

SSIE 660: Stochastic Systems
Dr. Sung H. Chung
Note 2
Chapter 2. Random Variables

2.1 Random Variables

Random variables are real-valued functions defined on the sample space. Random variable are either discrete or continuous.

Ex.: Let X denote the random variable defined as the sum of two fair dice. (Discrete, Continuous)

$$P\{X = 5\} =$$

Ex.: Let X denote the life time of a car. (Discrete, Continuous)

$$P\{X = 5\} =$$

$$P\{2 \leq X \leq 5\} =$$

2.2 Discrete Random Variables

A random variable that can take on at most a countable number of possible values is said to be discrete.

Probability Mass Function of X :

$$p(a) = P\{X = a\}$$

If X must assume one of the values x_1, x_2, \dots then,

$$\sum_{i=1}^n p(x_i) = 1$$

Also,

$$p(x_i) > 0 \text{ for some } i$$

$$p(x_i) = 0 \text{ for all other } i$$

Cumulative Distribution Function of X:

$$F(a) = \sum_{x_i \leq a} p(x_i)$$

Ex.: Let X have a probability mass function given by

$$p(1) = 1/2, p(2) = 1/3, p(3) = 1/6$$

Then,

$$F(a) = \begin{cases} 0, & a < 1 \\ 1/2, & 1 \leq a < 2 \\ 5/6, & 2 \leq a < 3 \\ 1, & 3 \leq a \end{cases}$$

2.2.1 The Bernoulli Random Variables

If an outcome is either a "success" or a "failure" and if we let $X = 1$ for success, $X = 0$ for failure, then the probability mass function of X is given by

$$\begin{aligned} p(0) &= P(X = 0) = 1 - p \\ p(1) &= P(X = 1) = p \end{aligned}$$

where p is the probability of a success and $0 \leq p \leq 1$. Such random variable X is said to be a Bernoulli random variable.

2.2.2 The Binomial Random Variables

Binomial experiment conditions

1. fixed number of trials
2. either a "success" with probability p or a "failure" with probability of $1 - p$.
3. p is fixed for all trials.

Let X represent the number of success out of n trials, then X is a binomial random variable with parameters (n, p) .

Probability mass function:

$$p(i) = \binom{n}{i} p^i (1 - p)^{n-i}, i = 0, 1, \dots, n$$

where

$$\binom{n}{i} = {}_n C_i = \frac{n!}{(n-i)!i!} \quad (\text{Combination})$$

Note that

$${}_n P_i = \frac{n!}{(n-i)!} \quad (\text{Permutation})$$

Example 1. Four fair coins are flipped. If the outcomes are assumed independent, what is the probability that two heads and two tails are obtained?

Example 2. According to the Experian Automotive, 35% of all car-owning households have three or more cars. In a random sample of 400 car-owning households, what is the probability that 150 households have three or more cars?

Example 3. Assume that the probability that a plane arrives on time is 75% for all arriving planes in Albuquerque airport. Out of 15 planes, what is the probability that

1. 8 planes arrive on time?

2. more than 12 planes arrive on time?

2.2.3 The Geometric Random Variables

Suppose each independent trial has a fixed probability of "success", p . If X represents the number of trial required until the first success, then X is said to be a geometric random variable.

Probability mass function:

$$p(n) = P(X = n) = (1 - p)^{n-1}p, n = 1, 2, \dots$$

2.2.3 The Poisson Random Variables

If a random variable X takes on one of the values $0, 1, 2, \dots$, then X is said to be a Poisson random variable with parameter λ , if for some $\lambda > 0$ the probability mass function is defined as follows.

Probability mass function:

$$p(i) = P(X = i) = e^{-\lambda} \frac{\lambda^i}{i!}, i = 0, 1, 2, \dots$$

- The Poisson random variable has a wide range of applications
- If n is large and p is small, then it can be used to approximate a binomial random variable. That is,

$$P(X = i) \approx e^{-\lambda} \frac{\lambda^i}{i!}$$

where X is a binomial random variable.

Example 4. Suppose that the number of typographical errors on a single page of a book has a Poisson distribution with parameter $\lambda = 1$. Calculate the probability that there is at least one error on a certain page.

Example 5. If the number of accidents occurring on a highway each day is a Poisson random variable with parameter $\lambda = 3$, what is the probability that no accidents occur today?

2.3 Continuous Random Variables

A random variable X is continuous if its set of possible values is uncountable.

Probability density function

- defined for all real $x \in (-\infty, \infty)$.
- for any set B of real numbers,

$$P\{X \in B\} = \int_B f(x)dx$$

- also,

$$P\{X \in (-\infty, \infty)\} = \int_{-\infty}^{\infty} f(x)dx = 1$$

Ex.

$$P\{X \in (a, b)\} = \int_a^b f(x)dx$$

$$P\{X = a\} = \int_a^a f(x)dx = 0$$

Cumulative distribution

$$F(a) = P\{X \in (-\infty, a)\} = \int_{-\infty}^a f(x)dx$$

such that

$$\frac{d}{da}F(a) = f(a)$$

2.3.1 The Uniform Random Variable

A random variable X is said to be uniformly distributed on the interval (α, β) if its probability density function is given by

$$f(x) = \begin{cases} \frac{1}{\beta - \alpha}, & \text{if } \alpha < x < \beta \\ 0, & \text{otherwise} \end{cases}$$

Example 6. Calculate the cumulative distribution function of a random variable uniformly distributed over (α, β) .

Example 7. If X is uniformly distributed over $(0, 10)$, calculate the probability that (a) $X < 3$, (b) $1 < X < 6$.

2.3.2 Exponential Random Variables

A continuous random variable whose probability density function is given, for some $\lambda > 0$, by

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}$$

is said to be an exponential random variable with parameter λ .

Cumulative distribution

$$F(a) = \int_0^a \lambda e^{-\lambda x} dx = 1 - e^{-\lambda a}, \quad a \geq 0$$

Note that $F(\infty) = 1$.

2.3.3 Gamma Random Variables

A continuous random variable whose probability density function is given by, for some $\alpha > 0, \lambda > 0$

$$f(x) = \begin{cases} \frac{\lambda e^{-\lambda x} (\lambda x)^{\alpha-1}}{\Gamma(\alpha)}, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}$$

is said to be a gamma random variable with parameters α, λ .

Gamma function

$$\Gamma(\alpha) = \int_0^{\infty} e^{-x} x^{\alpha-1} dx$$

2.3.4 Normal Random Variables

A continuous random variable whose probability density function is given by, for some μ, σ^2

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}, \quad -\infty < x < \infty$$

is said to be a normal random variable with parameters μ, σ^2 .

$Y = \alpha X + \beta$ will also be normally distributed with parameters $\alpha\mu + \beta$, and $\alpha^2\sigma^2$.

Ex. Let us define Y as $Y = (X - \mu)/\sigma$, then Y will be normally distributed with parameters 0 and 1. Y is said to have the standard normal distribution.