SSIE 660: Stochastic Systems
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Exam 1, Nov. 16, 2016

Name: Key

1. A box in a certain supply room contains four 40w lightbulbs, five 60w bulbs, and six 75w bulbs. Suppose that two bulbs are randomly selected. What is the probability that one 60w and one 75w bulbs are selected? (4 points)

$$\frac{{}_5C_1 *_6 C_1}{{}_{15}C_2}$$

2. The joint density of X and Y is

$$f(x,y) = \frac{e^{-y}}{y},\ 0 < x < y,\ 0 < y < \infty$$

Calculate $E[X^3|Y=y]$. (7 points)

$$f_{X|Y}(x|y) = \frac{e^{-y}/y}{\int_0^y e^{-y}/y dx} = \frac{1}{y},\ 0 < x < y$$

$$E[X^3|Y=y] = \int_0^y x^3 \frac{1}{y} dx = y^3/4$$

3. The joint probability distribution of X and Y are given as follows: (2 points each, total 14 points)

X	1	1	-1	-1
Y	1	-1	1	-1
$P_{X,Y}(x,y)$	c	$\frac{1}{2}$	$\frac{1}{8}$	$\frac{1}{8}$

(a) What is the value of c?

$$c + 1/2 + 1/8 + 1/8 = 1 \rightarrow c = 1/4$$

(b) Find the pmf of X.

X	1	-1
$P_X(x)$	$\frac{3}{4}$	$\frac{1}{4}$

(c) Find the pmf of Y.

Y	1	-1
$P_Y(y)$	$\frac{3}{8}$	$\frac{5}{8}$

(d) Calculate $E[X]$.

$E[X] = 1 \cdot \frac{3}{4} + (-1) \cdot \frac{1}{4} = \frac{1}{2}$

(e) Calculate $E[Y]$.

$E[Y] = 1 \cdot \frac{3}{8} + (-1) \cdot \frac{5}{8} = -\frac{1}{4}$

(f) Calculate $E[XY]$.

$E[XY] = 1 \cdot \frac{1}{4} + (-1) \cdot \frac{1}{2} - 1 \cdot \frac{1}{8} + 1 \cdot \frac{1}{8} = -\frac{1}{4}$

(g) Calculate $Var[X]$.
$E[X^2] = 1$
$Var[X] = E[X^2] - (E[X])^2 = 1 - 1/4 = 3/4$

4. Each of the members of a seven judge panel will independently make a correct decision with probability .7.

(a) If the panel's decision is made by majority rule, what is the probability that the panel makes the correct decision? (5 points)

Let C be the event that the jury makes the correct decision, and let F be the event that four of the judges agreed. Then,

$$P(C) = \sum_{i=4}^{7} {}_7C_i(.7)^i(.3)^{7-i} \approx 0.874$$

(b) Given that 4 of the judges agreed, what is the probability that the jury made the correct decision? (5 points)

$$P(C|F) = \frac{P(CF)}{P(F)} = \frac{{}_7C_4(.7)^4(.3)^3}{{}_7C_4(.7)^4(.3)^3 + {}_7C_3(.7)^3(.3)^4} = .7$$

5. The joint density function of X and Y is

$$f_{X,Y}(x,y) = \begin{cases} x + ay & 0 < x < 1, 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

(a) Calculate a. (5 points)

$$\begin{aligned} \int_0^1 \int_0^1 (x+ay)dxdy &= \int_0^1 \left[\frac{1}{2}x^2 + ayx\right]_0^1 dy \\ &= \int_0^1 \left(\frac{1}{2} + ay\right) dy \\ &= \left[\frac{1}{2}y + \frac{1}{2}ay^2\right]_0^1 = \frac{1}{2} + \frac{1}{2}a = 1 \end{aligned}$$

Therefore, $a = 1$.

(b) Find the density function of Y. (5 points)

$$\begin{aligned} f_Y(y) = \int_0^1 (x+y)dx &= \left[\frac{1}{2}x^2 + yx\right]_0^1 \\ &= \frac{1}{2} + y, \quad 0 < y < 1 \end{aligned}$$

(c) Find $P\{X + Y < 1/2\}$. (5 points)

$$\begin{aligned} P\{X+Y < 1/2\} &= \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}-x} (x+y)dydx \\ &= \int_0^{\frac{1}{2}} \left[xy + \frac{1}{2}y^2\right]_0^{\frac{1}{2}-x} dx \\ &= \int_0^{\frac{1}{2}} \left(\frac{1}{8} - \frac{1}{2}x^2\right) dx \\ &= \left[\frac{1}{8}x - \frac{1}{6}x^3\right]_0^{\frac{1}{2}} = \frac{1}{24} \end{aligned}$$

6. Buses arrive at s specified stop at 20-minute intervals starting at 7am. That is, they arrive at 7, 7:20, 7:40, 8, 8:20, and so on. If a passenger arrives at the stop at a time that is uniformly distributed between 7:30 and 7:50, find the probability that he waits less than 5 minutes for a bus. (5 points)

$$f(x) = \begin{cases} \dfrac{1}{20} & 30 \le x \le 50 \\ \\ 0 & o.w \end{cases}$$

$$P(5 \le x \le 10) = \frac{1}{4}$$

7. Consider a Markov chain with states 1, 2, and 3 having the following transition probability matrix.

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0.3 & 0.3 & 0.4 \\ 0 & 0.6 & 0.4 \end{bmatrix}$$

   (a) Identify the transient states and identify each class of recurrent states. (5 points)

   States 1, 2, and 3 constitute a class of recurrent states. That is, $\{1, 2, 3\}$.

   (b) Identify a period (aperiodic if a state does not have any) for each state. (5 points)

   All states are aperiodic.

   Consider the number of steps (transitions) required to go from 1 to 1. This could be 1,2,3,4.... . The greatest common divisor = 1.
   Same applies to 2 and 3.
   Thus, all states are aperiodic

   (c) If the Markov chain has steady state probabilities, find them. If not, explain why. (5 points)

   In order for the Markov chain to have steady state probabilities, it should be:
   a. Ergodic : which means being both recurrent and aperiodic
   b. Irreducible.
   Since both apply, the Markov Chain has steady state probabilities.

$$\pi_1 = .1216, \pi_2 = .4054, \pi_3 = .4730$$

8. Suppose that a system (say, a bridge) changes states in accordance with the following transition probability matrix:

$$\begin{bmatrix} 3/4 & 1/4 & 0 \\ 0 & 1/2 & 1/2 \\ 3/4 & 1/8 & 1/8 \end{bmatrix}$$

Assume that state 1 is "Acceptable' and classified as an 'Up' state and that states 2 and 3 are "Unacceptable' and classified as 'Down' states.

(a) Find the rate at which the system goes from 'Up' to 'Down'. (5 points)

Looking at the transition probability matrix, going from UP to Down means either going from state 1 to 2 or going from state 1 to 3, with probabilities P_{12} and P_{13}. The chance for the latter,however, is 0. Thus, the rate of breakdown is the multiplication of P_{12} by the steady state probability for the first state.

Steady state probabilities: $\pi_1 = 12/23, \pi_2 = 7/23, \pi_3 = 4/23$.

Rate of breakdown:

$$\pi_1 * (P_{12} + P_{13}) = 12/23 * (1/4) = 3/23 \approx .13$$

(b) What is the average length of time the system remains Down when it goes Down? (5 points)

$$\frac{\pi_2 + \pi_3}{\pi_1 * (P_{12} + P_{13})} = \frac{11/23}{3/23} = \frac{11}{3}$$

(c) What is the average length of time the system remains Up when it goes Up? (5 points)

$$\frac{\pi_1}{\pi_2 * P_{21} + \pi_3 * P_{31}} = \frac{\frac{12}{23}}{\frac{4}{23} \cdot \frac{3}{4}} = 4$$

9. Consider an inventory problem. The demand of a certain item per day is a random variable with the following probability distribution.

Demand, d	Probability
0	0.1
1	0.4
2	0.3
3	0.2

The ordering policy is as follows: If the inventory at the end of the day is zero, order three items; If the inventory at the end of the day is one, order 2 items; If the inventory at the end of the day is two, order 1 item; Otherwise do not order. Assume that the replenishment is instantaneous for the sake of simplicity. Also assume that the initial inventory level is either 1, 2, or 3.

(a) Identify transient states and identify each class of recurrent states. (5 points)

$\{1\}$: Transient, $\{2\}$: Transient, $\{3\}$: Recurrent

(b) Find the transition probability matrix. (5 points)

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

(c) Find the steady state probabilities, if any. If there is no steady state probabilities, explain why. (5 points)

No, because only irreducible ergodic Markov chains have steady state probabilities.