SSIE 660: Stochastic Systems
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Quiz 3
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Name: Key

1. A box in a certain supply room contains four 40w lightbulbs, five 60w bulbs, and six 75w bulbs. Suppose that three bulbs are randomly selected.

   (a) What is the probability that exactly two of the selected bulbs are rated 75w?

   $$\frac{{}_6C_2 *_9 C_1}{{}_{15}C_3} = \frac{15*9}{455} \approx 0.297.$$

   (b) What is the probability that all three of the selected bulbs have the same rating?

   $$\frac{{}_4C_3 +_5 C_3 +_6 C_3}{{}_{15}C_3} = \frac{34}{455} \approx 0.0747.$$

2. In flipping a fair coin, let X be the number of tosses to get the first head. Find the moment generating function $M_X(t)$ and use it to get $E(X)$ and $V(X)$.

   Note that X is a geometric random variable with P(head) =0.5, thus the probability mass function of X is

   $$p(k) = p(1-p)^{k-1} = 0.5^k, k = 1, 2, 3, ...$$

   Then, the mgf of X is

   $$M_X(t) = E[e^{tX}] = \sum_x e^{xt} p(x) = \sum_{k=1}^{\infty} e^{kt} p(1-p)^{k-1} = pe^t \sum_{k=1}^{\infty} [(1-p)e^t]^{k-1}$$
   $$= \frac{pe^t}{1-(1-p)e^t}$$

   Differentiating $M_X(t)$ gives:

   $$M_X^{(1)} = \frac{pe^t[1-(1-p)e^t] - pe^t[-(1-p)e^t]}{[1-(1-p)e^t]^2} = \frac{pe^t}{[1-(1-p)e^t]^2}$$

   $$M_X^{(2)} = \frac{pe^t[1+(1-p)e^t]}{[1-(1-p)e^t]^3}$$

Setting $t = 0$ gives

$$E[X] = \frac{1}{p}$$

$$E[X^2] = \frac{2-p}{p^2}$$

$$Var(X) = E[X^2] - (E[X])^2] = \frac{1-p}{p^2}$$

Thus, with $p = 0.5$,

$$E[X] = 2, Var(X) = 2$$

3. The joint density of X and Y is

$$f(x,y) = \frac{(y^2 - x^2)}{8} e^{-y},\ 0 < y < \infty,\ -y \le x \le y$$

Calculate $E[X|Y = y]$.

$$\begin{aligned}
E[X|Y = y] &= \int_{-y}^{y} x f_{X|Y}(x|y) dx \\
&= \int_{-y}^{y} x \frac{f(x,y)}{f_Y(y)} dx \\
&= \int_{-y}^{y} x \frac{f(x,y)}{f_Y(y)} dx \\
&= C \int_{-y}^{y} x(y^2 - x^2) dx \\
&= 0
\end{aligned}$$

where $C = \frac{3}{4y^3}$ and

$$\begin{aligned}
f_Y(y) &= \int_{-y}^{y} \frac{y^2 - x^2}{8} e^{-y} dx \\
&= \left[e^{-y} \left(\frac{y^2 x}{8} - \frac{x^3}{24} \right) \right]_{-y}^{y} = \frac{y^3}{6} e^{-y}
\end{aligned}$$

$$
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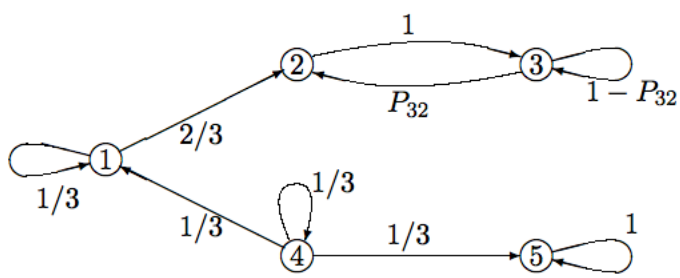
E[X|Y=y] &= \int_{-y}^{y} x f_{X|Y}(x|y)dx \\
&= \int_{-y}^{y} x\frac{f(x,y)}{f_Y(y)}dx \\
&= \int_{-y}^{y} x\frac{f(x,y)}{f_Y(y)}dx \\
&= C\int_{-y}^{y} x(y^2-x^2)dx \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\text{where } C &= \frac{3}{4y^3} \text{ and} \\
f_Y(y) &= \int_{-y}^{y} \frac{y^2-x^2}{8}e^{-y}dx \\
&= \left[e^{-y}\left(\frac{y^2x}{8}-\frac{x^3}{24}\right)\right]_{-y}^{y} = \frac{y^3}{6}e^{-y}
\end{aligned}
$$

4. Consider the following finite Markov chain.



(a) Identify the transient states and identify each class of recurrent states.

States 1 and 4 are transient. States 2 and 3 constitute a class of recurrent states and state 5 constitutes another class (a singleton class) of recurrent states.

(b) Find the following $n-$step transition probabilities, $P_{ij}^n = P[X_n = j|X_0 = i]$ as a function of n.

i. P_{44}^n

$\left(\frac{1}{3}\right)^n$ since n successive self-loop transitions, each of probability $1/3$, are required.

ii. P_{45}^n

$P_{45}^n = (1/3) + (1/3)^2 + \cdots + (1/3)^n = \frac{1}{2}\left(1 - 3^{-n}\right)$. The reason for this is that there are n walks going from 4 to 5 in n steps; each such walk contains the 4 to 5 transition at a different time. If it occurs at time i then there are $i - 1$ self-transitions from 4 to 4, so the probability of that walk is $(1/3)^i$. As a check, note that $\lim_{n\to\infty} P_{45}^n = 1/2$ which can be verified from the symmetry in leaving the transient state, node 4.

iii. P_{41}^n

$P_{41}^n = n(1/3)^n$ since there are n walks that go from 4 to 1 in n steps, one for each step in which the $4 \to 1$ transition can be made. Each walk has probability $(1/3)^n$.

5. Consider a Markov chain having the following transition probability matrix.

$$P = \begin{bmatrix} 0.1 & 0.2 & 0.7 \\ 0.3 & 0.3 & 0.4 \\ 0.2 & 0.4 & 0.4 \end{bmatrix}$$

(a) Identify the transient states and identify each class of recurrent states.

States 1, 2, and 3 constitute a class of recurrent states.

(b) Identify a period (aperiodic if a state does not have any) for each state.

All states are aperiodic.

(c) If the Markov chain has steady state probabilities, find them.

Since the Markov chain is ergodic and irreducible, it has has steady state probabilities, which can be found by solving the following:

$$\begin{aligned} .1\pi_1 + .3\pi_2 + .2\pi_3 &= \pi_1 \\ .2\pi_1 + .3\pi_2 + .4\pi_3 &= \pi_2 \\ .7\pi_1 + .4\pi_2 + .4\pi_3 &= \pi_3 \\ \pi_1 + \pi_2 + \pi_3 &= 1 \end{aligned}$$

Thus, $\pi_1 = .2114, \pi_2 = .3252, \pi_3 = .4634$.

6. Consider an inventory problem. The demand of a certain item per day is a random variable with the following probability distribution.

Demand, d	Probability
0	0.2
1	0.5
2	0.2
3	0.1

The ordering policy is as follows: If the inventory at the end of the day is zero, order three items; If the inventory at the end of the day is one, order 2 items; Otherwise do not order. Assume that the replenishment is instantaneous for the sake of simplicity. Also assume that the initial inventory level is either 1, 2, or 3.

(a) Identify transient states and identify each class of recurrent states.

States 1 is transient. States 2 and 3 constitute a class of recurrent states.

(b) Find the transition probability matrix.

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0.2 & 0.8 \\ 0 & 0.5 & 0.5 \end{bmatrix}$$

(c) Find the steady state probabilities, if any. If there is no steady state probabilities, explain why.

The Markov chain is not irreducible and, thus, there exists no steady state probabilities.