SSIE 660: Stochastic Systems
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Note 16
Chapter 7. Renewal Theory - Cont'd

Relationship between $m(t)$, mean number of renewals by time t, and $E[S_{N(t)+1}]$, the expected time of the first renewal after t.

- Let $g(t) = E[S_{N(t)+1}]$.
- The first renewal occurs by time x. That is,

$$X_1 = x$$

- Conditioning on the time of the first renewal,

$$g(t) = \int_0^\infty E[S_{N(t)+1} | X_1 = x] f_{X_1}(x) dx$$

(i) $x < t$.

(ii) $x > t$.

$$g(t) = \int_0^t [x + g(t-x)] f_X(x)dx + \int_t^\infty x f_X(x)dx$$

$$=$$

Renewal equation:

$$m(t) = F_X(t) + \int_0^t m(t-x) f_X(x)dx$$

Let $g_1(t) = \dfrac{g(t)}{\mu} - 1 =$

Proposition 1 (7.2). $E[S_{N(t)+1} = \mu[m(t)+1]$

Figure:

$$Y(t) =$$
$$S_{N(t)+1} =$$

Taking expectations:

$$g(t) = E[S_{N(t)+1}] =$$

Example 2. Consider the renewal process whose inter-arrival distribution is the convolution of two exponentials. Find $m(t)$.

$$F_i(t) =$$

$$F = F_1 * F_2$$

$$F(t) =$$

Figure:

$$EX] =$$

$$E[Y(t)] =$$

Figure:

- Now the problem is to find $p(t)$:

  Prob

- Let us define a stochastic process $\{X(t), t \geq 0\}$ such that

$$X(t) = \begin{cases} 1 & \text{if component 1 is employed at } t \\ 2 & \text{if component 2 is employed at } t \end{cases}$$

- What 'stochastic process' does $X(t)$ follow?

- Consider the following problem:

- We found $P_{00}(t)$ and $P_{10}(t)$ (We could also find $P_{01}(t)$ and $P_{11}(t)$)

$$P_{00}(t) = \frac{\mu}{\mu+\lambda} + \frac{\lambda}{\mu+\lambda}e^{-(\mu+\lambda)t}, \;\; P_{10}(t) = \frac{\mu}{\mu+\lambda} - \frac{\mu}{\mu+\lambda}e^{-(\mu+\lambda)t}$$

$E[Y(t)] =$

Renewal Reward Process

- A reward is received each time a renewal occurs. Specifically, a reward of R_n is received at the time of the n^{th} renewal.
- $R_n, n \geq 1$ are independent and identically distributed.
- But R_n may depend on X_n, the length of the n^{th} renewal interval.
- The total reward earned by time t is $R(t)$.

$$R(t) =$$

- Let $E[R] = E[R_n]$ and $E[X] = E[X_n]$.

Proposition 3. *If $E[R] < \infty$ and $E[X] < \infty$, then*

1. *with probability 1,* $\lim_{t\to\infty} \dfrac{R(t)}{t} = \dfrac{E[R]}{E[X]}$
2. $\lim_{t\to\infty} \dfrac{E[R(t)]}{t} = \dfrac{E[R]}{E[X]}$

Proof. $\dfrac{R(t)}{t} =$

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Remark:

1. If we say that a cycle is completed every time a renewal occurs, then Proposition 3 states that the long-run average reward per unit time is equal to

2. This result is valid when the reward is earned gradually throughout the renewal cycle, also.

Example 4 (A Car Buying Model)**.** The life time of a car is continuous random variable having a distribution H and probability density h. Mr. Brown has a policy that he buys a new car as soon as his old one either breaks down or reaches age of T years. Suppose that a new car costs C_1 dollars and also that an additional cost of C_2 dollars is incurred whenever Mr. Brown's car breaks down. Under the assumption that a used car has no resale value, what is Mr. Brown's long-run average cost?