discrete time : state of the system $\equiv X_n$ (at n^{th} epoch) $n=0,1,\dots$

$P[X_{n+1}=j \mid X_n=i, X_{n-1}, \dots, X_0]$
$= P[X_{n+1}=j \mid X_n=i]$

SSIE 660: Stochastic Systems
Dr. Sung H. Chung
Note 13
Chapter 6. Continuous-Time Markov Chain

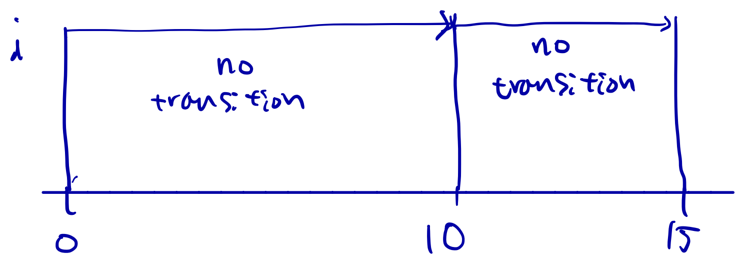
<u>Continuous-Time Markov Chain</u>

$$P[X(t+s)=j|X(s)=i, X(u)=i_1, ...] = P[X(t+s)=j|X(s)=i]$$

where $0 \le u \le s$.

If $P[X(t+s)=j|X(s)=i] = P_{ij}(t)$, then a Markov chain is said to have homogeneous or stationary transition probabilities.

- Consider a continous-time Markov chain. Assume that it enters state i at time 0 and that it does not leave state i during the next ten minutes. (That is, no transition during the next ten minutes.) What is the probability that the process will not leave state i during the following next five minutes?



$P[X(15)=i \mid X(10)=i]$: prob. of no transition during 5 min.

Let T_i denote the amount of time the process stays in state i before making a transition into a different state.

- Required probability is: $P[X(15)=i|X(10)=i] =$ $P[T_i > 15 \mid T_i > 10]$

  $= P[T_i > 5]$

- In general,
  $P[T_i > s + t | T_i > s] =$ $P[T_i > t]$ for all $s, t \geq 0$
- Hence, $T_i \sim$ exponential dist memoryless

Alternate way of defining Continuous-Time Markov Chain

It is a stochastic process having the property that each time it enters state i,

1. The amount of time it spends in that state before making a transition into a different state is *exponential* with mean $1/v_i$.
2. When the process leaves state i, it next enters state j with some probability, say P_{ij}. (Note that P_{ij} has no meaning unless the process leaves state i. It is *not* the probability of going from i to j within an interval of length t, which is $P_{ij}(t)$)
   Thus,

$$P_{ii} = 0 \qquad\qquad P_{ii}(t) \neq P_{ii}$$

$$\sum_j P_{ij} = 1$$

3. The amount of time the process spends in state i and the next state it visits must be independent.

   Mean length of time the process spends in state i = $\frac{1}{v_i}$

Poisson process

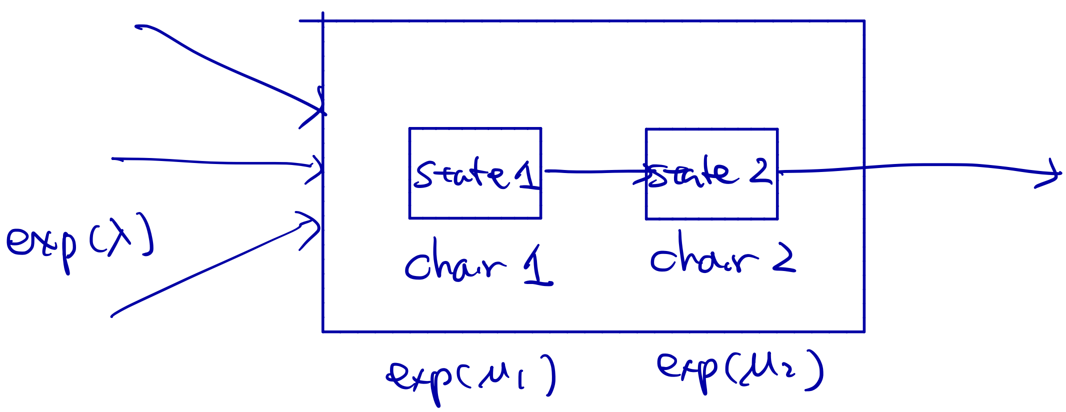
$v_i = \lambda$

$P_{i,i+1} = 1$

v_i : mean rate of transactions out of state i

Note: In Continuous-Time Markov Chains, it is important to define v_i (and $1/v_i$), for every state i, after defining the states. In fact, the feasibility of defining v_i (and $1/v_i$) must be taken into considerations, while defining the states. Also, all $P'_{ij}s$ must be defined.

Example 1. Consider a shoeshine establishment consisting of two chairs - chair 1 and chair 2. A customer upon arrival goes initially to chair 1 where his shoes are cleaned and polish is applied. After this is done, the customer moves on to chair 2, where polish is buffed. The service times at the two chairs are assumed to be independent random variables which are exponentially distributed with respective rates μ_1 and μ_2. Suppose that potential customers arrive in accordance with a Poisson process having rate λ, and that a potential customer will only enter the system if both chairs are empty. Is this a continuous Markov chain?



Definition of states:

(A) Number of customers in the system:

State (i)	Interpretation
0	0 customer in the system
1	1 "

Let us consider state $i = 1$.

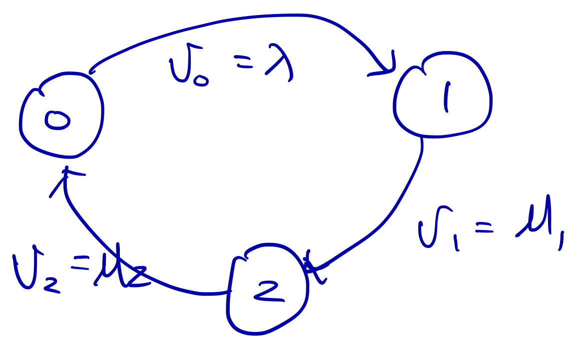
$\frac{1}{v_i} = \frac{1}{v_1}$ = mean length of time during which there is one customer in the system.

can't be defined, unless the stage of service (1 or 2) is known.

(B) State definition must include the states of service of customers in the system.

State (i)	Interpretation
0	0 customer in the system
1	1 " in stage 1
2	1 " " 2

<u>Transition Graph:</u>



<u>i=0:</u>

$$P_{01} = 1 \quad ; \quad P_{02} = 0$$

$$\frac{1}{v_0} = \frac{1}{\lambda} \quad ; v_0 = \lambda$$

i=1:

$$P_{12} = 1 \quad ; \quad P_{10} = 0$$

$$\frac{1}{v_1} = \frac{1}{\mu_1} \quad ; v_1 = \mu_1$$

i=2:

$$P_{20} = 1 \quad ; \quad P_{21} = 0$$

$$\frac{1}{v_2} = \frac{1}{\mu_2} \quad ; v_2 = \mu_2$$

Example 2. Consider two machines that are maintained by a single repairman. Machine i functions for an exponential time with rate μ_i before breaking down, i = 1, 2. The repair times (for either machine) are exponential with rate μ. Can we analyze this as a continuous Markov chain?

State Definition A:

State (i)	Interpretation
0	Both machines working
1	machine #1 up, machine #2 down and in service
2	machine #2 up, machine #1 down, in service
3	Both machines are down

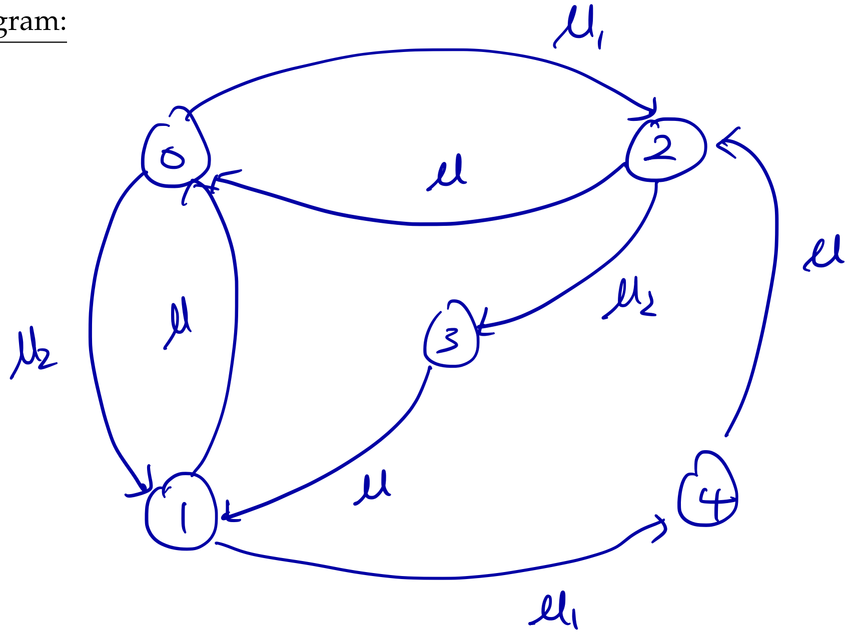
Problem with the definition:

P_{31}, P_{32} can't be given unless which machine in state 3 being serviced is known

State Definition B:

State (i)	Interpretation
0	Both machines working
1	m #1 up , 2 down
2	2 up 1 down
3	both down 1 in service
4	" 2 in service

Diagram:



why? $\min(X_1, X_2)$

$v_0 = \mu_1 + \mu_2$; $P_{01} = \dfrac{\mu_2}{\mu_1 + \mu_2}$; $P_{02} = \dfrac{\mu_1}{\mu_1 + \mu_2}$

$v_1 = \mu + \mu_1$; $P_{10} = \dfrac{\mu}{\mu + \mu_1}$; $P_{14} = \dfrac{\mu_1}{\mu + \mu_1}$

$v_2 = \mu + \mu_2$; $P_{20} = \dfrac{\mu}{\mu + \mu_2}$; $P_{23} = \dfrac{\mu_2}{\mu + \mu_2}$

$v_3 = \mu$; $P_{31} = 1$

$v_4 = \mu$; $P_{42} = 1$

Birth and Death Processes

Characteristics:

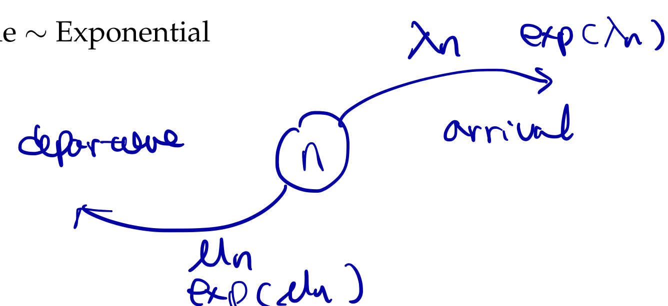
State: the number of 'people' in the system at any time.

Arrival rate when there are n 'people' in the system = λ_n *(birth rate).*

Inter-arrival time $\sim$ Exponential

Departure rate when there are n 'people' in the system = μ_n *(death rate).*

Inter-departure time $\sim$ Exponential



- It is a continuous-time Markov chain with states { 0, 1, 2, ... } .
- Transitions from state n may go only to either $(n-1)$ or $(n+1)$.

Diagram

$v_0 = \lambda_0$; $v_1 = \lambda_1 + \mu_1$; $v_2 = \lambda_2 + \mu_2$ ...

$$v_i = \lambda_i + \mu_i$$

$$P_{01} = 1$$

$$P_{10} = \frac{\mu_1}{\lambda_1 + \mu_1} \quad ; P_{12} = \frac{\lambda_1}{\lambda_1 + \mu_1}$$

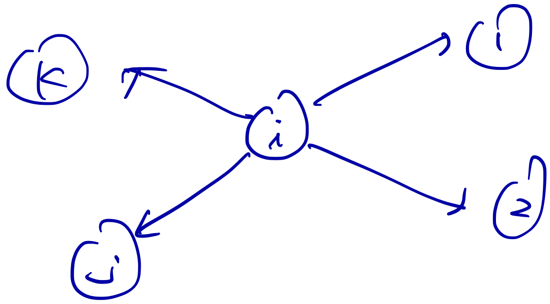
$$P_{21} = \frac{\mu_2}{\lambda_2 + \mu_2} \quad ; P_{23} = \frac{\lambda_2}{\lambda_2 + \mu_2}$$

$$P_{i,i-1} = \frac{\mu_i}{\lambda_i + \mu_i} \quad ; P_{i,i+1} = \frac{\lambda_i}{\lambda_i + \mu_i}$$

- It a system has arrivals (births) only, it has a pure birth process and it has departures (deaths) only, it has a pure death process.

<u>The Kolmogorov Differential Equations</u>

- Consider state i and the states to which the process could move from i.



- Let v_i be the total rate of transitions out of i and P_{ij} be the probability that the process moves to state j, once it gets out of state i.
- Let q_{ij} be the rate at which the process moves from i to j.

$$q_{ij} = v_i \times P_{ij}$$

$$v_i = \sum_j q_{ij} = \sum_j v_i \times P_{ij} = v_i \sum_j P_{ij} = v_i$$

$$P_{ij} = q_{ij} / v_i$$

- Our next task is to find $P_{ij}(t)$, which is

$$P_{ij}(t) = P[X(t+s) = j \mid X(s) = i)$$

using v_i, P_{ij}, and q_{ij}.

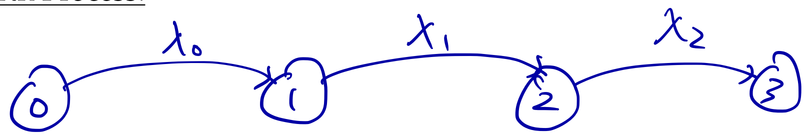
Lemma 3. 1. $\lim_{h \to 0} \dfrac{1 - P_{ii}(h)}{h} = v_i$.

2. $\lim_{h \to 0} \dfrac{P_{ij}(h)}{h} = q_{ij}$.

Theorem 4 (Kolmogorov's Backward Equations.)**.** *For all states i, j and times $t \geq 0$,*

$$P'_{ij}(t) = \sum_{k \neq i} q_{ik} P_{kj}(t) - v_i P_{ij}(t)$$

Pure Birth Process:



$v_i = \lambda_i$

$q_{ij} = \lambda_i = V_i$ all i & $j = i+1$

$q_{ij} = 0$ for all $j \neq i+1$

$\sum_{k \neq i} q_{ik} P_{kj}(t) = q_{i,1} P_{1,j}(t) + q_{i,2} P_{2,j}(t) + \cdots + q_{i,i-1} P_{i-1,j}(t) + q_{i,i+1} P_{i+1,j}(t) + \cdots$

As the only possible transition from i is to $i+1$ in a pure birth process,

$\sum_{k \neq i} q_{ik} P_{kj}(t) = q_{i,i+1} P_{i+1,j}(t) = \lambda_i P_{i+1,j}(t)$

So, the backward equation is

$P'_{ij}(t) = \lambda_i \left[P_{i+1,j}(t) - P_{ij}(t) \right]$

Birth and Death Process:

$v_0 = \lambda_0$ $\quad ; v_i = \lambda_i + \mu_i$

$$P_{ij} = \begin{cases} \dfrac{\lambda_i}{\mu_i + \lambda_i} & j = i+1 \qquad P_{01} = 1 \\ \dfrac{\mu_i}{\mu_i + \lambda_i} & j = i-1 \end{cases}$$

In a birth and death process, the only transitions possible from i are to $i-1$ and $i+1$. Therefore,

$\sum_{k \neq i} q_{ik} P_{kj}(t) = q_{i,i-1} P_{i-1,j}(t) + q_{i,i+1} P_{i+1,j}(t)$

$$q_{i,i-1} = v_i P_{i,i-1} = \begin{cases} 0 & \text{if } i = 0 \\ (\lambda_i + \mu_i) \dfrac{\mu_i}{\mu_i + \lambda_i} = \mu_i & i > 0 \end{cases}$$

$$q_{i,i+1} = v_i P_{i,i+1} = \begin{cases} \lambda_0 \cdot 1 = \lambda_0 & i = 0 \\ (\lambda_i + \mu_i) \dfrac{\lambda_i}{\mu_i + \lambda_i} = \lambda_i, & i \geq 1 \end{cases}$$

So, the backward equation is

$\underline{i = 0}$: $P'_{0j}(t) = \sum_{k \neq 0} q_{0k} P_{k,j}(t) - v_0 P_0(t)$

$$= q_{01} P_{1j}(t) - v_0 P_{0j}(t)$$

$$= \lambda_0 P_{1j}(t) - \lambda_0 P_{0j}(t) = \lambda_0 [P_{1j}(t) - P_{0j}(t)]$$

$\underline{i > 0}$: $P'_{ij}(t) = \sum_{k \neq i} q_{ik} P_{kj}(t) - \cancel{v_i P_{ij}(t)} - v_i P_{ij}(t)$

$$= q_{i,i-1} P_{i-1,j}(t) + q_{i,i+1} P_{i+1,j}(t) - v_i P_{ij}(t)$$

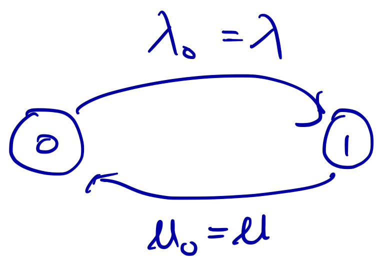
$$= \mu_i P_{i-1,j}(t) + \lambda_i P_{i+1,j}(t) - (\lambda_i + \mu_i) P_{ij}(t)$$

Example of obtaining transition probabilities using Kolmogorov's backward equations.

Example 5. Consider a machine that works for an exponential amount of time having mean $1/\lambda$ before breaking down; and suppose that it takes an exponential amount of time having mean $1/\mu$ to repair the machine. If the machine is in working condition at time 0, then what is the probability that it will be working at time t=10?

State (i)	Interpretation
0	Machine working
1	Machine down and in service

Diagram:



$v_0 = \lambda \quad v_1 = \mu$

$P_{01} = 1 \quad P_{10} = 1$

- It is a Birth and Death process with
- We want to find $P_{00}(t)$

The Kolmogorov's backward equations for a birth and death process is:

$$P'_{0j}(t) = \lambda_0[P_{1j}(t) - P_{0j}(t)] \quad a$$

$$P'_{ij}(t) = \lambda_i P_{i+1,j}(t) + \mu_i P_{i-1,j}(t) - (\lambda_i + \mu_i)P_{ij}(t) \quad b$$

Hence, the Kolmogorov's backward equations for this system are:

j=0: in ⓐ

$$P'_{00}(t) = \lambda_0[P_{10}(t) - P_{00}(t)] \qquad (c)$$

$i=1,\ j=0$ in ⓑ $\quad P'_{10}(t) = \lambda_1 P_{20}(t) + \mu_1 P_{00}(t) - (\lambda_1+\mu_1)P_{10}(t)$

$$= \mu P_{00}(t) - \mu P_{10}(t) \qquad (e)$$

$$(c) \rightarrow P'_{00}(t) = \lambda[P_{10}(t) - P_{00}(t)] \quad -(g) \quad \times \mu$$

$$P'_{10}(t) = \mu[P_{00}(t) - P_{10}(t)] \quad -(h) \quad \times \lambda$$

$$\mu P'_{00}(t) + \lambda P'_{10}(t) = 0$$

$$\mu \int_0^t P'_{00}(t)dt + \lambda \int P'_{10}(t)dt = \int_0^t 0\, dt + C = C$$

$$\mu P_{00}(t) + \lambda P_{10}(t) = C$$

$$t=0 \quad \mu \cdot 1 + \lambda \cdot 0 = C$$

$$\therefore \mu P_{00}(t) + \lambda P_{10}(t) = \mu \quad \Rightarrow \lambda P_{10}(t) = \mu[1 - P_{00}(t)]$$

plug into (c) $\quad P'_{00}(t) = \lambda[P_{10}(t) - P_{00}(t)]$

$$= \lambda P_{10}(t) - \lambda P_{00}(t)$$

$$= \mu[1 - P_{00}(t)] - \lambda P_{00}(t)$$

$$= \mu - P_{00}(t)(\lambda+\mu)$$

Let $h(t) = \frac{\mu}{\lambda+\mu} - P_{00}(t) \Rightarrow P_{00}(t) = \frac{\mu}{\lambda+\mu} - h(t)$

$$h'(t) = -P'_{00}(t) = -\mu + (\lambda+\mu) P_{00}(t) = -\mu + (\lambda+\mu)\left[\frac{\mu}{\lambda+\mu} - h(t)\right]$$

$$= -(\lambda+\mu) h(t)$$

$$\therefore \frac{h'(t)}{h(t)} = -(\lambda+\mu)$$

$$\int \frac{h'(t)}{h(t)} = -\int (\lambda+\mu)dt + K$$

$$\ln h(t) = -(\lambda+\mu)t + K$$

$$h(t) = e^{-(\lambda+\mu)t+K} \qquad \therefore\ P_{00}(t) = \frac{\mu}{\lambda+\mu} - e^{-(\lambda+\mu)t+K}$$

$$P_{00}(0) = 1 = \frac{\mu}{\lambda+\mu} - e^{K} = \frac{\mu}{\lambda+\mu} - \bar{K}$$

$$\bar{K} = -1 + \frac{\mu}{\lambda+\mu} = \frac{-\lambda}{\lambda+\mu} = e^{K}$$

$$P_{00}(t) = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t}$$

Similarly, $P_{10}(t) = \dfrac{\mu}{\lambda+\mu} - \dfrac{\mu}{\mu+\lambda} e^{-(\lambda+\mu)t}$

$P_{01}'(t)$: $j=1$ in (a)

$P_{11}'(t)$: $i=j=1$ in (b)

$P_{00}(t)$ $\quad P_{10}(t)$

$P_{01}(t)$ $\quad P_{11}(t)$

backword $P_{ij}'(t) = \sum_{k \neq i} q_{ik} P_{kj}(t) - v_i P_{ij}(t)$

The Kolmogorov's forward equation:

$$P'_{ij}(t) = \sum_{k \neq i} q_{kj} P_{i,k}(t) - v_j P_{ij}(t)$$