SSIE 660: Stochastic Systems
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Note 4
Chapter 2. Random Variables

2.5 Jointly Distributed Random Variables

Probability Distribution (Density) Function of the Sum of Two Independent Random Variables

Discrete Random Variables

Example 1. Given that:

x	1	2	3	4	5	6	7
$p_X(x)$	1/8	1/16	1/4	3/16	1/32	3/32	1/4
y	2	3	4	5	6	7	8
$p_Y(y)$	2/24	4/24	6/24	5/24	4/24	2/24	1/24

Let $Z = X + Y$, then what is the probability distribution of Z?

Let us first consider $P[Z = 5]$.

Writing the right hand side as a summation,
$P[Z = 5]$ =

In general: $P[Z = z] =$

Determining LL and UL

Let us consider all combinations of Y and X which yield $Z = 5$.

From the previous page

$$P[Z = 5] = \sum_{y=2}^{4} P[Y = y] * P[X = 5 - y]$$

In this, LL of Y =2, which is

Let us consider another example.

Example 2. Given that:

x	-2	-1	0	1	2	3	4
$p_X(x)$	1/10	1/10	2/10	2/10	2/10	1/10	1/10
y	0	1	2	3	4	5	6
$p_Y(y)$	2/24	4/24	6/24	5/24	4/24	2/24	1/24

In this example, find $P[Z = 5]$, where $Z = X + Y$.

$P[Z = 5] = (P[Y = 1] * P[X = 5 - 1]) + (P[Y = 2] * P[X = 5 - 2]) + (P[Y = 3] * P[X = 5 - 3]) + (P[Y = 4] * P[X = 5 - 4]) + (P[Y = 5] * P[X = 5 - 5]) + (P[Y = 6] * P[X = 5 - 6])$

In this example, LL of Y is:

Therefore, in general we have two candidates for LL:

How do we select the LL to be used in any problem?

The general formula for the LL is:

$$LL = \max(\qquad , \qquad)$$

Using the same approach, we will devise a general formula for UL.

The general formula for the UL is:

$$LL = \min(\qquad , \qquad)$$

Now we can write the general formula for $P[Z = z]$ as:

$$P[Z = z] =$$

<u>Continuous Random Variables</u> The formula for the density function $Z = X + Y$ can be written as

$$X \sim f_X(x), X_{min} < X < X_{max}$$
$$Y \sim f_Y(y), Y_{min} < Y < Y_{max}$$
$$Z \sim f_Z(z), Z_{min} < Z < Z_{max}$$
$$Z_{max} = X_{max} + Y_{max}, \;\; Z_{min} = X_{min} + Y_{min}$$

Then,

$$f_Z(z) = \int_{LL}^{UL} f_Y(y) f_X(z - y) dy$$

Example 3. Find the probability density function $Z = X + Y$, where X and Y follow the density functions given below:

$$f_X(x) = 1/4, \;\; 6 < X < 10$$
$$f_Y(y) = \lambda e^{-\lambda y}, \;\; y > 0$$

Range of Z:

$$f_Z(z) =$$

$LL = max(Y_{min}, z - X_{max})$

$UL = min(Y_{max}, z - X_{min})$

2.6 Moment Generating Functions

Definition: The r-th moment about the origin of the random variable X is:

Special cases (which we have studied so far):

Mean of X is $E(X)$

$E(X^2) =$

Hence $Var(X) =$

Definition: The Moment-Generating Function (MGF) of the random variable X, denoted by $\phi(t)$ is the expected value of e^{tx}.

(the sum of the integral has to converge; otherwise, the moment generating function does not exist.)

If the MGF of a random variable X exists, then it can be used to generate all the moments of X.

Example 4. Find the Moment Generating Function of the Exponential Density Function with mean $1/\lambda$. Using the MGF, find the mean and the variance of the exponential density function. The exponential density function is as follows:

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}$$

Example 5. The MGF of the Binomial Distribution with parameters n and p is:

$$\phi(t) = (pe^t + q)^n$$

where $q = 1 = p$. Using this MGF, find the mean and variance of binomial distribution.

Some results related to MGF:

1. Let $phi(t)$ be the MGF of X and $Y = X + a$, where a is a constant, then MGF of Y is:

   Example 6. The MGF of the binomial distribution with parameters n and p is

   $$\phi(t) = (pe^t + q)^n$$

   where $q = 1 - p$. What is the MGF of $Y = X + 5$?

2. $Y = X_1 + X_2 + \cdots + X_n$ where $X_1, \cdots, X_n$ are independent random variables with MGFs $\phi_{X_1}(t), \cdots, \phi_{X_n}(t)$, respectively. Then the MGF of Y is:

   Example 7. Let X be the binomial random variable with parameters n_1 and p_1 and U be a binomial random variable with parameters n_2 and p_2. Find the MGF of Y where $Y = X + U$.