SSIE660: Stochastic Systems

Homework assignment 4

1. Find the MGF of a continuous uniform density function in the range (A, B). Obtain $E(X)$ and $\text{Var}(X)$, by differentiating the MGF.

$$f(x) = \begin{cases} \dfrac{1}{B-A}, & A < x < B \\ 0, & \text{elsewhere} \end{cases}$$

$$\phi(t) = \int_A^B e^{tx} \frac{1}{B-A} dx = \frac{1}{(B-A)t} \left[e^{tx}\right]_A^B = \frac{e^{Bt} - e^{At}}{(B-A)t}$$

$$\frac{d\phi(t)}{dt} = \frac{1}{B-A}\left[\frac{Be^{Bt} - Ae^{At}}{t} - \frac{e^{Bt} - e^{At}}{t^2}\right] = \frac{1}{B-A}\left[\frac{tBe^{Bt} - tAe^{At} - e^{Bt} + e^{At}}{t^2}\right]$$

$$E[X] = \left.\frac{d\phi(t)}{dt}\right|_{t=0} = \lim_{t\to 0} \frac{1}{B-A}\left[\frac{tB^2e^{Bt} + Be^{Bt} - tA^2e^{At} - Ae^{At} - Be^{Bt} + Ae^{At}}{2t}\right] \text{ (l'Hospital's rule)}$$

$$= \lim_{t\to 0} \frac{1}{B-A}\left[\frac{B^2e^{Bt} - A^2e^{At}}{2}\right] = \frac{B^2 - A^2}{2(B-A)} = \frac{B+A}{2}$$

$$\frac{d^2\phi(t)}{dt^2} = \frac{1}{B-A}\left[\frac{tB^2e^{Bt} + Be^{Bt} - tA^2e^{At} - Ae^{At} - Be^{Bt} + Ae^{At}}{t} - \frac{2(tBe^{Bt} - tAe^{At} - e^{Bt} + e^{At})}{t^3}\right]$$

$$= \frac{1}{B-A}\left[\frac{tB^2e^{Bt} - tA^2e^{At} - 2tBe^{Bt} + 2tAe^{At} + 2e^{Bt} - 2e^{At}}{t^3}\right]$$

$$E[X^2] = \left.\frac{d^2\phi(t)}{dt^2}\right|_{t=0}$$

$$= \lim_{t\to 0}\left[\frac{2tB^2e^{Bt} + t^2B^3e^{Bt} - 2tA^2e^{At} - t^2A^3e^{At} - 2Be^{Bt} - 2tB^2e^{Bt} + 2Ae^{At} + 2tA^2e^{At} + 2Be^{Bt} - 2Ae^{At}}{3t^2}\right]\frac{1}{B-A}$$

$$= \frac{B^3 - A^3}{3(B-A)} = \frac{B^2 + AB + A^2}{3}$$

$$Var(X) = E[X^2] - (E[X])^2 = \frac{B^2 + AB + A^2}{3} - \frac{(B+A)^2}{4} = \frac{(B-A)^2}{12}$$

2. Let X_1 and X_2 be two independent random variables with the density function

$$f(x_i) = \begin{cases} e^{-x_i}, & x_i > 0 \text{ for } i = 1, i = 2 \\ 0, & \text{elsewhere} \end{cases}$$

(a) Find the MGF of $Y = 5X_1$.

$$\phi_{x_i}(t) = \int_0^\infty e^{tx_i} e^{-x_i} dx_i = -\frac{1}{1-t}\left[e^{-x_i(1-t)}\right]_0^\infty = \frac{1}{1-t}$$

$$Y = 5X_1 \rightarrow f(y) = \frac{1}{5}e^{-y/5}, y > 0$$

which is exponential with $\lambda = 1/5$. Therefore,

$$\phi_y(t) = \int_0^\infty e^{ty}\frac{1}{5}e^{-y/5}dy = -\frac{1}{5}\left[\frac{e^{-y(1/5-t)}}{1/5-t}\right] = \frac{1}{1-5t}$$

(b) Find the MGF of $V = X_1 + X_2$.

$$V = X_1 + X_2 \rightarrow \phi_y(t) = \phi_{X_1}(t) * \phi_{X_2}(t) = \frac{1}{1-t}\frac{1}{1-t} = \frac{1}{(1-t)^2}$$

3. A total of 11 people, including you, are invited to a party. The times at which people arrive at the party are independent uniform (0,1) random variables. Find the expected number of people who arrive before you. Find the variance of the number of people who arrive before you.

   Let X be the number of people who arrive before you. Because you are equally likely to be the first, second, or third .... , or even eleventh arrival,

$$P[X = i] = 1/11, i = 0, 1, ..., 10$$
$$E[X] = \frac{1}{11}(1 + 2 + \cdots + 10) = 5$$
$$E[X^2] = \frac{1}{11}\left(1^2 + 2^2 + \cdots + 10^2\right) = 35$$
$$Var(X) = 35 - 25 = 10$$

4. Solve Chapter 3. Problem 1.
   If X and Y are both discrete, show that $\sum_x Px|y(x|y) = 1$ for all y such that $p_y(y) > 0$

$$\sum_x P_X|_Y(x|y) = \frac{\sum\limits_x P(x,y)}{P_Y(y)} = \frac{P_Y(y)}{P_Y(y)} = 1$$

5. Solve Chapter 3. Problem 3.

$$E[X|Y = 1] = 2$$
$$E[X|Y = 2] = \frac{5}{3}$$
$$E[X|Y = 3] = \frac{12}{5}$$

6. Solve Chapter 3. Problem 7.

Given $Y = 2$, the conditional distribution of X and Z is
$P\{(x, z) = (1, 1)|Y = 2\} = \frac{1}{5}$
$P\{(1, 2)|Y = 2\} = 0$
$P\{(2, 1)|Y = 2\} = 0$
$P\{(2, 2)|Y = 2\} = \frac{4}{5}$
So,
$E[X|Y = 2] = \frac{1}{5} + \frac{8}{5} = \frac{9}{5}$
$E[X|Y = 2, Z = 1] = 1$