SSIE 660: Stochastic Systems

Homework assignment 8 - Key

1. Solve Chapter 5. Problem 4.
   B and C must finish service before A.

   (a) 0

   (b) Service time for A: 3, Service time for B and C: 1, respectively.
   $\frac{1}{27}$

   (c) (The probability that B finishes the service before A) multiplied by (The probability that C finishes the service before A)
   $\frac{1}{4}$

2. Solve Chapter 5. Problem 6.

   Condition on which server initially finishes first. Now,

   $$P[\text{Smith is not last}|\text{server 1 finishes first}] = P[\text{server 1 finishes before server 2}] = \frac{\lambda_1}{\lambda_1 + \lambda_2}$$

   Similarly,

   $$P[\text{Smith is not last}|\text{server 2 finishes first}] = \frac{\lambda_2}{\lambda_1 + \lambda_2}$$

   and thus

   $$P[\text{Smith is not last}] = \left[\frac{\lambda_1}{\lambda_1 + \lambda_2}\right]^2 + \left[\frac{\lambda_2}{\lambda_1 + \lambda_2}\right]^2$$

3. Solve Chapter 5. Problem 15.

   Let T_i denote the time between $(i-1)^{th}$ and the i^{th} failure. Then the T_i are independent with T_i being exponential with rate $(101 - i)/200$. Thus,

   $$E[T] = \sum_{i=1}^{5} E[T_i] = \sum_{i=1}^{5} \frac{200}{101 - i}$$

   $$Var(T) = \sum_{i=1}^{5} Var(T_i) = \sum_{i=1}^{5} \frac{200^2}{(101 - i)^2}$$

4. Solve Chapter 5. Problem 16.

(a) Suppose i and j are initially begun, with k waiting for one of them to be completed. Then,

$$E[T_i] + E[T_j] + E[T_k] = \frac{1}{\mu_i} + \frac{1}{\mu_j} + \frac{1}{\mu_i + \mu_j} + \frac{1}{\mu_k} = \sum_{i=1}^{3} \frac{1}{\mu_i} + \frac{1}{\mu_i + \mu_j}$$

Hence, the preceding is minimized when $\mu_i + \mu_j$ is as large as possible, showing that it is optimal to begin processing on jobs 2 and 3. Consequently,to minimize the expected sum of the completion times the jobs having largest rates should be initiated first.

(b) Letting X_i be the processing time of job i, this follows from the identity

$$2(M - S) + S = \sum_{i=1}^{3} X_i$$

which follows because if we interpret X_i as the work of job i then the total amount of work is $\sum_{i=1}^{3} X_i$, whereas work is processed at rate 2 per unit time when both servers are busy and at rate 1 per unit time when only a single processor is working.

(c) $E[S] = \frac{1}{\mu}P(\mu) + \frac{1}{\lambda}P(\lambda)$

(d) $P_{1,2}(\mu) = \frac{\lambda}{\mu + \lambda} < \frac{\lambda}{\mu + \lambda} + \frac{\mu}{\mu + \lambda}\frac{\lambda}{\mu + \lambda} = P_{1,3}(\mu)$

(e) If $\mu > \lambda$, then $E[S]$ is minimized when $P(\mu)$ is as large as possible. Hence, because minimizing $E[S]$ is equivalent to minimizing $E[M]$, it follows that $E[M]$ is minimized when jobs 1 and 3 are initially processes.

(f) In this case $E[M]$ is minimized when jobs 1 and 2 are initially processed. In all cases $E[M]$ is minimized when the jobs having smallest rates are initiated first.

5. Solve Chapter 5. Problem 37.
This is the summation of 1) expected time of failure and 2) expected time until the machine is found to be failed, once it fails. Consider the memoryless effect.

$$\frac{1}{\mu} + \frac{1}{\lambda}$$

6. Solve Chapter 5. Problem 42.

(a) $E[S_4]$ = Expected time of T_1 + Expected time of T_2 + Expected time of T_3 + Expected time of T_4
$E[S_4] = 4/\lambda$

(b) $E[S_4|N(1) = 2] = 1 + E[\text{time for 2 more events}]$ $E[S_4|N(1) = 2] = 1 + E[\text{time for 2 more events}] = 1 + 2/\lambda$

(c) $E[N(4) - N(2)|N(1) = 3] = E[N(4) - N(2)] = 2\lambda$