SSIE 660: Stochastic Systems
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Note 10
Chapter 4. Discrete Markov Chains

- As per Theorem 4.1, only irreducible ergodic Markov Chains can have steady state probabilities. That means only Markov Chain (A) have steady state probabilities and (B), (C), and (D) cannot have $\pi_j's$. Let us use the above equation, anyway, to obtain $\pi_j's$ in case of (B), (C), and (D).

(B)

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \pi_1 & \pi_2 & \pi_3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \pi_1 & \pi_2 & \pi_3 \end{bmatrix}$$

$$\begin{aligned} \pi_3 &= \pi_1 \\ \pi_1 &= \pi_2 \\ \pi_2 &= \pi_3 \\ \pi_1 + \pi_2 + \pi_3 &= 1 \end{aligned}$$

Solution is: $\pi_1 = \frac{1}{3}$; $\pi_2 = \frac{1}{3}$; $\pi_3 = \frac{1}{3}$. From previous note, we know that this Markov Chain does not have steady state probabilities. Then, what do π_1, π_2, and π_3 we obtained represent? These are

average long-run proportions of time that the System is in state 1, 2, & 3

(C)

$$P = \begin{bmatrix} 0.3 & 0.3 & 0.4 \\ 0.4 & 0.2 & 0.4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \pi_1 & \pi_2 & \pi_3 \end{bmatrix} \begin{bmatrix} 0.3 & 0.3 & 0.4 \\ 0.4 & 0.2 & 0.4 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \pi_1 & \pi_2 & \pi_3 \end{bmatrix}$$

$$
\begin{aligned}
0.3\pi_1 + 0.4\pi_2 &= \pi_1 \\
0.3\pi_1 + 0.2\pi_2 &= \pi_2 \\
0.4\pi_1 + 0.4\pi_2 + \pi_3 &= \pi_3 \\
\pi_1 + \pi_2 + \pi_3 &= 1
\end{aligned}
$$

Then,

$$\pi_1 = 0, \pi_2 = 0, \pi_3 = 1$$

These are the same values we obtained earlier.

(D)

$$
P = \begin{bmatrix} 0.2 & 0.8 & 0 & 0 \\ 0.4 & 0.6 & 0 & 0 \\ 0 & 0 & 0.3 & 0.7 \\ 0 & 0 & 0.5 & 0.5 \end{bmatrix}
$$

$$
\begin{bmatrix} \pi_1 & \pi_2 & \pi_3 & \pi_4 \end{bmatrix} \begin{bmatrix} 0.2 & 0.8 & 0 & 0 \\ 0.4 & 0.6 & 0 & 0 \\ 0 & 0 & 0.3 & 0.7 \\ 0 & 0 & 0.5 & 0.5 \end{bmatrix} = \begin{bmatrix} \pi_1 & \pi_2 & \pi_3 & \pi_4 \end{bmatrix}
$$

$$
\begin{aligned}
0.2\pi_1 + 0.4\pi_2 &= \pi_1 \\
0.8\pi_1 + 0.6\pi_2 &= \pi_2 \\
0.3\pi_3 + 0.5\pi_4 &= \pi_3 \\
0.7\pi_3 + 0.5\pi_4 &= \pi_4 \\
\pi_1 + \pi_2 + \pi_3 + \pi_4 &= 1
\end{aligned}
$$

$$
\boxed{\begin{aligned} -.8\,\pi_1 + .4\pi_2 &= 0 \\ .8\,\pi_1 - .4\pi_2 &= 0 \end{aligned}}
$$

$$-.7\pi_3 + .5\pi_4 = 0$$

How can we solve this? It depends on Starting state

If 1 or 2

$$\pi_1 = .333, \pi_2 = .667, \pi_3 = 0, \pi_4 = 0$$

If 3 or 4

$$\pi_1 = 0, \pi_2 = 0, \pi_3 = .417, \pi_4 = .563$$

From previous note,

$$P^{16} = P^{17} = \begin{bmatrix} 0.333 & 0.667 & 0 & 0 \\ 0.333 & 0.667 & 0 & 0 \\ 0 & 0 & 0.417 & 0.583 \\ 0 & 0 & 0.417 & 0.583 \end{bmatrix}$$

It can be seen that even though the rows do not change from transition to transition, they are not the same, within the same P^n. This implies that the steady state probability of being in a state depends on the row state (starting state), which is contrary to the definition of steady state. Therefore, this Markov Chain does not have steady state probabilities.

Interpretation of π_j :

1. Steady state probability that the system will be in state j in the long run. In the inventory problem, $\pi_1 = 0.280$, $\pi_2 = 0.262$, $\pi_3 = 0.458$. After the system has been in operation for a long time, $P[\text{the inventory level at the beginning of any day is } 2] = 0.262$.

2. π_j is the long-run proportion of time that the process/system will be in state j. In the inventory example, the long-run proportion of days at the beginning of which, the inventory level is 2 is 0.262.

3. Let m_{jj} be the expected number of transitions until the process, starting in state j, returns to the state j. Then,

$$\pi_j = \frac{1}{m_{jj}}$$

Use of π_j's in optimizing or analyzing systems
Let's see the inventory example.

Probability distribution of demand:

Demand	Probability
0	0.3
1	0.4
2	0.2
3	0.1

Ordering policy: Order 3 units if the inventory at the end of a day is 0; otherwise, do not oder.

Objective: Find the optimal ordering policy which minimizes the total cost, consisting of the expected inventory cost and the expected ordering cost.

Unit Costs:

1. The company spends \$10.00 per day for every unit at the beginning of the day. (carrying cost).
2. It costs the company \$50.00 to place an order.

Analysis: Let us find the total cost.

1. Expected Inventory Cost: Recall the steady state probabilities:

$$\pi_1 = 0.280, \pi_2 = 0.262, \pi_3 = 0.458$$

   Then, the expected inventory cost per day is = $10 \times$ expected inventory

   $= 10(1 \times .28 + 2 \times .262 + 3 \times .458)$

   $= 21.78$

2. Expected Ordering Cost:
   The expected ordering cost per day is =

   $50 \times P[$ordering at the end of a day$]$

Probability of ordering at the end of a day

Beginning inventory	Steady state prob.	Demand	Prob.	Ending Inv.	Prob.
1	.28	0	.3	1	
		1	.4	0	$.28(.4+.2+.1)$
		2	.2	0	
		3	.1	0	$= .196$
2	.262	0	.3	2	
		1	.4	1	
		2	.2	0	$.262(.2+.1)$
		3	.1	0	$= .0786$
3	.458	0	.3	3	
		1	.4	2	
		2	.2	1	
		3	.1	0	$\rightarrow .458(.1) = .0458$

Prob.[Ordering] = Prob.[Inventory at the end of the day is 0] = $.196 + .0786 + .0458$

$= .3704$

Expected valued of ordering cost =

$50(.3704) = 16.02$

Total cost = $21.78 + 16.02 = 37.80$

Note: The optimal ordering policy can be obtained by 'trial and error'. That is, by finding the costs for various ordering policies and selecting the policy which yields the minimum total cost.

Example 1. Suppose that a production process changes states in accordance with an irreducible, positive recurrent Markov chain having transition probabilities $P_{ij}, i, j, 1, ..., n$, and suppose that certain of the states are considered acceptable and the remaining unacceptable. Let A denote the acceptable states and A^C the unacceptable ones. If the production process is said to be "UP" when in an acceptable state and "DOWN" when in an unacceptable state, determine

1. the rate at which the production process goes from up to down (that is, the rate of breakdowns);
2. the average length of time the process remains down when it goes up; and
3. the average length of time the process remains up when it goes up.

given the following transition matrix

$$P = \begin{bmatrix} 1/4 & 1/4 & 1/2 & 0 \\ 0 & 1/4 & 1/2 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 0 & 1/2 \end{bmatrix}$$

where the acceptable (up) states are 1, 2 and the unacceptable states (down) are 3, 4. That is, $A = \{1, 2\}$, $A^C = \{3, 4\}$.

Using the following equations:

$$\sum_{i=1}^{m} \pi_i P_{ij} = \pi_j \quad j = 1, \ldots, m$$

$$\sum_{j=1}^{m} \pi_j = 1$$

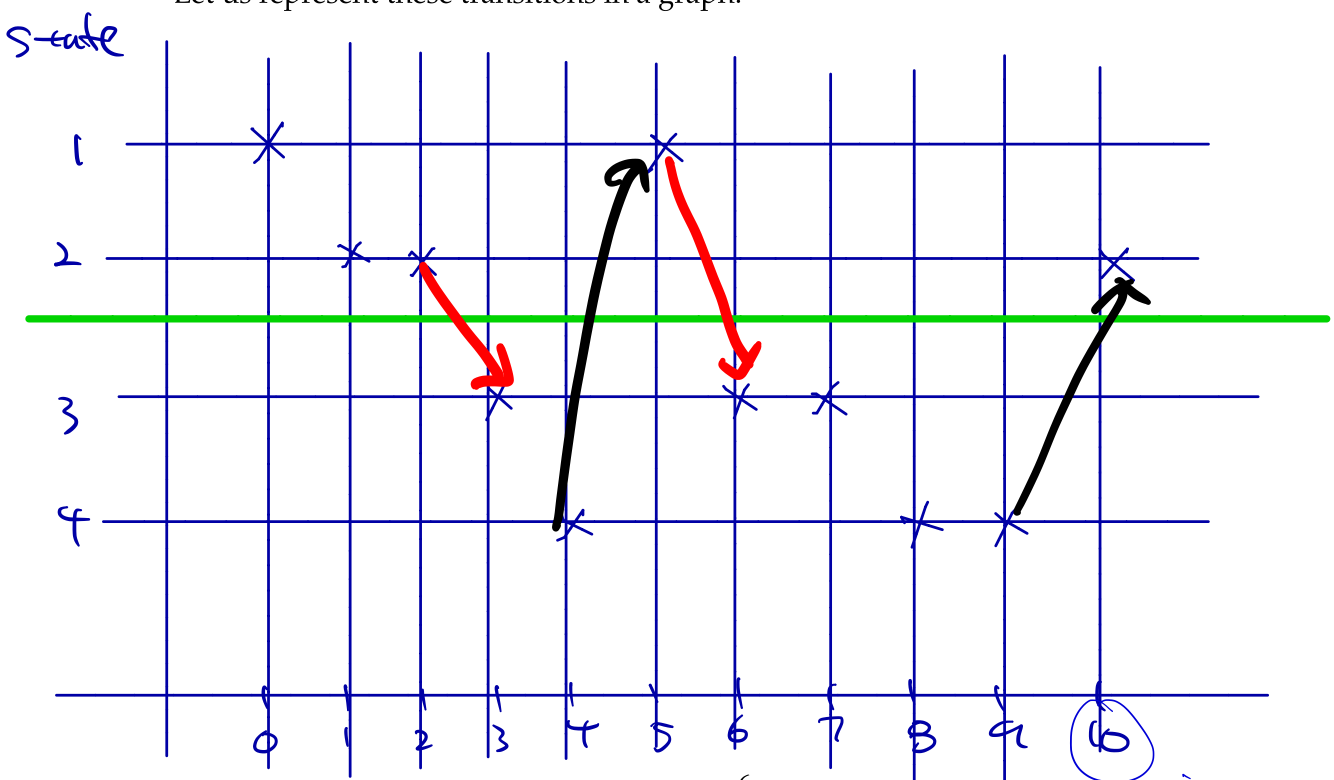
The steady state probabilities can be found as follows.

$$\pi_1 = \frac{3}{16} \quad , \pi_2 = \frac{1}{4} \quad , \pi_3 = \frac{14}{48} \quad , \pi_4 = \frac{13}{48}$$

Let us assume that we start observing this machine after it has been 'in operation' for a long time (i.e., it has reached the steady state) and record the state of the machine at observation epochs.

Time of observation	State	Group
0	1	Up
1	2	Up
2	2	Up
3	3	Down
4	4	Down
5	1	Up
6	3	Down
7	3	Down
8	4	Down
9	4	Down
10	2	Up

Let us represent these transitions in a graph.



ignore to simplify calculation

Based on these observations and the above sample path, what is our estimate of the rate of break downs?

Rate of break down= rate of going from up to down

= prob. that the sample path crosses the line in a downward direction = prob. that the machine goes from up to down in a period

Possible events in a 'Down Crossing'

$1 \to 3$ or 4

$2 \to 3$ or 4

Estimate = $\dfrac{\text{\# of down crossings}}{\text{total \# of transitions}} = \dfrac{2}{10}$

Probability = $\pi_1 (P_{13} + P_{14}) + \pi_2 (P_{23} + P_{24})$

$= 9/32$

(Theoretical, Not an estimate)

Rate of going from 'Down' to 'Up' = rate of upcrossing

= prob that the sample path crosses the line in an upword direction

= prob that the machine goes from down to up in a period.

Estimate= $\frac{2}{10}$

In the estimates of the Rates of "Down" crossing and "Up" crossing, the denominators are the same.

The numerators are = # of down crossings & # of up crossings

What is the maximum difference between these two numbers?
If the denominator is large, then

$$\frac{\text{\# of Down Crossing}}{\text{Total \# of transitions}} = \frac{\text{\# of Up Crossing}}{\text{Total \# of transitions}}$$

In theory, rate of down crossings = rate of up crossings.

- This is called the Theory of Level Crossing, developed by Dr. Percy Brill and Dr. George Shanthikumar in the late seventies.

Let us check this in our problem.
Possible events in an 'Up' Crossing = $3 \to 1$ or 2 $\quad 4 \to 1$ or 2

3 and 1 or 2, OR 4 and 1 or 2.

Probability = $\pi_3 (P_{31} + P_{32}) + \pi_4 (P_{41} + P_{42}) = 9/32$

(Theoretical, not an estimate)
Possible events in an 'Down' Crossing = $1 \to 3$ or 4 $\quad 2 \to 3$ or 4

1 and 3 or 4, OR 2 and 3 or 4.

Probability = $9/32$

(Theoretical, not an estimate)

States	Estimate		Theoretical		
1	$\frac{2}{10}$	$\frac{4}{10}$	$\frac{3}{16} = \pi_1$	$\frac{7}{16}$	up
2	$\frac{2}{10}$		$\frac{1}{4} = \pi_2$		
3	$\frac{3}{10}$	$\frac{6}{10}$	$\frac{14}{48} = \pi_3$	$\frac{27}{48}$	down
4	$\frac{3}{10}$		$\frac{13}{48} = \pi_4$		

Average length of time the machine remains down, when it goes down. ($\bar{D}$)

Estimate: $\frac{2+4}{2} = 3$ $\qquad \dfrac{\frac{6}{10}}{\frac{2}{10}}$

prob. the machine is in state 3 or 4

rate of down crossing

Theoretical:

$$\frac{\text{prob(machine down)}}{\text{rate of down crossing}} = \frac{\pi_3 + \pi_4}{\pi_1(P_{13} + P_{14}) + \pi_2(P_{23} + P_{24})}$$

Average length of time the machine remains up, when it goes up. ($\bar{U}$)

Estimate: $\frac{3+1}{2} = 2$ $\qquad \frac{\frac{4}{10}}{\frac{2}{10}} = 2$

Theoretical: $\frac{\text{prob [machine up]}}{\text{rate of upcrossing}}$

$$= \frac{\pi_1 + \pi_2}{\pi_3 (P_{31} + P_{32}) + \pi_4 (P_{41} + P_{42})}$$

$$= 14/9$$