SSIE 660: Applied Stochastic Processes
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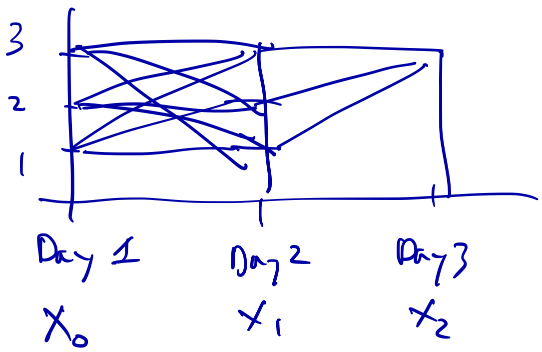
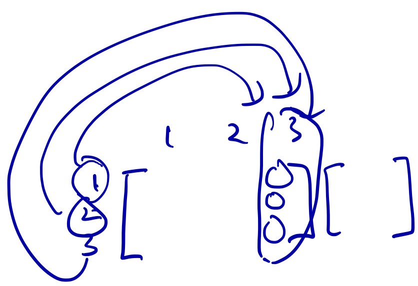
Note 8
Chapter 4. Discrete Markov Chains

Unconditional Probabilities

Example: What is the probability that the inventory at the beginning of the 3rd day is 3? That is $P[X_2 = 3]$?





Beginning step is '0', not '1'

$$P^2 = \begin{bmatrix} 0.23 & 0.28 & 0.49 \\ 0.3 & 0.21 & 0.49 \\ 0.30 & 0.28 & 0.42 \end{bmatrix}$$

We cannot find the unconditional probability $P[X_2 = 3]$, unless we know

$P[X_0 = i] \quad i = 1, 2, 3$

let's denote $\alpha_1 \quad \alpha_2 \quad \alpha_3$

$P[X_2 = 3] = \sum_{i=1}^{3} P[X_2 = 3|x_0 = i]P[X_0 = i]$

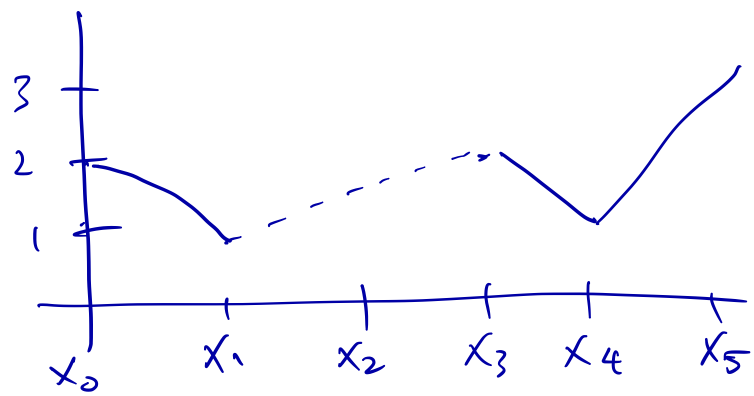
Let $\alpha_1 = 0.3, \alpha_2 = 0.6, \alpha_3 = 0.1$. Then, $P[X_2 = 3] = \begin{pmatrix} .49 \\ .49 \\ .42 \end{pmatrix}^T \begin{pmatrix} .3 \\ .6 \\ .1 \end{pmatrix} = .483.$

In general,

$P[X_n = j] = \sum_i P[X_j = j | X_0 = i] P[X_0 = i]$

Other Probabilities

$P[X_5 = 3, X_4 = 1, X_3 = 2, X_1 = 1 | X_0 = 2] =$



$= P_{21} * P_{12}^{(2)} * P_{21} * P_{13} = .4 * .28 * .4 * .7 = .0314$ → Conditional probability

$P[X_4 = 3, X_3 = 1, X_2 = 1] =$
Recall: $(\alpha_1 = 0.3, \alpha_2 = 0.6, \alpha_3 = 0.1)$

$$\sum_{i=1}^{3} P[X_4 = 3, X_3 = 1, X_2 = 1 | X_0 = i] P[X_0 = i]$$

$$= P_{11}^{2} \cdot P_{11} \cdot P_{13} \alpha_1 + P_{21}^{2} \cdot P_{11} \cdot P_{13} \alpha_2 + P_{31}^{2} P_{11} P_{13} \alpha_3$$

$$= .0586$$

Classification of States:
(A) Transition probability matrix in our inventory example.

$$P = \begin{bmatrix} 0.3 & 0 & 0.7 \\ 0.4 & 0.3 & 0.3 \\ 0.2 & 0.4 & 0.4 \end{bmatrix}, P^4 = \begin{bmatrix} 0.284 & 0.26 & 0.456 \\ 0.279 & 0.265 & 0.456 \\ 0.279 & 0.260 & 0.461 \end{bmatrix}$$

$$P^8 = \begin{bmatrix} 0.280 & 0.262 & 0.458 \\ 0.280 & 0.262 & 0.458 \\ 0.280 & 0.262 & 0.458 \end{bmatrix}, P^{12} = \begin{bmatrix} 0.280 & 0.262 & 0.458 \\ 0.280 & 0.262 & 0.458 \\ 0.280 & 0.262 & 0.458 \end{bmatrix}$$

(B)

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, P^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, P^{12} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P^{13} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, P^{14} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

(C)

$$P = \begin{bmatrix} 0.3 & 0.3 & 0.4 \\ 0.4 & 0.2 & 0.4 \\ 0 & 0 & 1 \end{bmatrix}, P^2 = \begin{bmatrix} 0.21 & 0.15 & 0.64 \\ 0.20 & 0.16 & 0.64 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P^{12} = \begin{bmatrix} 0.001 & 0.0009 & 0.998 \\ 0.001 & 0.0009 & 0.998 \\ 0 & 0 & 1 \end{bmatrix}, P^{16} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

(D)

$$P = \begin{bmatrix} 0.2 & 0.8 & 0 & 0 \\ 0.4 & 0.6 & 0 & 0 \\ 0 & 0 & 0.3 & 0.7 \\ 0 & 0 & 0.5 & 0.5 \end{bmatrix}, P^2 = \begin{bmatrix} 0.36 & 0.64 & 0 & 0 \\ 0.32 & 0.68 & 0 & 0 \\ 0 & 0 & 0.44 & 0.56 \\ 0 & 0 & 0.40 & 0.60 \end{bmatrix}$$

$$P^{16} = \begin{bmatrix} 0.333 & 0.667 & 0 & 0 \\ 0.333 & 0.667 & 0 & 0 \\ 0 & 0 & 0.417 & 0.583 \\ 0 & 0 & 0.417 & 0.583 \end{bmatrix}, P^{17} = \begin{bmatrix} 0.333 & 0.667 & 0 & 0 \\ 0.333 & 0.667 & 0 & 0 \\ 0 & 0 & 0.417 & 0.583 \\ 0 & 0 & 0.417 & 0.583 \end{bmatrix}$$

- It can be seen that in some Markov Chains, multiplying P by itself a large number of times will result in identical rows (A and C) - independent of starting states.

- This means that n-step transition probability does not depend on starting state.
- These probabilities are called steady state or stationary probabilities.
- In B and D, the rows are not identical; this means that only certain type of Markov Chains have steady state probabilities. This depends upon the types of states a Markov Chain has, which we will study next.

Classification of States:

1. State j is "accessible" from state i, if

$$P_{ij}^{n} > 0, \text{ for some } n \geq 0.$$

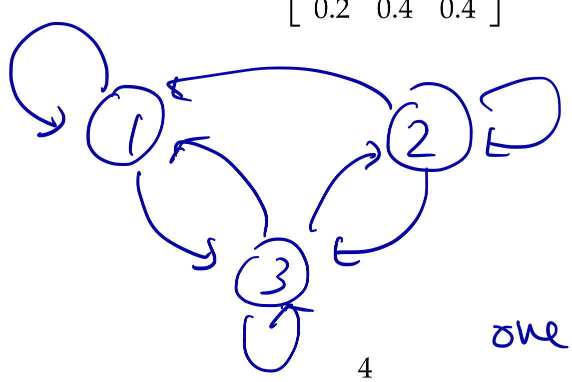
   That is, process or system will enter state j, at some time in the future (some n), starting in state i now.

2. Two states, i and j, that are accessible to each other are said to "communicate each other"
   (a) If $i \leftrightarrow j$, then $j \leftrightarrow i$.
   (b) If $i \leftrightarrow j$, and $j \leftrightarrow k$, then $i \leftrightarrow k$.
   (c) A state communicates with itself if $i \leftrightarrow i$.
3. If $i \leftrightarrow j$, then i, j belong to the same class.
4. A Markov Chain could have more than one class of states.
5. If a Markov Chain has only one class, then it is said to be irreducible.

Example 1.

$$P = \begin{bmatrix} 0.3 & 0 & 0.7 \\ 0.4 & 0.3 & 0.3 \\ 0.2 & 0.4 & 0.4 \end{bmatrix}$$

$1 \leftrightarrow 2$ $(2 \leftrightarrow 1)$



$2 \leftrightarrow 3$ $(3 \leftrightarrow 2)$

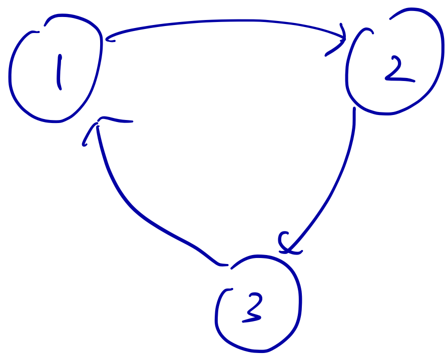
$1 \leftrightarrow 3$ $(3 \leftrightarrow 1)$

one class $\{1, 2, 3\}$

irreducible

Example 2.

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$



$1 \leftrightarrow 2$

$2 \leftrightarrow 3$

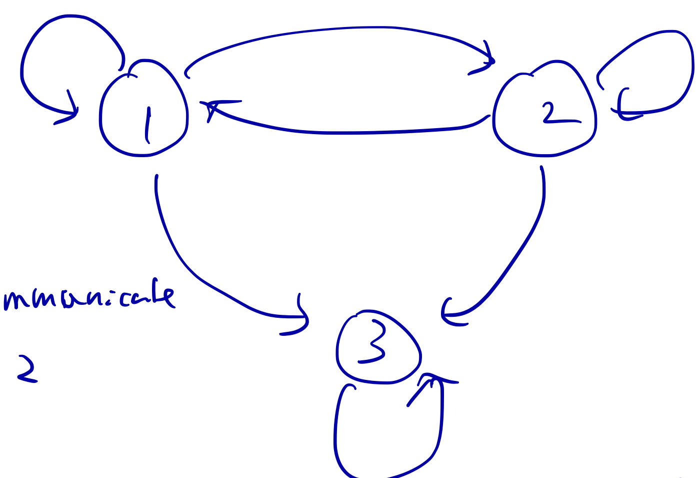
$3 \leftrightarrow 1$

$\{1, 2, 3\}$

irreducible

Example 3.

$$P = \begin{bmatrix} 0.3 & 0.3 & 0.4 \\ 0.4 & 0.2 & 0.4 \\ 0 & 0 & 1 \end{bmatrix}$$



3 does not communicate with 1 or 2

$1 \leftrightarrow 2$

$\cancel{1 \leftrightarrow 3}$

$2 \not\leftrightarrow 3$

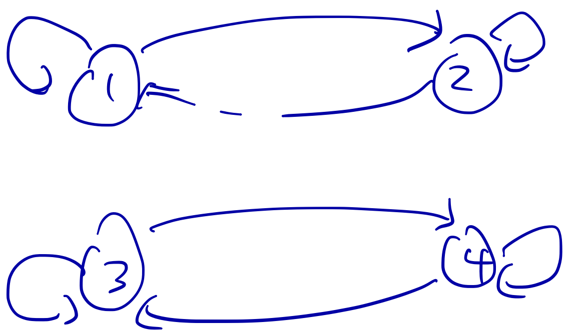
$\{1, 2\}$ $\{3\}$

two classes

not irreducible

Example 4.

$$P = \begin{bmatrix} 0.2 & 0.8 & 0 & 0 \\ 0.4 & 0.6 & 0 & 0 \\ 0 & 0 & 0.3 & 0.7 \\ 0 & 0 & 0.5 & 0.5 \end{bmatrix}$$



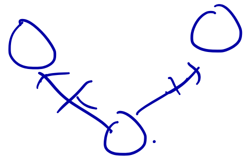
$1 \leftrightarrow 2$

$3 \leftrightarrow 4$

$\{1, 2\}$

$\{3, 4\}$

not irreducible



Closed Sets:

1. No state outside the set can be reached from it.
2. If only one state forms a closed set, then it is called an *absorbing state*. That is if j is an absorbing state, then $P_{jj} =$ 1

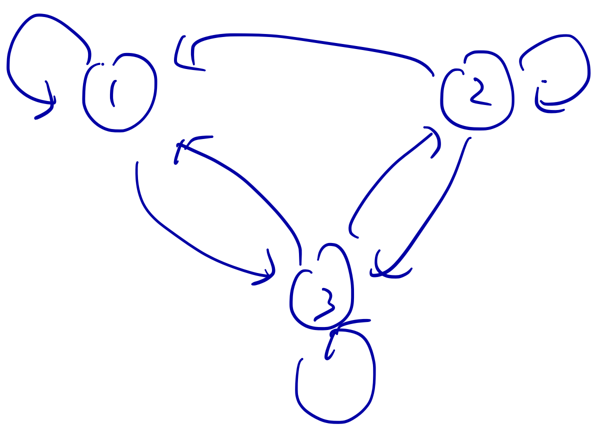
Definition:

Probability [the system will *ever* revisit state i, after it leaves it] = f_i

- A state is said to be *recurrent* if $f_i = 1$.
- A state is said to be *transient* if $f_i < 1$.

Example 5.

$$P = \begin{bmatrix} 0.3 & 0 & 0.7 \\ 0.4 & 0.3 & 0.3 \\ 0.2 & 0.4 & 0.4 \end{bmatrix}$$

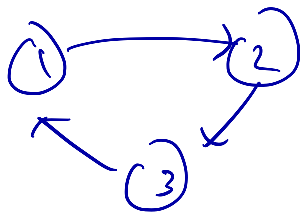


{1, 2, 3}
only one state

recurrent
{1, 2, 3}
transient
X.

Example 6.

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

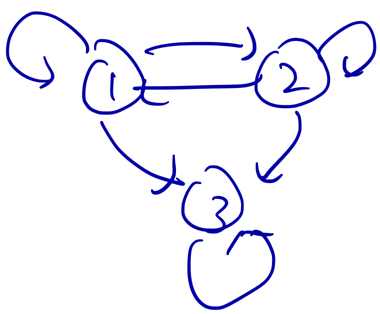


recurrent $\{1, 2, 3\}$

transient $\times$

Example 7.

$$P = \begin{bmatrix} 0.3 & 0.3 & 0.4 \\ 0.4 & 0.2 & 0.4 \\ 0 & 0 & 1 \end{bmatrix}$$



$\{1, 2\}$ state

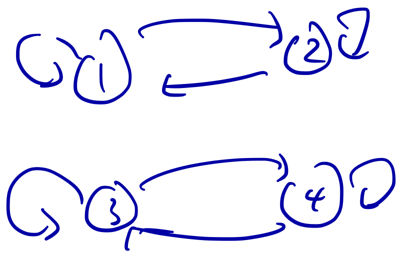
$\{3\}$ state

recurrent $\{3\}$

transient $\{1, 2\}$

Example 8.

$$P = \begin{bmatrix} 0.2 & 0.8 & 0 & 0 \\ 0.4 & 0.6 & 0 & 0 \\ 0 & 0 & 0.3 & 0.7 \\ 0 & 0 & 0.5 & 0.5 \end{bmatrix}$$



recurrent

$\{1, 2\}$ $\{3, 4\}$

transient X.

- If state i is recurrent, then the number of times the process/system visits state i is ∞
- If state i is transient, then the number of times the process/system visits state i is finite
- Let us consider a transient state i and derive the mean number of times the system visites i, before it leaves state i forever.
- After each visit to state i, there are two outcomes.
  1. It will come back with probability f_i
  2. It will not come back with probability $1-f_i$
- The number of times the system visits i, before it leaves i forever

$N = 1 \qquad f_i(1-f_i) \qquad$ let $p = 1-f_i$

$\vdots$

$N = n \qquad f_i^n(1-f_i) \qquad : P(N=n) = (1-p)^n p$

geometric.

- Mean number of times the system visits i, before it leaves i forever

$P_i[N_i = \infty] = 0$

$$\boxed{\text{mean} = E[N] = \frac{1}{p} = \frac{1}{1-f_i}}$$

- We can also derive an alternative expression for the expected (mean) number of visits to state i, before the system leaves i forever.
- Let us define an indicator (0-1 variable) I_n as follows.

$$I_n = \begin{cases} 1, & \text{if } X_n = 1 \\ 0, & \text{o.w} \end{cases}$$

- Total number of visits to i after leaving i

$$\sum_{n=0}^{\infty} I_n | X_0 = i$$

- Expected number of visits to i, after leaving i

$$E\left[\sum_{n=0}^{\infty} I_n | X_0 = i\right] = \sum_{n=0}^{\infty} E[I_n | X_0 = i]$$

$$= \sum_{n=0}^{\infty} P[X_n = i | X_0 = i] = \sum_{n=0}^{\infty} P_{ii}^n$$

- State i is recurrent if $\sum_{n=0}^{\infty} P_{ii}^n = \infty$
- State i is transient if $\sum_{n=0}^{\infty} P_{ii}^n < \infty$

In summary: State i is

- recurrent if $f_i = 1$ $\quad \sum_{n=0}^{\infty} P_{ii}^{\infty} := \infty$

- transient if $f_i < 1$ $\quad \sum_{n=0}^{\infty} P_{ii}^{n} < \infty$

Also,

- If i is recurrent and if $i \leftrightarrow j$, then j is recurrent

- If i is transient and if $i \leftrightarrow j$, then j is transient

- Recurrence and Transience are Class properties.