SSIE 660: Stochastic Systems
Dr. Sung H. Chung
Note 12
Chapter 5. The Exponential Distribution and the Poisson Process

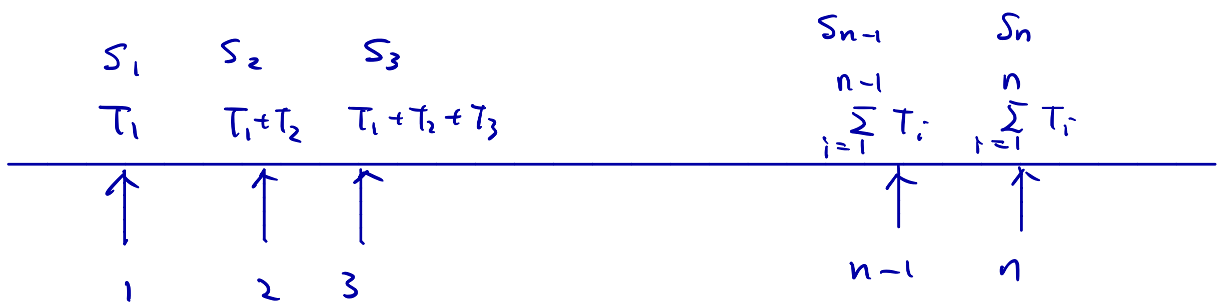
Interarrival and Waiting Time Distributions

Consider a Poisson process $P[N(t) = n] = e^{-\lambda t}\dfrac{(\lambda t)^n}{n!}$

Time of the first event = T_1

Time elapsed between the $(n-1)^{th}$ and n^{th} event = T_n (for n >1)

Then, the sequence $\{T_1, T_2, ..., T_n\}$ is the sequence of inter arrival times.



Probability distribution of T_n: $f_{T_1}(t_1)$

$1 - F_{T_1}(t_1) = e^{-\lambda t_1}$

$F_{T_1}(t_1) = 1 - e^{-\lambda t_1}$

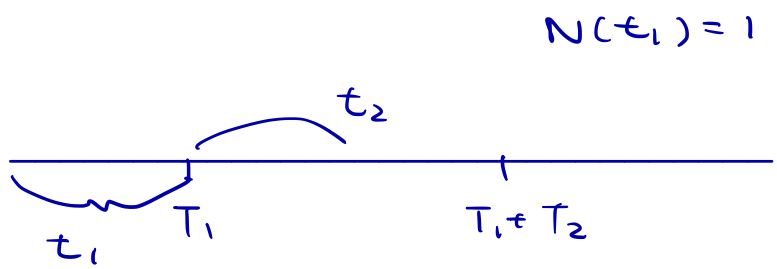
$F_{T_1}(t_1) = P[T_1 \le t_1] =$ $P[N(t_1) \geq 1]$

$1 - F_{T_1}(t_1) = P[T_1 > t_1] =$ $P[N(t_1) < 1] = P[N(t_1) = 0]$

$P[N(t_1) = 0] =$ $\dfrac{e^{-\lambda t_1}(\lambda t_1)^0}{0!} = e^{-\lambda t_1}$

$f_{T_1}(t_1) =$ $\dfrac{d}{dt}\left[1 - e^{-\lambda t_1}\right] = \lambda e^{-\lambda t_1}$ exponential w/

Now let us find $f_{T_2}(t_2)$.



$P[T_2 > t_2] = E[P\{T_2 > t_2|T_1\}]$

double check

However,

$$\begin{aligned}P[T_2 > t_2|T_1 = t_1] &= P[\,0 \text{ event in } (t_1, t_1 + t_2)|T_1 = t_1] \\ &= P[\,0 \text{ event in } (t_1, t_1 + t_2)] \\ &= e^{-\lambda t_2}\end{aligned}$$

$$\frac{e^{-\lambda t_2}(\lambda t_2)^0}{0!}$$

$F_{T_2}(t_2) = \quad 1 - e^{-\lambda t_2}$

$f_{T_2}(t_2) = \quad \lambda e^{-\lambda t_2}$

Proposition 1. *$T_i, i = 1, 2, \ldots$ are independent and identically distributed exponential random variables having mean $1/\lambda$.*

Probability distribution of the time until the n^{th} arrival.

Let $S_n = \sum_{i=1}^{n} T_i$

Probability distribution: $f_{S_n}(t)$

$f_{S_n}(t) \longleftarrow F_{S_n}(t) \leftarrow (1 - F_{S_n}(t))$

$$F_{S_n}(t) = P(S_n \le t) = P[N(t) \ge n]$$

$$= \sum_{j=n}^{\infty} \frac{e^{-\lambda t}(\lambda t)^j}{j!}$$

$$f_{S_n}(t) = \frac{d}{dt}[F_{S_n}(t)]$$

$$= \sum_{j=n}^{\infty} \frac{(-\lambda) e^{-\lambda t}(\lambda t)^j}{j!} + \sum_{j=n}^{\infty} \frac{e^{-\lambda t}\, j \cdot \lambda \cdot (\lambda t)^{j-1}}{j!}$$

$$= -\sum_{j=n}^{\infty} \frac{\lambda e^{-\lambda t}(\lambda t)^j}{j!} + \sum_{j=n}^{\infty} \frac{\lambda e^{-\lambda t}(\lambda t)^{j-1}}{(j-1)!}$$

$$= \underbrace{-\sum_{j=n}^{\infty} \frac{\lambda e^{-\lambda t}(\lambda t)^j}{j!} + \sum_{j=n+1}^{\infty} \frac{\lambda e^{\lambda t}(\lambda t)^{j-1}}{(j-1)!}}_{\text{same}} + \frac{\lambda e^{-\lambda t}(\lambda t)^{n-1}}{(n-1)!}$$

$$= \frac{\lambda e^{-\lambda t}(\lambda t)^{n-1}}{(n-1)!} \quad \sim \text{gamma dist w/ parameters } n \,\&\, \lambda.$$

$$E[S_n] = \frac{n}{\lambda}$$

$$Var[S_n] = \frac{n}{\lambda^2}$$

Further properties of Poisson process:

1. Consider two independent Poisson processes $\{N_1(t), t \geq 0\}$ with rate λ_1 and $\{N_2(t), t \geq 0\}$ with rate λ_2. Let S_n^1 denote the time of the n^{th} event of the first process and S_m^2 denote the time of the m^{th} event of the second process. Find $P[S_n^1 < S_m^2]$. That is,

   P [n events of type 1 occurs before m events of type 2 occurs]

   - We know that the inter arrival times of these processes follow exponential density functions with means $1/\lambda_1$ and $1/\lambda_2$, respectively.
   - Let us consider the case when $n = 1, m = 1$.
   - We know from earlier results that

   $$P[S_1^1 < S_1^2] = \frac{\lambda_1}{\lambda_1 + \lambda_2}$$
   $$P[S_1^2 < S_1^1] = \frac{\lambda_2}{\lambda_1 + \lambda_2}$$

   - How about this probability with $n = 2, m = 1$? That is, 2 events of type 1 should occur before 1 event of type 2. ($P[S_2^1 < S_1^2]$).
   - This will happen if 1^{st} event of type 1 occurs before type 2 event and 2^{nd} event of type 1 occurs before type 2 event.

   $$\frac{\lambda_1}{\lambda_1 + \lambda_2} \frac{\lambda_1}{\lambda_1 + \lambda_2}$$

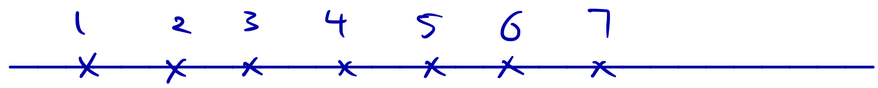
   because, after the 1^{st} event of type 1 occurs, the process starts all over again.

   $$P[S_2^1 < S_1^2] = \left(\frac{\lambda_1}{\lambda_1 + \lambda_2}\right)^2$$

   - Now let us consider the case with $n = 2, m = 5$.

   $$P[S_2^1 < S_5^2]$$

- The events included in the event { 2 events of type 1 before 5 events of type 2 } are:



2 or more events of type 1 must occur within 2+5−1 events

$P[Type\ 1] = \dfrac{\lambda_1}{\lambda_1+\lambda_2}$ $\qquad$ $P[Type\ 2] = \dfrac{\lambda_2}{\lambda_1+\lambda_2}$

\# of Type 1 $\quad 2$ $\qquad$ \# of Type 2 $\quad \le 5$

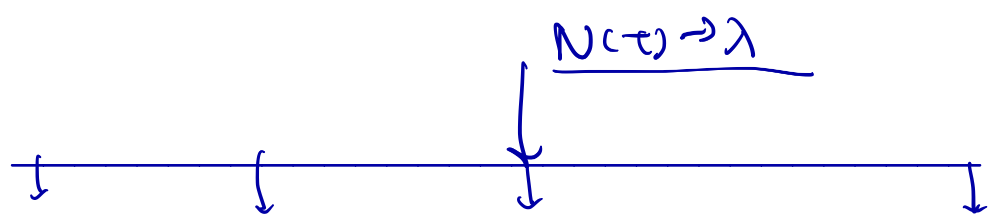
$$P[S^1_2 < S^2_5] = \sum_{k=2}^{2+5-1} \binom{2+5-1}{k} \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{2+5-1-k}$$

In general,

$$P[S^1_n < S^2_m] = \sum_{k=n}^{n+m-1} \binom{n+m-1}{k} \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n+m-1-k}$$

2. Decomposition of a Poisson process: (Proposition 5.2)

Let $\{N(t), t \geq 0\}$ be a Poisson process with rate λ. Suppose that each time an event occurs. It is classified as type $1, 2, ...$ and k with probabilities $p_1, p_2, ...$ and p_k, respectively. Let $N_i(t)$ be the number of events of type i in time t, $i = 1, 2, ...k$. Then $\{N_i(t), t \geq 0\}$ is a Poisson process with rate $\lambda * p_i$ and is *independent* of $N_1(t)$, $N_2(t)$, ..., $N_{i-1}(t)$, $N_{i+1}(t)$, ... $N_k(t)$ for $i = 1, 2, ..., k$. $p_1 + p_2 + \cdots p_k = 1$.

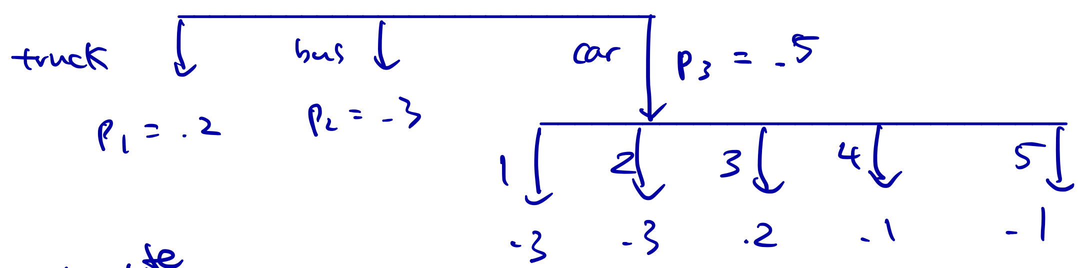


p_1 : type 1 $\quad$ p_2 : type 2 $\quad$ p_i : type i $\quad$ p_k : type k

$N_1(t) = \lambda p_1$ $\quad$ $N_2(t) = \lambda p_2$ $\quad$ $N_k(t) \to \lambda p_k$

Example 2. Vehicles stopping at a roadside restaurant form a Poisson process $\{N(t), t \geq 0\}$ with rate $\lambda = 30$/hour. Twenty percent of these vehicles are trucks, thirty percent are buses, and the remaining are passenger cars. The number of passengers in a truck is one. the number of passengers in a passenger car is equal to 1, 2, 3, 4, and 5 with probabilities 0.3, 0.3, 0.2, 0.1, and 0.1, respectively. The number of passengers in a bus is more than 20. What is the probability that within a period of 2 hours, four cars with one passenger will stop at the restaurant?

$\lambda = 30/h$



decomposed rate

$$P[\# \text{ of cars w/ one passenger within two hours}] = \frac{e^{-4.5\times 2}(4.5\times 2)^4}{4!}$$

$\lambda' = 30 \times .5 \times .3 = 4.5$

$t = 2$

$n = 4.$

3. Superposition of Poisson processes (Reverse of the decomposition property, not in the book).

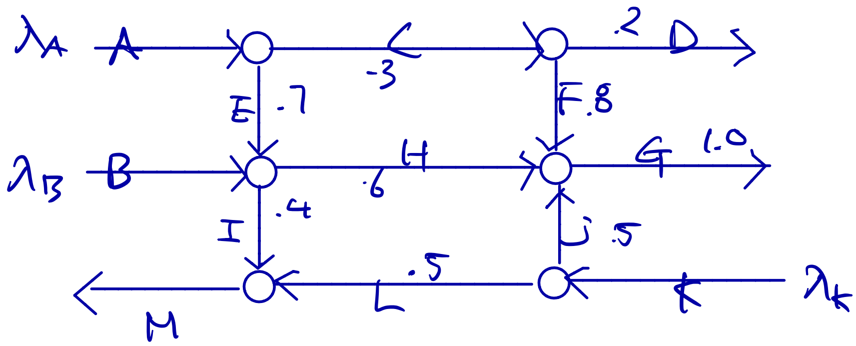
Let $\{N_1(t), t \geq 0\}$,$\{N_2(t), t \geq 0\}$, ... ,$\{N_k(t), t \geq 0\}$ be k independent Poisson processes with rates $\lambda_1, \lambda_2, ...\lambda_k$, repectively. Then

$$N(t) = N_1(t) + N_2(t) + \cdots + N_k(t)$$

is a Poisson process with rate

$$\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_k$$

Example 3. Consider the road network pictured below. The inputs into streets A, B and K are Poisson processes (independent) with the rates indicated. The probabilities of a vehicle choosing the indicated directions are written parentheses along the arcs. What is the probability that the number of vehicles using street M within 3 hours is 20?



$\lambda_E = \lambda_A \times .7$

$\lambda_I = (\lambda_B + \lambda_E) \times .4$

$\lambda_L = \lambda_K \times .5$

$\lambda_M = \lambda_I + \lambda_L$

$P[N_M(3) = 20] = \dfrac{e^{-\lambda_M \cdot 3} (\lambda_M \cdot 3)^{20}}{20!}$

Conditional Distribution of Arrival Times

To find the conditional p.d.f. of the time at which an event occurred, given that exactly only one event has occurred by time t.

exactly one event has occurred.

Let s be the time at which it occurred.

To find $f_S(s|N(t)=1)$, it is better to find $F_S(S|N(t)=1)$.

Probability distribution/Density function known:

(i) $P[N(t)=n]$ Poisson

(ii) $f_S(s)$

$$
\begin{aligned}
F_S(s|N(t)=1] &= P[S \le s \mid N(t)=1] \\
&= \frac{P[S \le s,\ N(t)=1]}{P[N(t)=1]} \\
&= \frac{P[N(s)=1,\ N(t-s)=0]}{P[N(t)=1]} \\
&= \frac{e^{-\lambda s}\cdot \lambda s}{1!}\cdot\frac{e^{-\lambda(t-s)}(\lambda(t-s))^0}{0!} \Big/ \frac{e^{-\lambda t}(\lambda t)}{1!} \\
&= \frac{e^{-\lambda s}\lambda s\, e^{-\lambda(t-s)}}{e^{-\lambda t}\lambda t} = \frac{s}{t}
\end{aligned}
$$

$$f_S(s|N(t)=1] = \frac{d\{F_S[s \mid N(t)=1]\}}{ds}$$

$$= \frac{d}{ds}\left(\frac{s}{t}\right)$$

$$= \frac{1}{t}$$

Uniform in the range of $(0, t)$.

Each interval of equal length in the interval $(0, t)$ has the same probability of containing the event.