SSIE 660: Stochastic Systems
Dr. Sung H. Chung
Note 14
Chapter 6. Continuous-Time Markov Chain

Limiting Probabilities

As in the case of discrete-time Markov chains, continuous-time Markov chains also have steady state probabilities, after the system has been working for a long time. The steady state probability of being in state j is denoted by P_j (independent of the initial state).

$$P_j = \lim_{t \to \infty} P_{ij}(t)$$

Equations required for finding $P_j's$

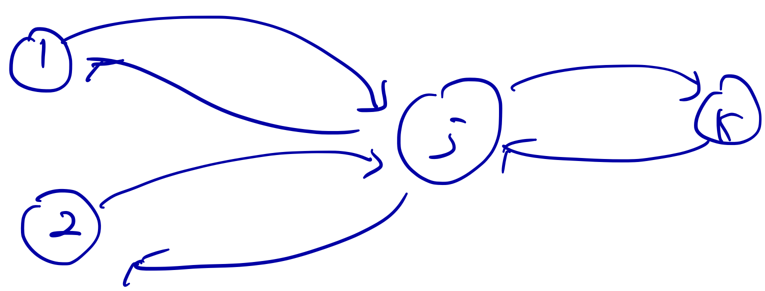
Review

v_k: Rate out of state k, given that it is in k.

P_{kj}: Probability of going from k to j, given that the system leaves state k.

$v_k P_{kj} = q_{kj}$: Total rate of transition from k to j.

Consider state j:



Total rate out of j = $P_j V_j = P_j \sum q_{jk} = P_j \sum_k V_j P_{jk}$

Total rate into j (from all other states) = $\sum_k P_k q_{kj}$

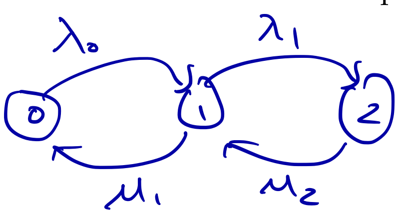
When the system reaches steady state:

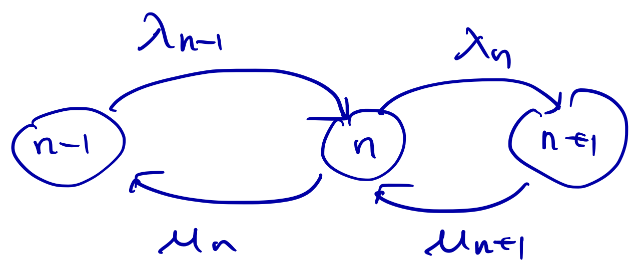
rate in = rate out of each state

These are called balance equations.

- If there are m states, balance equations will give m equations (m unknowns).
- As in the case of discrete-time Markov chains, we need to use: $\sum_{j=1}^{m} P_j = 1$
- This means that one of the m balance equations is redundant.
- In most cases, the steady state probabilities will be expressed as functions of one of the probabilities (usually P_0). Then the equation $\sum_{\text{all} j} P_j = 1$ will be used to find that one probability and the rest.

Consider a birth and death process.





<u>State 0:</u>

$$P_0\lambda_0 = P_1 \mu_1 \qquad P_1 = P_0 \frac{\lambda_0}{\mu_1}$$

<u>State 1:</u>

$$P_1(\lambda_1 + \mu_1) = P_0 \lambda_0 + P_2 \mu_2$$

$$P_0 \frac{\lambda_0}{\mu_1}(\lambda_1 + \mu_1) = P_0\lambda_0 + P_2\mu_2$$

$$\Rightarrow \quad P_2 = \frac{\lambda_0 \lambda_1}{\mu_1 \mu_2} P_0$$

<u>State 2:</u>

$$P_3 = \frac{\lambda_0\lambda_1\lambda_2}{\mu_1\mu_2\mu_3} P_0$$

$$P_n = \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n} P_0 = \left(\prod_{j=1}^{n} \frac{\lambda_{j-1}}{\mu_j} \right) P_0$$

prove it using induction

$$\sum_{n=0}^{\infty} P_n = 1 = P_0 + \sum_{n=1}^{\infty} \prod_{j=1}^{n} \frac{\lambda_{j-1}}{\mu_j} P_0 = 1$$

$$P_0 = 1 \Big/ \left[1 + \sum_{n=1}^{\infty} \prod_{j=1}^{n} \frac{\lambda_{j-1}}{\mu_j} \right]$$

$$P_n = \prod_{j=1}^{n} \frac{\lambda_{j-1}}{\mu_j} \Big/ \left[1 + \sum_{n=1}^{\infty} \prod_{j=1}^{n} \frac{\lambda_{j-1}}{\mu_j} \right]$$

Necessary and significant condition for the existence of steady state probabilities:

$$\sum_{n=1}^{\infty} \prod_{j=1}^{n} \frac{\lambda_{j-1}}{\mu_j} < \infty$$

Example 1. Single Server Queueing Systems with Poisson Arrivals and Exponential Service Times.

- It's a birth and death process with

$$\lambda_n = \lambda \quad \text{for all } n \geq 0$$

$$\mu_n = \mu \quad \text{for all } n \geq 1$$

$$\sum_{n=1}^{\infty} \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n} = \sum_{n=1}^{\infty} \left(\frac{\lambda}{\mu}\right)^n = \frac{1}{1-\frac{\lambda}{\mu}} - \left(\frac{\lambda}{\mu}\right)^0 \text{ if } \frac{\lambda}{\mu} < 1$$

<u>Known result:</u>

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \quad \text{if } r < 1$$

$$\sum_{n=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^n = \frac{1}{1-\frac{\lambda}{\mu}} \quad \text{if } \frac{\lambda}{\mu} < 1$$

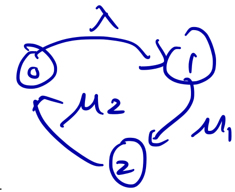
We need:

$$\sum_{n=1}^{\infty} \left(\frac{\lambda}{\mu}\right)^n = \frac{1}{1-\frac{\lambda}{\mu}} - \left(\frac{\lambda}{\mu}\right)^0 = \frac{\lambda}{\mu - \lambda}$$

$$P_0 = \frac{1}{1 + \sum_{n=1}^{\infty} \left(\frac{\lambda}{\mu}\right)^n} = 1 - \frac{\lambda}{\mu}$$

$$P_n = \prod_{j=1}^{n} \frac{\lambda_{j-1}}{\mu_j} P_0 = \prod_{j=1}^{n} \frac{\lambda}{\mu} P_0 = \left(\frac{\lambda}{\mu}\right)^n P_0$$

Example 2. Consider a shoeshine establishment consisting of two chairs - chair 1 and chair 2. A customer upon arrival goes initially to chair 1 where his shoes are cleaned and polish is applied. After this is done, the customer moves on to chair 2, where polish is buffed. The service times at the two chairs are assumed to be independent random variables which are exponentially distributed with respective rates μ_1 and μ_2. Suppose that potential customers arrive in accordance with a Poisson process having rate λ, and that a potential customer will only enter the system if both chairs are empty. Is this a continuous Markov chain?



Definition of states:

State (i)	Interpretation
0	0 customer in the system
1	1 customer in chair 1
2	1 customer in chair 2

Is it a birth and death process?

No $\sim$ b&d. state i should be connected only to i+1, i-1.

State 0:

$$P_0\lambda = P_2\mu_2 \qquad P_2 = P_0\frac{\lambda}{\mu_2}$$

State 1:

$$P_1\mu_1 = P_0\lambda \qquad P_1 = P_0\frac{\lambda}{\mu_1}$$

$$P_0 + P_1 + P_2 = 1$$

$$P_0 = \frac{1}{1 + \frac{\lambda}{\mu_1} + \frac{\lambda}{\mu_2}}$$

$$P_1 = \frac{\frac{\lambda}{\mu_1}}{1 + \frac{\lambda}{\mu_1} + \frac{\lambda}{\mu_2}}$$

$$P_2 = \frac{\frac{\lambda}{\mu_2}}{1 + \frac{\lambda}{\mu_1} + \frac{\lambda}{\mu_2}}$$

Example 3. Machine Repairman Problem: Consider a job shop that consists of M machines and one serviceman. Suppose that the amount of time each machine runs before breaking down is exponentially distributed with mean $1/\lambda$, and suppose that the amount of time that it takes for the serviceman to fix a machine is exponentially distributed with mean $1/\mu$.

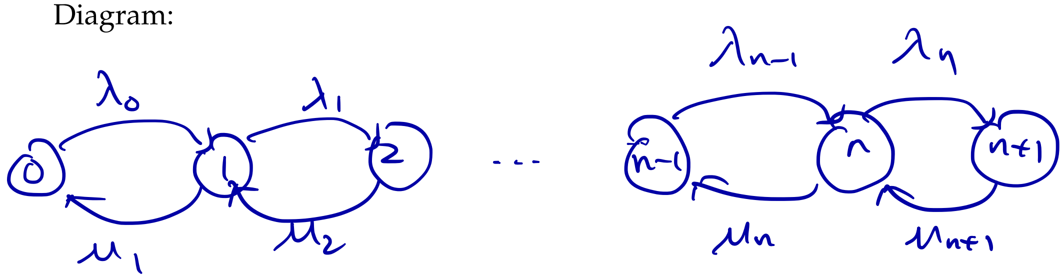
1. What is the average number of machines not in use?
2. What proportion of time is each machine in use?

State:

The number of machines down = n

The number of machines working = $M - n$

Diagram:



$$\lambda_i = \begin{cases} (M-i)\lambda & \text{if } i < M \\ 0 & i \geq M \end{cases}$$

$$\mu_i = \begin{cases} \mu & \text{if } i \geq 1 \\ 0 & \text{if } i = 0 \end{cases}$$

From earlier results:

$$= \frac{1}{1 + \sum_{n=1}^{M} \left(\frac{\lambda}{\mu}\right)^n \left(\frac{M!}{(M-n)!}\right)}$$

$\uparrow$

$$P_0 = \frac{1}{1 + \sum_{n=1}^{\infty} \prod_{j=1}^{n} \frac{\lambda_{j-1}}{\mu_j}} = \frac{1}{1 + \sum_{n=1}^{M} \frac{(M\lambda)(M-1)\lambda \cdots (M-n+1)\lambda}{\mu^n}}$$

$$P_n = P_0 \prod_{j=1}^{n} \frac{\lambda_{j-1}}{\mu_j} = \frac{\left(\frac{\lambda}{\mu}\right)^n \frac{M!}{(M-n)!}}{1 + \sum_{n=1}^{\infty} \left(\frac{\lambda}{\mu}\right)^n \frac{M!}{(M-n)!}}$$

Average number of machines not in use =

$$\sum_{n=0}^{M} n \cdot P_n = \sum_{n=0}^{M} \frac{n \left(\frac{\lambda}{\mu}\right)^n \frac{M!}{(M-n)!}}{1 + \sum_{n=1}^{M} \left(\frac{\lambda}{\mu}\right)^n \frac{M!}{(M-n)!}}$$

Proportion of time each machine is not in use = $\sum_{n=0}^{M} \frac{n}{M} P_n$

Proportion of time each machine is in use = $1 - \sum_{n=0}^{M} \frac{n}{M} P_n$

$\Downarrow$

utilization of the machine

Example 4. Service Facility with Limited Waiting Room:

Inter-arrival times of customers at the facility $\sim$ exponential $(1/\lambda)$

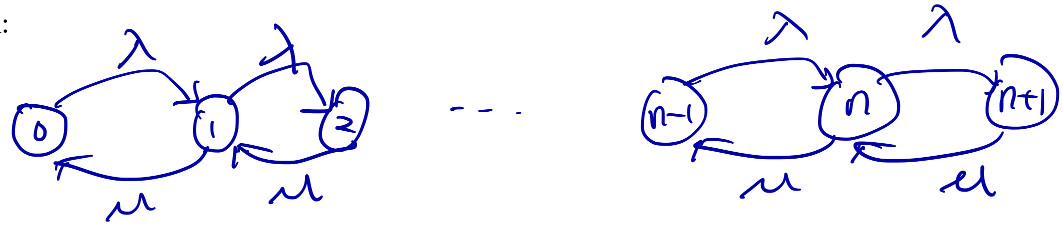
Single server: Service times of customers $\sim$ exponential $(1/\mu)$

Facility has a maximum capacity of N (waiting + service)

Customers arriving when the facility is full will not enter the facility and are lost to the system.

<u>States:</u> # of customers in the service facility

Diagram:



$$\lambda_n = \begin{cases} \lambda & \text{if } n \leq N-1 \\ 0 & \text{o.w.} \end{cases}$$

$$\mu_n = \begin{cases} \mu & \text{if } n \geq 1 \\ 0 & \text{o.w.} \end{cases}$$

$$P_0 = \frac{1}{1 + \sum_{n=1}^{\infty} \prod_{j=1}^{n} \frac{\lambda_{j-1}}{\mu_j}}$$

$$P_n = P_0 \prod_{j=1}^{n} \frac{\lambda_{j-1}}{\mu_j} = \left(\frac{\lambda}{\mu} \right)^n P_0$$