SSIE 660: Stochastic Systems
Dr. Sung H. Chung
Note 6
Chapter 3. Conditional Probability/Expectation

3.4 Computing Expectations by Conditioning

Let $E[X|Y] = E[X|Y = y]$: a R.V.

Note: $E[X] = E[E[X|Y]]$. That is:

$$E[X] = \sum_y E[X|Y = y]P\{Y = y\}, \quad y : \text{discrete}$$
$$E[X] = \int_{-\infty}^{\infty} E[X|Y = y]f_Y(y)dy, \quad y : \text{continuous}$$

Proof:

$$\sum_y E[X|Y]P[y] = \sum_y \frac{\sum_x x \cdot P(x,y)}{\cancel{P(y)}} \cancel{P(y)}$$

$$= \sum_y \sum_x x \, P(x,y)$$

$$= \sum_x \sum_y x \, P(x,y)$$

$$= \sum_x x \cdot \sum_y P(x,y)$$

$$= \sum_x x \cdot P(x) = E[X]$$

Example 1. Sam will read either one chapter of his probability book or one chapter of his history book. The number of misprints in a chapter is Poisson distributed with mean 2 (probability book) and 5 (history book), respectively. If Sam is equally likely to choose either book, what is the expected number of misprints that Sam will come across?

X: the number of misprints
Y: 1 (history book), 2 (probability book)

$E[X] =?$ $\quad E[X] = E[X \mid Y=1]\, P(Y=1) + E[X \mid Y=2]\, P(Y=2)$

$$= 5 \cdot \left(\tfrac{1}{2}\right) + 2\left(\tfrac{1}{2}\right) = 7/2$$

Example 2. A coin, having probability p of coming up heads, is to be successively flipped until the first head appears. What is the expected number of flips required?

$Y = \begin{cases} 1 & \text{head} \\ 0 & \text{tail} \end{cases}$ $\quad X$: # of flips $\quad \begin{matrix} p \\ 1-p \end{matrix}$

$$E[X] = E[X \mid Y=1]\, P(Y=1) + E[X \mid Y=0]\, P(Y=0)$$

$$= E[X \mid Y=1] \cdot p + E[X \mid Y=0]\,(1-p)$$

$$= 1 \cdot p + (1 + E[X])(1-p)$$

$$= \cancel{p} + 1 - \cancel{p} + \cancel{E[X]} - p\,E[X] = \cancel{E[X]}$$

$$\Rightarrow 1 - p\,E[X] = 0 \qquad E[X] = \frac{1}{p}.$$

Example 3. A miner is trapped in a mine containing three doors. The first door leads to a tunnel that takes him safety after two hours of travel. The second door leads to a tunnel that returns him to the mine after three hours of travel. The third door leads to a tunnel that returns him to his mine after five hours. Assuming that the miner is at all times equally likely to choose any one of the doors, what is the expected length of time until the miner reaches safety?

$Y = 1$ 1st $\quad X$: time

$\quad\;\; 2$ 2nd

$\quad\;\; 3$ 3rd

$$E[X] = E[X \mid Y=1]\, p(Y=1) + E[X \mid Y=2]\, p(Y=2) + E[X \mid Y=3]\, p(Y=3)$$

$$= (2)\cdot\frac{1}{3} + (3 + E[X])\cdot\frac{1}{3} + (5 + E[X])\cdot\frac{1}{3}$$

$$= \frac{2}{3} + 1 + \frac{1}{3}E[X] + \frac{5}{3} + \frac{1}{3}E[X]$$

$$= \frac{10}{3} + \frac{2}{3}E[X] = E[X]$$

$$\boxed{\therefore\ E[X] = 10.}$$

3.5 Computing Probabilities by Conditioning

Suppose that X and Y are independent random variables. Find $P[X \leq Y]$.
As $P[X \leq Y]$ depends upon the value of Y, we have to condition on Y.

Discrete: $P[X \leq Y] = \sum_y P[X \leq y]P[Y = y]$

Continuous: $P[X \leq Y] = \int_y F_X(y) f_y(y) dy$

Example 4.

x	$P[X = x]$	y	$P[Y = y]$
1	0.2	0	0.1
2	0.4	2	0.5
3	0.3	3	0.3
4	0.1	6	0.1

Find $P[X \leq Y]$

y	$P[Y = y]$	$P[X \leq Y]$
0	.1	0
2	.5	.6
3	.3	.9
6	.1	1

Therefore, $P[X \leq Y] =$ $.5 \times .6 + .3 \times .9 + .1 \times 1$

$= .3 + .27 + .1 = .67$

Example 5.

$$f_X(x) = \begin{cases} \frac{1}{4}, & 2 \le x \le 6 \\ 0, & \text{elsewhere} \end{cases}$$

$$f_Y(y) = \begin{cases} \frac{1}{5}, & 7 \le y \le 12 \\ 0, & \text{elsewhere} \end{cases}$$

Find $P[X \le Y]$.

Recall $P[X \le Y] = \int_y F_X(y) f_y(y) dy$

$F_X(y) = \int_2^y \frac{1}{4} dx = \left[\frac{1}{4} x \right]_2^y = \frac{y}{4} - \frac{1}{2}$ $\boxed{2 \le y \le 6}$

$P[X \le Y] = \int_7^{12} \left(\frac{1}{2} - \frac{y}{4} \right)\left(\frac{1}{5} \right) dy$

$= 1$

Example 6.

$$f_X(x) = \begin{cases} \frac{1}{4}, & 2 \le x \le 6 \\ 0, \text{ elsewhere} \end{cases}$$

$$f_Y(y) = \begin{cases} \frac{1}{7}, & 1 \le y \le 8 \\ 0, \text{ elsewhere} \end{cases}$$

Find $P[X \le Y]$.

Recall $P[X \le Y] = \int_y F_X(y) f_y(y) dy$

$$F_X(y) = \int_2^y \frac{1}{4}\, dx = \left[\frac{1}{4}\right]_2^y = \frac{y}{4} - \frac{1}{2} \qquad 2 \le y \le 6$$

$$P[X \le Y] = \int_1^2 F_x(y) f_y(y)\, dy + \int_2^6 F_x(y) f_y(y)\, dy + \int_6^8 F_x(y) f_y(y)\, dy$$

$$= 0 + \int_2^6 \left(\frac{y}{4} - \frac{1}{2}\right) \cdot \frac{1}{7}\, dy + \int_6^8 1 \cdot \frac{1}{7}\, dy$$

$$= \frac{1}{7} \left[\frac{y^2}{8} - \frac{1}{2} y\right]_2^6 + \frac{2}{7}$$

$$= \frac{\cancel{32}^{\,4}}{\cancel{8}} - \frac{\cancel{4}}{\cancel{2}}\, 2$$

$$= \frac{2}{7} + \frac{2}{7} = \frac{4}{7}$$

Example 7.

$$f_{X,Y}(x,y) = \begin{cases} \dfrac{6-x-y}{8}, & 0 < x < 2,\ 2 < y < 4, \\ 0, \text{ elsewhere} \end{cases}$$

Find $P[2 < Y < 3 | 1 < X < 2]$.

$$= \frac{P(2<y<3,\ 1<x\le 2)}{P_x(1 \le x \le 2)} = \frac{1/4}{3/8} = 2/3$$

$$f_x(x) = \int_2^4 \frac{6-x-y}{8}\,dy = \left[\frac{6-x}{8}\cdot y - \frac{y^2}{16}\right]_2^4$$

$$= \frac{6-x}{4} - \frac{16-4}{16} = \frac{6-x-3}{4} = \frac{3-x}{4}$$

$$\therefore P(1 \le x \le 2] = \int_1^2 \left(\frac{3-x}{4}\right)dx = \frac{3}{4} - \left[\frac{x^2}{8}\right]_1^2$$

$$= \frac{3}{4} - \frac{3}{8} = \frac{3}{8}$$

$$P(2 \le y \le 3,\ 1 \le x \le 2) = \int_1^2 \int_2^3 \frac{6-x-y}{8}\,dy\,dx$$

$$= \frac{1}{4}.$$

$\therefore$

Example 8 (The expectations of the sum of a random number of random variables)**.** Suppose that the expected number of accidents per week at an industrial plant is four. Suppose also that the number of workers injured in each accident are independent random variables with a common mean of 2. Assume also that the number of workers injured in each accident is independent of the number of accident that occur. What is the expected number of injuries during a week?

N : # of accidents $\quad X_i$: # of injured in the ith accident

$$E\left[\sum_{i=1}^{N} X_i\right] = E\left[E\left[\sum_{i=1}^{N} X_i \mid N\right]\right]$$

$\because$ independence

But $E\left[\sum_{i=1}^{N} X_i \mid N=n\right] = E\left[\sum_{i=1}^{n} X_i \mid N=n\right] = E\left[\sum X_i\right]$

$= n E[X]$

which yields

$$E\left[\sum_{i=1}^{N} X_i \mid N=n\right] = \underline{N E[X]}$$

$$\therefore E\left[\sum_{i=1}^{N} X_i\right] = E[N \cdot E[X]] = E[N] \cdot E[X] = 8$$

Proposition 9. *The conditional variance formula*

$$Var(X) = E[Var(X|Y)] + Var(E[X|Y])$$

① $E[Var(X|Y)] = E\left[E[X^2|Y] - (E[X|Y])^2\right]$

$= E\left[E[X^2|Y]\right] - E\left[(E[X|Y])^2\right]$

$= E[X^2] - E\left[(E[X|Y])^2\right]$

② $Var(E[X|Y]) = E\left[(E[X|Y])^2\right] - \left(E[E[X|Y]]\right)^2$

$= E\left[(E[X|Y])^2\right] - (E[X])^2$

$\therefore$ ① + ② $= E[X^2] - (E[X])^2$

Example 10. Let $X_1, X_2, ...$ be i. i. d. random variables with mean μ and variance σ^2. Also assume that they are independent of the nonnegative integer valued random variable N. As noted in the previous Example, where its expected values was determined, the random variable $S = \sum_{i=1}^{N} X_i$ is called a compound random variable. Find its variance.

$$Var(S) = E[Var(S|N)] + Var(E[S|N])$$

$$Var(S|N) = Var(S|N=n) = Var\left(\sum_{i=1}^{N} X_i \,\middle|\, N\right)$$

$$= Var\left(\sum_{i=1}^{n} X_i \,\middle|\, N=n\right) = Var\left(\sum_{i=1}^{n} X_i\right) = n\sigma^2$$

By the same reasoning,

$$E[S|N] = N.\mu.$$

$$\therefore Var(S) = E[N\sigma^2] + Var(n\mu)$$

$$= \sigma^2 E[N] + \mu^2 Var(N)$$