SSIE 660: Stochastic Systems

Homework assignment 5: Key

1. Solve Chapter 3. Problem 8.

   (a) $E[X] = E[X|\text{first roll is 6}]P[\text{first roll is 6}] + E[X|\text{first roll is not 6}]P[\text{first roll is not 6}]$

   $$E[X] = E[X|\text{first roll is 6}]\frac{1}{6} + E[X|\text{first roll is not 6}]\frac{5}{6}$$

   $$= \frac{1}{6} + (1 + E[X])\frac{5}{6}$$
   $$E[X] = 6$$

   (b) $E[X|Y = 1]$: This is the case where the first outcome is a five.

   $$E[X|Y = 1] = 1 + E[X] = 7$$

   (c) $E[X|Y = 5]$

   $$= 1\left[\frac{1}{5}\right] + 2\left[\frac{4}{5}\right]\left[\frac{1}{5}\right] + 3\left[\frac{4}{5}\right]^2\left[\frac{1}{5}\right] + 4\left[\frac{4}{5}\right]^3\left[\frac{1}{5}\right] + 6\left[\frac{4}{5}\right]^4\left[\frac{1}{6}\right] + 7\left[\frac{4}{5}\right]^4\left[\frac{5}{6}\right]\left[\frac{1}{6}\right] + ...$$

2. Solve Chapter 3. Problem 15.

   $$f_{X|Y=y}(x|y) = \frac{\frac{1}{y}exp^{-y}}{f_y(y)} = \frac{\frac{1}{y}exp^{-y}}{\int_0^y \frac{1}{y}exp^{-y}dx}$$

   $$= \frac{1}{y},\ \ 0 < x < y$$

   $$E[X^2|Y = y] = \frac{1}{y}\int_0^y x^2 dx = \frac{y^2}{3}$$

3. Solve Chapter 3. Problem 20.

   (a) $f(x|\text{disease}) = \dfrac{P(\text{disease}|x)f(x)}{\displaystyle\int P(\text{disease}|x)f(x)dx}$

   $$= \frac{p(x)f(x)}{\int P(x)f(x)dx}$$

(b) $f(x|\text{no disease}) = \dfrac{[1-P(x)]f(x)}{\displaystyle\int [1-P(x)]f(x)dx}$

(c) $\dfrac{f(x|\text{disease})}{f(x|\text{no disease})} = C\dfrac{P(x)}{1-P(x)}$, where C does not depend on x.

4. Solve Chapter 3. Problem 30.

$$E[N] = \sum_{j=1}^{m} E[N|X_0 = j]p(j) = \sum_{j=1}^{m} \frac{1}{p(j)} p(j) = m$$

5. Solve Chapter 3. Problem 36.

$$E[X] = E[X|X \neq 0](1-p_0) + E[X|X=0]p_0$$

yielding

$$E[X|X \neq 0] = \frac{E[X]}{1-P_0}$$

Similarly,

$$E[X^2] = E[X^2|X \neq 0](1-p_0) + E[X^2|X=0]P_0$$

yielding

$$E[X^2|X \neq 0] = \frac{E[X^2]}{1-p_0}$$

Hence,

$$Var(X|X \neq 0) = \frac{E[X^2]}{1-p_0} - \frac{E^2[X]}{(1-p_0)^2} = \frac{\mu^2+\sigma^2}{1-p_0} - \frac{\mu^2}{(1-p_0)^2}$$

6. Solve Chapter 3. Problem 40.
Let X denote the number of the door chosen, and let N be the total number of days spent in jail.
$E[N] = \sum_1^3 E\{N|X=i\}P\{X=1\}$

(a) The process results each time the prisoner returns to his cell. Therefore,
$E(N|X=1) = 2 + E(N)$
$E(N|X=2) = 3 + E(N)$
$E(N|X=3) = 0$

and

$E(N) = (.5)(2+E(N)) + (.3)(3+E(N)) + (.2)(0)$

or
E(N) = 9.5 days

(b) Let N_i denote the number of additional days the prisoner spends after having initially chosen cell i.

$E[N] = \frac{1}{3}(2 + E[N_1]) + \frac{1}{3}(3 + E[N_2]) + \frac{1}{3}(0).$

$= \frac{5}{3} + \frac{1}{3}(E[N_1] + E[N_2])$

Now to find the variance for part a ,

$E[N_1] = \frac{1}{2}(3) + \frac{1}{2}(0) = \frac{3}{2}$

$E[N_2] = \frac{1}{2}(2) + \frac{1}{2}(0) = 1$

and so, $E[N] = \frac{5}{3} + \frac{1}{3}\frac{5}{2} = \frac{5}{2}$

To find the variance,

Now,

$E[N^2] = 0.5E[(2+N)^2] + 0.3E[(3+N)^2] + 0.2E[0^2]$

$E[N^2] = 0.5E[(4+4N+N^2)] + 0.3[(9+6N+N^2)]$

$E[N^2] = 2 + 2E[N] + 0.5E[N^2] + 2.7 + 1.8E[N] + 0.3E[N^2]$

Solve for $E[N^2]$,

$E[N^2] = 204$

$V[N] = E[N^2] - E[N]^2$

$V[N] = 113.75$

To find $V[N^2]$ for part b, same way as part a

$E[N^2] = 10.5$

$V[N] = 4.25$