**SSIE 660: Stochastic Systems**
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**Note 3**
**Chapter 2. Random Variables**

## 2.4 Expectation of a Random Variables

Discrete case: Let $X$ be a discrete random variable with a probability mass function $p(x)$, then the expected value of $X$ is defined by

$$E[X] = \sum_{x:p(x)>0} xp(x)$$

The weighted average of the possible values of $X$ can take on, each being weighted by the probability that $X$ assumes that value.

**Example 1.** Find $E[X]$ where $X$ is the outcome when we roll a fair die.

| $x$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $p(x)$ | 1/6 | 1/6 | 1/6 | 1/6 | 1/6 | 1/6 |

$$\therefore\ E[X] = \sum x p(x) = 1\times \tfrac{1}{6} + 2\times \tfrac{1}{6} + 3\times \tfrac{1}{6} + \cdots + 6\times \tfrac{1}{6} = 7/2$$

**Example 2.** Calculate $E[X]$ where $X$ is binomially distributed with parameters $n$ and $p$.

$$E[X] = \sum_{i=0}^{n} i\,p(i) = \sum_{i=0}^{n} i\binom{n}{i} p^i (1-p)^{n-i} = \sum \frac{i\,n!}{(n-i)!\,i!} p^i (1-p)^{n-i}$$

$$= \sum_{i=1}^{n} \frac{n!}{(n-i)!\,(i-1)!} p^i (1-p)^{n-i} = np \sum_{i=1}^{n} \frac{(n-1)!}{(n-i)!\,(i-1)!} p^{i-1} (1-p)^{n-i}$$

$$= np \sum_{k=0}^{n-1} \binom{n-1}{k} p^k (1-p)^{n-1-k} = np\,[p + (1-p)]^{n-1} = np$$

Continuous case: Let $X$ be a continuous random variable with a probability density function $f(x)$, then the expected value of $X$ is defined by

$$E[X] = \int_{-\infty}^{\infty} x f(x)\,dx$$

**Example 3.** Find $E[X]$ where $X$ is a random variable uniformly distributed over $(\alpha, \beta)$.

$$E[X] = \int_{\alpha}^{\beta} \frac{x}{\beta-\alpha}\,dx = \frac{\beta^2-\alpha^2}{2(\beta-\alpha)} = \frac{\beta+\alpha}{2}$$

Expectation of a Function of a Random Variable: Sometimes we may want to calculate $E[g(X)]$ where $g(X)$ is a function of a random variable $X$.

**Example 4.** Calculate $E[X^2]$ when $X$ has the following probability mass function.

$$p(0) = 0.2, p(1) = 0.5, p(2) = 0.3$$

$P(X=0) = .2$

$P(X=1) = .5$

$P(X=2) = .3$

$E[X^2] = \sum x^2 p(x)$

$= 0 \times .2 + 1 \times .5 + 4 \times .3$

$= 1.7$

$E[X] = 0 \times .2 + 1 \times .5 + 2 \times .3 = 1.1 \quad \therefore (E[X])^2 = 1.21$

**Proposition 5.** *(a) If $X$ is a discrete random variable with probability mass function $p(x)$, then for any real-valued function $g$*

$$E[g(X)] = \sum_{x:p(x)>0} g(x)p(x)$$

*(b) If $X$ is a continuous random variable with probability density function $f(x)$, then for any real-valued function $g$*

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)dx$$

**Example 6.** Calculate $E[X^3]$ when $X$ is uniformly distributed over (0,1).

$$\int_0^1 x^3\,dx = 1/4$$

**Corollary 7.** *If $a$ and $b$ are constants, then*

$$E[aX+b] = aE[X]+b$$

Note that $E[X]$ is called the first moment of $X$ and $E[X^n], n \geq 1$ is called the $n$th moment of $X$.

Also,

$$E[a_1X_1 + ... + a_nX_n] = a_1E[X_1] + ... + a_nE[X_n]$$

Variance of a Random Variable is denoted by Var($X$), and defined by

$$\text{Var}(X) = E[(X - E[X])^2]$$

**Example 8.** Calculate Var($X$) when $X$ is continuous with density $f$ and $E[X] = \mu$.

$$\begin{aligned} Var(X) &= E[(X-\mu)^2] \\ &= E[X^2 - 2\mu X + \mu^2] \\ &= E[X^2] - 2\mu E[X] + \mu^2 \\ &= E[X^2] - 2\mu^2 + \mu^2 \\ &= E[X^2] - \mu^2 = \underline{E[X^2] - (E[X])^2} \end{aligned}$$

Note: Var($X$) = $E[X^2] - (E[X])^2$.

**Example 9.** Find Var$(X)$ where $X$ is the outcome when we roll a fair die.

$$E[X^2] = 1 \cdot \left(\tfrac{1}{6}\right) + 2^2\left(\tfrac{1}{6}\right) + \cdots + 6^2\left(\tfrac{1}{6}\right) = \frac{91}{6}$$

$$\therefore Var(X) = E[X^2] - (E(X))^2 = \frac{91}{6} - \left(\tfrac{7}{2}\right)^2 = \frac{35}{12}$$

**2.5 Jointly Distributed Random Variables**

ex) $X$ : 0 or 1 coin
$Y$ : 1, 2, 3, 4, 5, 6. die

$P_{X,Y}(1,3) = \frac{1}{2}\cdot\frac{1}{6} = \frac{1}{12}$

$P_X(x) = \begin{cases} 0 & 1/2 \\ 1 & 1/2 \end{cases}$ $\quad P_Y(y) = \begin{cases} 1 \\ 2 \\ \vdots \\ 6 \end{cases}$

Joint Distribution of Random Variables: Let $X$ and $Y$ be two ramdom variables.

**Discrete**

Joint probability mass function: $p_{X,Y}(x,y) = P\{X = x, Y = y\}$

Marginal Probability Distributions:

$X$ :
$$p_X(x) = \sum_{y:p(x,y)>0} p(x,y)$$

$Y$ :
$$p_Y(y) = \sum_{x:p(x,y)>0} p(x,y)$$

Conditional Probability:

$$P(X = x|Y = y) = \frac{P(X=x, Y=y)}{P(Y=y)} = \frac{P_{XY}(x,y)}{P_Y(y)}$$

$$P(X=1 \mid Y=6) = \frac{P(1,6)}{P_Y(6)} = \frac{\frac{1}{12}}{1/6} = 1/2.$$

$$P(Y = y|X = x) = \frac{P(X=x, Y=y)}{P(X=x)} = \frac{P_{XY}(x,y)}{P_X(x)}$$

**Continuous**

$X$ and $Y$ are jointly continuous if there exists a joint probability density function: $f_{X,Y}(x,y)$, for all sets $A$ and $B$ of real numbers

$$P\{X \in A, Y \in B\} = \int_B \int_A f(x,y)dxdy$$

Marginal: $P\{a_1 \le X \le a_2,\ b_1 \le Y \le b_2\} = \int_{b_1}^{b_2}\int_{a_1}^{a_2} f_{X,Y}(x,y)\,dx\,dy.$

Marginal density function of $X$:
$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy$$

Marginal density function of $Y$:
$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx$$

Conditional Probability:
$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

**Example 10.** Let $X$ and $Y$ represent the proportions of time the drive-in and walk-in facilities of a fast food restaurant are in use, respectively. The joint density function of $X$ and $Y$ is

$$f_{X,Y}(x,y) = \begin{cases} \frac{6(x+y^2)}{5}, \ 0 \le x \le 1, 0 \le y \le 1 \\ 0, \text{ elsewhere} \end{cases}$$

1. Find the marginal probability density functions of $X$ and $Y$.
2. Find the conditional probability that the drive-in facility is used less than $1/2$ the time, given that the walk-in facility is used $1/4$ time.

$$f_X(x) = \int_0^1 \frac{6(x+y^2)}{5} dy = \left[ \frac{6}{5}xy + \frac{6}{5}\frac{1}{3}y^3 \right]_0^1 = \frac{6}{5}x + \frac{2}{5} \quad 0 \le x \le 1$$

$$f_Y(y) = \int_0^1 \frac{6(x+y^2)}{5} dx = \left[ \frac{6}{5}\frac{1}{2}x^2 + \frac{6}{5}y^2 x \right]_0^1 = \frac{3}{5} + \frac{6}{5}y^2, \ 0 \le y \le 1$$

$$P(X < \tfrac{1}{2} | y = \tfrac{1}{4}) = \frac{P(X \le \frac{1}{2}, Y = 1/4)}{P_Y(\frac{1}{4})} = \frac{\frac{6}{5}x + \frac{6}{5}y^2}{\frac{3}{5} + \frac{6}{5}y^2} = \frac{6x + 6y^2}{3 + 6y^2}$$

$$\int_0^{1/2} \frac{6x + 6(\frac{1}{4})^2}{3 + 6(\frac{1}{4})^2} dx$$

Independence of two RVs:

$X$ and $Y$ are independent if $F(x,y) = F(x)F(y)$.

$X$ and $Y$ are independent if and only if for all $x$ and $y$,

$$P(X = x, Y = y) = P(X = x)P(Y = y)$$

or

$$f_{XY}(x,y) = f_X(x)f_Y(y)$$

If $X$ and $Y$ are independent, then

$$E[g(x), h(y)] = E[g(x)]E[h(y)]$$

Variance:

$$\text{Var}[a_1X_1 + a_2X_2 + ...a_nX_n] = a_1^2\text{Var}(X_1) + ... + a_n^2\text{Var}(X_n) + 2\sum_{i=1}^{n}\sum_{j<i} a_ia_j\text{Cov}(X_i, X_j)$$

Covariance of $X$ and $Y$ = $\text{Cov}(X,Y) = E[(X - E[X])(Y - E[Y])] = E[XY] - E[X]E[Y]$

If $X$ and $Y$ are independent, then $\text{Cov}(X,Y)$ =0. However, $\text{Cov}(X,Y)$ =0 does not mean $X$ and $Y$ are independent.

$\text{Cov}(X,Y) = \text{Cov}(Y,X)$

$\text{Cov}(X,X) = \text{Var}(X)$

$\text{Cov}(X,Y) = \sigma_X\sigma_y\rho_{XY}$ where $\rho_{XY}$ is the correlation coefficient. $\rho_{XY} \in [-1,1]$.

**Example 11.** Let $X$ denote the reaction time, in seconds, to a certain stimulant and $Y$ denote the temprature ($^0$F) at which a certain reaction starts to take place. Suppose that two random variables $X$ and $Y$ have the joint density

$$\frac{2x}{4} - \frac{\cancel{x}}{\cancel{16}\,8} = \frac{3}{8}x$$

$$f_{X,Y}(x,y) = \begin{cases} 4xy, \ 0 \le x \le 1, 0 \le y \le 1 \\ 0, \ \text{elsewhere} \end{cases}$$

Find

1. $P\left(0 < X < \frac{1}{2} \text{ and } \frac{1}{4} < Y < \frac{1}{2}\right)$
2. $P(X < Y)$
3. Are $X$ and $Y$ independent?

$$\int_0^{\frac{1}{2}}\int_{\frac{1}{4}}^{\frac{1}{2}} 4xy\,dy\,dx = \int_0^{\frac{1}{2}} \left[2xy^2\right]_{\frac{1}{4}}^{\frac{1}{2}} dx = \int_0^{\frac{1}{2}} \frac{3}{8}x\,dx$$

$$= \left[\frac{3}{16}x^2\right]_0^{\frac{1}{2}} = \frac{3}{64}$$

$$\int_{\frac{1}{4}}^{\frac{1}{2}}\int_0^{\frac{1}{2}} 4xy\,dx\,dy = \int_{\frac{1}{4}}^{\frac{1}{2}} \left[2x^2y\right]_0^{\frac{1}{2}} dy = \int_{\frac{1}{4}}^{\frac{1}{2}} \frac{1}{2}y\,dy$$

$$= \left[\frac{y^2}{4}\right]_{\frac{1}{4}}^{\frac{1}{2}} = \frac{1}{16} - \frac{1}{64} = \frac{3}{64}$$

2.

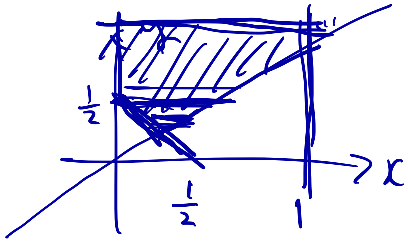
$$\int_0^1\int_0^y 4xy\,dx\,dy$$

$$= \int_0^1 \left[2x^2y\right]_0^y dy$$

$$= \int_0^1 2y^3\,dy = \left[\frac{2}{4}y^4\right]_0^1 = \frac{1}{2}$$

$f_X(x) = 2x$ $\therefore$ independent

$f_Y(y) = 2y$



$x<y$

**Example 12.** Let $X$ denote the diameter of an armored electric cable and $Y$ denote the diameter of the ceramic mold that makes the cable. Both $X$ and $Y$ are scaled so that they range between 0 and 1. Suppose that two random variables $X$ and $Y$ have the joint density

1. $y \geq \frac{1}{2} - x$

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{y}, & \boxed{0 < x < y < 1} \\ 0, & \text{elsewhere} \end{cases}$$

~~note $0 \leq x \leq 1$, $0 \leq y \leq 1$~~

1. Find $P(X + Y \geq \frac{1}{2})$
2. Are $X$ and $Y$ independent?

$1 - \frac{1}{2y} + 1$

$1'$

$x + y \geq \frac{1}{2}$ $\quad \boxed{x \geq \frac{1}{2} - y.}$

$$\int_{\frac14}^{\frac12}\int_{\frac12 - y}^{y} \frac{1}{y}\,dx\,dy = \int_{\frac14}^{\frac12} \left[\frac{x}{y}\right]_{\frac12 - y}^{y} dy = \int_{\frac14}^{\frac12} 1 - \frac{\frac12 - y}{y}\,dy = \int_{\frac14}^{\frac12} \left(2 - \frac{1}{2y}\right) dy$$

$$= \left[2y - \frac12 \ln y\right]_{\frac14}^{\frac12} = 2\left(\frac14\right) - \frac12\left(\ln\frac12 - \ln\frac14\right)$$

$$+\ \Big) = 1 - \frac12 \ln\frac12 + \frac12 \ln\frac14$$

$$\int_{\frac12}^{1}\int_0^{y} \frac{1}{y}\,dx\,dy = \int_{\frac12}^{1} \left(\frac{x}{y}\right)_0^{y} dy = \int_{\frac12}^{1} 1\,dy = \frac12$$

$= .653$

2. $f_X(x) = \int_x^1 \frac{1}{y}\,dy = \left[\ln y\right]_x^1 = \ln 1 - \ln x = -\ln x.$ $\quad \underline{0 < x < y}$

$f_Y(y) = \int_0^y \frac{1}{y}\,dx = \left[\frac{x}{y}\right]_0^y = 1 \quad x < y < 1$