SSIE 660: Stochastic Systems
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Note 3
Chapter 2. Random Variables

2.4 Expectation of a Random Variables

Discrete case: Let X be a discrete random variable with a probability mass function $p(x)$, then the expected value of X is defined by

$$E[X] = \sum_{x:p(x)>0} xp(x)$$

The weighted average of the possible values of X can take on, each being weighted by the probability that X assumes that value.

Example 1. Find $E[X]$ where X is the outcome when we roll a fair die.

Example 2. Calculate $E[X]$ where X is binomially distributed with parameters n and p.

Continuous case: Let X be a continuous random variable with a probability density function $f(x)$, then the expected value of X is defined by

$$E[X] = \int_{-\infty}^{\infty} xf(x)dx$$

Example 3. Find $E[X]$ where X is a random variable uniformly distributed over (α, β).

Expectation of a Function of a Random Variable: Sometimes we may want to calculate $E[g(X)]$ where $g(X)$ is a function of a random variable X.

Example 4. Calculate $E[X^2]$ when X has the following probability mass function.

$$p(0) = 0.2, p(1) = 0.5, p(2) = 0.3$$

Proposition 5. *(a) If X is a discrete random variable with probability mass function $p(x)$, then for any real-valued function g*

$$E[g(X)] = \sum_{x:p(x)>0} g(x)p(x)$$

(b) If X is a continuous random variable with probability density function $f(x)$, then for any real-valued function g

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)dx$$

Example 6. Calculate $E[X^3]$ when X is uniformly distributed over (0,1).

Corollary 7. *If a and b are constants, then*

$$E[aX + b] = aE[X] + b$$

Note that $E[X]$ is called the first moment of X and $E[X^n], n \geq 1$ is called the nth moment of X.

Also,

$$E[a_1 X_1 + ... + a_n X_n] = a_1 E[X_1] + ... + a_n E[X_n]$$

Variance of a Random Variable is denoted by Var(X), and defined by

$$\text{Var}(X) = E[(X - E[X])^2]$$

Example 8. Calculate Var(X) when X is continuous with density f and $E[X] = \mu$.

Note: Var(X) = $E[X^2] - (E[X])^2$.

Example 9. Find $\text{Var}(X)$ where X is the outcome when we roll a fair die.

2.5 Jointly Distributed Random Variables

Joint Distribution of Random Variables: Let X and Y be two ramdom variables.

Discrete

Joint probability mass function: $p_{X,Y}(x,y) = P\{X = x, Y = y\}$

Marginal Probability Distributions:

X :

$$p_X(x) =$$

Y :

$$p_Y(y) =$$

Conditional Probability:

$$P(X = x|Y = y) =$$

$$P(Y = y|X = x) =$$

Continuous

X and Y are jointly continuous if there exists a joint probability density function: $f_{X,Y}(x,y)$, for all sets A and B of real numbers

$$P\{X \in A, Y \in B\} = \int_B \int_A f(x,y)dxdy$$

Marginal:

Marginal density function of X:

$$f_X(x) =$$

Marginal density function of Y:

$$f_Y(y) =$$

Conditional Probability:

$$f_{X|Y}(x|y) =$$

$$f_{Y|X}(y|x) =$$

Example 10. Let X and Y represent the proportions of time the drive-in and walk-in facilities of a fast food restaurant are in use, respectively. The joint density function of X and Y is

$$f_{X,Y}(x,y) = \begin{cases} \dfrac{6(x+y^2)}{5}, \ 0 \le x \le 1, 0 \le y \le 1 \\ 0, \text{ elsewhere} \end{cases}$$

1. Find the marginal probability density functions of X and Y.
2. Find the conditional probability that the drive-in facility is used less than 1/2 the time, given that the walk-in facility is used 1/4 time.

Independence of two RVs:

X and Y are independent if $F(x,y) = F(x)F(y)$.

X and Y are independent if and only if for all x and y,

$$P(X = x, Y = y) = P(X = x)P(Y = y)$$

or

$$f_{XY}(x,y) = f_X(x)f_Y(y)$$

If X and Y are independent, then

$$E[g(x), h(y)] = E[g(x)]E[h(y)]$$

Variance:

$$\mathrm{Var}[a_1X_1 + a_2X_2 + ...a_nX_n] = a_1^2\mathrm{Var}(X_1) + ... + a_n^2\mathrm{Var}(X_n) + 2\sum_{i=1}^{n}\sum_{j<i} a_ia_j\mathrm{Cov}(X_i, X_j)$$

Covariance of X and Y = $\mathrm{Cov}(X, Y) = E[(X - E[X])(Y - E[Y])] = E[XY] - E[X]E[Y]$

If X and Y are independent, then $\mathrm{Cov}(X, Y) = 0$. However, $\mathrm{Cov}(X, Y) = 0$ does not mean X and Y are independent.

$\mathrm{Cov}(X, Y) = \mathrm{Cov}(Y, X)$

$\mathrm{Cov}(X, X) = \mathrm{Var}(X)$

$\mathrm{Cov}(X, Y) = \sigma_X\sigma_y\rho_{XY}$ where ρ_{XY} is the correlation coefficient. $\rho_{XY} \in [-1, 1]$.

Example 11. Let X denote the reaction time, in seconds, to a certain stimulant and Y denote the temprature (^{0}F) at which a certain reaction starts to take place. Suppose that two random variables X and Y have the joint density

$$f_{X,Y}(x, y) = \begin{cases} 4xy,\ 0 \le x \le 1, 0 \le y \le 1 \\ 0,\ \text{elsewhere} \end{cases}$$

Find

1. $P\left(0 < X < \frac{1}{2} \text{ and } \frac{1}{4} < Y < \frac{1}{2}\right)$
2. $P(X < Y)$
3. Are X and Y independent?

Example 12. Let X denote the diameter of an armored electric cable and Y denote the diameter of the ceramic mold that makes the cable. Both X and Y are scaled so that they range between 0 and 1. Suppose that two random variables X and Y have the joint density

$$f_{X,Y}(x,y) = \begin{cases} \dfrac{1}{y}, \ 0 < x < y < 1 \\ 0, \text{ elsewhere} \end{cases}$$

1. Find $P(X + Y \geq \frac{1}{2})$
2. Are X and Y independent?