SSIE 660: Stochastic Systems
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Note 12
Chapter 5. The Exponential Distribution and the Poisson Process

Interarrival and Waiting Time Distributions

Consider a Poisson process $P[N(t) = n] = e^{-\lambda t} \frac{(\lambda t)^n}{n!}$

Time of the first event = T_1

Time elapsed between the $(n-1)^{th}$ and n^{th} event = T_n (for n >1)

Then, the sequence $\{T_1, T_2, ..., T_n\}$ is the sequence of inter arrival times.

Probability distribution of T_n: $f_{T_1}(t_1)$

$F_{T_1}(t_1) = P[T_1 \leq t_1] =$

$1 - F_{T_1}(t_1) = P[T_1 > t_1] =$

$P[N(t_1) = 0] =$

$f_{T_1}(t_1) =$

Now let us find $f_{T_2}(t_2)$.

$P[T_2 > t_2] = P\{T_2 > t_2 | T_1\}$

However,

$$\begin{aligned} P[T_2 > t_2 | T_1 = t_1] &= P[\text{ 0 event in } (t_1, t_1 + t_2) | T_1 = t_1] \\ &= P[\text{ 0 event in } (t_1, t_1 + t_2)] \\ &= e^{-\lambda t_2} \end{aligned}$$

$F_{T_2}(t_2) =$

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Proposition 1. *$T_i, i = 1, 2, ...$ are independent and identically distributed exponential random variables having mean $1/\lambda$.*

<u>Probability distribution of the time until the n^{th} arrival.</u>

Let $S_n = \sum_{i=1}^{n} T_i$

Probability distribution: $f_{S_n}(t)$

$f_{S_n}(t) \longleftarrow F_{S_n}(t)$

$$F_{S_n}(t) =$$

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Further properties of Poisson process:

1. Consider two independent Poisson processes $\{N_1(t), t \geq 0\}$ with rate λ_1 and $\{N_2(t), t \geq 0\}$ with rate λ_2. Let S_n^1 denote the time of the n^{th} event of the first process and S_m^2 denote the time of the m^{th} event of the second process. Find $P[S_n^1 < S_m^2]$. That is,

   P

   - We know that the inter arrival times of these processes follow exponential density functions with means $1/\lambda_1$ and $1/\lambda_2$, respectively.
   - Let us consider the case when $n = 1, m = 1$.
   - We know from earlier results that

   $$P[S_1^1 < S_1^2] =$$

   $$P[S_1^2 < S_1^1] =$$

   - How about this probability with $n = 2, m = 1$? That is, 2 events of type 1 should occur before 1 event of type 2. ($P[S_2^1 < S_1^2]$).
   - This will happen if 1^{st} event of type 1 occurs before type 2 event and 2^{nd} event of type 1 occurs before type 2 event.

   $$\frac{\lambda_1}{\lambda_1 + \lambda_2} \frac{\lambda_1}{\lambda_1 + \lambda_2}$$

   because, after the 1^{st} event of type 1 occurs, the process starts all over again.

   $P[S_2^1 < S_1^2] =$

   - Now let us consider the case with $n = 2, m = 5$.

   $$P[S_2^1 < S_5^2]$$

- The events included in the event { 2 events of type 1 before 5 events of type 2 } are:

$P[Type\,1] =$ $P[Type\,2] =$

# of Type 1 # of Type 2

$P[S_2^1 < S_5^2] =$

In general,

$P[S_n^1 < S_m^2] =$

2. Decomposition of a Poisson process: (Proposition 5.2)

   Let $\{N(t), t \geq 0\}$ be a Poisson process with rate λ. Suppose that each time an event occurs. It is classified as type $1, 2, ...$ and k with probabilities $p_1, p_2, ...$ and p_k, respectively. Let $N_i(t)$ be the number of events of type i in time t, $i = 1, 2, ...k$. Then $\{N_i(t), t \geq 0\}$ is a Poisson process with rate $\lambda * p_i$ and is *independent* of $N_1(t)$, $N_2(t)$, ..., $N_{i-1}(t)$, $N_{i+1}(t)$, ... $N_k(t)$ for $i = 1, 2, ..., k$. $p_1 + p_2 + \cdots p_k = 1$.

Example 2. Vehicles stopping at a roadside restaurant form a Poisson process $\{N(t), t \geq 0\}$ with rate $\lambda = 30/\text{hour}$. Twenty percent of these vehicles are trucks, thirty percent are buses, and the remaining are passenger cars. The number of passengers in a truck is one. the number of passengers in a passenger car is equal to 1, 2, 3, 4, and 5 with probabilities 0.3, 0.3, 0.2, 0.1, and 0.1, respectively. The number of passengers in a bus is more than 20. What is the probability that within a period of 2 hours, four cars with one passenger will stop at the restaurant?

3. Superposition of Poisson processes (Reverse of the decomposition property, not in the book).

   Let $\{N_1(t), t \geq 0\}$,$\{N_2(t), t \geq 0\}$, ... ,$\{N_k(t), t \geq 0\}$ be k independent Poisson processes with rates $\lambda_1, \lambda_2, ...\lambda_k$, repectively. Then

$$N(t) = N_1(t) + N_2(t) + \cdots + N_k(t)$$

   is a Poisson process with rate

$$\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_k$$

Example 3. Consider the road network pictured below. The inputs into streets A, B and K are Poisson processes (independent) with the rates indicated. The probabilities of a vehicle choosing the indicated directions are written parentheses along the arcs. What is the probability that the number of vehicles using street M within 3 hours is 20?

$\lambda_E =$

$\lambda_I =$

$\lambda_L =$

$\lambda_M =$

$P[N_M(3) = 20] =$

Conditional Distribution of Arrival Times

To find the conditional p.d.f. of the time at which an event occurred, given that exactly only one event has occurred by time t.

Let s be the time at which it occurred.

To find $f_S(s|N(t) = 1)$, it is better to find $F_S(S|N(t) = 1)$.

Probability distribution/Density function known:

(i) $P[N(t) = n]$

(ii) $f_S(s)$

$F_S(s|N(t) = 1] =$

$f_S(s|N(t) = 1] =$

Uniform in the range of $(0, t)$.

Each interval of equal length in the interval $(0, t)$ has the same probability of containing the event.