SSIE 660: Stochastic Systems
Dr. Sung H. Chung
Note 7
Chapter 4. Discrete Markov Chains

- In many business, math, engineering, and scientific problems, the change of some measure of a process with respect to time might be of interest to the analyst. This measure is a random variable if the exact value that it will take cannot be predicted with certainty.

- As an example, consider an inventory problem. The demand of a certain item per day is a random variable with the following probability distribution.

Demand, d	Probability
0	0.3
1	0.4
2	0.2
3	0.1

- The ordering policy is as follows: If the inventory at the end of the day is zero, order three items; otherwise do not order. Assume that the replenishment is instantaneous for the sake of simplicity.

- Now consider the inventory at the beginning of each day. It is a random variable because its value cannot be predicted with certainty before the inventory at the beginning of the previous day and the actual demand of the previous day are known.

- At the same time, the probability distribution of the inventory at the beginning of any day, $n+1$, can be determined once the inventory at the beginning of the previous day, n, is known. This is due to the ordering policy used that makes the inventory at the beginning of any day a function of the inventory at the beginning of the previous day and the actual demand during the previous day only.

- The inventories at the beginning of any other preceding days, that is day $n-1, n-2, ..., 1$, do not affect the probability distribution of the inventory at the beginning of the day $n+1$.

- This is a random variable, which changes at specific points in time, and the probability distribution of which at any point in time depends only upon the value it takes at the immediately preceding time instant.

- It can be easily seen that the inventory at the beginning of any given day can be one of 1, 2, or 3 only. The random variable representing the inventory at the beginning of day n is denoted by

- $X_n = 1, 2, ...,$ and a state number, j_n, is assigned to the value that it takes where $j_n = 1, 2, 3, ...$ The sequence of random variables $X_1, X_2, ..., X_n, X_{n+1}, ...,$ is a special stochastic process, which will be defined later.

Table. Inventory

Inventory at the beginning of the n^{th} day	demand	prob. $D=d$	Inven. at the end of n^{th} day	order quantity	Inventory at the beginning of n^{th} day
1	0	.3	1	0	1 (.3)
	1	.4	0	3	3 (.7)
	2	.2	0		
	3	.1	0		
2	0	.3	2	0	2 (.3)
	1	.4	1		1 (.4)
	2	.2	0	3	3 (.3)
	3	.1	0		
3	0	.3	3	0	3 .3
	1	.4	2	0	2 .4
	2	.2	1	0	1 .2
	3	.1	0	3	3 .1

.4

Probability of X_{n+1} depends only on X_n and demand

not on $X_{n-1}, X_{n-2}, ...$

This is a special Stochastic Process called Markov Chain.

$P[X_{n+1} = 3 | X_n = 2, X_{n-1} = 1, X_{n-2} = 3, ..., X_0 = 1] =$ $P[X_{n+1} = 3 | X_n = 2] = .3$

The above statement means that the conditional probability of any future state X_{n+1}, given the past states $X_0, X_1, X_2, ..., X_{n-1}$ and present state X_n is

$P(X_{n+1} = j | X_n = i, X_{n-1}, ... X_0\} = P(X_{n+1} = j | X_n = i) = p(i, j)$

Definition 1. *The stochastic process consisting of $X_0, X_1, ..., X_{n-1}, X_n, X_{n+1}, ...$ is called a Markov Chain if and only if*

$$P[X_{n+1} = j | X_0, X_1, ..., X_{n-1}, X_n = i] = P(X_{n+1} = j \mid X_n = i)$$

for all i and j.

- If the conditional probability $P[X_{n+1} = j | X_n = i]$ is the same for all values of n (that is $P[X_{2+1} = j | X_2 = i] = P[X_{1000+1} = j | X_{1000} = i]$) then the Markov Chain is said to be **time-homogenous**.

- The conditional probability $P[X_{n+1} = j | X_n = i]$ in time-homogenous Markov Chains is usually denoted by P_{ij}, $P_{i,j}$ or $P(i, j)$, and is called the **One-Step Transition Probability** . Here, i and j are called the **states of the process** and they need not have any physical meaning.

- For example in our inventory problem, the states are

  The numbers used to represent the states need not be equal to the possible values of inventories.

- In the notation for the one-step transition probability $P_{i,j}$,

  i denotes the starting state

  j denotes the ending state

- As $P_{i,j}$ are probabilities, the following conditions must be met.

  $\sum_j P_{ij} = 1 \quad \forall i \qquad 0 \leq P_{ij} \leq 1$ for all i, j

- In our example, we will denote the states by 1, 2, and 3 (for inventories 1, 2, and 3, respectively).

  Then, $P_{32} =$ .4 ; $P_{12} =$ 0

Calculating probabilities:

$P_{11} = .3$
$P_{12} = 0$
$P_{13} = .7$
$P_{21} = .4$
$P_{22} = .3$
$P_{23} = .3$
$P_{31} = .2$
$P_{32} = .4$
$P_{33} = .4$

- It is customary to arrange the one-step probabilities P_{ij} in a square matrix, called the

Transition Probability Matrix

denoted by P. In this matrix, the rows represent the starting state i and the columns represent the ending state j

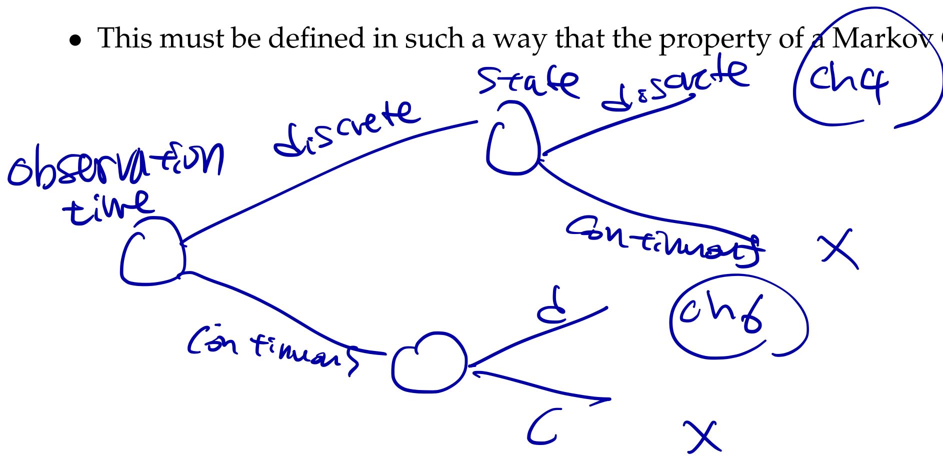
- In P, the order of row and column states must be the same. In our example, P is

$$P = \begin{bmatrix} .3 & 0 & .7 \\ .4 & .3 & .3 \\ .2 & .4 & .4 \end{bmatrix}$$

- There are two characteristics associated with each Markov Chain. These are:

  1. time . (observation time instant)
  2. state of the system

- This must be defined in such a way that the property of a Markov Chain is satisfied:



Example 2 (Transforming a Process into a Markov Chain)**.** Suppose that whether or not it rains today depends on previous weather conditions through the last two days. Specifically, suppose that if it has rained for the past two days, then it will rain tomorrow with probability 0.7; if it rained today but not yesterday, then it will rain tomorrow with probability 0.5; if it rained yesterday but not today, then it will rain tomorrow with probability 0.4; if it has not rained in the past two days, then it will rain tomorrow with probability 0.2.

Observation Epochs: beginning of a day

States: (definition) weather condition during the last two consecutive days

States			Numbering of states (i, j)
RR	(1 1)	.7	0
NR	(0 1)	.5	1
RN	(1, 0)	.4	2
NN	(0, 0)	.2	3

0 RR → RR .7 0
→ RN .3 2

1 NR → RR .5 0
→ RN .5 2

2 RN → NR .4 1
→ NN .6 3

3 NN → NR .2 1
NN .8 3

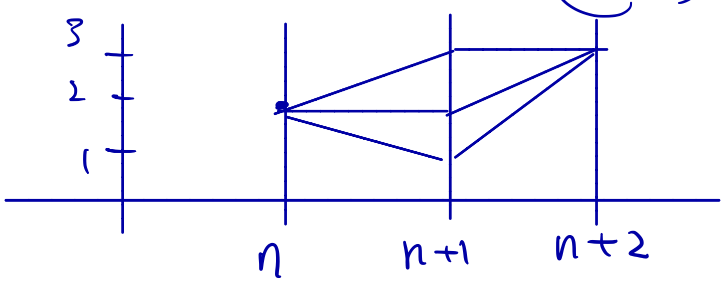
Transition Probability Matrix:

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} .7 & 0 & .3 & 0 \\ .5 & 0 & .5 & 0 \\ 0 & .4 & 0 & .6 \\ 0 & .2 & 0 & .8 \end{bmatrix} \end{array}$$

n-step transition: Let us revisit our example (inventory problem). We obtained the one-step transition probabilities, which are the probabilities of change in the inventories at the beginning of a day given to some value within one day. Now let us obtain the probabilities of changes in the inventory levels in two days.

Probabilities of going from State 2 to State 3 in 2 days = $P(X_{n+2} = 3 \mid X_n = 2)$

$= P(X_2 = 3 \mid X_0 = 2) = P^2_{23} \neq P_{23} \times P_{23}$



$P^2_{23} = P_{21} \cdot P_{13} + P_{22} P_{23} + P_{23} \cdot P_{33}$

In general, $P^2_{ij} = \sum_k P_{ik} P_{kj}$

These two step probabilities can be obtained from

P^2= P · P (matrix)

$$P = \begin{bmatrix} 0.3 & 0 & 0.7 \\ 0.4 & 0.3 & 0.3 \\ 0.2 & 0.4 & 0.4 \end{bmatrix}; P^2 = \begin{bmatrix} 0.3 & 0 & 0.7 \\ 0.4 & 0.3 & 0.3 \\ 0.2 & 0.4 & 0.4 \end{bmatrix} \begin{bmatrix} 0.3 & 0 & 0.7 \\ 0.4 & 0.3 & 0.3 \\ 0.2 & 0.4 & 0.4 \end{bmatrix} = \begin{bmatrix} 0.23 & 0.28 & 0.49 \\ 0.3 & 0.21 & 0.49 \\ 0.30 & 0.28 & 0.42 \end{bmatrix}$$

- The probability of going from 1 to 3 in 2 days is .49
- The probability of going from i to j in n transitions (or n steps) is denoted by P^n_{ij}