SSIE 660: Stochastic Systems
Dr. Sung H. Chung
Quiz 4 (Takehome, Due 8:30am Nov 8, 2016)

Name: Key

You may use your book and notes, but NO COLLABORATION.

1. Let $\{N(t), t \geq 0\}$ be a Poisson process with rate λ. Find $P[N(5) = 4|N(2) = 3]$.

   $P[N(5) = 4|N(2) = 3] = \dfrac{P[N(5) = 4, N(2) = 3]}{P[N(2) = 3]} = \dfrac{P[N(3) = 1]P[N(2) = 3]}{P[N(2) = 3]} =$
   $P[N(3) = 1] = e^{-3\lambda}\dfrac{3\lambda}{1}$

2. Let's assume that a male customer arrival follows a Poisson process with rate 10 and a female customer arrival follows a Poisson process with rate 15.

   (a) Find the probability that one male customer arrives before the first female customer arrives.

   Let S^1_n denote the time that the n^{th} male customer arrival and S^2_n denote the time that the n^{th} female customer arrival. The desired probability is:

   $$P[S^1_1 < S^2_1] = \frac{10}{10+15}$$

   (b) Find the probability that three male customers arrive before six female customers arrive.

   The desired probability is:

   $$P[S^1_3 < S^2_6] = \sum_{k=1}^{3+6-1} {}_{3+6-1}C_k \left(\frac{10}{10+15}\right)^k \left(\frac{15}{10+15}\right)^{3+6-1-k}$$

3. In a certain system, a customer must first be served by server 1, and then by server 2. The service times at server i are exponential with rate $\mu_i, i = 1, 2$. An arrival finding server 1 is busy waits in line for that server. Upon completion of service at server 1, a customer either enters service with server 2 if that server is free or else remains with server 1 (blocking any other customer from entering service) until server 2 is free. Customers depart the system after being served by server 2. Suppose that you

arrive to find two others in the system, one being served by server 1 and one by server 2.

What is the expected total time you spend in the system?

E[total time] = E[waiting time before server 1] + E[service time at server 1] + E[waiting time before server 2] + E[service time at server 2]

E[waiting time before server 1]=
E[service time for the customer already being served by server 1] + E[blocking time]
= $\frac{1}{\mu_1} + \frac{1}{\mu_2} * P$[server 1 completes before server 2] = $\frac{1}{\mu_1} + \frac{1}{\mu_2}\frac{\mu_1}{\mu_1 + \mu_2}$

E[service time at server 1] = $\frac{1}{\mu_1}$

E[waiting time before server 2]= $\frac{1}{\mu_2} * P$[server 1 completes before server 2] = $\frac{1}{\mu_2}\frac{\mu_1}{\mu_1 + \mu_2}$

E[service time at server 1] = $\frac{1}{\mu_2}$

E[total time] =
E[waiting time before server 1] + E[service time at server 1] + E[waiting time before server 2] + E[service time at server 2] =
$\frac{1}{\mu_1} + \frac{1}{\mu_2}\frac{\mu_1}{\mu_1 + \mu_2} + \frac{1}{\mu_1} + \frac{1}{\mu_2}\frac{\mu_1}{\mu_1 + \mu_2} + \frac{1}{\mu_2}$
$= \frac{2}{\mu_1} + \frac{2}{\mu_2}\frac{\mu_1}{\mu_1 + \mu_2} + \frac{1}{\mu_2}$

4. A subway station has both local and express service, on opposite sides of the same platform. Local and express trains arrive independently according to a Poisson process with rates 1/5 and 1/15 per minute, respectively. Both trains stop at your destination, with transit times of 17 minutes for a local train and 11 minutes for an express train.

   (a) What is the waiting time distribution until the next local train arrives?

   The waiting time distribution follows an exponential distribution with rate λ. Therefore,

$$f_{local}(t) = \frac{1}{5}e^{-\frac{1}{5}t}$$

(b) What is the waiting time distribution until the next express train arrives?

$$f_{express}(t) = \frac{1}{15}e^{-\frac{1}{15}t}$$

(c) What is the waiting time distribution until the next train (either local or express) arrives?

$$f_{either}(t) = \left(\frac{1}{5} + \frac{1}{15}\right) e^{-\left(\frac{1}{5}+\frac{1}{15}\right)t} = \frac{4}{15}e^{-\frac{4}{15}t}$$

(d) What is the probability that the next local train arrives before the express?

$$P[Local < Express] = \frac{1/5}{1/5 + 1/15} = \frac{3}{4}$$

(e) If the next train that arrives is a local, should you board that train or wait for an express, assuming that your objective is to minimize your expected travel time?

E[total time for local] = 17 min.

E[total time for express] =E[waiting time for express] + 11 min = $1/\lambda_{express}$+ 11 = 15 + 11 = 26 min.
It's better to take a local train.

5. Suppose that potential customers arrive at a single-server bank in accordance with a Poisson process having rate $1/4$. The potential customer will enter the bank only if the server is free when he arrives. Otherwise, (s)he will go home rather than entering the bank. If we assume that the amount of time spent in the bank by an arriving customer is exponential with mean 5, then

(a) What is the rate at which customers enter the bank?

Let us define the state as the number of customers in the system.

$$\lambda = 1/4, \mu = 1/5$$

$$P_0\lambda = P_1\mu$$
$$P_0 + P_1 = 1$$
$$\text{Therefore, } P_0 = \frac{\mu}{\lambda + \mu},\ P_1 = \frac{\lambda}{\lambda + \mu}$$

The rate at which customers enter the bank is

$$\lambda P_0 = \lambda \frac{\mu}{\lambda + \mu} = \frac{1}{4} * \frac{\frac{1}{5}}{\frac{1}{4} + \frac{1}{5}} = \frac{1}{9}$$

(b) What proportion of potential customers actually enter the bank?

$$\frac{\lambda P_0}{\lambda} = P_0 = \frac{\mu}{\lambda + \mu} = \frac{\frac{1}{5}}{\frac{1}{4} + \frac{1}{5}} = \frac{20}{45} = \frac{4}{9}$$

6. If the mean-value function of the renewal process $\{N(t), t \geq 0\}$ is given by $m(t) = t/4, t \geq 0$, what is $P\{N(2) = 0\}$?

Note that $m(t)$ is the mean value function of a Poisson process, and $m(t) = t/4$ implies that the Poisson process has rate $1/4$. Therefore,

$$P\{N(2) = 0\} = e^{-(1/4)*2} = e^{-1/2}$$

7. There are N terminals that share the same main computer. All the terminals are same. Every terminal works for a time period of S then sends a request to the main computer and waits until it gets a reply from the main computer. The main computer takes a time period of T to process one process and at most one at a time. Assume that S and T are exponential with rates λ and μ, respectively. Find out that on average, how many terminals are working and how many terminals are waiting for reply from the main computer.

Let us define the state as the number of terminals waiting for reply. Then, the balance equations are:

$$
\begin{aligned}
P_0 N\lambda &= \mu P_1 \\
\to P_1 &= \frac{N\lambda}{\mu} P_0 \\
P_2 &= \frac{(N-1)\lambda}{\mu} P_1 = N(N-1)\left(\frac{\lambda}{\mu}\right)^2 P_0 \\
&\vdots \\
P_n &= \frac{N!}{(N-n)!}\left(\frac{\lambda}{\mu}\right)^n P_0,\ n = 1, 2, \ldots, N
\end{aligned}
$$

Since,

$$\sum_{n=0}^{N} P_n = 1$$

We get

$$P_0 = 1 / \sum_{n=0}^{N} \left[\frac{N!}{(N-n)!}\left(\frac{\lambda}{\mu}\right)^n \right]$$

The average number of terminals working can be found by calculating

$$NP_0 + (N-1)P_1 + (N-2)P_2 + \cdots + (N-N)P_N = \sum_{n=0}^{N} (N-n)P_n$$

The average number of terminals waiting can be found by calculating

$$0P_0 + 0P_1 + 1P_2 + \cdots + (N-1)P_N = \sum_{n=1}^{N} (n-1)P_n$$