**SSIE 660: Stochastic Systems**
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**In-class Exercise: Chapters 4 and 5**

Key

1. Consider the following transition matrix for states 1, 2, 3, and 4.

$$P = \begin{bmatrix} 1-p & 0 & p & 0 \\ q & 0 & q & 0 \\ 0 & 0 & 1-p & p \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

   (a) Find the value of $q$.

   From row 2, we know that $q + q = 1$. Thus, $q = 1/2$.

   (b) Classify the states of the chain. You may wish to consider different values of $p$.
   We need to consider three cases: 1) $p$=0, 2) $p = 1$, and 3) $0 < p < 1$.

   i. If $p = 0$, then the transition matrix is:

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

   There are four classes: $\{1\}$, $\{2\}$,$\{3\}$,$\{4\}$. States 1 and 3 are recurrent, and states 1 and 4 are transient.

   ii. If $p = 1$, then the transition matrix is:

$$P = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

   There are two classes: $\{1, 3, 4\}$, $\{2\}$. States 1, 2, and 3 are recurrent, and state 4 is transient.

   iii. If $0 < p < 1$, then the transition matrix is:

$$P = \begin{bmatrix} 1-p & 0 & p & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1-p & p \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

There are two classes: {1, 3, 4}, {2}. States 1, 2, and 3 are recurrent, and state 4 is transient.

(c) What is the probability that the chain is in state 4 at the $4^{th}$ step given that it started in state 2?

$$P^2 = \begin{bmatrix} (1-p)^2 & 0 & 2p(1-p) & p^2 \\ 1/2(1-p) & 0 & 1/2 & .5p \\ p & 0 & (1-p)^2 & (1-p)p \\ 1-p & 0 & p & 0 \end{bmatrix}$$

Thus,

$$P^4_{24} = 0.5(1-p)p^2 + 0.5(p-p^2) = 0.5p(1-p^2)$$

2. At the beginning of each day, a machine is inspected to determine its working condition. The equipment can be found in one of four working conditions indexed by 1, 2, 3, and 4. Working condition $i$ is better than $i+1, i = 1, 2, 3$. The equipment deteriorates over time. If the present working condition is $i$ and if no repair is done, then its working condition at the beginning of the next day is $j$ with probability $q_{ij}$ given below.

| | | $j$ | | | |
|---|---|---|---|---|---|
| | | 1 | 2 | 3 | 4 |
| | 1 | 0.75 | 0.2 | 0.05 | 0 |
| $i$ | 2 | 0 | 0.5 | 0.2 | 0.3 |
| | 3 | 0 | 0 | 0.6 | 0.4 |

Working condition 4 represents a malfunction which requires a repair that will restore the machine to working condition 1, at the beginning of the next day. If the machine is found to be working condition 3, then a preventive maintenance is done 60% of the time, which will restore the machine to working condition 1, at the beginning of the next day.

(a) Define the states and find the one-step transition probability matrix.

The states are the working conditions at the begging of the day. One step transition matrix is:

$$P = \begin{bmatrix} .75 & .2 & .05 & 0 \\ 0 & .5 & .2 & .3 \\ .6 & 0 & .24(= .6 * .4) & .16(= .4 * .4) \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

(b) Assume that the machine starts in working condition 1 at the beginning of today ($X_0 = 1$). Then, what is $P[X_3 = 2, X_2 = 1, X_1 = 3]$ ?

$$P[X_3 = 2, X_2 = 1, X_1 = 3] = P_{12}P_{31}P_{13} = .2 * .6 * .05 = .006.$$

(c) What is $P[X_4 = 1, X_3 = 4, X_2 = 3|X_1 = 2]$?

$$P[X_4 = 1, X_3 = 4, X_2 = 3|X_1 = 2] = P_{41}P_{34}P_{23} = 1 * .16 * .2 = .032.$$

(d) What is $P[X_4 = 1, X_3 = 2|X_1 = 2]$?

$$P[X_4 = 1, X_3 = 2|X_1 = 2] = P_{21}P_{22}^2 = 0$$

3. The demand per day of a certain item follows the probability distribution given below:

| Demand | Probability |
|---|---|
| 0 | 0.2 |
| 1 | 0.5 |
| 2 | 0.3 |

The ordering policy is as follows: " Order two items, if the inventory at the end of day is 0; otherwise, do not order"

No backorders are allowed. The states of the system are the inventories at the beginning of a day after receipt of ordered quantities.

(a) Obtin all the one-step transition probabilities, using the information given above.

All the one-step transition probabilities can be represented in the transition probability matrix:

$$P = \begin{bmatrix} .2 & .8 \\ .5 & .5 \end{bmatrix}$$

(b) You are given the following. $\pi_1 = 0.385$. The carrying cost per day per unit at the end of a day is \$50.00. and the ordering cost per order is \$100.00. What is the long-run expected cost per day consisting of the expected carrying cost per day and the expected ordering cost per day?

$$\pi_2 = 1 - \pi_1 = .615$$

Carrying cost:

$$(1 * (\pi_1 * .2 + \pi_2 * .5) + 2 * (\pi_2 * .2)) * 50 = 31.525$$

Ordering cost:

$$(\pi_1 * .8 + \pi_2 * .3) * 100 = 49.25$$

Total cost per day:

$$31.525 + 49.25 = 80.775$$

4. In a certain system, a customer must first be served by server 1, and then by server 2. The service times at server $i$ are exponential with rate $\mu_i, i = 1, 2$. An arrival finding server 1 is busy waits in line for that server. Upon completion of service at server 1, a customer either enters service with server 2 if that server is free or else remains with server 1 (blocking any other customer from entering service) until server 2 is free. Customers depart the system after being served by server 2. Suppose that you arrive to find two others in the system, one being served by server 1 and one by server 2.

   What is the expected total time you spend in the system?

   $E$[total time] = $E$[waiting time before server 1] + $E$[service time at server 1] + $E$[waiting time before server 2] + $E$[service time at server 2]

   $E$[waiting time before server 1]=
   $E$[service time for the customer already being served by server 1] + $E$[blocking time]
   = $\frac{1}{\mu_1} + \frac{1}{\mu_2} * P$[server 1 completes before server 2] = $\frac{1}{\mu_1} + \frac{1}{\mu_2}\frac{\mu_1}{\mu_1 + \mu_2}$

   $E$[service time at server 1] = $\frac{1}{\mu_1}$

   $E$[waiting time before server 2]= $\frac{1}{\mu_2} * P$[server 1 completes before server 2] = $\frac{1}{\mu_2}\frac{\mu_1}{\mu_1 + \mu_2}$

   $E$[service time at server 1] = $\frac{1}{\mu_2}$

$E$[total time] =
$E$[waiting time before server 1] + $E$[service time at server 1] + $E$[waiting time before server 2] + $E$[service time at server 2] =

$$\frac{1}{\mu_1} + \frac{1}{\mu_2}\frac{\mu_1}{\mu_1+\mu_2} + \frac{1}{\mu_1} + \frac{1}{\mu_2}\frac{\mu_1}{\mu_1+\mu_2} + \frac{1}{\mu_2}$$
$$= \frac{2}{\mu_1} + \frac{2}{\mu_2}\frac{\mu_1}{\mu_1+\mu_2} + \frac{1}{\mu_2}$$

5. A subway station has both local and express service, on opposite sides of the same platform. Local and express trains arrive independently according to a Poisson process with rates 1/5 and 1/15 per minute, respectively. Both trains stop at your destination, with transit times of 17 minutes for a local train and 11 minutes for an express train.

   (a) What is the waiting time distribution until the next local train arrives?

   The waiting time distribution follows an exponential distribution with rate $\lambda$. Therefore,

   $$f_{local}(t) = \frac{1}{5}e^{-\frac{1}{5}t}$$

   (b) What is the waiting time distribution until the next express train arrives?

   $$f_{express}(t) = \frac{1}{15}e^{-\frac{1}{15}t}$$

   (c) What is the waiting time distribution until the next train (either local or express) arrives?

   $$f_{either}(t) = \left(\frac{1}{5} + \frac{1}{15}\right)e^{-\left(\frac{1}{5}+\frac{1}{15}\right)t} = \frac{4}{15}e^{-\frac{4}{15}t}$$

   (d) What is the probability that the next local train arrives before the express?

   $$P[Local < Express] = \frac{1/5}{1/5 + 1/15} = \frac{3}{4}$$

   (e) If the next train that arrives is a local, should you board that train or wait for an express, assuming that your objective is to minimize your expected travel time?

   $E$[total time for local] = 17 min.

$E$[total time for express] $=E$[waiting time for express] + 11 min = $1/\lambda_{express}$+ 11
= 15 + 11 = 26 min.

It's better to take a local train.