### SSIE 660: Applied Stochastic Processes Dr. Sung H. Chung Note 5

## Chapter 2. Random Variables, Chapter 3. Conditional Probability/Expectation

## **2.7 Moment Generating Functions**

## Summary

Table 2.1

Discrete probability distribution	Probability mass function, $p(x)$	Moment generating function, $\phi(t)$	Mean	Variance
Binomial with parameters $n, p$ , $0 \le p \le 1$	$\binom{n}{x}p^{x}(1-p)^{n-x},$ $x = 0, 1, \dots, n$	$(pe^t + (1-p))^n$	пр	np(1-p)
Poisson with parameter $\lambda > 0$	$e^{-\lambda} \frac{\lambda^x}{x!},$ $x = 0, 1, 2, \dots$	$\exp\{\lambda(e^t-1)\}$	λ	λ
Geometric with parameter $0 \le p \le 1$	$p(1-p)^{x-1},$ $x = 1, 2, \dots$	$\frac{pe^t}{1-(1-p)e^t}$	$\frac{1}{p}$	$\frac{1-p}{p^2}$

Table 2.2

Continuous probability distribution	Probability density function, $f(x)$	Moment generating function, $\phi(t)$	Mean	Variance
Uniform over (a, b)	$f(x) = \begin{cases} \frac{1}{b-a}, & a < x < b \\ 0, & \text{otherwise} \end{cases}$	$\frac{e^{tb} - e^{ta}}{t(b-a)}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Exponential with parameter $\lambda > 0$	$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & x < 0 \end{cases}$	$rac{\lambda}{\lambda-t}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
Gamma with parameters $(n, \lambda), \lambda > 0$	$f(x) = \begin{cases} \frac{\lambda e^{-\lambda x} (\lambda x)^{n-1}}{(n-1)!}, & x \ge 0\\ 0, & x < 0 \end{cases}$	$\left(\frac{\lambda}{\lambda-t}\right)^n$	$rac{n}{\lambda}$	$\frac{n}{\lambda^2}$
Normal with parameters $(\mu, \sigma^2)$	$f(x) = \frac{1}{\sqrt{2\pi}\sigma}$ $\times \exp\{-(x-\mu)^2/2\sigma^2\},$	$\exp\left\{\mu t + \frac{\sigma^2 t^2}{2}\right\}$	$\mu$	$\sigma^2$
	$-\infty < x < \infty$			

#### 2.8 Limit Theorems

**Proposition 1.** Let X be a non-negative random variable, then for any value a>0,  $P[X\geq a]\leq$  It is called a Markov's Inequality Proof.  $\square$  **Proposition 2.** If X is a random variable with mean  $\mu$  and variance  $\sigma^2$ , then, for any value k>0,  $P[|x-\mu|\geq k]\leq$  It is called Chebyshev's Inequality. Proof.

These inequalities help us to obtain bounds for the probabilities, without knowing the actual probability density function - only the mean and variance need to be known.

**Example 3.** Suppose we know that the number of items produced in a factory during a week is a random variable with a mean of 500.

1. What can be said about the probability that this week's production will be at least 1000?

2. If the variance of a week's production is known to equal 100, then what can be said about the probability that this week's production will be between 400 and 600?

**Theorem 4** (Strong Law of Large Numbers). Let  $X_1, X_2, ...$  be a sequence of independent random variables having a common distribution, and let  $E[X_i] = \mu$ . Then with probability 1,

$$\frac{X_1 + X_2 + \dots + X_n}{n} \to \mu \quad as \ n \to \infty$$

**Example 5.** Let  $X_1, X_2, ...$  be a sequence of independent binomial random variables with parameters n and p. Given that  $E[X_i] = np$  for all i, we have by the strong law of large numbers that, with probability 1,

$$\frac{X_1 + X_2 + \dots + X_m}{m} \to np \text{ as } m \to \infty$$

**Theorem 6** (Central Limit Theorem: Linderberg). Let  $X_1, X_2, ...$  be a sequence of independent, (not necessarily) identically distributed random variables. Let  $E[X_i] = \mu_i$ ,  $Var(X_i) = \sigma_i^2$ . Let

$$S_n = \frac{X_1 + X_2 + \dots + X_n - \sum_i \mu_i}{\sqrt{\sum_i \sigma_i^2}}$$

then the probability function  $S_n$  converges to a standard normal distribution N(0,1) as  $n \to \infty$ .

**Theorem 7** (Central Limit Theorem: Special Case). Let  $X_1, X_2, ...$  be a sequence of independent, identically distributed (iid) random variables, each with mean  $\mu$  and variance  $\sigma^2$ . Then the distribution of

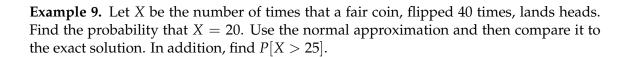
$$\frac{X_1 + X_2 + \dots + X_n - n\mu}{\sigma\sqrt{n}}$$

tends to the standard normal as  $n \to \infty$ . That is,

$$P\left\{\frac{X_1 + X_2 + \dots + X_n - n\mu}{\sigma\sqrt{n}} \le a\right\} \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^a e^{-x^2/2} dx \text{ as } n \to \infty$$

**Example 8.** Let X be a binomial random variable with parameters n and p. Then  $X = X_1 + \cdots + X_n$  where  $X_i$  is a Bernoulli random variable with a parameter p.

- 1. Find  $E[X_i]$
- 2. Find  $Var(X_i)$ .
- 3. As  $n \to \infty$ , does *X* follow N(0,1)? Why?



**Example 10.** From an urn containing 10 identical balls numbered 0 to 9, n balls are drawn with replacement. Use the Central Limit Theorem to find the probability that among n numbers chosen, the numbers 5 will appear between  $\frac{n-3\sqrt{n}}{10}$  and  $\frac{n+3\sqrt{n}}{10}$  times, if (i) n=25, (ii) n=100.

#### 2.9 Stochastic Processes

What is a Stochastic Process

• A stochastic process is a collection of random variables (RVs), usually denoted by

$$X = \{X(t) : t \in T\}$$

- *t* is a call index, most commonly it denotes times.
- *T* is called the index set.
- X(t) is a random variable (RV) associated with an index t. We say X(t) is the state of the system at time t.
- If X(t) takes values in a set S for every  $t \in T$ , then S is called the sample space of X.
- For example, X(t) may represent the total number of customers that have entered a supermarket by time t; or the number of the customers in the super market at time t; or the total amount of sales that have been recorded in the market by time t, etc.

Thus, a stochastic process is a family of random variables that describes the evolution through time of some (physical) processes.

#### Chapter 3. Conditional Probability and Conditional Expectations

#### 3.1 Introduction

- i. We are often interested in calculating probabilities and expectations when some partial information is available.
- ii. In calculating a desired probability or expectation it is often extremely useful to first condition on some appropriate random variables.

#### 3.2 The Discrete Case

Conditional probability mass function:

$$P[X = x | Y = y] =$$

Conditional expectation:

$$E[X|Y = y] =$$

## Example 11.

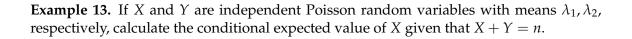
y x	0	1	2	Marginal $p_X(x)$
1	1/16	3/16	1/16	5/16
2	2/16	3/16 1/16	0	3/16
3	5/16	1/16	2/16	8/16
Marginal $p_Y(y)$	8/16	5/16	3/16	1

Find E[X|Y=0]

**Example 12.** Suppose that p(x,y), the joint probability mass function of X and Y, is given by

$$p(1,1) = 0.5, p(1,2) = 0.1, p(2,1) = 0.1, p(2,2) = 0.3$$

Calculate the conditional probability mass function of X given that Y = 1.



### 3.3 The Continuous Case

Conditional probability density function:

$$f_{X|Y}(x|y) =$$

Conditional expectation:

$$E[X|Y = y] =$$

# Example 14. Given

$$f_{X,Y}(x,y) = \begin{cases} x^2 + xy/3, & 0 < x < 1, \ 0 < y < 2 \\ 0, \text{ elsewhere} \end{cases}$$

Find E[X|Y=y].

# Example 15. Given

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{2} y e^{-xy}, & 0 < x < \infty, \ 0 < y < 2 \\ 0, \text{ elsewhere} \end{cases}$$

Find  $E[e^{X/2}|Y = 1]$ .