SSIE 660: Stochastic Systems
Dr. Sung H. Chung
Note 16
Chapter 7. Renewal Theory - Cont'd

Relationship between $m(t)$, mean number of renewals by time t, and $E[S_{N(t)+1}]$, the expected time of the first renewal after t.

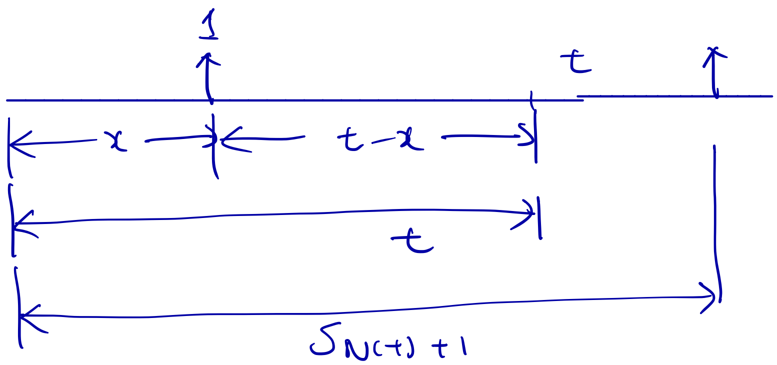
- Let $g(t) = E[S_{N(t)+1}]$.
- The first renewal occurs by time x. That is,

$$X_1 = x$$

- Conditioning on the time of the first renewal,

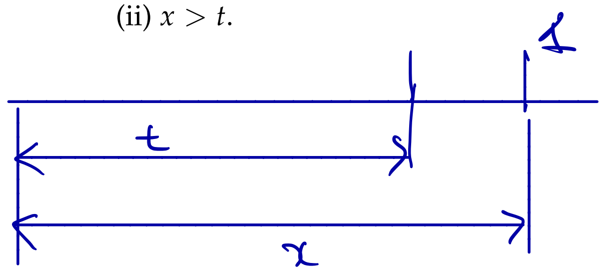
$$g(t) = \int_0^\infty E[S_{N(t)+1} | X_1 = x] f_{X_1}(x) dx$$

(i) $x < t$.



$$E[S_{N(t)+1} | X_1 = x] = x + E[S_{N(t-x)+1}] = x + g(t-x)$$

(ii) $x > t$.



$$E[S_{N(t)+1} | X_1 = x] = x$$

$$g(t) = \int_0^t [x + g(t-x)] f_X(x) dx + \int_t^\infty x f_X(x) dx = \int_0^\infty x f_X(x) dx + \int_0^t g(t-x) f_X(x) dx$$

$$= E[X] + \int_0^t g(t-x) f_X(x) dx$$

Renewal equation:

$g(t-x) = \mu g_1(t-x) + \mu$

$$m(t) = F_X(t) + \int_0^t m(t-x) f_X(x) dx$$

Let $g_1(t) = \frac{g(t)}{\mu} - 1 = \frac{E[S_{N(t)+1}]}{\mu} - 1 = \frac{1}{\mu}\int_0^t g(t-x) f_X(x) dx$

$$= \int_0^t [1 + g_1(t-x)] f_X(x) dx$$

$$\therefore \quad g_1(t) = F_X(t) + \int_0^t g_1(t-x) f_X(x) dx$$

g_1 satisfies renewal equation uniqueness

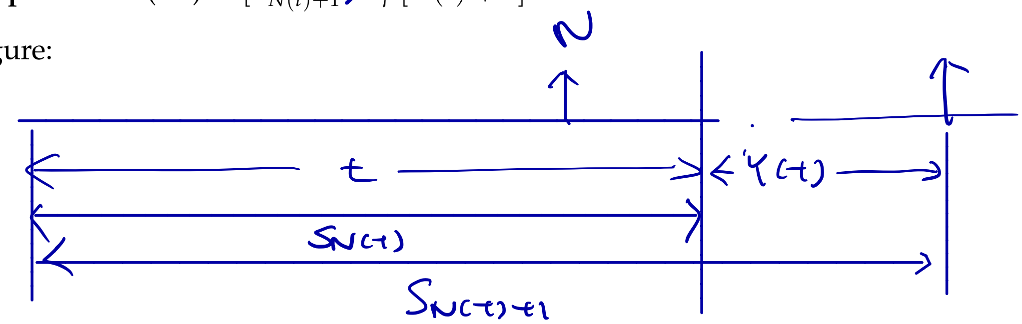
$$\therefore g_1(t) = m(t)$$

$$\underline{E[S_{N(t)+1}] = \mu[m(t)+1] = g(t)}$$

↳ proposition 7.2

Proposition 1 (7.2). $E[S_{N(t)+1}] = \mu[m(t)+1]$

Figure:



$$Y(t) = \text{time from } t \text{ until the next arrival}$$
$$S_{N(t)+1} = t + Y(t)$$

Taking expectations:

$$g(t) = E[S_{N(t)+1}] = t + E[Y(t)] = \mu[m(t)+1]$$

divide by μt

$$\frac{m(t)}{t} + \frac{1}{t} = \frac{1}{\mu} + \frac{E[Y(t)]}{t}\cdot\frac{1}{\mu}$$

if $\underline{\frac{E[Y(t)]}{t} \to 0 \text{ as } t\to\infty}$ $\quad \frac{m(t)}{t} \to \frac{1}{\mu}$ as $t\to\infty$

show this to prove the elementary renewal theorem

Example 2. Consider the renewal process whose inter-arrival distribution is the convolution of two exponentials. Find $m(t)$.

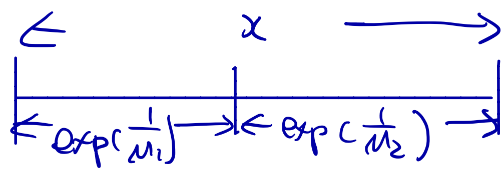
do this by determining $E[Y(t)]$: $\underline{Y(t): \text{remaining life}}$.

$$F_i(t) = 1 - e^{-\mu_i(t)} \qquad i = 1, 2$$

$$F = F_1 * F_2$$

$$F(t) = \int_0^t F_1(\tau) F_2(t-\tau)\, d\tau$$

Figure:



1st component 2nd component

renewal = new machine
each machine has two compo.
initially 1st compo is used.
then 2nd. after 2nd fails.
another machine will be put. (renewal)

$$EX] = \mu = \frac{1}{\mu_1} + \frac{1}{\mu_2}$$

$$E[Y(t)] = E[\text{remaining life of the machine part}]$$

Figure:

1st component working w/ prob. $p(t)$ 2nd component working at t. $1-p(t)$

- Now the problem is to find $p(t)$:

Prob $\underline{[\text{1st component is employed at } t\,]}$

- Let us define a stochastic process $\{X(t), t \geq 0\}$ such that

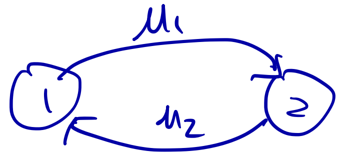
$$X(t) = \begin{cases} 1 & \text{if component 1 is employed at } t \\ 2 & \text{if component 2 is employed at } t \end{cases}$$

if 1st is still in use : remaining life is $\frac{1}{\mu_1} + \frac{1}{\mu_2}$

if 2nd is being used : " $\frac{1}{\mu_2}$

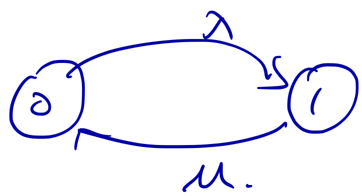
$$\therefore E[Y(t)] = \left(\frac{1}{\mu_1} + \frac{1}{\mu_2}\right) p(t) + \frac{1}{\mu_2}(1 - p(t))$$

- What 'stochastic process' does $X(t)$ follow? Continuous time Markov Chain



$$P_{11}(t) = \frac{\mu_1}{\mu_1+\mu_2} e^{-(\mu_1+\mu_2)t} + \frac{\mu_2}{\mu_1+\mu_2}$$

- Consider the following problem:



- We found $P_{00}(t)$ and $P_{10}(t)$ (We could also find $P_{01}(t)$ and $P_{11}(t)$)

$$P_{00}(t) = \frac{\mu}{\mu+\lambda} + \frac{\lambda}{\mu+\lambda}e^{-(\mu+\lambda)t}, \quad P_{10}(t) = \frac{\mu}{\mu+\lambda} - \frac{\mu}{\mu+\lambda}e^{-(\mu+\lambda)t}$$

$$E[Y(t)] = \frac{1}{\mu_2} + \frac{1}{\mu_1}\left[\frac{\mu_2}{\mu_1+\mu_2} + \frac{\mu_1}{\mu_1+\mu_2} e^{-(\mu_1+\mu_2)t}\right]$$

$$= \frac{1}{\mu_2} + \frac{\mu_2}{\mu_1(\mu_1+\mu_2)} + \frac{1}{\mu_1+\mu_2} e^{-(\mu_1+\mu_2)t}$$

$$\mu[m(t)+1] = t + E[Y(t)]$$

$$m(t) = \frac{t}{\mu} - 1 + \frac{E[Y(t)]}{\mu} \qquad \mu = E[X] = \frac{1}{\mu_1} + \frac{1}{\mu_2} = \frac{\mu_1+\mu_2}{\mu_1\mu_2}$$

$$m(t) = t\frac{\mu_1\mu_2}{\mu_1+\mu_2} - 1 + \frac{\mu_1\mu_2}{(\mu_1+\mu_2)\mu_2} + \frac{\mu_2 \, \mu_1\mu_2}{\mu_1(\mu_1+\mu_2)(\mu_1+\mu_2)}$$

$$+ \frac{\mu_1\mu_2}{(\mu_1+\mu_2)^2} e^{-(\mu_1+\mu_2)t}$$

$$= t\frac{\mu_1\mu_2}{\mu_1+\mu_2} - \frac{\mu_1\mu_2}{(\mu_1+\mu_2)^2}\left[1 - e^{-(\mu_1+\mu_2)t}\right]$$

Renewal Reward Process

- A reward is received each time a renewal occurs. Specifically, a reward of R_n is received at the time of the n^{th} renewal.
- $R_n, n \geq 1$ are independent and identically distributed.
- But R_n may depend on X_n, the length of the n^{th} renewal interval.
- The total reward earned by time t is $R(t)$.

$$R(t) = \sum_{n=1}^{N(t)} R_n$$

- Let $E[R] = E[R_n]$ and $E[X] = E[X_n]$.

Proposition 3. *If $E[R] < \infty$ and $E[X] < \infty$, then*

1. *with probability 1,* $\lim_{t\to\infty} \dfrac{R(t)}{t} = \dfrac{E[R]}{E[X]}$
2. $\lim_{t\to\infty} \dfrac{E[R(t)]}{t} = \dfrac{E[R]}{E[X]}$

Proof.
$$\frac{R(t)}{t} = \frac{\sum_{n=1}^{N(t)} R_n}{t} = \underbrace{\left(\frac{\sum_{n=1}^{N(t)} R_n}{N(t)}\right)}\left(\frac{N(t)}{t}\right)$$

$\downarrow$ $E[R]$ by strong law of large numbers

as $t \to \infty$

and by proposition 2.1

$$\frac{N(t)}{t} \to \frac{1}{\mu} = \frac{1}{E[X]} \text{ as } t \to \infty$$

Remark:

1. If we say that a cycle is completed every time a renewal occurs, then Proposition 3 states that the long-run average reward per unit time is equal to the expected reward earned during a cycle divided by the expected length of a cycle
2. This result is valid when the reward is earned gradually throughout the renewal cycle, also.

Example 4 (A Car Buying Model)**.** The life time of a car is continuous random variable having a distribution H and probability density h. Mr. Brown has a policy that he buys a new car as soon as his old one either breaks down or reaches age of T years. Suppose that a new car costs C_1 dollars and also that an additional cost of C_2 dollars is incurred whenever Mr. Brown's car breaks down. Under the assumption that a used car has no resale value, what is Mr. Brown's long-run average cost?

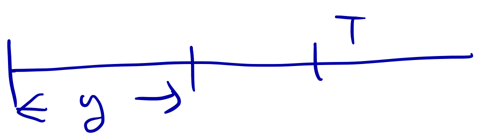
Find optimal T to minimize the long-run average cost.

Renewal = Brown buying a new car

a cycle completes every time Brown buying a new car

$$\frac{\text{long-run average cost}}{\text{unit time}} = \frac{E[\text{cost per cycle}]}{E[\text{length of a cycle}]}$$

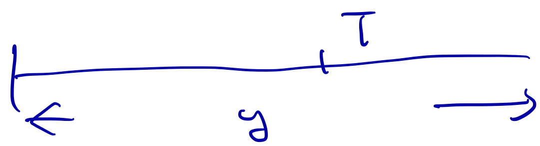
$y \equiv$ life of the car $\sim h_Y(y)$



(i) $y \le T$

cycle length : y

cost $C_1 + C_2$



(ii) $y > T$

cycle length : T

cost C_1

$$E[\text{length of a cycle}] = \int_0^T y \cdot h_Y(y)\,dy + \int_T^\infty T h_Y(y)\,dy$$

length $\begin{cases} Y & \text{if } Y \le T \\ T & \text{if } Y > T \end{cases}$

$$= \int_0^T y \cdot h_Y(y)\,dy + T(1 - H_Y(T))$$

$$E[\text{cost for cycle}] = C_1 P[y > T] + (C_1 + C_2) P[y \le T]$$

$$= C_1 (P[y > T] + P[y \le T]) + C_2 P[y \le T]$$

$$= C_1 + C_2 H_Y(T)$$

$\therefore$ Mr Brown's long-run average cost will be

$$\frac{C_1 + C_2 H(T)}{\int_0^T y\, h(y)\,dy + T[1 - H(T)]}$$

Now, suppose that $h(y) \sim$ uniform$(0, 10)$ $\quad h_Y(y) = \frac{1}{10}$

$C_1 = 3k \qquad C_2 = \frac{1}{2}K \qquad\qquad H(T) = \int_0^T \frac{1}{10}\,dy = \frac{T}{10}.$

if $T \le 10$, then $\displaystyle \frac{3 + \frac{1}{2}(T/10)}{\int_0^T \left(\frac{y}{10}\right)dy + T\left(1 - \frac{T}{10}\right)}$

$$= \frac{3 + T/20}{T^2/20 + (10T - T^2)/10} = \frac{60 + T}{20T - T^2}$$

let $g(T) = \dfrac{60 + T}{20T - T^2} \qquad g'(T) = \dfrac{(20T - T^2) - (60 + T)(20 - 2T)}{(20T - T^2)^2} = 0$

$\rightarrow 20T - T^2 = (60 + T)(20 - 2T)$

$\rightarrow T^2 + 120T - 1200 = 0$

$T \doteq 9.25$ or -129.25