SSIE 660: Stochastic Systems
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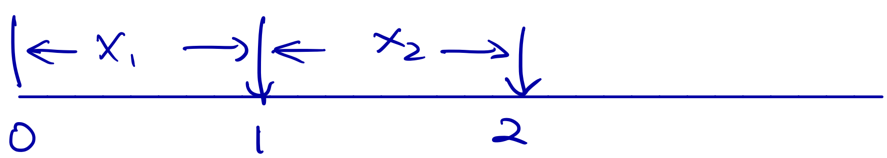
Note 15
Chapter 7. Renewal Theory

- Poisson process is a counting process for which the time between successive events are IID exponential random variables.
- Now, we will relax the assumption regarding the distribution of time between successive events.
- Let $\{N(t), t \geq 0\}$ be a counting process and X_n *be the time between the* $(n-1)^{th}$ *and* n^{th} *events,* $n \geq 1$.

Definition 1. *If the sequence of non-negative variables* $\{X_1, X_2, X_3, ...\}$ *is independent and identically distributed, then the counting process* $\{N(t), t \geq 0\}$ *is said to be a renewal process.*



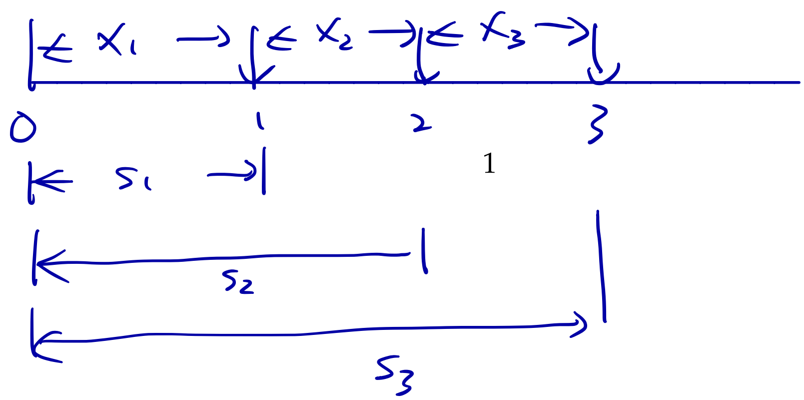
Example 2. Failures of light bulbs and their immediate replacements, if the life time of the bulbs are I.I.D. random variables.

$N(t) =$ # of failed (and replaced) light bulbs by t.

$X_n =$ life time of the n^{th} bulb.

Let $S_n = \sum_{i=1}^{n} X_i$: time of the n^{th} renewal from $t=0$

Figure:



Let $E[X_n] = \mu, n \geq 1$

expected lifetime

It is assumed that $P[X_n = 0] < 1$, for all $n \longrightarrow \mu > 0$.

It can be shown that an infinite amount of renewals cannot occur in a finite amount of time.

Probability Distribution of $N(t)$

$$N(t) \geq n \longleftrightarrow \quad S_n \leq t$$

$$P[N(t) = n] = P(N(t) \geq n) - P(N(t) \geq n+1)$$

$$= \underbrace{P(S_n \leq t)}_{} - \underbrace{P(S_{n+1} \leq t)}_{}$$

$$= F_n(t) \qquad F_{n+1}(t)$$

Example 3. Suppose that the inter-arrival distribution is geometric, with probability of success = p.

$S_1 = X_1$: # of trials required until ~~the~~ first success

$$P[X_n = i] = p(1-p)^{i-1} \qquad i \geq 1$$

S_n : # of trials required for obtaining n successes

$\rightarrow$ negative binomial.

$$P[S_n = k] = \begin{cases} \binom{k-1}{n-1} p^n (1-p)^{k-n} & k \geq n \\ 0 & k < n \end{cases}$$

$(k-1)$ trials
$(n-1)$ successes
$p^{n-1}(1-p)^{k-n} \cdot p$

Now we have $P[N(t) = n] = \sum_{k=n}^{[t]} \binom{k-1}{n-1} p^n (1-p)^{k-n} - \sum_{k=n+1}^{[t]} \binom{k-1}{n} p^n (1-p)^{k-n-1}$

equivalently, since an event independently occurs w/ prob. p at each of the times $1, 2, \dots$

$$P(N(t) = n) = \binom{[t]}{n} p^n (1-p)^{[t]-n}$$

$[t] = \max[n \in \mathbb{N}, n \leq t]$

$[t] = \max\{t : N(t) \leq n+1\}$

Let $m(t)$ be the mean value of $N(t)$. That is,

$$m(t) = E[N(t)]$$

$$E[X] = \sum_{k=1}^{\infty} k \cdot P(X=k) = P[X=1] + P[X=2] + P[X=3] + \cdots$$
$$+ P[X=2] + P[X=3] + \cdots$$
$$+ P[X=3] + \cdots$$
$$+ \cdots$$

$$m(t) = \sum_{n=1}^{\infty} P[X \geq n] = \sum_{n=1}^{\infty} P(N(t) \geq n)$$
$$= \sum_{n=1}^{\infty} P(S_n \leq t)$$
$$= \sum_{n=1}^{\infty} F_n(t)$$

$m(t)$ is called the mean-value or the renewal function.

1. It uniquely determines the renewal process.
2. One to one corresponding between the inter-arrival distributions F and $m(t)$.
3. $m(t) < \infty$ for all $t < \infty$.

m(t) → mean value function of a poisson process w/ rate 2

$X_i \sim \exp(1/2)$ $\quad P(N(10)=n) = \dfrac{e^{-2\times 10}(2\times 10)^n}{n!}, n \geq 0$

Example 4. Suppose that we have a renewal process whose mean-value function is

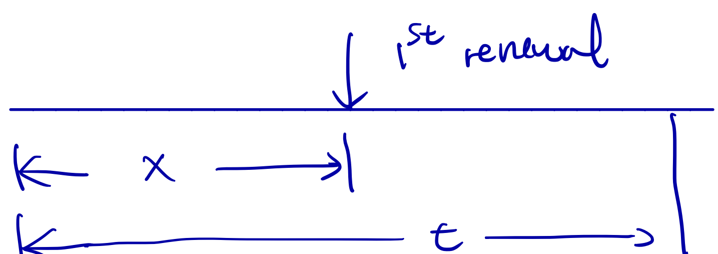
$$m(t) = 2t, t \geq 0$$

What is the distribution of the number of arrivals occurring by time 10?

- $m(t) < \infty$, for all $t < \infty$.
- Since $m(t)$ uniquely determines the inter-arrival distribution, it follows that the Poisson process is the only renewal process having a linear mean-value function.
- The first renewal occurs at X_1. Then

  $$m(t) = E[N(t)] = \int_0^\infty E[N(t) \mid X_1 = x] f_{X_1}(x)\,dx$$

- Suppose that the first renewal occurs at a time x that is less than t.



- A renewal process probabilistically starts over when a renewal occurs. Hence (the number of renewals by time t) and (1+ the number of renewals in the $t-x$ time units have the same distribution.

  $$E[N(t)|X_1 = x] = 1 + E[N(t-x)] \qquad x<t$$

  When $x > t, E[N(t)|X_1 = x] = 0$

  $$m(t) = \int_0^t \{1 + E[N(t-x)]\} f_X(x)\,dx$$
  $$= \int_0^t \{1 + m(t-x)\} f_X(x)\,dx$$
  $$= \int_0^t f_X(x)\,dx + \int_0^t m(t-x) f_X(x)\,dx$$
  $$= F_X(t) + \int_0^t m(t-x) f_X(x)\,dx.$$

renewal eq. $\quad$ only numerical solution possible

Example 5. Let the inter-arrival distribution be uniform (0,1).

$$f_X(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{o-w} \end{cases}$$

$$m(t) = F_X(t) + \int_0^t m(t-x) f_X(x)\, dx$$

$$F_X(x) = \int_0^t 1\, dt = t \qquad t \in (0,1)$$

$$y = t - x \qquad x = 0 \to y = t$$

$$x = t \to y = 0$$

$$dy = -dx$$

$$m(t) = t + \int_0^t m(y)\, dy$$ differentiating both sides by t

$m'(t) = 1 + m(t)$. let $h(t) = 1 + m(t)$ $h'(t) = m'(t)$

$$h'(t) = h(t) \quad \therefore \quad \frac{h'(t)}{h(t)} = 1 \qquad \int \frac{h'(t)}{h(t)}\, dt = \int 1\, dt = t + C$$

$$\therefore \ln h(t) = t + C, \quad h(t) = e^{t+C} = e^t \cdot e^C = e^t \cdot K$$

$$m(t) = h(t) - 1 = e^t \cdot K - 1$$

$$m(0) = 0 \to K = 1 \qquad \therefore m(t) = e^t - 1$$

- The renewal equation

$$m(t) = F_X(t) + \int_0^t m(t-x)f(x)dx$$

has application in Reliability.

- The expected cost incurred by a manufacturer during the warranty period of a product depends upon

# of replacement (failures) services during the warranty period

- Let us assume that the life time of a product which has a warranty period of W time units follows exponential density function with mean $1/\lambda$. This means the failure rate of the product is λ. Assume also that a failed product is instantaneously replaced. Then the expected number of products replaced during the warranty period is

expected #: λW

$$m(W) = F_X(w) + \int_0^W m(w-x) f_X(x)dx$$

$$= 1 - e^{-\lambda w} + \int_0^w m(w-x)\lambda e^{-\lambda x}dx \qquad y = w - x$$

$$= 1 - e^{-\lambda w} + \lambda\int_0^w m(y) e^{-\lambda(w-y)}dy$$

$$= 1 - e^{-\lambda W} + \lambda e^{-\lambda w}\int_0^w m(y) e^{\lambda y}dy$$

$$= 1 - e^{-\lambda W} + \lambda e^{-\lambda W}\int_0^w \lambda y e^{\lambda y}dy \qquad \int uv' = uv - \int u'v$$

$$= 1 - e^{-\lambda w} + \lambda e^{-\lambda w} \cdot \lambda \int_0^w y e^{\lambda y}dy$$

$$= 1 - e^{-\lambda W} + \lambda \cdot e^{-\lambda w} \cdot \lambda \left[\frac{y \cdot e^{\lambda y}}{\lambda}\Big|_0^w - \frac{\int e^{\lambda y}}{\lambda} \right]$$

$$= 1 - e^{-\lambda W} + \lambda e^{-\lambda w}\left[y e^{\lambda y}\Big|_0^w - \frac{e^{\lambda y}}{\lambda}\Big|_0^w \right] = \lambda w$$

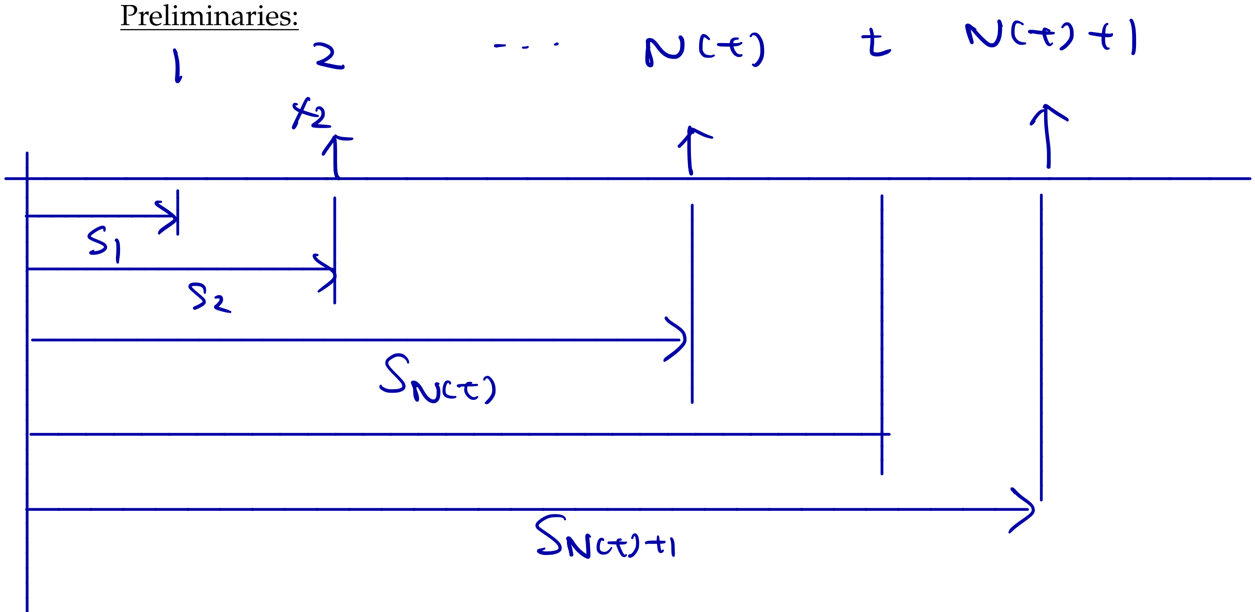
$S_{N(t)}$: time of the $N(t)^{th}$ renewal (time of the last renewal prior to or at time t)

$S_{N(t)+1}$: time of the first renewal after t.

Proposition 6.

$$\frac{N(t)}{t} \to \frac{1}{\mu} \ as \ t \to \infty$$

Preliminaries:



$$\frac{S_{N(t)}}{N(t)} \leq \frac{t}{N(t)} \leq \frac{S_{N(t)+1}}{N(t)}$$

Strong law of large numbers:

Let $X_1, X_2, ..., X_n$ be a sequence of independent random variables having a common distribution and let $E[X_i] = \mu$. Then, with probability 1,

$$\frac{X_1 + X_2 + \cdots + X_n}{n} \to \mu \ \text{as} \ n \to \infty$$

- $S_{N(t)} = \sum_{i=1}^{N(t)} X_i$
- Hence,

$$\frac{S_{N(t)}}{N(t)} \to \mu \quad as \quad N(t) \to \infty$$

- As $t \to \infty, N(t) \to \infty$.
- Hence,

$$\frac{S_{N(t)}}{N(t)} \to \mu \quad \text{as } t \to \infty$$

- $\frac{S_{N(t)+1}}{N(t)} \to \frac{S_{N(t)+1}}{N(t)+1} \; \frac{N(t)+1}{N(t)}$
- $\frac{S_{N(t)+1}}{N(t)+1} \to \mu$ as $t \to \infty$, and $\frac{N(t)+1}{N(t)} \to 1$ as $t \to \infty$
- Hence,

$$\frac{S_{N(t)+1}}{N(t)} \to \mu \quad \text{as } t \to \infty$$

- Consider the relation:

$$\frac{S_{N(t)}}{N(t)} \leq \frac{t}{N(t)} \leq \frac{S_{N(t)+1}}{N(t)}$$

- Hence, $\frac{t}{N(t)} \to \mu$ as $t \to \infty$
- The number $\frac{1}{\mu}$ is called the rate of the renewal process.
- How about $\frac{m(t)}{t}$ as $t \to \infty$, where $m(t) = E[N(t)]$?

$$\frac{m(t)}{t} = \frac{1}{\mu} \quad \text{as } t \to \infty$$

- This result is known as the Elementary Renewal Theorem.

Summary of the results studied so far:

1. $N(t) \geq n \longleftrightarrow S_n \leq t \quad \{S_n = \sum_{i=1}^{n} X_i\}$.
2. $P[N(t) = n] = P[S_n \leq t] - P[S_{n+1} \leq t] = F_n(t) - F_{n+1}(t)$
3. Mean-value of Renewal Function: $m(t) = E[N(t)] = \sum_{n=1}^{\infty} F_n(t)$

4. Renewal Equation: $m(t) = F_X(t) + \int_0^t m(t-x) f_X(x) dx$, where $f_X(x)$ is the probability function of $X_i's$.

5. $\frac{N(t)}{t} \to \frac{1}{\mu}$ as $t \to \infty$, where $\frac{1}{\mu}$ is called the Rate of the Renewal Process.

6. Elementary Renewal Theorem: $\frac{m(t)}{t} \longrightarrow \frac{1}{\mu}$ as $t \to \infty$.

- Weibull Density Function:

$$f(x) = \begin{cases} \alpha \beta^\alpha x^{\alpha-1} e^{-(\beta x)^\alpha} & x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

α = shape parameter; β = scale parameter.

$$\Gamma(\alpha) = \int_0^\infty e^{-x} x^{\alpha-1} dx$$

$$\mu$$

$$E[X] = \cancel{1/\mu} = \frac{1}{\beta}\Gamma\left(1 + \frac{1}{\alpha}\right)$$

$$V(X) = \frac{1}{\beta^2}\left\{\Gamma\left(1 + \frac{2}{\alpha}\right) - \Gamma^2\left(1 + \frac{1}{\alpha}\right)\right\}$$

Example 7. The time between failures of a repairable unit is supposed to follow a Weibull distribution with $\alpha = 10$ and $\beta = 0.01$. Assuming that repairs are perfect, i.e., the unit is renewed to its original state upon a failure, asses the mean number of repairs during mission time $t = 1,000$ hours.

$$\frac{m(1000)}{1000} = \frac{1}{\mu} = \frac{1}{95.13}$$

$$\mu = E[X] = \frac{1}{.01}\Gamma\left(1 + \frac{1}{10}\right) = \frac{.9513}{.01} = 95.13$$

$$\therefore m(1000) = \frac{1000}{95.13} = 10.51 \text{ repairs}/1000h$$

Bounds on the renewal function $m(t)$ (Lorden, 1970)

$$\frac{t}{\mu} - 1 \le m(t) \le \frac{t}{\mu} + \frac{\sigma^2}{\mu^2}$$

where μ, σ^2 are the mean and variance of X (time between successive renewals), respectively.

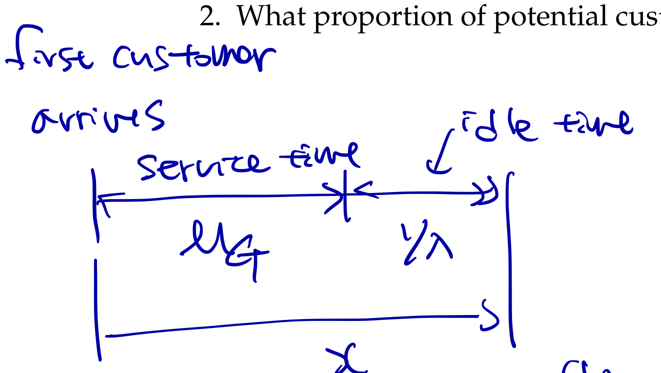
$$10.51 - 1 \le m(t) \le 10.51 + \frac{\sigma^2}{95.13^2}$$

$$\sigma^2 = \frac{1}{.01^2}\left\{ \Gamma\left(1+\frac{2}{10}\right) - \Gamma^2\left(1+\frac{1}{10}\right) \right\} = 132.28$$

$$\therefore \quad 9.51 \le m(1000) \le 10.525$$

Example 8. Suppose that potential customers arrive at a single-server bank in accordance with a Poisson process having rate λ. However, suppose that the potential customer will enter the bank only if the server is free when he arrives. That is, if there is already a customer in the bank, then our arrive rather than entering the bank, will go home. If we assume that the amount of time spent in the bank by an arriving customer is a random variable having distribution G, then

1. What is the rate at which customers enter the bank?
2. What proportion of potential customers actually enter the bank?



a) $E[X] = \mu_X = \mu_G + \frac{1}{\lambda}$

rate entering $= \frac{1}{\mu_X} = \frac{\lambda}{1+\mu_G \cdot \lambda} < \lambda$

(b) proportion $= \frac{\frac{1}{\mu_X}}{\lambda} = \frac{1}{1+\lambda \cdot \mu_G}$