CSCI 4830 / 5722
Computer Vision

University of Colorado **Boulder**

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Computer Vision

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Lecture 3

Reminders

Submissions:
- Homework 1: Sat 1/26 at 6 pm

Moodle, Matlab, Piazza
Disabilities forms

Readings:
- Szeliski Ch. 3
- P&F Ch. 1 (or 1&2 in 1st edition)

Welcome!

- Administrative issues:
  - Register on Moodle (password:'*vision*')
    - use your colorado.edu email address
    - use your official name
  - Matlab install (this week) – instructions on Moodle
  - Piazza.

- Office hours calendar - will be on Moodle

Recommended Books

- *Computer Vision: Algorithms and Applications* by Richard Szeliski
- *Computer Vision: A Modern Approach* by D. Forsyth and J. Ponce
- *Multiple View Geometry in Computer Vision* by R. Hartley and A. Zisserman
- *An Invitation to Computer Vision* by Ma, Soatto, Kosecka, and Sastry

Linear Independence

- A set of vectors is **linearly dependent** if one of the vectors can be expressed as a linear combination of the other vectors. (non-zero coeficients)
- Example:

$$\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}-3\\2\end{bmatrix}, \begin{bmatrix}2\\4\end{bmatrix}$$

$$a1 * \begin{bmatrix}1\\1\end{bmatrix} + a2 * \begin{bmatrix}-3\\2\end{bmatrix} + a3 * \begin{bmatrix}2\\4\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

Linear Independence

- Example:

$$\begin{bmatrix} 1 & -3 & 2 \\ 1 & 2 & 4 \end{bmatrix} * \begin{bmatrix} a1 \\ a2 \\ a3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & 2 \\ 0 & 5 & 2 \end{bmatrix} * \begin{bmatrix} a1 \\ a2 \\ a3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 16/5 \\ 0 & 1 & 2/5 \end{bmatrix} * \begin{bmatrix} a1 \\ a2 \\ a3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} * \begin{bmatrix} a1 \\ a2 \end{bmatrix} = \begin{bmatrix} a1 \\ a2 \end{bmatrix} = -a3 * \begin{bmatrix} 16/5 \\ 2/5 \end{bmatrix}$$

Linear Independence

- A set of vectors is **linearly independent** if none of the vectors can be expressed as a linear combination of the other vectors.

  Example:

$$a1 * \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + a2 * \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, + a3 * \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

, if and only if

a1 = a2 = a3 = 0. No other solution exists.

Linear Independence - True or False?

If the zero vector is part of a set of vectors that set is dependent.

If a set of vectors is dependent so is any larger set which contains it.

Rank of a matrix

The **rank** of a matrix is the number of linearly independent columns of the matrix.

Note:

- When all the columns of a matrix are linearly independent the matrix is said to be *full rank*

Theorem:

- If det(A) = 0, then the vectors are linearly dependent
- Can only be computed for n x n matrices

Computing Rank of an n x n matrix

Let's start with a 2 x 2 matrix A:

- If $\det(A) \neq 0$, then $\text{rank}(A) = 2$, since both column vectors are linearly independent
- If $\det(A) = 0$, but $A \neq 0$, then $\text{rank}(A) = 1$, since both column vectors are linearly dependent, but one of the columns is non-zero, hence this column is linearly independent
- If $A = 0$, then $\text{rank}(A) = 0$

Computing Rank of an n x n matrix

Now for an *n x n* (or an *m x n*) matrix *A*:

1) Find minors of order k, where k = n-p. Start with minors of rank (n-1)

Note: minor of order k is a k x k sub-matrix of A

2) Compute determinants of minors. If one exists and its determinant is not zero, then rank(*A*) = k
3) If not, then move on to minors of (k-1) order and repeat from step 2

Rank of a matrix

Examples:

det(A) = 0, hence after computing determinants for minors of order 2 we conclude Rank(A) = 2

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

det(A) ≠ 0, hence Rank(A) = 3

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Singular Matrix

- All of the following conditions are equivalent. We say a square ($n \times n$) matrix is **singular** if any one of these conditions (and hence all of them) is satisfied.
  - The columns are linearly dependent
  - The rows are linearly dependent
  - The determinant = 0
  - The matrix is not invertible
  - The matrix is not full rank (i.e., rank < n)

Linear Spaces

- A set of vectors **V** is considered a **linear space** over the field **R** if its elements, called vectors, are closed under to basic operations: scalar multiplication and vector summation.
- Same as: given <u>any</u> two vectors v1 and v2 and <u>any</u> two scalars α and β in **R**, the linear combination:

v = α*v1 + β*v2 is also a vector in **V**

Spanned subspace

- Given a set of ***m*** vectors **S**, the subspace spanned by **S** is the set of all finite linear combinations of these ***m*** vectors, for all $\alpha_{1,}\ \alpha_{2,}$ **…** α_m in **R**

Example: $\mathbf{R}^3$, 3-dimensional Euclidean space, given *v1* and *v2* (below), the spanned subspace consists of vectors with the general form of *v:*

$$v1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, v2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, v = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

Basis

- A set of ***m*** vectors **B** of a linear space **V** is said to be a **basis** if **B** is linearly independent set and B spans the entire space **V**

Example: $\mathbf{R}^3$, 3-dimensional Euclidean space, is spanned by the following basis:

$$\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$$

Linear Subspaces

- A **linear subspace** is the space spanned by a subset of the vectors in a linear space.
  - The space spanned by the following vectors is a 2D subspace of $\mathbf{R}^3$.

$$\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}$$

What does it look like?

$$\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$$

What does it look like?

Linear transformations

A linear transformation from a linear vector space $\mathbf{R}^n$ to $\mathbf{R}^m$ is defined as a map $\mathbf{L}: \mathbf{R}^n \rightarrow \mathbf{R}^m$ such that

- $L(x + y) = L(x) + L(y)$
- $L(\alpha * x) = \alpha * L(x)$

L can be represented as a matrix **A,** such that

$L(x) = A*x$

Linear transformations

Examples:

<u>Identity:</u>

- identity operation:

  x' = x

  y' = y

- With a linear transformation matrix:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} * \begin{bmatrix} x \\ y \end{bmatrix}$$

identity matrix

Linear transformations

Examples:

Inversion:

- inversion operation:

  x' = -x

  y' = -y

- With a linear transformation matrix:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} * \begin{bmatrix} x \\ y \end{bmatrix}$$

inversion matrix

Linear transformations

<u>Scaling:</u>

- Scaling operation:

  x’ = a * x

  y’ = b * y

- With a linear transformation matrix:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}}_{\textbf{scaling matrix}} * \begin{bmatrix} x \\ y \end{bmatrix}$$

Other Linear transformations matrices

Mirror / reflection:

- About y axis:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} * \begin{bmatrix} x \\ y \end{bmatrix}$$

- Over (0,0):

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} * \begin{bmatrix} x \\ y \end{bmatrix}$$

Other Linear transformations matrices

Rotation:

- 2D rotation about (0,0):

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} * \begin{bmatrix} x \\ y \end{bmatrix}$$

What about ...?

<u>Translation:</u>

- 2D translation operations:

  x' = x + a

  y' = y + b

- Is it possible to have a linear transformation?

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} * \begin{bmatrix} x \\ y \end{bmatrix}$$