SSIE 660: Applied Stochastic Processes
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Note 9
Chapter 4. Discrete Markov Chains

True / False Test

1. (T F) All states in a finite Markov Chain (with finite number of states) can be transient.

2. (T F) All states in a finite irreducible Markov Chain are recurrent.

3. (T F) No transient state can be reached from any recurrent state.

4. (T F) From a recurrent state, only recurrent states can be reached.

- A recurrent state i is said to be *positive recurrrent*, if starting in i, the expected time until the system returns to state i is

  Otherwise, the recurrent state is

- Positive recurrence is a class property. If any one state in a class is positive recurrent, then

- In a finite Markov Chain,

- Let δ be the greatest common divisor of the set of all $n \geq 1$, for which $P_{jj}^n > 0$.
  1. If $\delta = 1$, then

  2. If $\delta \geq 2$, then

Let us recall the previous five examples.

(A)

$$P = \begin{bmatrix} 0.3 & 0 & 0.7 \\ 0.4 & 0.3 & 0.3 \\ 0.2 & 0.4 & 0.4 \end{bmatrix}$$

(B)

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

(C)

$$P = \begin{bmatrix} 0.3 & 0.3 & 0.4 \\ 0.4 & 0.2 & 0.4 \\ 0 & 0 & 1 \end{bmatrix}$$

(D)

$$P = \begin{bmatrix} 0.2 & 0.8 & 0 & 0 \\ 0.4 & 0.6 & 0 & 0 \\ 0 & 0 & 0.3 & 0.7 \\ 0 & 0 & 0.5 & 0.5 \end{bmatrix}$$

One more,
(E)

$$P = \begin{bmatrix} 0 & 0.5 & 0.5 \\ 1 & 0 & 0 \\ 0 & 0.5 & 0.5 \end{bmatrix}$$

- Ergodic: A state which is positive recurrent and aperiodic is called ergodic.

- If all the states in a Markov Chain are ergodic, then the Markov Chain is said to be ergodic.

- Now let us check which of the four Markov Chains we have been using as examples are irreducible and ergodic Markov Chains.

(A)

$$P = \begin{bmatrix} 0.3 & 0 & 0.7 \\ 0.4 & 0.3 & 0.3 \\ 0.2 & 0.4 & 0.4 \end{bmatrix}$$

(B)

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

(C)

$$P = \begin{bmatrix} 0.3 & 0.3 & 0.4 \\ 0.4 & 0.2 & 0.4 \\ 0 & 0 & 1 \end{bmatrix}$$

(D)

$$P = \begin{bmatrix} 0.2 & 0.8 & 0 & 0 \\ 0.4 & 0.6 & 0 & 0 \\ 0 & 0 & 0.3 & 0.7 \\ 0 & 0 & 0.5 & 0.5 \end{bmatrix}$$

Theorem 1 (Thm 4.1 in the text book)**.** *For an irreducible ergodic Markov Chain,* $\lim_{n\to\infty} P_{ij}^n$ *exists and is independent of i (starting state).* $\lim_{n\to\infty} P_{ij}^n = \pi_j$. *These are called stationary or steady state probabilities.*

As per this theorem, (   ) can have steady state probabilities.

How to find steady state probabilities

Let us look at P^{12} and P^{13} of (A).

$$P^{12} = \begin{bmatrix} 0.280 & 0.262 & 0.458 \\ 0.280 & 0.262 & 0.458 \\ 0.280 & 0.262 & 0.458 \end{bmatrix}, P^{13} = \begin{bmatrix} 0.280 & 0.262 & 0.458 \\ 0.280 & 0.262 & 0.458 \\ 0.280 & 0.262 & 0.458 \end{bmatrix}$$

(Here, the system reaches steady state when $n = 12$, subjecting to rounding off. Most Markov Chains ready steady state when $n < \infty$. Theoretically, steady state is achieved only when $n = \infty$.)

- As we noted earlier:
  - Rows of P^n are the same.
  - Row of f P^{n+1} are the same as those of P^n.
- We know that $P^{n+1} =$ or
- In (A),

$$\begin{bmatrix} 0.280 & 0.262 & 0.458 \\ 0.280 & 0.262 & 0.458 \\ 0.280 & 0.262 & 0.458 \end{bmatrix} * \begin{bmatrix} 0.3 & 0 & 0.7 \\ 0.4 & 0.3 & 0.3 \\ 0.2 & 0.4 & 0.4 \end{bmatrix} = \begin{bmatrix} 0.280 & 0.262 & 0.458 \\ 0.280 & 0.262 & 0.458 \\ 0.280 & 0.262 & 0.458 \end{bmatrix}$$

- The row of P^{12} and P^{13} are the same and hence any row of P^{12} and P^{13} can be considered. Let us look at row 1.

$$\begin{bmatrix} 0.280 & 0.262 & 0.458 \end{bmatrix} * \begin{bmatrix} 0.3 & 0 & 0.7 \\ 0.4 & 0.3 & 0.3 \\ 0.2 & 0.4 & 0.4 \end{bmatrix} = \begin{bmatrix} 0.280 & 0.262 & 0.458 \end{bmatrix}$$

- In this example, $\pi_1 =$, $\pi_2 =$, $\pi_3 =$.
- In general,

$$\pi = [\pi_1 \ \ \pi_2 \ \ \cdots \ \ \pi_m] \tag{1}$$

- From (1), we obtain the following equations:

(equations here)

- The equations to be solved are:

$$-0.7\pi_1 + 0.4\pi_3 = 0 \tag{2}$$
$$0.7\pi_1 + 0.3\pi_2 - 0.6\pi_3 = 0 \tag{3}$$

and

- We have already obtained the following relations using (2) and (3):

$$\pi_2 = \frac{4}{7}\pi_3, \quad \pi_1 = \frac{3}{4.9}\pi_3$$

- In general,

$$[\pi_1 \;\; \pi_2 \;\; \cdots \;\; \pi_m] \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1m} \\ P_{21} & P_{22} & \cdots & P_{2m} \\ \vdots & & & \vdots \\ P_{m1} & P_{m2} & \cdots & P_{mm} \end{bmatrix} = [\pi_1 \;\; \pi_2 \;\; \cdots \;\; \pi_m]$$

and