SSIE 660: Stochastic Systems

Homework assignment 2 – Key

1. Exercise 1.1 in the textbook
   Sample space = {(R,B),(R,G),(G,R),(G,B),(B,R),(B,G),(R,R),(B,B),(G,G)}
   Probability of each point of the sample space, knowing that each marble in the box is equally likely to be selected.
   P(E) = $\frac{1}{9}$

2. Exercise 1.4 in the textbook

   a) Only F occurs: $E^c \cap F \cap G^c$
   b) Both E and F but not G occurs: $E \cap F \cap G^c$
   c) At least one event occurs: $E \cup F \cup G$
   d) At least two events occur $(E \cap F) \cup (F \cap G) \cup (E \cap G)$
   e) All three events occur: $E \cap F \cap G$
   f) none occurs: $E^c \cap F^c \cap G^c$
   g) At most one occurs: $(E \cap F)^c \cap (E \cap G)^c \cap (F \cap G)^c$
   h) At most two occurs: $(EFG)^c$

3. Exercise 1.18 in the textbook
   Let B be the event both are girls; E= event oldest is girl; L= event at least one is a girl

   a) $P(B \mid E) = \frac{P(B \cap E)}{P(E)} = \frac{1/4}{2/4} = \frac{1}{2}$

   b) $P(L) = 1 - P(\text{no girls}) = 1 - \frac{1}{4} = \frac{3}{4}$
   $P(B \mid L) = \frac{P(BL)}{P(L)} = \frac{1/4}{3/4} = \frac{1}{3}$

4. Exercise 1.28 in the textbook
   Yes.
   $P(\text{A}|\text{B}) > P(\text{A})$ is equivalent to $P(\text{AB}) > P(\text{A})P(\text{B})$, which is equivalent to $P(\text{B}|\text{A}) > P(B)$.

5. Exercise 1.42 in the textbook
   let B = event biased coin was flipped; F and U (same as above).

   $$P(U \mid H) = \frac{P(H \mid U)P(U)}{P(H \mid U)P(U) + P(H \mid B)P(B) + P(H \mid F)P(F)}$$

$$= \frac{1 \cdot \frac{1}{3}}{1 \cdot \frac{1}{3} + \frac{3}{4} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{3}} = \frac{4}{9}$$

6. Exercise 1.46 in the textbook

   Let X(=B or =C) denote the jailer's answer to prisoner A. Now for instance,

   P { A to be executed | X = B }

   $$= \frac{P\{ \text{ A to be executed}, X = B\}}{P\{X = B\}}$$
   $$= \frac{P\{ \text{ A to be executed}\}P\{X = B \mid \text{ A to be executed}\}}{P\{X = B\}}$$
   $$= \frac{(1/3)P\{ \text{ } X = B \mid \text{ A to be executed}\}}{1/2}$$

   Now it is reasonable to suppose that if A is to be executed, then the jailor is equally likely to answer either B or C. That is,

   $$\text{P\{ X = B | A to be executed } \} = \frac{1}{2}$$

   and so,
   $$\text{P\{ A to be executed} \mid X = B\} = \frac{1}{3}$$

   Similarly,

   $$\text{P\{ A to be executed} \mid X = C\} = \frac{1}{3}$$

   and thus the jailer's reasoning is invalid. (It is true that if the jailer were to answer B, then A knows that the condemned is either himself or C, but it is twice as likely to be C)

7. Given the joint probability density function

   $$f_{X,Y}(x,y) = \begin{cases} \frac{6-x-y}{8}, & 0 < x < 2,\ 2 < y < 4 \\ 0, & \text{elsewhere} \end{cases}$$

   Find $f_X(x), f_Y(y), f_{X|Y}(x|y)$, and $f_{Y|X}(y|x)$.

   $f_X(x) = \int_{y=2}^{4} \frac{6-x-y}{8} dy = \frac{6-2x}{8},\ 0 < x < 2; 0$, elsewhere.

   $f_Y(y) = \int_{x=0}^{2} \frac{6-x-y}{8} dx = \frac{10-2y}{8},\ 2 < y < 4; 0$, elsewhere.

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{\frac{6-x-y}{8}}{\frac{10-2y}{8}} = \frac{6-x-y}{10-2y},\ 0 < x < 2,\ 2 < y < 4; 0, \text{elsewhere.}$$

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{\frac{6-x-y}{8}}{\frac{6-2x}{8}} = \frac{6-x-y}{6-2x},\ 0 < x < 2,\ 2 < y < 4; 0, \text{elsewhere.}$$

8. In the above problem, find $P[1 < Y < 3|X = 2]$.

$P[1 < Y < 3|X = 2] = \int_{y=1}^{3} f_{Y|X}(y|x = 2)dy = \int_{y=2}^{3} f_{Y|X}(y|x = 2)dy$ (because $2 < y$).

$= \int_{y=2}^{3} f_{Y|X}(y|x = 2)dy = \int_{y=2}^{3} \frac{6-2-y}{6-4}dy = \frac{1}{2}\left[4y - \frac{y^2}{2}\right]_2^3 = 0.75$