SSIE 660: Stochastic Systems
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Note 1

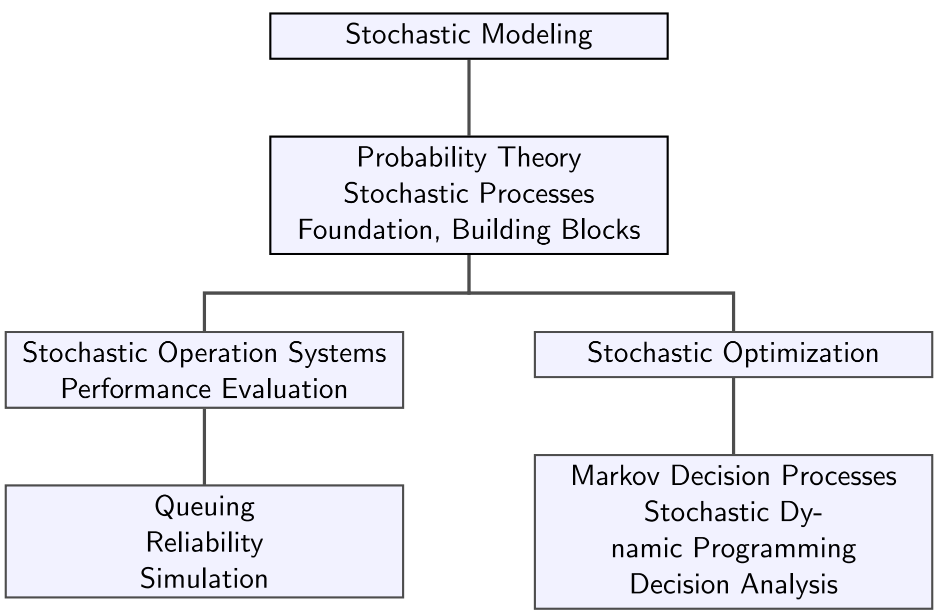
What are stochastic processes?

- Stochastic processes: A subject about modeling and analysis of random phenomena of certain system evolving over time.
- Stochastic processes are the corner stone in the analysis of stochastic system.

Why are stochastic processes important?

- The study of many operating system is through mathematical modeling.
- Optimization and uncertainty are two dominant themes in mathematical modeling.
- Two major categories of math methods used in Operations Research/Management Science (OR/MS):
  - Deterministic Optimization: Math Programming
  - Stochastic Modeling: Applied Probability

Stochastic Modeling



What is a Stochastic Process

- A stochastic process is a collection of random variables (RVs), usually denoted by

$$X = \{X(t) : t \in T\}$$

- t is a call index, most commonly it denotes times.
- T is called the index set.
- $X(t)$ is a random variable (RV) associated with an index t. We say $X(t)$ is the state of the system at time t.
- If $X(t)$ takes values in a set S for every $t \in T$, then S is called the sample space of X.

Types of stochastic processes

- If index set T is countable, then X is called a discrete-time SP.
- If index set T is continuous, then X is called a continuous-time SP.
- If sample space S is countable, then X is said to have a discrete state space.
- If sample space S is continuous, then X is said to have a continuous state space.

Chapter 1. Introduction to Probability Theory

Sample space: The set of all possible outcomes of an experiment, usually denoted by S.

EX.

1. an experiment of flipping a coin: $S = \{H, T\}$.
2. an experiment of rolling a die: $S = \{1, 2, 3, 4, 5, 6\}$.
3. an experiment of measuring the lifetime of a car: $S = [0, \infty)$.

Event: any subset E of the sample space S.

EX.

- $E = \{H\}$, $E = \{T\}$ in the above example 1. E is the event that a head, and a tail appears, respectively.
- $E = \{2, 4, 6\}$ in the above example 2. E is the event that an even number appears on the roll.
- $E = (2, 6)$ in the above example 3. E is the event that the car lasts between two and six years.

Event $E \cup F$: either E or F. If there are more than two events, we use the notation $\bigcup_{n=1}^{\infty} E_n$

EX.

- If $E = \{H\}$ and $F = \{T\}$, then $E \cup F = \{H, T\}$

Event $E \cap F$ or EF: intersection of E and F. If there are more than two events, we use the notation $\bigcap_{n=1}^{\infty} E_n$

EX.

- If $E = \{1, 3, 5\}$ and $F = \{1, 2, 3\}$, then $EF = \{1, 3\}$

Null event $\emptyset$: event consisting of no outcomes. IF $EF = \emptyset$, then E and F are said to be mutually exclusive.

Complement of E, denoted by E^C: all outcomes in the sample space S that are not in E.

Probability of the event E, denoted by $P(E)$: satisfies the following three conditions.

1. $0 \leq P(E) \leq 1$
2. $P(S) = 1$.
3. For any sequence of events $E_1, E_2, ...$ that are mutually exclusive, that is, events for which $E_n E_m = \emptyset$ when $n \neq m$, then $P\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} P(E_n)$

EX.

- a fair coin, $P(\{H\}) = 1/2$.
- a fair die, $P(\{1\}) = 1/6$.
- a fair die, $P(\{1, 4, 5\}) = P(\{1\}) + P(\{4\}) + P(\{5\}) = 1/2 \because$ mutually exclusive events.

Probability of the union of events, e.g., $E \cup F$, denoted by $P(E \cup F)$: $P(E \cup F) = P(E) + P(F) - P(E \cap F)$. In general,

$$P(E_1 \cup E_2 \cup ... \cup E_n) = \sum_i P(E_i) - \sum_{i<j} P(E_i E_j) + \sum_{i<j<k} P(E_i E_j E_k) - ... + (-1)^{n+1} P(E_1 E_2 ... E_n)$$

which is called the inclusion-exclusion identity.

Conditional probability, usually denoted by $P(E|F)$: probability that the event E occurs given that the event F has occurred. That is,

$$P(E|F) = \frac{P(EF)}{P(F)}$$

Note that the above formula is only well-defined when $P(F) > 0$.

Example 1. Suppose cards numbered one through ten are placed in a hat, mixed up, and then one of the cards is drawn. If we are told that the number on the drawn card is at least five, then what is the conditional probability that it is ten?

Example 2. A family has two children. What is the conditional probability that both are boys given that at least one of them is a boy? Assume that all outcomes in the sample space are equally likely.

Example 3. Suppose that each of three men at a party throws his hat into the center of the room. The hats are first mixed up and then each man randomly selects a hat. What is the probability that none of the three men selects his own hat?

<u>Independent events:</u> Two events E and F are said to be independent if $P(EF) = P(E)P(F)$ or if $P(E|F) = P(E)$.

Example 4. Let a ball be drawn from an urn containing four balls, numbered 1,2,3,4. Let $E = \{1, 2\}$, $F = \{1, 3\}$, $E = \{1, 4\}$. If all four outcomes are assumed equally likely, then

$$\begin{aligned} P(EF) &= \\ P(EG) &= \\ P(FG) &= \\ P(EFG) &= \end{aligned}$$

Therefore, the events E, F, G are ( ) independent but they are not ( ) independent.

Baye's formula

Let E and F be events, We may express E as:

$$E = EF \cup EF^C$$

Since EF and EF^C are mutually exclusive, we have that

$$\begin{aligned} P(E) &= P(EF) + P(EF^C) \\ &= P(E|F)P(F) + P(E|F^C)P(F^C) \\ &= P(E|F)P(F) + P(E|F^C)(1 - P(F)) \end{aligned}$$

Example 5. Consider two urns. The first contains two white and seven black balls, and the second contains five white and six black balls. We flip a fair coin and then draw a ball from the first urn or the second urn depending on whether the outcome was heads or tails. What is the conditional probability that the outcome of the toss was heads given that a white ball was selected?

Example 6. In answering a question on a multiple-choice test a student either knows the answer or guesses. Let p be the probability that she knows the answer and $1 - p$

the probability that she guesses. Assume that a student who guesses at the answer will be correct with probability $1/m$, where m is the number of multiple-choice alternatives. What is the conditional probability that a student know the answer to a question given that she answered it correctly?

Now suppose $F_1, F_2, ..., F_n$ are mutually exclusive events such that $\bigcup_{i=1}^{n} F_i = S$. Then,

$$E = \bigcup_{i=1}^{n} EF_i$$

Using the fact that $EF_i, i = 1, ..., n$ are mutually exclusive, we obtain:

$$P(E) = \sum_{i=1}^{n} P(EF_i) = \sum_{i=1}^{n} P(E|F_i)P(F_i)$$

which states that $P(E)$ is equal to a weighted average of $P(E|F_i)$, each term being weighted by the probability of the event on which it is conditioned.

Suppose now that E has occurred and we are interested in determining which one of the F_i also occurred.

$$P(F_i|E) = \frac{P(EF_i)}{P(E)} = \frac{P(E|F_i)P(F_i)}{\sum_{i=1}^{n} P(E|F_i)P(F_i)}$$

The above equation is known as *Baye's formula*.

Example 7. You know that a certain letter is equally likely to be in any one of three different folders. Let α_i be the probability that you will find your letter upon making a quick examination of folder i if the letter is, in fact, in folder $i, i = 1, 2, 3$. (We may have $\alpha_i < 1$.) Suppose you look in folder 1 and do not find the letter. What is the probability that the letter is in folder 1?