SSIE 660: Stochastic Systems

Homework assignment 1 - key

1. An interval of length 1, say (0,1), is divided into three segments by choosing two points at random. What is the probability that the three line segments form a triangle?
   (Hint: triangle $\rightarrow$ any segment $<$ the sum of the other two segments).

   Sol) Let the lengths of the segments be x and y. Then,

   $$S = \{x, y | 0 \leq x \leq 1, 0 \leq y \leq 1 - x\}$$

   and the event E is "the three segments form a triangle". There are three constraints.

   (a) $x < y + (1 - x - y)$
   (b) $y < x + (1 - x - y)$
   (c) $1 - x - y < x + y$

   Summarizing the above three constraints, we get

   $$E = \{x, y | 0 \leq x \leq 1/2,\ 0 \leq y \leq 1/2,\ x + y \geq 1/2\}$$

   and

   $$P(E) = \frac{\text{Area of } E}{\text{Area of } S} = \frac{1}{4}$$

2. Find the probability that the sum of two randomly selected positive numbers, both ≤ 1, will not exceed 1 and that their product will not exceed $1/4$.

   Sol) $S = \{x, y | 0 \leq x \leq 1,\ 0 \leq y \leq 1\}$
   $E = \{x, y | x, y | 0 \leq x \leq 1,\ 0 \leq y \leq 1, x + y \leq 1, xy \leq 1/4\}$ Therefore,

   $$P(E) = \frac{\text{Area of } E}{\text{Area of } S} = \frac{1}{2}$$

3. In a class there are four freshman boys, six freshman girls, and six sophomore boys. How many sophomore girls must be present if gender and class are to be independent when a student is selected at random?

   Sol) Let F represent freshmen, A sophomore, B boy, and G girl. Then, in order for class and gender to be independent, the following must hold.

   $$P(B|F) = P(B)$$

Let x be the number of sophomore girls. Then, $P(B) = \frac{\text{number of boys}}{\text{total number of students}} = \frac{4+6}{4+6+6+x} = \frac{10}{16+x}$. Also,

$$P(B|F) = \frac{P(BF)}{P(F)} = \frac{\frac{4}{16+x}}{\frac{10}{16+x}} = \frac{4}{10}$$

Therefore,

$$\frac{4}{10} = \frac{10}{16+x} \rightarrow x = 9$$

We can get the same result ($x = 9$) if we solve the following equation (check yourself). $P(G|F) = P(G)$ or $P(B|A) = P(B)$.

4. Suppose that we have ten coins such that if the i^{th} if flipped, then heads will appear with probability $i/10, i = 1, 2, ..., 10$. When one of the coins is randomly selected and flipped, it shows heads. What is the probability that it was the fifth coin?

   Sol) $E_i \equiv$ "ith coin flipped", $H \equiv$ "Outcome is head". Then, the probability we want to find is:

   $$P(E_5|H) = \frac{P(E_5H)}{P(H)}$$

   Now, $P(H) = P(E_1H) + \cdots + P(E_{10}H) = \frac{1}{10} * \left(\frac{1}{10} + \frac{2}{10} + \frac{3}{10} + \cdots + \frac{10}{10}\right)$ and $P(E_5H) = P(E_5)P(H) = \frac{1}{10} * \frac{5}{10}$

   Therefore,

   $$P(E_5|H) = \frac{P(E_5H)}{P(H)} = \frac{\frac{1}{10} * \frac{5}{10}}{\frac{1}{10} * \left(\frac{1}{10} + \frac{2}{10} + \frac{3}{10} + \cdots + \frac{10}{10}\right)} = \frac{1}{11}$$

5. An urn contains b black balls and r red balls. One of the balls is drawn at random, but when it is put back in the urn, c additional balls of the same color are put with it. Now suppose that we draw another ball. Show that the probability that the first ball drawn is black, given that the second ball drawn was red is

   $$\frac{b}{b+r+c}$$

   Sol) $B_i \equiv$ "ith ball is black", $R_i \equiv$ "ith ball is red". Then, the probability we want to find is:

   $$P(B_1|R_2) = \frac{P(B_1R_2)}{P(R_2)}$$

   Note that,

   $$P(B_1) = \frac{b}{b+r}$$
   $$P(R_1) = \frac{r}{b+r}$$
   $$P(B_1R_2) = \frac{b}{b+r} * \frac{r}{b+r+c}$$
   $$P(R_2) = P(B_1R_2) + P(R_1R_2) = \frac{b}{b+r} * \frac{r}{b+r+c} + \frac{r}{b+r} * \frac{r+c}{b+r+c}$$

Therefore,

$$P(B_1|R_2) = \frac{P(B_1R_2)}{P(R_2)} = \frac{\frac{b}{b+r} * \frac{r}{b+r+c}}{\frac{b}{b+r} * \frac{r}{b+r+c} + \frac{r}{b+r} * \frac{r+c}{b+r+c}} = \frac{b}{b+r+c}$$