SSIE 660: Applied Stochastic Processes

Homework assignment 9 – Key

Nov. 30, 2016
Due: Dec 7, 2016 Before class starts

1. Solve Chapter 6. Problem 4.

   Let $N(t)$ denote the number of customers in the station at time t. Then $\{N(t)\}$ is a birth and death process. The entering rate is the arrival rate multiplied by the probability of entering the system with $\lambda_n = \lambda\alpha_n, \mu_n = \mu$.

2. Solve Chapter 6. Problem 8.

   The number of failed machines is a birth and death process with

$$\begin{aligned}
&\lambda_0 = 2\lambda, \lambda_1 = \lambda, \\
&\lambda_n = 0, n > 1 \\
&\mu_1 = \mu_2 = \mu, \\
&\mu_n = 0, n \neq 1, 2
\end{aligned}$$

   Find more parameters such as v_0, q_{01}, etc. Then write the Kolmogorov backward equations using such parameters. That is,

$$\begin{aligned}
&v_0 = q_{01} = 2\lambda \\
&v_1 = (\lambda + \mu) \\
&v_2 = q_{21} = \mu \\
&q_{01} = 2\lambda \\
&q_{10} = \mu \\
&q_{12} = \lambda \\
&q_{21} = \mu \\
&\text{All other } v_j, q_{ij} = 0
\end{aligned}$$

The Kolmogorov backward equations are then:

$$
\begin{aligned}
&i=0, j=0;\ P'_{00}(t) = 2\lambda[P_{10}(t) - P_{00}(t)]\\
&i=0, j=1;\ P'_{01}(t) = 2\lambda[P_{11}(t) - P_{01}(t)]\\
&i=0, j=2;\ P'_{02}(t) = 2\lambda[P_{12}(t) - P_{02}(t)]\\
&i=1, j=0;\ P'_{10}(t) = \mu P_{00}(t) + \lambda P_{20}(t) - (\lambda+\mu)P_{10}(t)\\
&i=1, j=1;\ P'_{11}(t) = \mu P_{01}(t) + \lambda P_{21}(t) - (\lambda+\mu)P_{11}(t)\\
&i=1, j=2;\ P'_{12}(t) = \mu P_{02}(t) + \lambda P_{22}(t) - (\lambda+\mu)P_{12}(t)\\
&i=1, j=2;\ P'_{12}(t) = \mu P_{02}(t) + \lambda P_{22}(t) - (\lambda+\mu)P_{12}(t)\\
&i=2, j=0;\ P'_{20}(t) = \mu[P_{10}(t) - P_{20}(t)]\\
&i=2, j=1;\ P'_{21}(t) = \mu[P_{11}(t) - P_{21}(t)]\\
&i=2, j=2;\ P'_{22}(t) = \mu[P_{12}(t) - P_{22}(t)]
\end{aligned}
$$

3. Solve Chapter 6. Problem 13.

With the number of customers in the shop as the state, we get a birth and death process with

$$\lambda_0 = \lambda_1 = 3, \mu_1 = \mu_2 = 4$$

Therefore

$$P_1 = \frac{3}{4}P_0, P_2 = \frac{3}{4}P_1 = \left(\frac{3}{4}\right)^2 P_0$$

And since $\sum_0^2 P_i = 1$, we get

$$P_0 = \left(1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2\right)^{-1} = \frac{16}{37}$$

(a) The average number of customers in the shop is

$$
\begin{aligned}
P_1 + 2P_2 &= \left(\frac{3}{4} + 2\left(\frac{3}{4}\right)^2\right) P_0\\
&= \frac{30}{16}\left(1 + \frac{3}{4} + 2\left(\frac{3}{4}\right)^2\right) P_0 = \frac{30}{37}
\end{aligned}
$$

(b) The proportion of customers that enter the shop is

$$\frac{\lambda(1-P_2)}{\lambda} = 1 - P_2 = 1 - \frac{9}{16}\cdot\frac{16}{37} = \frac{28}{37}$$

(c) Now $\mu = 8$, and so

$$P_0 = \left(1 + \frac{3}{8} + \left(\frac{3}{8}\right)^2\right)^{-1} = \frac{64}{97}$$

So the proportion of customers who now enter the shop is

$$1 - P_2 = 1 - \left(\frac{3}{8}\right)^2 \frac{264}{97} = 1 - \frac{9}{97} = \frac{88}{97}$$

The rate of added customers is therefore

$$\lambda\left(\frac{88}{97}\right) - \lambda\left(\frac{28}{37}\right) = 3\left(\frac{88}{97} - \frac{28}{37}\right) = 0.45$$

The business he does would improve by .45 customers per hour.

4. Solve Chapter 6. Problem 14.

Letting the number of cars in the station be the states, we have a birth and death process with

$$\begin{aligned}
&\lambda_0 = \lambda_1 = \lambda_2 = 20 \\
&\lambda_i = 0, i > 2 \\
&\mu_1 = \mu_2 = 12 \text{ (12 cars per hour)}
\end{aligned}$$

Hence,

$$\begin{aligned}
P_1 &= \frac{5}{3}P_0 \\
P_2 &= \frac{5}{3}P_1 = \left(\frac{5}{3}\right)^2 P_0 \\
P_3 &= \frac{5}{3}P_2 = \left(\frac{5}{3}\right)^3 P_0
\end{aligned}$$

In addition,

$$\sum_{i=0}^{3} P_i = 1$$

Therefore,

$$P_0 = \left(1 + \frac{5}{3} + \left(\frac{5}{3}\right)^2 + \left(\frac{5}{3}\right)^3\right)^{-1} = \frac{27}{272}$$

(a) The fraction of the attendant's time spent servicing cars is equal to the fraction of time there are cars in the system and is therefore $1 - P_0 = 245/272$.

(b) The fraction of potential customers that are lost is equal to the fraction of customers that arrive when there are three cars in the station and is therefore

$$P_3 = \left(\frac{5}{3}\right)^3 P_0 = \frac{125}{272}$$

5. Solve Chapter 6. Problem 15.
   With the number of customers in the system as the state, we get a birth and death process with

$$\lambda_0 = \lambda_1 = \lambda_2 = 3$$
$$\lambda_i = 0, i > 2$$
$$\mu_1 = 2, \mu_2 = \mu_3 = 4$$

   Therefore, the balance equations reduce to

$$P_1 = \frac{3}{2}P_0, P_2 = \frac{3}{4}P_1 = \frac{9}{8}P_0, P_3 = \frac{3}{4}P_2 = \frac{27}{32}P_0$$

   And therefore,

$$P_0 = \left(1 + \frac{3}{2} + \frac{9}{8} + \frac{27}{32}\right)^{-1} = \frac{32}{143}$$

   (a) The fraction of potential customers that enter the system is

$$1 - P_3 = 1 - \frac{27}{32} * \frac{32}{143} = \frac{116}{143}$$

   (b) With a server working twice as fast we would get

$$P_1 = \frac{3}{4}P_0,\ P_2 = \frac{3}{4}P_1 = \left(\frac{3}{4}\right)^2 P_0,\ P_3 = \left(\frac{3}{4}\right)^3 P_0$$

   And therefore,

$$P_0 = \left(1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3\right)^{-1} = \frac{64}{175}$$

   so that now

$$1 - P_3 = 1 - \frac{27}{64} * \frac{64}{175} = \frac{148}{175}$$

6. Solve Chapter 6. Problem 19.

   There are four states. Let state 0 mean that no machines are down, state 1 that machine 1 is down and 2 is up, state 2 that machine 1 is up and 2 is down, and 3 that both machines are down (machine 1 is in service).

   The balance equations are as follows:

$$(\lambda_1 + \lambda_2)P_0 = \mu_1 P_1 + \mu_2 P_2$$
$$(\mu_1 + \lambda_2)P_1 = \lambda_1 P_0$$
$$(\lambda_1 + \mu_2)P_2 = \lambda_2 P_0 + \mu_1 P_3$$
$$\mu_1 P_3 = \lambda_2 P_1 + \lambda_1 P_2$$
$$\sum_{i=0}^{3} P_i = 1$$

   These equations are easily solved and the proportion of time machine 2 is down is $P_2 + P_3$.