SSIE 660: Stochastic Systems

Homework assignment 3

1. Solve Chapter 2. Problem 1.

   Sample space = $\{BB, RR, OO, RB, OB, OR\}$

   $P\{X = 0\} = \frac{\binom{7}{2}}{\binom{10}{2}} = \frac{14}{30}$

2. Solve Chapter 2. Problem 9

   $P\{X = 0\} = \frac{1}{2}$
   $P\{X = 1\} = \frac{1}{10}$
   $P\{X = 2\} = \frac{1}{5}$
   $P\{X = 3\} = \frac{1}{10}$
   $P\{X = 3.5\} = \frac{1}{10}$

3. Solve Chapter 2. Problem 20.

   $= \frac{5!}{2!1!2!} \left[\frac{1}{5} \right]^2 \left[\frac{3}{10} \right]^2 \left[\frac{1}{2} \right]^1 = .054$

4. Solve Chapter 2. Problem 33.

$$c \int_{-1}^{1} (1 - x^2) dx = 1$$

$$c - \left[x - \frac{x^3}{3} \right] \Big|_{-1}^{1} = 1$$

$$c = \frac{3}{4}$$

$$F(X) = \frac{3}{4} \int_{-1}^{1} (1 - x^2) dx$$

$$= \frac{3}{4} \left[x - \frac{x^3}{3} + \frac{2}{3} \right]$$

5. Solve Chapter 2. Problem 54.
(a) Using the fact that $E[X+Y]=0$, we see that $0=2(p(1,1)-2p(-1,-1)$, which gives the result.

(b) This follows since

$$0 = E[X-Y] = 2p(1,-1) - 2p(-1,1)$$

(c) Var(X) = $E[X^2]=1$.

(d) Var(Y) = $E[Y^2]=1$.

(e) Since

$$\begin{aligned} 1 &= p(1,1)+p(-1,1)+p(1,-1)+p(-1,-1) \\ &= 2p(1,1)+2p(1,-1) \end{aligned}$$

we see that if $p=2p(1,1)$ then

$$1-p = 2p(1,-1)$$

Now,

$$\begin{aligned} Cov(X,Y) &= E[XY] \\ &= p(1,1)+p(-1,-1)-p(1,-1)-p(-1,1) \\ &= p-(1-p) = 2p-1 \end{aligned}$$

6. Solve Chapter 2. Problem 76.

$$\begin{aligned} &P\left\{\left|\frac{X_1+\cdots+X_n-n\mu}{n}\right|>\epsilon\right\} \\ &= P\{|X_1+\cdots+X_n-n\mu|>n\epsilon\} \\ &\leq Var(X_1+\cdots+X_n)/n^2\epsilon^2 \\ &= n\sigma^2/n^2\epsilon^2 \to 0 \text{ as } n\to\infty \end{aligned}$$

7. Given that X and Y are two independent random variables with the following joint probability distributions,

x	0	1	2	3	4
$p(X=x)$	1/8	1/8	1/4	1/3	1/6
y	1	2	3	4	5
$p(Y=y)$	1/4	1/8	1/8	1/4	1/4

Find $P[X\leq Y+1]$.

$P[X\leq Y+1] = P[Y=1]*P[X\leq 2]+P[Y=2]*P[X\leq 3]+P[Y=3]*P[X\leq 4]+P[Y=4]*P[X\leq 5]+P[Y=5]*P[X\leq 6] = \frac{1}{4}\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{4}\right)+\frac{1}{8}\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{4}+\frac{1}{3}\right)+$

$$\frac{1}{8}\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{4}+\frac{1}{3}+\frac{1}{6}\right)+\frac{1}{4}\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{4}+\frac{1}{3}+\frac{1}{6}\right)+\frac{1}{4}\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{4}+\frac{1}{3}+\frac{1}{6}\right)=\frac{41}{48}=0.8542$$

8. X and Y are two independent random variables with the following probability density functions.

$$f_X(x) = \begin{cases} 1/4, & 2 < x < 6 \\ 0, & \text{elsewhere} \end{cases}$$

$$f_Y(y) = \begin{cases} 1/6, & 1 < y < 7 \\ 0, & \text{elsewhere} \end{cases}$$

Find $P[X \leq Y]$.

$P[X \leq Y] = \int_1^7 F_X(y) f_Y(y) dy = \int_1^7 \frac{1}{6} F_X(y) dy.$

Now we have to find $F_X(y)$ using the density function of X.

$y < 2 : F_X(y) = 0$
$2 < y < 6 : F_X(y) = \int_2^y \frac{1}{4} dx = \frac{y-2}{4}$
$6 < y : F_X(y) = 1$

$P[X \leq Y] = \int_1^7 F_X(y) f_Y(y) dy = \int_1^2 \frac{1}{6}(0) dy + \int_2^6 \frac{1}{6}\frac{y-2}{4} dy + \int_6^7 \frac{1}{6}(1) dy = 0.5.$