SSIE 660: Stochastic Systems
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Note 11
Chapter 4. Discrete Markov Chains

Time Reversible Markov Chains

- Consider an irreducible and ergodic Markov Chain with one-step transition probabilities, P_{ij}, that has reached steady state, with steady state probabilities π_i.
- Let us trace the sequence of states backward in time. That is, starting at time n, we consider the sequence of states, $X_n, X_{n-1}, X_{n-2}, \ldots$. This sequence of states is itself a Markov Chain with transition probabilities Q_{ij}, given by

  $Q_{ij} =$

- What should we prove so that the reversed process is indeed a Markov Chain?

- If $Q_{ij} = P_{ij}$, then the Markov Chain is said to be Time Reversible.

Chapter 5. The Exponential Distribution and the Poisson Process

The Exponential Distribution

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & otherwise \end{cases}$$

$$P(X \leq a) = F_X(a) = \int_0^a \lambda e^{-\lambda x} dx =$$

$$P(X > a) = \begin{cases} & a > 0 \\ & otherwise \end{cases}$$

$$E[X] =$$

$$Var(X) =$$

Properties (I) Assume that the lifetime of a component follows exponential density function. Let us find the probability that the component lasts for at least $(s+t)$ hours, given that it has survived t hours.

$$P[X > t + s | x > t] =$$

- That is, if the component is working at time t, then the distribution of the remaining amount of time that it survives is
- This is called the memoryless property of exponential distribution. (The component does not *remember* that it has already been in use for time t.)

Example 1. Suppose that the amount of time a customer spends in a bank is exponentially distributed with a mean of 5 minutes ($\lambda = 1/5$).

1. What is the probability that a customer spends more than 10 minutes in the bank?
2. What is the probability that a customer spends an additional 10 minutes in the bank, given that she/he is still in the bank after 5 minutes?

$X \sim$ amount of time the customer spends in the bank.

1. $P[X > 10] =$

2. $P[X > 10 + 5 | X > 5] =$

Properties (II) Let $X_1, X_2, ..., X_n$ be independent and identically distributed (iid) exponential random variables having mean $1/\lambda$. Let $S = \sum_{i=1}^{n} X_i$. Then S follows gamma distribution with parameters n and λ.

$$f_S(s) = \begin{cases} \lambda e^{-\lambda s} \dfrac{(\lambda s)^{n-1}}{(n-1)!}, & s > 0 \\ 0, & otherwise \end{cases}$$

Properties (III) Let X_1 and X_2 be two exponential random variables with means $1/\lambda_1$ and $1/\lambda_2$, respectively. X_1 and X_2 are independent to each other. What is $P[X_1 < X_2]$?

$$f_{X_1}(x_1) = \begin{cases} \lambda_1 e^{-\lambda_1 x_1}, & x_1 > 0 \\ 0, & otherwise \end{cases}$$
$$f_{X_2}(x_2) = \begin{cases} \lambda_2 e^{-\lambda_2 x_2}, & x_2 > 0 \\ 0, & otherwise \end{cases}$$

$P[X_1 < X_2] =$

$F_{X_1}(x_2) =$

Properties (IV) The failure rate (or hazard rate) function $r(t)$ is

$$r(t) = \frac{f(t)}{1 - F(t)}$$

For exponential density function,

$$r(t) =$$

Let the density function of X be $f_X(x)$ and the density function of Y be $f_Y(y)$.

$$f_X(x) = \begin{cases} \lambda_1 e^{-\lambda_1 x}, & x > 0 \\ 0, & otherwise \end{cases}$$

$$f_Y(y) = \begin{cases} \lambda_2 e^{-\lambda_2 y}, & y > 0 \\ 0, & otherwise \end{cases}$$

Let $Z = \min\{X, Y\}$. What is the conditional density function of Z, given that $Z = X$?

In problems like this, it is easier to obtain the cumulative distribution function of Z, $F_Z(z)$ and derive $f_Z(z)$ by differentiating $F_Z(z)$ with respect to z.

$P[z < x < y] = P[x > z, x < y] =$

The Poisson Process

Counting Processes: A stochastic process $\{N(t), t \geq 0\}$ is said to be a counting process, if $N(t)$ represents the *total number* of 'events' that have occurred up to time t.

Example 2. The number of customers entered a store *up to* time t.

$N(2) =$; $N(3) =$; $N(7) =$; $N(5.2) =$.

Requirements of a Counting Process

1. $N(t) \geq 0$.
2. $N(t)$ is integer valued.
3. If $s < t$, $N(s) \leq N(t)$.
4. For $s < t$, $N(t) - N(s)$ equals the number of events that have occurred in the interval (s, t).

Example 3. N(5.8) - N(2.2) =

Definition 4. *A counting process is said to possess* independent increments*, if the number of events which occur in disjoint time intervals are independent.*

Example 5. If $\{N(7) - N(5.5)\}$ is independent of $\{N(4.6) - N(1.9)\}$, then $N(t)$ possesses independent increments.

Definition 6. *A counting process is said to possess* stationary increments*, if the probability distribution of the number of events which occur in any interval of time depends only on the* length of time *interval.*

Example 7. If $P[N(7) - N(5.5) = n] = P[N(20) - N(18.5) = n]$ for all n, then $N(t)$ possesses stationary increments.

Definition 8 (Poisson Process)**.** *A counting process $\{N(t), t \geq 0\}$ is said to be a Poisson process having rate $\lambda > 0$, if*

1. $N(0) = 0.$
2. *The process has independent increments, and*
3. *For all $s, t \geq 0$,*

$$P[N(t+s) - N(s) = n] = P[N(t) = n] = \frac{e^{-\lambda t}(\lambda t)^n}{n!},\ n = 0, 1, 2, \ldots$$

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$E[N(t)] = \lambda t$. *(Mean of Poisson process)*

Definition 9 (Alternative Definition)**.** *A counting process $\{N(t), t \geq 0\}$ is said to be a Poisson process having rate $\lambda > 0$, if*

1. $N(0) = 0.$
2. *The process has* stationary *and independent increments, and*
3. $P[N(h) = 1] = \lambda h + o(h)$*, and*

4. $P[N(h) \geq 2] = o(h)$.

Note that $f(\cdot)$ is $o(h)$ if $\lim_{h \to 0} = \dfrac{f(h)}{h} = 0$.

Example 10. Let $\{N(t), t \geq 0\}$ be a Poisson process with rate $\lambda = 15$. Compute:

1. $P[N(6) = 9] =$

2. $P[N(6) = 9, N(20) = 13, N(56) = 27] =$

3. $P[N(20) = 13 | N(6) = 9] =$