Math 660: Applied Stochastic Processes
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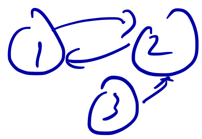
Note 9
Chapter 4. Discrete Markov Chains

True / False Test

1. (T F) All states in a finite Markov Chain (with finite number of states) can be transient.



$$\begin{bmatrix} 11 & 12 & 13 \\ 21 & 22 & 23 \\ 31 & 32 & 33 \end{bmatrix} \quad \boxed{\sum_j P_{1j} = 1}$$

2. (T F) All states in a finite irreducible Markov Chain are recurrent.

   only one state

3. (T F) No transient state can be reached from any recurrent state.

4. (T F) From a recurrent state, only recurrent states can be reached.

   has the same properties

- A recurrent state i is said to be *positive recurrrent*, if starting in i, the expected time until the system returns to state i is finite

  Otherwise, the recurrent state is Null recurrent

  state 0 ±1, ±2, --

  $P_{i,i+1} = p = 1 - P_{i,i-1}$

  $i \xrightarrow{p} i+1$, $i \xrightarrow{1-p} i-1$

  $\sum_{n=1}^{\infty} P_{00}^{n} = \infty$ iff $p = \frac{1}{2}$

- Positive recurrence is a class property. If any one state in a class is positive recurrent, then all states within that class are also positive recurrent.

- In a finite Markov Chain,

  all the recurrent states are positive recurrent.

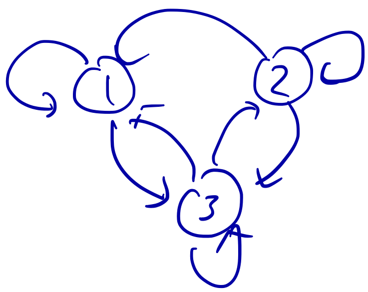
- Let δ be the greatest common divisor of the set of all $n \geq 1$, for which $P_{jj}^n > 0$.
  1. If $\delta = 1$, then j = aperiodic
  2. If $\delta \geq 2$, then j is periodic w/ period δ

periodicity is a class property.

Let us recall the previous five examples.

(A)

$$P = \begin{bmatrix} 0.3 & 0 & 0.7 \\ 0.4 & 0.3 & 0.3 \\ 0.2 & 0.4 & 0.4 \end{bmatrix}$$



class {1, 2, 3}

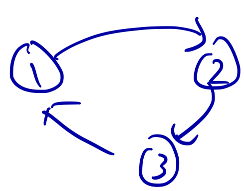
consider # of steps / transitions required to go from 1 to 1

1, 2, 3, 4... G.C.D = 1

aperiodic

(B)

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$



class {1, 2, 3}

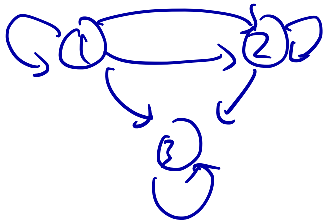
1 → 1 : 3, 6, 9

GCD = 3

period = 3

(C)

$$P = \begin{bmatrix} 0.3 & 0.3 & 0.4 \\ 0.4 & 0.2 & 0.4 \\ 0 & 0 & 1 \end{bmatrix}$$



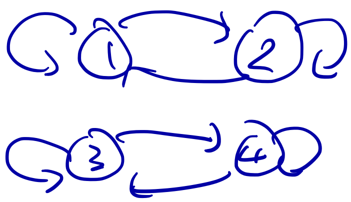
$\{1, 2\}$ aperiodic

$\{3\}$: absorbing state :

no period

(D)

$$P = \begin{bmatrix} 0.2 & 0.8 & 0 & 0 \\ 0.4 & 0.6 & 0 & 0 \\ 0 & 0 & 0.3 & 0.7 \\ 0 & 0 & 0.5 & 0.5 \end{bmatrix}$$

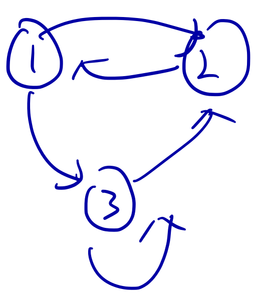


$\{1, 2\}$ aperiodic

$\{3, 4\}$ "

One more,
(E)

$$P = \begin{bmatrix} 0 & 0.5 & 0.5 \\ 1 & 0 & 0 \\ 0 & 0.5 & 0.5 \end{bmatrix}$$



1 : 2, 3, .. GCD = 1

3 1, 2, 3 " 1

aperiodic

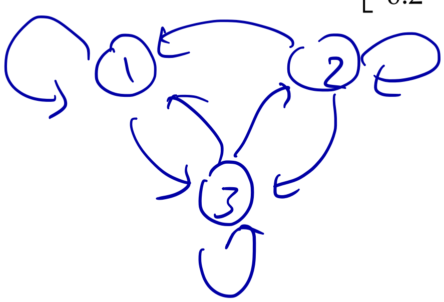
- Ergodic: A state which is positive recurrent and aperiodic is called ergodic.

- If all the states in a Markov Chain are ergodic, then the Markov Chain is said to be ergodic.

- Now let us check which of the four Markov Chains we have been using as examples are irreducible and ergodic Markov Chains.

(A)

$$P = \begin{bmatrix} 0.3 & 0 & 0.7 \\ 0.4 & 0.3 & 0.3 \\ 0.2 & 0.4 & 0.4 \end{bmatrix}$$



irreducible

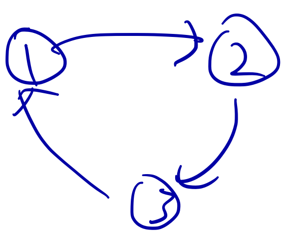
aperiodic

recurrent

ergodic

(B)

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$



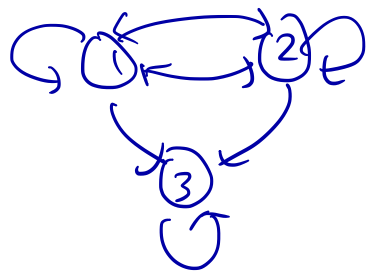
irreducible

period = 3

not ergodic

(C)

$$P = \begin{bmatrix} 0.3 & 0.3 & 0.4 \\ 0.4 & 0.2 & 0.4 \\ 0 & 0 & 1 \end{bmatrix}$$

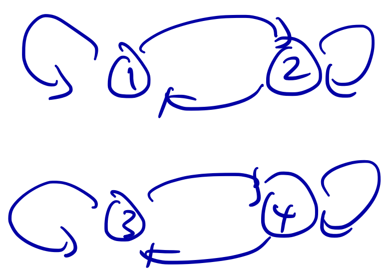


not irreducible

not ergodic

(D)

$$P = \begin{bmatrix} 0.2 & 0.8 & 0 & 0 \\ 0.4 & 0.6 & 0 & 0 \\ 0 & 0 & 0.3 & 0.7 \\ 0 & 0 & 0.5 & 0.5 \end{bmatrix}$$



not irreducible

ergodic

Theorem 1 (Thm 4.1 in the text book)**.** *For an irreducible ergodic Markov Chain,* $\lim_{n\to\infty} P_{ij}^n$ *exists and is independent of i (starting state).* $\lim_{n\to\infty} P_{ij}^n = \pi_j$*. These are called stationary or steady state probabilities.*

As per this theorem, (A) can have steady state probabilities.

<u>How to find steady state probabilities</u>

Let us look at P^{12} and P^{13} of (A).

$$P^{12} = \begin{bmatrix} 0.280 & 0.262 & 0.458 \\ 0.280 & 0.262 & 0.458 \\ 0.280 & 0.262 & 0.458 \end{bmatrix}, P^{13} = \begin{bmatrix} 0.280 & 0.262 & 0.458 \\ 0.280 & 0.262 & 0.458 \\ 0.280 & 0.262 & 0.458 \end{bmatrix}$$

(Here, the system reaches steady state when $n = 12$, subjecting to rounding off. Most Markov Chains ready steady state when $n < \infty$. Theoretically, steady state is achieved only when $n = \infty$.)

- As we noted earlier:
  - Rows of P^n are the same.
  - Row of f P^{n+1} are the same as those of P^n.
- We know that $P^{n+1} =$ $P^n \cdot P$ or $P^n \cdot P = P^{n+1}$
- In (A),

$$\begin{bmatrix} 0.280 & 0.262 & 0.458 \\ 0.280 & 0.262 & 0.458 \\ 0.280 & 0.262 & 0.458 \end{bmatrix} * \begin{bmatrix} 0.3 & 0 & 0.7 \\ 0.4 & 0.3 & 0.3 \\ 0.2 & 0.4 & 0.4 \end{bmatrix} = \begin{bmatrix} 0.280 & 0.262 & 0.458 \\ 0.280 & 0.262 & 0.458 \\ 0.280 & 0.262 & 0.458 \end{bmatrix}$$

- The row of P^{12} and P^{13} are the same and hence any row of P^{12} and P^{13} can be considered. Let us look at row 1.

$$\begin{bmatrix} 0.280 & 0.262 & 0.458 \end{bmatrix} * \begin{bmatrix} 0.3 & 0 & 0.7 \\ 0.4 & 0.3 & 0.3 \\ 0.2 & 0.4 & 0.4 \end{bmatrix} = \begin{bmatrix} 0.280 & 0.262 & 0.458 \end{bmatrix}$$

- In this example, $\pi_1 = .28$, $\pi_2 = .262$, $\pi_3 = .458$
- In general,

$$\pi = [\pi_1 \ \ \pi_2 \ \ \cdots \ \ \pi_m \,] \tag{1}$$

- From (1), we obtain the following equations:

$\pi \cdot P = \pi$

(equations here)

$$\begin{bmatrix} \pi_1 & \pi_2 & \pi_3 \end{bmatrix} \begin{bmatrix} .3 & 0 & .7 \\ .4 & .3 & .3 \\ .2 & .4 & .4 \end{bmatrix} = \begin{bmatrix} \pi_1 & \pi_2 & \pi_3 \end{bmatrix}$$

$$\begin{cases} .3\pi_1 + .4\pi_2 + .2\pi_3 = \pi_1 & (1) \\ .3\pi_2 + .4\pi_3 = \pi_2 & (2) \\ .7\pi_1 + .3\pi_2 + .4\pi_3 = \pi_3 & (3) \\ \pi_1 + \pi_2 + \pi_3 = 1 \end{cases}$$

$(\pi_1 \ \ \pi_2 \ \ \pi_3) \overset{?}{=} (0 . 0 . 0)$

$$-.7\pi_1 + .4\pi_2 + .2\pi_3 = 0 \quad (4)$$

$$-.7\pi_2 + .4\pi_3 = 0 \quad (5)$$

$$.7\pi_1 + .3\pi_2 - .6\pi_3 = 0 \quad (6)$$

(5) : $\pi_2 = \frac{4}{7}\pi_3$ (7)

$(7) \to (4)$: $-.7\pi_1 + \left(\frac{4}{7}\right)(.4)\,\pi_3 + .2\pi_3 = 0$

$$\to \quad \pi_1 = \frac{3}{4.9}\pi_3 \quad (8)$$

$$\frac{3}{4.9}\pi_3 + \frac{4}{7}\pi_3 + \pi_3 = 1$$

$$\therefore \ \pi_3 = .458 \quad \to \quad \pi_1 = .28 \quad \pi_2 = .262$$

- The equations to be solved are:

$$-0.7\pi_1 + 0.4\pi_3 = 0 \tag{2}$$

$$0.7\pi_1 + 0.3\pi_2 - 0.6\pi_3 = 0 \tag{3}$$

and

$$\pi_1 + \pi_2 + \pi_3 = 1$$

- We have already obtained the following relations using (2) and (3):

$$\pi_2 = \frac{4}{7}\pi_3, \quad \pi_1 = \frac{3}{4.9}\pi_3$$

- In general,

$$[\pi_1 \;\; \pi_2 \;\; \cdots \;\; \pi_m] \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1m} \\ P_{21} & P_{22} & \cdots & P_{2m} \\ \vdots & & & \vdots \\ P_{m1} & P_{m2} & \cdots & P_{mm} \end{bmatrix} = [\pi_1 \;\; \pi_2 \;\; \cdots \;\; \pi_m]$$

$$\sum_{i=1}^{n} \pi_i P_{ij} = \pi_j \qquad \text{— (a)}$$

and

$$\sum_{j=1}^{n} \pi_j = 1 \qquad \text{— (b)}$$

One of (a) can be discarded.