) + P(A|B^C)P(B^C) \because A, B \text{ independent} \end{aligned}$$

Therefore,

$$P(A|B^C)P(B^C) = P(A) - P(A)P(B) = P(A)(1 - P(B)) = P(A)P(B^C)$$

Thus, A and B^C are independent.

    b) Show that A^C and B^C are independent:

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\begin{aligned}
&P(A^C \cap B^C) = P((A \cup B)^C) \\
&= 1 - p(A \cup B) \\
&= 1 - P(A) - P(B) + P(A \cap B),\ \textit{A and B are independent} \\
&= 1 - P(A) - P(B) + P(A).P(B) \\
&= 1 - P(A) - P(B)(1 - P(A)) \\
&= P(A^C) - P(B)P(A^C) \\
&= P(A^C)(1 - P(B)) \\
&= P(A^C)P(B^C)\ \textit{Thus, they are independent.}
\end{aligned}
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