**SSIE 660: Stochastic Systems**
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**Note 13**
**Chapter 6. Continuous-Time Markov Chain**

Continuous-Time Markov Chain

$$P[X(t+s) = j | X(s) = i, X(u) = i_1, ...] = P[X(t+s) = j | X(s) = i]$$

where $0 \leq u \leq s$.

If $P[X(t+s) = j | X(s) = i] = P_{ij}(t)$, then a Markov chain is said to have homogeneous or stationary transition probabilities.

- Consider a continous-time Markov chain. Assume that it enters state $i$ at time 0 and that it does not leave state $i$ during the next ten minutes. (That is, no transition during the next ten minutes.) What is the probability that the process will not leave state $i$ during the following next five minutes?

Let $T_i$ denote the amount of time the process stays in state $i$ before making a transition into a different state.

- Required probability is: $P[X(15) = i | X(10) = i] =$

- In general,
  $P[T_i > s + t | T_i > s] =$

- Hence, $T_i \sim$

<u>Alternate way of defining Continuous-Time Markov Chain</u>

It is a stochastic process having the property that each time it enters state $i$,

1. The amount of time it spends in that state before making a transition into a different state is *exponential* with mean $1/v_i$.

2. When the process leaves state $i$, it next enters state $j$ with some probability, say $P_{ij}$. (Note that $P_{ij}$ has no meaning unless the process leaves state $i$. It is *not* the probability of going from $i$ to $j$ within an interval of length $t$, which is
   Thus,

   $$P_{ii} = \qquad\qquad\qquad\qquad P_{ii}(t) \neq$$

   $$\sum_j P_{ij} =$$

3. The amount of time the process spends in state $i$ and the next state it visits must be independent.

   Mean length of time the process spends in state $i$ =

Note: In Continuous-Time Markov Chains, it is important to define $v_i$ (and $1/v_i$), for every state $i$, after defining the states. In fact, the feasibility of defining $v_i$ (and $1/v_i$) must be taken into considerations, while defining the states. Also, all $P_{ij}'s$ must be defined.

**Example 1.** Consider a shoeshine establishment consisting of two chairs - chair 1 and chair 2. A customer upon arrival goes initially to chair 1 where his shoes are cleaned and polish is applied. After this is done, the customer moves on to chair 2, where polish is buffed. The service times at the two chairs are assumed to be independent random variables which are exponentially distributed with respective rates $\mu_1$ and $\mu_2$. Suppose that potential customers arrive in accordance with a Poisson process having rate $\lambda$, and that a potential customer will only enter the system if both chairs are empty. Is this a continuous Markov chain?

Definition of states:

(A) Number of customers in the system:

| State (i) | Interpretation |
|---|---|

Let us consider state $i = 1$.

$\frac{1}{v_i} = \frac{1}{v_1}$ = mean length of time during which there is one customer in the system.

(B) State definition must include the states of service of customers in the system.

| State (i) | Interpretation |
|---|---|

<u>Transition Graph:</u>

<u>i=0:</u>

$$P_{01} = \qquad ; \quad P_{02} =$$

$$\frac{1}{v_0} = \qquad ; v_0 =$$

i=1:

$$P_{12} = \qquad\qquad ; \; P_{10} =$$

$$\frac{1}{v_1} = \qquad\qquad ; v_1 =$$

i=2:

$$P_{20} = \qquad\qquad ; \; P_{21} =$$

$$\frac{1}{v_2} = \qquad\qquad ; v_2 =$$

**Example 2.** Consider two machines that are maintained by a single repairman. Machine $i$ functions for an exponential time with rate $\mu_i$ before breaking down, i = 1, 2. The repair times (for either machine) are exponential with rate $\mu$. Can we analyze this as a continuous Markov chain?

State Definition A:

| State (i) | Interpretation |
|---|---|
| 0 | Both machines working |
| 1 | |
| 2 | |
| 3 | |

Problem with the definition:

State Definition B:

| State (i) | Interpretation |
|---|---|
| 0 | Both machines working |
| 1 | |
| 2 | |
| 3 | |
| 4 | |

Diagram:

$v_0 =$ ; $P_{01} =$ ; $P_{02} =$

$v_1 =$ ; $P_{10} =$ ; $P_{14} =$

$v_2 =$ ; $P_{20} =$ ; $P_{23} =$

$v_3 =$ ; $P_{31} =$

$v_4 =$ ; $P_{42} =$

## Birth and Death Processes

Characteristics:

State: the number of 'people' in the system at any time.

*Arrival rate when there are n 'people' in the system =* $\lambda_n$ *(birth rate).*

Inter-arrival time $\sim$ Exponential

*Departure rate when there are n 'people' in the system =* $\mu_n$ *(death rate).*

Inter-departure time $\sim$ Exponential

- It is a continuous-time Markov chain with states $\{ 0, 1, 2, ... \}$ .
- Transitions from state $n$ may go only to either $(n-1)$ or $(n+1)$.

Diagram

$v_0 =$ ; $v_1 =$ ; $v_2 =$

$v_i =$

$P_{01} =$

$P_{10} =$ ; $P_{12} =$

$P_{21} =$ ; $P_{23} =$

$P_{i,i-1} =$ ; $P_{i,i+1} =$

- It a system has arrivals (births) only, it has a pure birth process and it has departures (deaths) only, it has a pure death process.

The Kolmogorov Differential Equations

- Consider state $i$ and the states to which the process could move from $i$.

- Let $v_i$ be the total rate of transitions out of $i$ and $P_{ij}$ be the probability that the process moves to state $j$, once it gets out of state $i$.
- Let $q_{ij}$ be the rate at which the process moves from $i$ to $j$.

  $q_{ij} =$

  $v_i =$

  $P_{ij} =$

- Our next task is to find $P_{ij}(t)$, which is

  $P_{ij}(t) =$

  using $v_i$, $P_{ij}$, and $q_{ij}$.

**Lemma 3.** *1.* $\lim_{h\to 0} \dfrac{1 - P_{ii}(h)}{h} = v_i$.

*2.* $\lim_{h\to 0} \dfrac{P_{ij}(h)}{h} = q_{ij}$.

**Theorem 4** (Kolmogorov's Backward Equations.)**.** *For all states $i, j$ and times $t \geq 0$,*

$$P'_{ij}(t) = \sum_{k \neq i} q_{ik} P_{kj}(t) - v_i P_{ij}(t)$$

<u>Pure Birth Process:</u>

$v_i =$

$q_{ij} =$

$\sum_{k \neq i} q_{ik} P_{kj}(t) = q_{i,1} P_{1,j}(t) + q_{i,2} P_{2,j}(t) + \cdots + q_{i,i-1} P_{i-1,j}(t) + q_{i,i+1} P_{i+1,j}(t) + \cdots$

As the only possible transition from $i$ is to $i+1$ in a pure birth process,

$\sum_{k \neq i} q_{ik} P_{kj}(t) = q_{i,i+1} P_{i+1,j}(t) =$

So, the backward equation is

$P'_{ij}(t) =$

<u>Birth and Death Process:</u>

$v_0 =$ $; v_i =$

$P_{ij} =$

In a birth and death process, the only transitions possible from $i$ are to $i-1$ and $i+1$. Therefore,

$\sum_{k \neq i} q_{ik} P_{kj}(t) = q_{i,i-1} P_{i-1,j}(t) + q_{i,i+1} P_{i+1,j}(t)$

$q_{i,i-1} = v_i P_{i,i-1} =$

$q_{i,i+1} = v_i P_{i,i+1} =$

So, the backward equation is

$\underline{i = 0}$: $P'_{0j}(t) =$

$\underline{i > 0}$: $P'_{ij}(t) =$

Example of obtaining transition probabilities using Kolmogorov's backward equations.

**Example 5.** Consider a machine that works for an exponential amount of time having mean $1/\lambda$ before breaking down; and suppose that it takes an exponential amount of time having mean $1/\mu$ to repair the machine. If the machine is in working condition at time 0, then what is the probability that it will be working at time t=10?

| State (i) | Interpretation |
|---|---|
| 0 | Machine working |
| 1 | Machine down and in service |

Diagram:

- It is a Birth and Death process with
- We want to find $P_{00}(t)$

The Kolmogorov's backward equations for a birth and death process is:

$$P'_{0j}(t) = \lambda_0[P_{1j}(t) - P_{0j}(t)]$$

$$P'_{ij}(t) = \lambda_i P_{i+1,j}(t) + \mu_i P_{i-1,j}(t) - (\lambda_i + \mu_i)P_{ij}(t)$$

Hence, the Kolmogorov's backward equations for this system are:

j=0:

$$P'_{00}(t) = \lambda_0[P_{10}(t) - P_{00}(t)]$$

The Kolmogorov's forward equation:

$$P'_{ij}(t) = \sum_{k \neq i} q_{kj} P_{i,k}(t) - v_j P_{ij}(t)$$