SSIE 660: Stochastic Systems
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Note 4
Chapter 2. Random Variables

2.5 Jointly Distributed Random Variables

Probability Distribution (Density) Function of the Sum of Two Independent Random Variables

Discrete Random Variables

Example 1. Given that:

x	1	2	3	4	5	6	7
$p_X(x)$	1/8	1/16	1/4	3/16	1/32	3/32	1/4
y	2	3	4	5	6	7	8
$p_Y(y)$	2/24	4/24	6/24	5/24	4/24	2/24	1/24

Let $Z = X + Y$, then what is the probability distribution of Z? X, Y independent.

Let us first consider $P[Z = 5]$.

$P[Y=2, X=5-2] + P[Y=3, X=5-3] + P[Y=4, X=5-4]$

$\frac{2}{24} \times \frac{1}{4} + \frac{4}{24} \cdot \frac{1}{16} + \frac{6}{24} \cdot \frac{1}{8} = \frac{1}{16}$

$= P[Y=2]P[X=3] + P[Y=3]P[X=2] + P[Y=4]P[X=1]$

Writing the right hand side as a summation,
$P[Z = 5] =$

$$\sum_{y=2}^{4} P[Y=y, X=5-y]$$

In general: $P[Z = z] =$

$$P[Z=z] = \sum_{y=LL}^{UL} P[Y=y, X=z-y] = \sum_{y=LL}^{UL} P[Y=y]P[X=z-y]$$

Determining LL and UL

Let us consider all combinations of Y and X which yield $Z = 5$.

From the previous page

$$P[Z = 5] = \sum_{y=2}^{4} P[Y = y] * P[X = 5 - y]$$

In this, LL of Y =2, which is Y_{min}

Let us consider another example.

Example 2. Given that:

x	-2	-1	0	1	2	3	4
$p_X(x)$	1/10	1/10	2/10	2/10	2/10	1/10	1/10
y	0	1	2	3	4	5	6
$p_Y(y)$	2/24	4/24	6/24	5/24	4/24	2/24	1/24

In this example, find $P[Z = 5]$, where $Z = X + Y$.

$P[Z=5] = (P[Y=1]*P[X=5-1]) + (P[Y=2]*P[X=5-2]) + (P[Y=3]*P[X=5-3]) + (P[Y=4]*P[X=5-4]) + (P[Y=5]*P[X=5-5]) + (P[Y=6]*P[X=5-6])$

In this example, LL of Y is 1

Therefore, in general we have two candidates for LL: $\underline{Y_{min}}$, $\underline{Z - X_{max}}$

How do we select the LL to be used in any problem?

The general formula for the LL is:

$$LL = \max(Y_{min}, Z - X_{max})$$

Using the same approach, we will devise a general formula for UL.

The general formula for the UL is:

$$\cancel{LL}\ UL = \min(Y_{max}, Z - X_{min})$$

Now we can write the general formula for $P[Z=z]$ as:

$$P[Z=z] = \sum_{y=LL}^{UL} P[Y=y, X=z-y]$$

<u>Continuous Random Variables</u> The formula for the density function $Z = X + Y$ can be written as

$$X \sim f_X(x), X_{min} < X < X_{max}$$
$$Y \sim f_Y(y), Y_{min} < Y < Y_{max}$$
$$Z \sim f_Z(z), Z_{min} < Z < Z_{max}$$
$$Z_{max} = X_{max} + Y_{max}, \;\; Z_{min} = X_{min} + Y_{min}$$

Then,

$$f_Z(z) = \int_{LL}^{UL} f_Y(y) f_X(z-y) dy$$

$$x = z - y$$

Example 3. Find the probability density function $Z = X + Y$, where X and Y follow the density functions given below:

$$f_X(x) = 1/4, \;\; 6 < X < 10$$
$$f_Y(y) = \lambda e^{-\lambda y}, \;\; y > 0$$

Range of Z: $\underline{(6, \infty)}$

$$f_Z(z) = \int_{y=LL}^{UL} f_Y(y) f_X(z-y)\, dy$$

$$= \underline{\int_{LL}^{UL} \lambda e^{-\lambda y} \frac{1}{4}\, dy}$$

$LL = max(Y_{min}, z - \underbrace{X_{max}}_{10})$

$UL = min(Y_{max}, z - X_{min})$

$$LL = \max(0, z - X_{max})$$
$$= \begin{cases} z - X_{max} & \text{if } z > 10 \\ 0 & \text{if } z \le 10 \end{cases}$$

$$UL = \min(Y_{max}, z - X_{min})$$
$$= \min(\infty, z - 6) = z - 6.$$

Summary

range of z	LL	UL
$6 \le z \le 10$	0	$z-6$
$10 < z$	$z - X_{max}$	$z-6.$

$$\therefore f_Z(z) = \begin{cases} \int_0^{z-6} \lambda e^{-\lambda y} \frac{1}{4}\, dy & \text{if } 6 \le z \le 10 \\ \int_{z - X_{max}}^{z-6} \lambda e^{-\lambda y} \frac{1}{4}\, dy & \text{if } 10 < z \end{cases}$$

2.6 Moment Generating Functions

Definition: The r-th moment about the origin of the random variable X is:

$$\mu_r = E[X^r] = \begin{cases} \sum_{x_{min}}^{x_{max}} x^r p_x(x) \\ \int_{x_{min}}^{x_{max}} x^r f_x(x)\,dx \end{cases}$$

Special cases (which we have studied so far):

Mean of X is $E(X)$

$E(X^2) =$ $\sum x^2 p(x)$ or $\int x^2 f(x) dx$

Hence $Var(X) =$ $E[X^2] - (E[X])^2$

Definition: The Moment-Generating Function (MGF) of the random variable X, denoted by $\phi(t)$ is the expected value of e^{tx}.

$$\phi(t) = \begin{cases} \sum e^{tx} p_x(x) \\ \int e^{tx} f_x(x)\,dx \end{cases}$$

$$\mu_r = \left. \frac{d^r \phi(t)}{dt^r} \right|_{t=0}$$

(the sum of the integral has to converge; otherwise, the moment generating function does not exist.)

If the MGF of a random variable X exists, then it can be used to generate all the moments of X.

$(A+B)' = A'B + AB'$

Example 4. Find the Moment Generating Function of the Exponential Density Function with mean $1/\lambda$. Using the MGF, find the mean and the variance of the exponential density function. The exponential density function is as follows:

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}$$

$$\phi(t) = \int_0^\infty e^{tx} \lambda e^{-\lambda x} dx = \int_0^\infty e^{(t-\lambda)x} \lambda dx = \underbrace{\left[\frac{\lambda}{t-\lambda} e^{(t-\lambda)x} \right]_0^\infty}_{\text{if } t<\lambda}$$

$$= 0 - \frac{\lambda}{t-\lambda} = \frac{\lambda}{\lambda - t}$$

$$\frac{d\phi(t)}{dt} = \left(\lambda \cdot (\lambda - t)^{-1}\right)' = -1 \cdot \lambda \cdot (\lambda - t)^{-1-1} (\lambda - t)'$$

$$= \frac{\lambda}{(\lambda - t)^2}\Big|_{t=0} = \frac{1}{\lambda}.$$

$$\frac{d^2\phi(\phi)}{dt^2} = \frac{\lambda(-2)(-1)}{(\lambda - t)^3}\Big|_{t=0} = \frac{2}{\lambda^2} \qquad \therefore Var(X) = E[X^2] - (E[X])^2$$

$$= \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$$

Example 5. The MGF of the Binomial Distribution with parameters n and p is:

$$\phi(t) = (pe^t + q)^n$$

where $q = 1 - p$. Using this MGF, find the mean and variance of binomial distribution.

$$\frac{d\phi(t)}{dt}\Big|_{t=0} = n(pe^t + q)^{n-1} \cdot pe^t \Big|_{t=0} = n(p+q)^{n-1} \cdot p$$

$$= np = E[X]$$

$$\frac{d^2\phi(t)}{dt^2} = n(n-1)(pe^t+q)^{n-2}(pe^t)^2 + n(pe^t+q)^{n-1}pe^t \Big|_{t=0}$$

$$= n(n-1)p^2 + np = E[X^2]$$

$$\therefore Var(X) = E[X^2] - (E[X])^2 = n(n-1)p^2 + np - n^2p^2$$

$$= np((n-1)p + 1 - np)$$

$$= np(\cancel{np} - p + 1 - \cancel{np})$$

$$= npq.$$

Some results related to MGF:

1. Let ~~phi~~(t) be the MGF of X and $Y = X + a$, where a is a constant, then MGF of Y is:
   $\phi(t)$
   $\phi_Y(t) = e^{at}\phi_X(t)$

   Example 6. The MGF of the binomial distribution with parameters n and p is

   $$\phi(t) = (pe^t + q)^n$$

   where $q = 1 - p$. What is the MGF of $Y = X + 5$? $\phi_Y(t) = e^{5t}(pe^t + q)^n$

2. $Y = X_1 + X_2 + \cdots + X_n$ where $X_1, \cdots, X_n$ are independent random variables with MGFs $\phi_{X_1}(t), \cdots, \phi_{X_n}(t)$, respectively. Then the MGF of Y is:

   $\phi_Y(t) = \phi_{X_1}(t) \cdot \phi_{X_2}(t) \cdots \phi_{X_n}(t)$

   Example 7. Let X be the binomial random variable with parameters n_1 and p_1 and U be a binomial random variable with parameters n_2 and p_2. Find the MGF of Y where $Y = X + U$.

   $\phi_X(t) = (p_1 e^t + q_1)^n$

   $Y = X + U$

   $\phi_U(t) = (p_2 e^t + q_2)^n$

   $\phi_Y(t) = (p_1 e^t + q_1)^n (p_2 e^t + q_2)^n$