SSIE 660: Stochastic Systems
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Note 2
Chapter 2. Random Variables

2.1 Random Variables

Random variables are real-valued functions defined on the sample space. Random variable are either discrete or continuous.

Ex.: Let X denote the random variable defined as the sum of two fair dice. (Discrete, Continuous)

$$P\{X = 5\} = P\{(1,4)\ (2,3)\ (3,2)\ (4,1)\} = 4/36$$

$X = \{2, 3, 4, 5, \cdots 12\}$

Ex.: Let X denote the life time of a car. (Discrete, Continuous)

$$P\{X = 5\} = 0$$

$$P\{2 \le X \le 5\} = \int_2^5 (\text{pdf})\, dt.$$

2.2 Discrete Random Variables

A random variable that can take on at most a countable number of possible values is said to be discrete.

Probability Mass Function of X:

$$p(a) = P\{X = a\}$$ notation

If X must assume one of the values $x_1, x_2, \ldots$ then,

$$\sum_{i=1}^{n} p(x_i) = 1$$

Also,

$$p(x_i) > 0 \text{ for some } i$$
$$p(x_i) = 0 \text{ for all other } i$$

Cumulative Distribution Function of X:

$$F(a) = \sum_{x_i \le a} p(x_i)$$

Ex.: Let X have a probability mass function given by

$$p(1) = 1/2, p(2) = 1/3, p(3) = 1/6$$

Then,

$$F(a) = \begin{cases} 0, & a < 1 \\ 1/2, & 1 \le a < 2 \\ 5/6, & 2 \le a < 3 \\ 1, & 3 \le a \end{cases}$$

2.2.1 The Bernoulli Random Variables

If an outcome is either a ”success” or a ”failure” and if we let $X = 1$ for success, $X = 0$ for failure, then the probability mass function of X is given by

$$p(0) = P(X = 0) = 1 - p$$
$$p(1) = P(X = 1) = p$$

where p is the probability of a success and $0 \le p \le 1$. Such random variable X is said to be a Bernoulli random variable.

2.2.2 The Binomial Random Variables

Binomial experiment conditions

1. fixed number of trials
2. either a ”success” with probability p or a ”failure” with probability of $1 - p$.
3. p is fixed for all trials.

Let X represent the number of success out of n trials, then X is a binomial random variable with parameters (n, p).

Probability mass function:

$$p(i) = \binom{n}{i} p^i (1-p)^{n-i}, i = 0, 1, ..., n$$

where

$$\begin{pmatrix} n \\ i \end{pmatrix} =_n C_i = \frac{n!}{(n-i)!i!} \quad (Combination)$$

Note that

$$_nP_i = \frac{n!}{(n-i)!} \quad (Permutation)$$

Example 1. Four fair coins are flipped. If the outcomes are assumed independent, what is the probability that two heads and two tails are obtained?

$\binom{4}{2} 1/2 \times 1/2 \times 1/2 \times 1/2$

Example 2. According to the Experian Automotive, 35% of all car-owning households have three or more cars. In a random sample of 400 car-owning households, what is the probability that 150 households have three or more cars?

$p = .35$

$\binom{400}{150}(.35)^{150}(.65)^{250}$

Example 3. Assume that the probability that a plane arrives on time is 75% for all arriving planes in Albuquerque airport. Out of 15 planes, what is the probability that

1. 8 planes arrive on time?

$\binom{15}{8}(.75)^{8}(.25)^{7}$

2. more than 12 planes arrive on time?

$$\binom{15}{12}(\quad)^{12}(\quad)^{3} + \binom{15}{13}(\quad)^{13}(\quad)^{2} + \binom{15}{14}(\quad)^{14}(\quad)^{1}$$
$$\binom{15}{15}(\quad)^{15}(\quad)^{0}$$

2.2.3 The Geometric Random Variables

Suppose each independent trial has a fixed probability of "success", p. If X represents the number of trial required until the first success, then X is said to be a geometric random variable.

Probability mass function:

$$p(n) = P(X = n) = (1-p)^{n-1}p,\ n = 1, 2, ...$$

2.2.3 The Poisson Random Variables

If a random variable X takes on one of the values 0, 1, 2, ..., then X is said to be a Poisson random variable with parameter λ, if for some $\lambda > 0$ the probability mass function is defined as follows.

Probability mass function:

$$p(i) = P(X = i) = e^{-\lambda}\frac{\lambda^i}{i!},\ \ i = 0, 1, 2, ...$$

- The Poisson random variable has a wide range of applications
- If n is large and p is small, then it can be used to approximate a binomial random variable. That is,
$$P(X = i) \approx e^{-\lambda}\frac{\lambda^i}{i!}$$
where X is a binomial random variable.

Example 4. Suppose that the number of typographical errors on a single page of a book has a Poisson distribution with parameter $\lambda = 1$. Calculate the probability that there is at least one error on a certain page.

$P\{X \geq 1\} = 1 - P\{X = 0\} = 1 - e^{-1} = .633$

Example 5. If the number of accidents occurring on a highway each day is a Poisson random variable with parameter $\lambda = 3$, what is the probability that no accidents occur today?

$P\{X=0\} = e^{-3} \doteq .05$

2.3 Continuous Random Variables

A random variable X is continuous if its set of possible values is uncountable.

Probability density function

- defined for all real $x \in (-\infty, \infty)$.
- for any set B of real numbers,

$$P\{X \in B\} = \int_B f(x)dx$$

- also,

$$P\{X \in (-\infty, \infty)\} = \int_{-\infty}^{\infty} f(x)dx = 1$$

Ex.

$$P\{X \in (a,b)\} = \int_a^b f(x)dx$$

$$P\{X = a\} = \int_a^a f(x)dx = 0$$

Cumulative distribution

$$F(a) = P\{X \in (-\infty, a)\} = \int_{-\infty}^{a} f(x)dx$$

such that

$$\frac{d}{da}F(a) = f(a)$$

2.3.1 The Uniform Random Variable

A random variable X is said to be uniformly distributed on the interval (α, β) if its probability density function is given by

$$f(x) = \begin{cases} \dfrac{1}{\beta - \alpha}, & \text{if } \alpha < x < \beta \\ 0, & \text{otherwise} \end{cases}$$

Example 6. Calculate the cumulative distribution function of a random variable uniformly distributed over (α, β).

$$F(a) = \begin{cases} 0 & a \le \alpha \\ \dfrac{a-\alpha}{\beta - \alpha} & \alpha \le a \le \beta \\ 1 & \alpha \geq 1 \end{cases}$$

Example 7. If X is uniformly distributed over (0, 10), calculate the probability that (a) $X < 3$, (b) $1 < X < 6$.

$$P(X < 3) = \int_0^3 \frac{1}{10}\, dx = 3/10$$

$$P(1 < x < 6) = 1/2$$

2.3.2 Exponential Random Variables

A continuous random variable whose probability density function is given, for some $\lambda > 0$, by

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}$$

is said to be an exponential random variable with parameter λ.

<u>Cumulative distribution</u>

$$F(a) = \int_0^a \lambda e^{-\lambda x} dx = 1 - e^{-\lambda a}, \;\; a \geq 0$$

Note that $F(\infty) = 1$.

2.3.3 Gamma Random Variables

A continuous random variable whose probability density function is given by, for some $\alpha > 0, \lambda > 0$

$$f(x) = \begin{cases} \dfrac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)}, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}$$

is said to be a gamma random variable with parameters α, λ.

Gamma function

$$\Gamma(\alpha) = \int_0^\infty e^{-x} x^{\alpha-1} dx$$

2.3.4 Normal Random Variables

A continuous random variable whose probability density function is given by, for some μ, σ^2

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}, \ -\infty < x < \infty$$

is said to be a normal random variable with parameters μ, σ^2.

$\underline{Y = \alpha X + \beta}$ will also be normally distributed with parameters $\alpha\mu + \beta$, and $\alpha^2\sigma^2$.

Ex. Let us define Y as $Y = (X - \mu)/\sigma$, then Y will be normally distributed with parameters 0 and 1. Y is said to have the standard normal distribution.