Exchange Rate Policies at the Zero Lower Bound*

Manuel Amador
Minneapolis Fed and U of Minnesota

Javier Bianchi
Minneapolis Fed

Luigi Bocola
Northwestern University

Fabrizio Perri
Minneapolis Fed

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Abstract

This paper studies how the Central Bank of a small open economy achieves an exchange rate objective in an environment that features a zero lower bound (ZLB) constraint on nominal interest rates and limits to arbitrage in international capital markets. If the nominal interest rate that is consistent with interest parity is positive, the Central Bank can achieve its exchange rate objective while giving up its monetary independence, a well known result in international finance. However, if the nominal interest rate consistent with interest rate parity is negative, the pursue of an exchange rate objective results in zero nominal interest rates, deviations from interest rate parity, capital inflows, and welfare costs associated with the accumulation of foreign reserves by the Central Bank. We characterize how these costs vary with the economic environment, analyze situations in which these interventions become optimal, and study the role for alternative policies that can complement/substitute foreign exchange interventions. We show that the recent break-downs in covered interest rate parity documented in the literature are associated to large foreign exchange interventions carried out by Central Banks operating at the ZLB.

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1 Introduction

In the aftermath of the global financial crisis of 2008, many advanced economies have experienced large capital inflows and appreciation pressures on their exchange rates. To avoid the resulting losses in competitiveness that could delay the economic recovery, Central Banks in these economies implemented policies geared towards containing the appreciation of their currencies. However, with interest rates close to their lower bound, Central Banks have found it challenging to weaken their currencies using conventional reductions in interest rates, and have resorted to massive interventions in currency markets. The case of Switzerland is emblematic in this respect. After experiencing a 35% appreciation of the Swiss Franc between 2008 and 2010, the Swiss National Bank (SNB) reduced interest rates to zero and increased by 100% of GDP the holdings of foreign assets between 2011 and 2015 in an attempt to prevent further appreciations of the exchange rate. Eventually, in January 2015, the SNB let the exchange rate appreciate, triggering an intense policy debate about the desirability and the effectiveness of the central banks's interventions.

Against this backdrop, a number of important questions regarding the conduct of monetary policy at the zero lower bound have not been addressed: How can a central bank depreciate its currency when it cannot lower interest rates any further? What is the role of foreign exchange intervention and what are the trade-offs involved? To address these questions, we develop a simple monetary model of a small open economy (SOE) that potentially features nominal interest rates at their zero lower bound (ZLB) and limits to international arbitrage. We consider the problem of a Central Bank that would like to temporarily depreciate its exchange rate, and analyze the role that foreign reserve accumulation plays for implementing this policy when nominal interest rates are at zero.

Our framework specializes the classic trilemma of international finance for an economy that operates at the ZLB. We show that engineering a temporary depreciation of the exchange rate when nominal interest rates are at zero necessarily entails violations of the interest parity condition. These deviations from arbitrage create an incentive for foreign investors to accumulate assets of the SOE. In absence of an intervention by the monetary authority, these capital inflows would put upward pressure on the exchange rate. In order to sustain its exchange rate target, the Central Bank needs to accumulate foreign assets, reversing the trades made by the foreigners. Therefore, foreign exchange interventions are the instrument through which the Central Bank weakens its currency when the ZLB constraint binds. Importantly, as identified by Cavallino

(2016) and Fanelli and Straub (2015), these interventions in the presence of deviations from interest rate parity are *costly* because the Central Bank takes the opposite side of arbitrage profits made by the foreign investors, leading to a loss for the SOE as a whole.

This result contrasts with what happens when nominal interest rates are positive. In this scenario, the Central Bank can always fight appreciation pressures by decreasing its policy rate, eliminating the possibility of arbitrage opportunities in international capital markets. Thus, when nominal interest rates are positive, our model replicates the textbook result that an exchange rate objective can be implemented without resorting to the accumulation of foreign assets. When the ZLB constraint binds, instead, the Central Bank needs to expand its balance sheet, and carry out costly interventions in foreign exchange markets.

We next turn to a comparative static analysis of the costs of sustaining an exchange rate target at the ZLB. Changes in the economic environment that are typically beneficial for a small open economy tend to increase these costs. For example, a deepening in international capital market integration is beneficial when nominal interest rates are positive, while it is detrimental when the ZLB constraint binds. Facing an increase in foreign wealth, the Central Bank needs to buy larger amount of foreign assets in order to keep the exchange rate depreciated at the ZLB, thus magnifying the losses. Similar comparative static results hold for reductions in foreign interest rates. If the economy is a net borrower, a decline in interest rates is beneficial away from the ZLB. At the ZLB, however, a decline in the interest rate reduces the return obtained by the Central Bank on the foreign assets, and increases the required intervention, again, magnifying the losses.

The ZLB constraint also makes exchange rate policies more vulnerable to shifts in the beliefs of private agents. When the ZLB constraint does not bind, the Central Bank can always exploit the incorrect beliefs of private agents regarding the path of the exchange rate. For example, if private agents were to expect a higher appreciation rate of the domestic currency relative to the actual policy, then the Central Bank could lower the domestic interest rate, accumulate foreign assets, and take advantage of the mistaken beliefs. At the ZLB, instead, the Central Bank lacks the ability to lower the nominal interest rate. Expecting further appreciation, private agents increase their demand for assets of the SOE, and the Central Bank only option to sustain its exchange rate policy is to accumulate foreign assets. These interventions, which occurs in presence of deviations from the interest rate parity condition, unambiguously increase the costs of sustaining the exchange rate target.

We also consider alternative policies that a Central Bank could use to achieve its exchange rate objectives at the ZLB. We first consider the role for capital controls. We show that both restrictions in the amount of foreign capital that can flow in the SOE and taxes on capital inflows allow the Central Bank to achieve the exchange rate target without the need to resort to foreign exchange interventions. Interestingly, taxing inflows achieve superior allocations than restricting quantities when the economy is a net borrower. When the Central Bank restricts the quantity of capital inflows, it faces a trade-off between reducing the losses associated to foreign exchange interventions and allowing for better consumption smoothing for the SOE. In contrast, when the Central Bank imposes taxes on capital inflows, it can completely eliminate the losses without restricting the amount of capital that flows beyond the efficient level.[1] We also study the role of negative nominal interest rates. We show that if the Central Bank has the ability to implement negative nominal interest rates, it could depreciate the exchange rate without the need to induce deviations from interest rate parity and accumulate foreign assets. Our framework can thus rationalize the behavior of Central Banks in Switzerland, Denmark and Sweden, which recently implemented negative nominal interest rates while facing severe appreciation pressures on their currencies.

After analyzing the costs and the fragilities involved in defending an exchange rate target at the ZLB, we turn on asking whether a Central Bank would ever want to engage in these costly interventions.To this end, we introduce nominal wage rigidities into our basic model. At the ZLB, the Central Bank now faces a trade-off: it can weaken its currency in order to correct output inefficiencies, but this requires costly foreign exchange interventions. If the nominal wage rigidities are severe enough, the benefits of depreciating the exchange rate dominate its costs, and the Central Bank finds it optimal to intervene. Consistent with our previous comparative static analysis, we show that the Central Bank may abstain from intervening when the capital available for international arbitrage increases or the world interest rate declines, as these factors increase the costs of carrying out foreign exchange interventions.

In the final part of the paper we examine whether some key implications of our framework are confirmed in the data. We collect data on foreign reserves held by Central Banks, nominal interest rates, and deviations from covered interest rate parity (CIP), for a panel of 9 advanced economies over the 2000-2015 period. We document a positive relation, both in the time series and in the cross-section, between the size of the foreign reserves held by Central Banks and the deviations from CIP. Moreover, we replicate the finding of Du et al. (2016) that countries with nominal

[1] At the ZLB, though, the Central Bank must be able to tax not only the holdings of domestic bonds, but also holdings of domestic currency.

interest rates close to zero experienced positive cross-currency bases during this period. We interpret these findings through the lenses of our model: Central Banks can sustain deviations in the interest rate parity condition by accumulating foreign assets, and these interventions become a necessity when trying to depreciate the exchange rate at the ZLB, but can be avoided otherwise.

One remaining question is how large these losses could be in practice. According to our model, the losses equal the product of foreign reserves and deviations from the covered interest rate parity condition. We apply this formula to the experience of the SNB during the 2010-2015 period. We document that the foreign exchange interventions by the SNB conducted over this period resulted in substantial losses, reaching around 0.8-1% of GDP in January of 2015.

Our paper is related to the literature on segmented international capital markets and exchange rate determination, such as Backus and Kehoe (1989), Alvarez et al. (2009) and Gabaix and Maggiori (2015). In an important contribution, Backus and Kehoe (1989) derive general conditions under which sterilized official purchases of foreign assets have no effects on exchange rates dynamics. Gabaix and Maggiori (2015) consider a model where capital flows across countries are intermediated by global financial intermediaries that face constraints on their leverage. In their model, changes in the equity of financial intermediaries contribute to determine exchange rates because these changes alter the relative demand of domestic and foreign assets. Therefore, purchases of foreign assets by a Central Bank, which cannot be replicated nor offset by local households, can have an effect on exchange rates.[2]

Following the work of Gabaix and Maggiori (2015), Cavallino (2016) and Fanelli and Straub (2015) study the optimality of foreign exchange rate interventions for small open economies that feature terms of trade externalities (Costinot et al., 2014). Cavallino (2016) considers an open economy New Keynesian model as in Gali and Monacelli (2005), and he shows that foreign exchange interventions are desirable in response to exogenous shifts in the demand for domestic bonds. Fanelli and Straub (2015) show that the deviations from interest parity induced by these interventions generate a cost in the inter-temporal resource constraint of the economy, which is proportional to size of the deviation.[3] This is an insight that we exploit in our analysis. These papers emphasize the role of foreign exchange interventions as an additional instrument

[2]A related paper, but in a closed economy setting, is Bassetto and Phelan (2015). They also explore how the limits of arbitrage interact with government policy while analyzing speculative runs on interest rate pegs.

[3]A related literature makes a similar point. Calvo (1991) first raised the warning about the potential costs of sterilizations by Central Banks in emerging markets. Subsequent papers have discussed and estimated the "quasi-fiscal" costs of these operations, and similarly identified the costs of sterilization as a loss in the inter-temporal budget constraint of the government, proportional to the interest parity deviations and the size of the accumulated reserves (see Kletzer and Spiegel (2004), Devereux and Yetman (2014), Liu and Spiegel (2015), and references therein).

to interest rate policy that a planner could use to correct inefficiencies.

Differently from the above papers, we consider a monetary economy and study the role of foreign exchange interventions when the pursue of an exchange rate policy conflicts with the zero lower bound constraint on nominal interest rates. We show these interventions arise as a necessary tool for the Central Bank to implement an exchange rate target when moving the nominal interest rate is not feasible. Our empirical analysis lends support to this insight, as Central Banks in advanced economies have historically resorted to large accumulation of foreign assets in periods of low nominal interest rates.

Our work is also related to the literature on capital controls, especially the one dealing with the Mundellian trilemma. Farhi and Werning (2014) and Schmitt-Grohé and Uribe (2016) show that in the presence of a fixed exchange rate regime, capital controls can partly substitute the use of interest rate policy to reallocate inter-temporally aggregate demand. In our model with limited international arbitrage, foreign exchange interventions is an alternative, yet costly, tool to manage exchange rates in the presence of a zero lower bound constraint.

The main mechanism at play in our model is related to the one highlighted in closed-economy New Keynesian models, such as in Eggertsson and Woodford (2003), Christiano et al. (2011) and Werning (2011). In both environments, there is "too much" desired saving in domestic asset markets. In New Keynesian models, the excess saving is typically driven by shocks to the households' discount rate, while in our set-up the excess saving is induced by the exchange rate policy, which coupled with the zero lower bound constraint makes domestic assets attractive. In both environments, restoring equilibrium in credit markets at the zero lower bound entails a costly reduction in the savings of domestic agents. In new Keynesian closed economy models, the reduction in savings arises because of declines in current output caused by nominal rigidities, and the cost is the output loss itself. In our model, the reduction in domestic savings is achieved through the accumulation of foreign assets by the Central Bank, and the cost arises because the intervention entails a transfer of resources from domestic to foreign agents. Recent contributions in the open economy literature are (Cook and Devereux (2013), Acharya and Bengui (2015), Caballero et al. (2015), Eggertsson et al. (2016), Fornaro (2015)).

Finally, our paper is related to recent research that emphasizes the role of unconventional fiscal instruments to achieve policy objectives when standard monetary instruments are not readily available. Notable examples include Correia et al. (2013), who study an economy with a zero lower bound constraint on nominal interest rates, and Farhi et al. (2014) and Schmitt-Grohé and Uribe (2016) that consider an economy with a fixed exchange rate. In these papers, fiscal instruments can help achieve the same outcomes that would prevail in the absence of constraints to monetary policy. We study an alternative monetary instrument, foreign exchange interventions, that can achieve the same policy objectives but at a cost for the economy.

The structure of the paper is as follows. Section 2 introduces the basic monetary setup. Section 3 discusses the implementation of an exchange rate path when the zero lower bound constraint on nominal interest rate is slack, and when it binds. Section 4 presents a comparative static analysis of the costs of foreign exchange interventions, while Section 5 explains the role of expectational mistakes of private agents. Section 6 considers the role for alternative policies, and in Section 7 we study optimal exchange rate interventions in an extension of the model with wage rigidities. Section 8 presents empirical evidence consistent with the mechanisms discussed in this paper and it measures the costs of foreign exchange rate interventions by the Swiss National Bank during the 2010-2015 period. Section 9 concludes.

2 The model

We consider a two-periods ($t = 1, 2$), two currencies (domestic and foreign), one-good, deterministic small open economy, inhabited by a continuum of domestic households, a monetary and a fiscal authority. The small open economy trades domestic assets with a continuum of foreign investors and foreign currency assets in the international financial market. We now proceed to describe the economy in detail.

2.1 Exchange rates, and interest rates

We denote by s_t the exchange rate in period t, i.e. the amount of domestic currency needed to purchase one unit of foreign currency in period t. We normalize the foreign price level (i.e. the amount of foreign currency needed to buy one unit of the good) to 1 in each period, and we assume that the law of one price holds. As a result, s_t is the domestic price level, i.e. the units of domestic currency needed to purchase one unit of the consumption good.

There are three assets available. First, there is a domestic nominal bond, which is traded within the domestic economy. This bond is denominated in domestic currency and has an interest rate which we denote by i. Domestic agents are also able to access the international financial markets and save in a foreign bond, denominated in foreign currency, with an interest rate denoted by i^*. In addition, there is domestic currency circulating in the domestic economy.

While the domestic interest rate will be determined endogenously on the domestic credit market, the foreign rate is exogenously given, in accord with the small open economy assumption.

2.2 Domestic households

Domestic households value consumption of the final good as well as from holding real money balances according to the following utility function:

$$U(c_1, c_2, m) = u(c_1) + h\left(\frac{m}{s_1}\right) + \beta u(c_2) \tag{1}$$

where $u(.)$ is a standard utility function, c_i is household consumption in period i, m is the nominal stock of money held by the household at the end of period 1 and $h(.)$ is an increasing and concave function, also displaying a satiation level $\bar{x}$ (i.e. there exists an $\bar{x}$ s.t. $h(x) = h(\bar{x})$, for all $x \geq \bar{x}$).

Domestic households are endowed with y_1 and y_2 units of the good in the two periods. The domestic households' budget constraints in periods 1 and 2 are

$$y_1 + T_1 = c_1 + \frac{m + a}{s_1} + f \tag{2}$$

$$y_2 + T_2 = c_2 - \frac{m + (1+i)a}{s_2} - (1 + i^*)f \tag{3}$$

where a and f represent the domestic holdings of domestic and foreign bonds and T_i represent the (real) transfer from the fiscal authority to the households in period i.

We assume that households cannot borrow directly in international financial markets, $f \geq 0$. This assumption will guarantee later on that domestic households cannot fully arbitrage the difference between domestic and foreign real interest rates.

The domestic households' problem is thus:

$$\max_{m,a,f,c_1,c_2} U(c_1, c_2, m)$$
$$\text{subject to equations (2), (3), and}$$
$$f \geq 0;\ m \geq 0$$

2.3 Monetary authority

We impose for now that the monetary authority has a given nominal exchange rate objective, which we denote by the pair (s_1, s_2). In general, an exchange rate objective would arise from the desire of achieving a particular inflation target or from the presence of nominal rigidities. In Section 6 we will study optimal exchange rate policies in a model with wage rigidities. For the moment, however, we simply assume that the monetary authority follows this objective, and define equilibrium for the economy given (s_1, s_2). This allows us to transparently illustrate the implementation of an exchange rate and the costs that will arise at the ZLB.

In period 1, the monetary authority issues monetary liabilities M. It uses these resources to purchase foreign and domestic bonds by amounts F and A, respectively, as well as to make a transfer, τ_1, to the fiscal authority.

In the second period, the monetary authority uses the proceeds from these investments to redeem the outstanding monetary liabilities at the exchange rate s_2, and to make a final transfer to the fiscal authority, τ_2.

Just as the domestic agents, we assume that the monetary authority cannot borrow in foreign bonds. As a result, the monetary authority faces the following constraints:

$$\begin{aligned} \frac{M}{s_1} &= F + \frac{A}{s_1} + \tau_1 \\ (1+i^\star)F + (1+i)\frac{A}{s_2} &= \frac{M}{s_2} + \tau_2 \\ M \geq 0;&\ \ F \geq 0 \end{aligned}$$

We will sometimes find it useful to analyze the case where the Central Bank cannot receive transfers from the fiscal authority in the first period, and cannot issue government bonds:

Assumption 1 (Lack of Fiscal Support)**.** *The monetary authority does not receive a positive transfer from the fiscal authority in the first period, and cannot issue interest paying liabilities:* $\tau_1 \geq 0$ *and* $A \geq 0$*.*

2.4 Fiscal authority

The fiscal authority makes transfers (T_1, T_2) to households in each period. It also receives transfers from the monetary authority, (τ_1, τ_2) in each period. The fiscal authority issues domestic nominal bonds B in period 1 and redeems them in period 2. The associated budget constraints are:

$$\frac{B}{s_1} + \tau_1 = T_1 \tag{4}$$

$$\tau_2 = T_2 + (1+i)\frac{B}{s_2} \tag{5}$$

Note that we assume that the fiscal authority does not borrow, or invest, in foreign markets.

Because public debt does not affect equilibrium outcomes due to Ricardian equivalence, we will treat the amount of bonds issued by the fiscal authority, B, as a fixed parameter.

2.5 Foreign investors and the international financial markets

A key assumption is that domestic and foreign markets are not fully integrated. In particular, there is a limit to the resources that foreign investors can channel to the domestic economy.[4] We assume that the only foreign capital that can be invested in the domestic economy is in the hands of a continuum of foreign investors, and is limited by a total amount $\bar{w}$, denominated in foreign currency.[5]

We assume that the foreign investors only value consumption in the second period. The investors cannot borrow in any of the financial markets, but can purchase both domestic and foreign assets.[6] In period 1, they decide how to allocate their wealth between foreign assets $f^\star$, domestic assets $a^\star$, and domestic currency $m^\star$; while in the second period they use the proceeds from their investments to finance their second period consumption, c^*. The foreign investor's problem is

$$\max_{f^\star, a^\star, m^\star} c^* \quad \text{subject to} \tag{6}$$

$$\bar{w} = f^\star + \frac{a^\star + m^\star}{s_1} \tag{7}$$

$$c^\star = (1+i^\star)f^\star + (1+i)\frac{a^\star}{s_2} + \frac{m^\star}{s_2} \tag{8}$$

$$f^\star \geq 0,\ a^\star \geq 0 \text{ and } m^\star \geq 0. \tag{9}$$

[4] There is a recent literature on segmented international asset markets, see for example Alvarez et al. (2009) and Gabaix and Maggiori (2015).

[5] This way of modeling foreign investors is different from Fanelli and Straub (2015). In that paper, foreign demand for domestic assets ends up been a linear function of the arbitrage return, that crosses the origin. In our model, instead, the foreign demand will be a step function of the arbitrage return. That is, there is always a strictly positive amount of foreign wealth ready to arbitrage away any profits from investing in the SOE.

[6] An alternative interpretation is that $\bar{w}$ already represents the total wealth available for investing in period 0, inclusive of any amount that could be borrowed.

Notice that unlike domestic investors, foreign investors do not enjoy a utility flow from holding domestic currency, so as expected, they will choose not to hold domestic currency when the domestic interest rate i is strictly positive.

2.6 Market clearing and the monetary equilibrium

Recall that our objective is to study whether a particular exchange rate policy can be attained as an equilibrium by the monetary authority, and to compute the costs of pursuing such a policy. Towards this goal, we will define equilibrium *for a given exchange rate policy* (s_1, s_2):

Definition 1. A monetary equilibrium, given an exchange rate policy (s_1, s_2), is a consumption profile for households, (c_1, c_2), and asset positions, (a, f, m); a consumption for investors, $c^\star$, and their asset positions $(a^\star, f^\star, m^\star)$; money supply, M; transfers from the fiscal to the monetary authority, (τ_1, τ_2); investments by the monetary authority, (A, F); transfers from the fiscal authority to the households, (T_1, T_2); and a domestic interest rate i, such that:

(i) the domestic households make consumption and portfolio choices to maximize utility, subject to their budget and borrowing constraints;

(ii) foreign investors make consumption and portfolio choices to maximize their utility, subject to their budget and borrowing constraints;

(iii) the purchases of assets by the monetary authority, its decision about the money supply and its transfers to the fiscal authority satisfy its budget constraints, as well as $F \geq 0$;

(iv) the fiscal authority satisfies its budget constraints;

(v) and the domestic asset market clears for both money and bond

$$\begin{aligned} m + m^\star &= M \\ a + a^\star + A &= B \end{aligned}$$

It is helpful to write down, using the market clearing conditions, the foreign asset position of the small open economy in any equilibrium. Using the household budget constraint in the first period, as well as the monetary authority and fiscal authority budget constraints, we obtain the following equality, linking the trade deficit to the net foreign asset position:

$$\underbrace{c_1 - y_1}_{\text{trade deficit}} = \underbrace{\frac{m^\star + a^\star}{s_1}}_{\text{foreign liabilities}} - \underbrace{[f + F]}_{\text{foreign assets}} \tag{10}$$

Similarly, using the budget constraint in the second period, we obtain the following equality:

$$c_2 - y_2 = (1 + i^\star)(f + F) - \frac{m^\star + (1 + i)a^\star}{s_2} \tag{11}$$

3 Implementing an exchange rate policy

We now study how the small open economy achieves an equilibrium given a policy for the exchange rate (s_1, s_2). We start in Section 3.1 by analyzing how foreign reserves affect the equilibrium in a real version of the model. We will see that the monetary authority can, by accumulating foreign reserves, generate a wedge between domestic and foreign real interest rates, and that such interventions will be costly from the point of view of the small open economy. We next turn in Section 3.2 to study the monetary equilibria given the exchange rate policy (s_1, s_2). The main result will be that a monetary authority that wishes to sustain a given exchange rate policy will *have to* engage in these costly interventions when the domestic nominal interest rate hits the zero lower bound constraint.

3.1 The accumulation of foreign reserves

In order to explain in the most transparent way how the accumulation of foreign reserves affects the equilibrium, we consider a version of the model without money. In this subsection, we assume that the Central Bank and the fiscal authority are just one single government agency.

We denote by r the rate of return on a real domestic bond. Because we have assumed that the foreign price level in constant, the return on the real foreign bond equals $i^\star$. In this environment, the only action of the central bank consists of choosing the amount of foreign reserves F in the first , which are financed with lump sum taxes. Because foreign reserves (plus interests) are rebated back to the representative household in the second period, an increase in F is equivalent to a shift of the domestic endowment from the first to the second period. It is convenient to define the households' endowment after the monetary authority sets the level of foreign reserves,

$$\tilde{y}_1 = y_1 - F,$$
$$\tilde{y}_2 = y_2 + (1 + i^\star)F.$$

The domestic households maximize utility $u(c_1) + \beta u(c_2)$ subject to the following budget constraints:

$$c_1 = \tilde{y}_1 - f - a$$
$$c_2 = \tilde{y}_2 + (1 + i^\star)f + (1 + r)a$$

where f and a represent their purchases of foreign and domestic assets, respectively. As in the monetary economy, we impose that they cannot borrow abroad, so $f \geq 0$.

The foreign investors are willing to invest up to the maximum of their wealth, $\overline{w}$, to maximize their returns. That is, their demand of domestic assets a^* satisfies,

$$\max_{0 \leq a^\star \leq \bar{w}} a^\star(r - i^\star) = \bar{w}(r - i^\star) \tag{12}$$

where the last equality follows from the maximization.

We will assume that $B = 0$, and market clearing in the domestic financial market implies that $a^* + a = 0$.

We can then define an equilibrium for a given policy of the monetary authority as follows

Definition 2. A *non-monetary equilibrium given* F for $F \geq 0$ is a consumption pair (c_1, c_2) and a domestic real interest rate, r, such that there exists a demand for domestic assets by foreign investors, $a^\star$, and bond holdings by domestic households, (a, f), with the properties that (i) (c_1, c_2) and (a, f) maximize the households' utility subject to the budget and borrowing constraints, (ii) $a^\star$ maximize the foreign investor's utility, and (iii) the domestic asset market clears.

To characterize an equilibrium, note that the first order conditions of the household imply

$$u'(c_1) = (1 + r)\beta u'(c_2) \tag{13}$$
$$r \geq i^\star \tag{14}$$

with $f = 0$ if the last inequality holds strictly.

The first condition is the standard Euler equation, while the second condition imposes that the real interest rate at home cannot be below the one abroad. If that was the case, the demand for domestic asset by households will be unbounded. Importantly, the converse is not true because we have assumed that households cannot borrow in foreign currency, $f \geq 0$, and because of the foreign investors' limited wealth.

We can eliminate a in the household's budget constraints, and obtain an inter-temporal resource constraint for the small open economy:

$$\tilde{y}_1 - c_1 + \frac{\tilde{y}_2 - c_2}{1+r} - f\left[\frac{r - i^\star}{1+r}\right] = 0.$$

From the household optimality condition stated above, we know that $f = 0$ if $r > i^\star$, so it follows then that the inter-temporal budget constraint simplifies to

$$\tilde{y}_1 - c_1 + \frac{\tilde{y}_2 - c_2}{1+r} = 0. \tag{15}$$

There is an additional equilibrium condition, constraining the trade deficit that the small open economy can run in the first period. Indeed, because $-a = a^\star \leq \bar{w}$, one must have that

$$c_1 = \tilde{y}_1 - f - a \leq \tilde{y}_1 + \bar{w}, \tag{16}$$

where the last inequality follows from the fact that $f \geq 0$. This expression tells us that the first period consumption of the households and the foreign reserves of the monetary authority cannot exceed the endowment of the small open economy and the wealth of foreigners $\bar{w}$. The non-monetary equilibrium is then fully characterized by conditions (13)-(16).

Before turning to the characterization of the equilibrium, it is useful to define the "first best" consumption allocation,

$$\begin{gathered}(c_1^{fb}, c_2^{fb}) \equiv \arg\max_{(c_1,c_2)} \{u(c_1) + \beta u(c_2)\} \\ \text{subject to:} \\ c_1 + \frac{c_2}{1+i^\star} = y_1 + \frac{y_2}{1+i^\star}.\end{gathered}$$

That is, (c_1^{fb}, c_2^{fb}) represents the equilibrium consumption allocation when the constraint on the first period trade balance does not bind.

We then have the following proposition.

Proposition 1. *Non-monetary equilibria given F are characterized as follows:*

(i) If $F \in [0, y_1 + \bar{w} - c_1^{fb}]$, there is a unique non-monetary equilibrium, and it features $r = i^\star$, $c_1 = c_1^{fb}$, and $c_2 = c_2^{fb}$.

(ii) If $F \in (y_1 + \bar{w} - c_1^{fb}, y_1 + \bar{w})$, there is a unique non-monetary equilibrium, and it features $c_1 = y_1 - F + \bar{w} < c_1^{fb}$, and c_2 solves

$$c_2 = y_2 - (1 + r)\bar{w} + (1 + i^\star)F, \tag{17}$$

with $r = \frac{u'(c_1)}{\beta u'(c_2)} - 1 > i^\star$.

(iii) If $F > y_1 + \bar{w}$, then there is no non-monetary equilibrium.

Proposition 1 tells us that there are only two possible equilibrium outcomes in the real economy, depending on the accumulation of foreign reserves by the Central Bank. We illustrate these two cases in Figure 1.

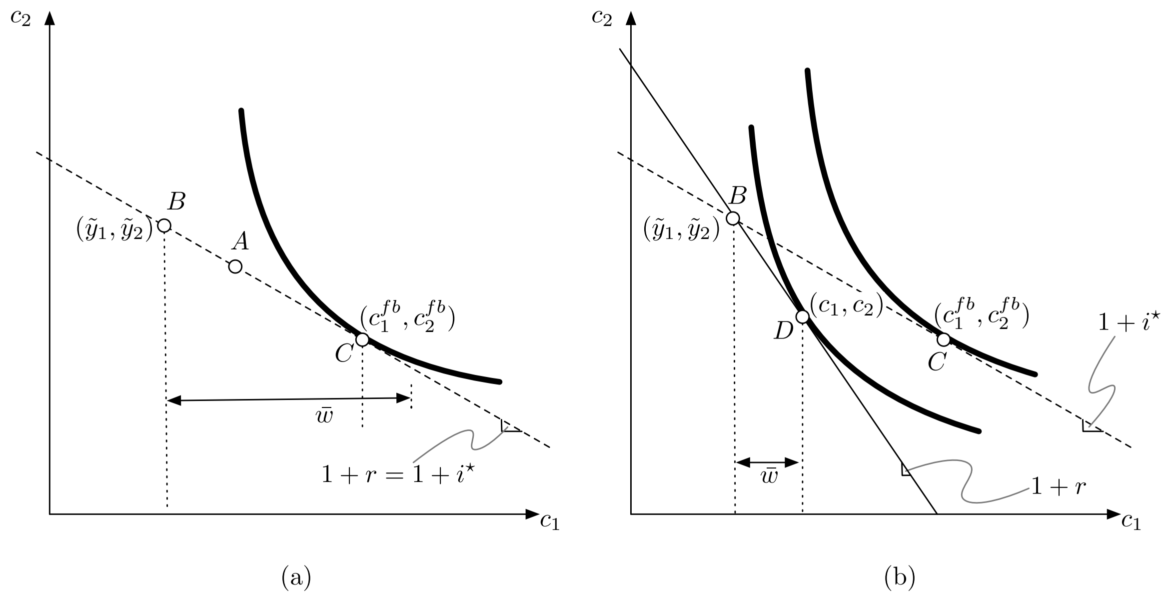
Panel (a) in the figure illustrates the first case. Point A represents the original endowment of the representative household, while point B is the households endowment after taking into account the foreign reserves accumulated by the Central Bank, F. Point C in the figure represents the first best consumption allocation, the one that would arise if the household could freely borrow and lend at the world interest rate $i^\star$. Importantly, point C is feasible for the small open economy only if there is sufficient foreign wealth to cover the first period trade balance, that is, if $y_1 - F - c_1^{fb} < \bar{w}$. This is precisely what happens in case (i) of proposition 1.

Panel (b) in the figure illustrates the second case. The accumulation of foreign reserves by the Central Bank is now so large that there is not enough foreign wealth to finance the trade deficit that would arise with the first best consumption allocation. Therefore, the constraint (16) binds, and consumption allocation is now in point D. Competition for these limited external resources results in a higher domestic real interest rate, which induces the household to consume less in the first period than what they would under the first best. In period 2, the household's consumption equals the endowment minus payments to foreigners, net of the proceeds from the accumulation of foreign reserves by the Central Bank.

We can now characterize the effects that foreign reserves have on the non-monetary equilibrium.

Corollary 1. *In the non-monetary equilibrium given F, for $F \in (y_1 + \bar{w} - c_1^{fb}, y_1 + \bar{w})$, the domestic real interest rate r is strictly increasing in F while the welfare of the domestic households*

Figure 1: **Non Monetary Equilibria given F**



is strictly decreasing in F. Foreign reserves have no impact on the domestic interest rate ($r = i^{\star}$), nor on domestic welfare, when $F \leq y_1 + \bar{w} - c_1^{fb}$.

The increase in F reduces $\tilde{y}_1$ and increases $\tilde{y}_2$. When F is small (that is $F < y_1 + \bar{w} - c_1^{fb}$) these interventions have no effects on the equilibrium because the private sector is able to undue the external position taken by the Central Bank: enough foreign wealth flows in from the rest of the world to equilibrate the domestic and foreign real rates. When F is large enough (that is, $F > y_1 + \bar{w} - c_1^{fb}$), however, the private sector cannot undue these interventions because the available foreign wealth is not large enough. In this case, the Central Bank interventions effectively make the small open economy "credit constrained", and induces an increase in the domestic real interest rate.

To understand the adverse consequences of this policy, let us rewrite the inter-temporal resource constraint for the small open economy, equation (15), as follows

$$BC \equiv (1 + r)(y_1 - c_1) + y_2 - c_2 - F(r - i^{\star}) = 0. \tag{18}$$

The term $F(r - i^{\star})$ captures the losses associated to foreign reserve accumulation by the Central Bank.[7] These losses appear because the Central Bank strategy consists in saving abroad, at a

[7] These losses correspond to the "quasi-fiscal" losses of Central Bank interventions highlighted in the sterilization literature.

low return, while the economy is in effect borrowing at a higher one.[8]

The welfare of the domestic household is given by the maximization of their utility subject to just (18), so we can read the effects on domestic welfare by understanding the effects of F on the budget constraint. Taking first order conditions (assuming that the equilibrium r is differentiable), we obtain that the marginal effect of F, for $F \in (y_1 + \bar{w} - c_1^{fb}, y_1 + \bar{w})$, is

$$\frac{dBC}{dF} = -(r - i^{\star}) - \underbrace{(c_1 + F - y_1)}_{\bar{w}} \frac{dr}{dF} < 0$$

From the above, we can see that there are two effects generated by an increase in F. First, one additional unit of reserves directly increases the budget constraint losses by the interest rate differential, $(r - i^{\star}) > 0$. But in addition, an increase in F also increases the equilibrium domestic real rate in this region, $dr/dF > 0$; and given that domestic households are net borrowers with respect to endowment point $\tilde{y}_1, \tilde{y}_2$, this induces a negative effect on the budget constraint.[9]

Figure 2 illustrates these welfare losses graphically. Without intervention, the equilibrium is denoted by point A, which in this case corresponds to the first best allocation. With a sufficiently large accumulation of foreign reserves, the Central Bank moves the economy from the income profile (y_1, y_2) to $(\tilde{y}_1, \tilde{y}_2)$. In this example, the first best allocation cannot be attained because foreign wealth is not large enough. The intervention leads to an increase in the equilibrium domestic real rate, which now exceeds $i^{\star}$, and a new consumption allocation that is now at point B.

We can see from Figure 2 the two effects associated with this intervention of the Central

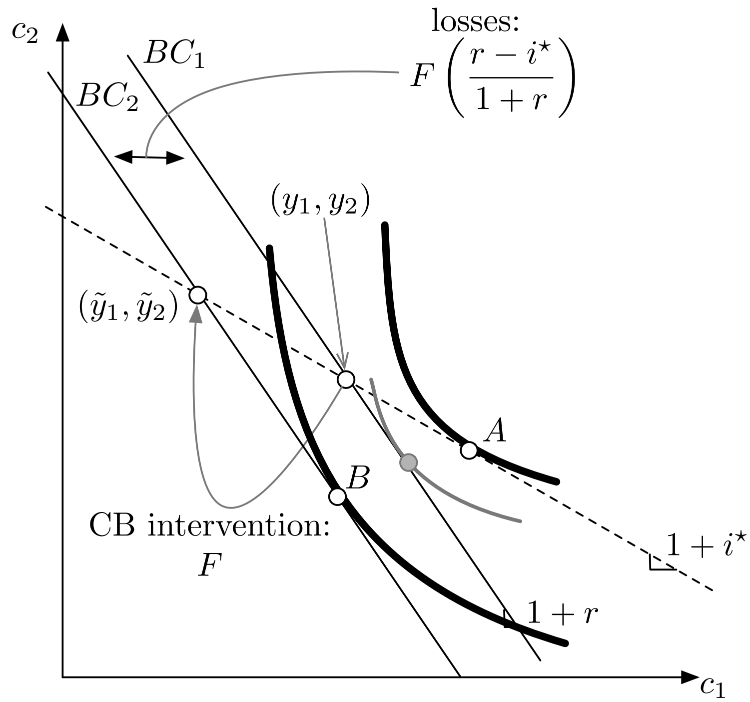
[8]This represents a loss to the entire small open economy. In Bassetto and Phelan (2015), an arbitrage gain also appears in the private agents budget constraint. But in their closed economy environment, the private agents' gains equal the government's losses; and as a result, the gains/losses do not affect the resource constraint of the economy.

[9]As done in Fanelli and Straub (2015), another way of representing the losses faced by domestic households is to rewrite the inter-temporal budget constraint solving out for foreign reserve holdings, using that $a^{\star}(r - i^{\star}) = \bar{w}(r - i^{\star})$ together with the market clearing condition, which leads to:

$$y_1 - c_1 + \frac{y_2 - c_2}{1 + i^{\star}} - \bar{w}\left[\frac{1+r}{1+i^{\star}} - 1\right] = 0 \tag{19}$$

The first two terms represent the standard inter-temporal resource constraint for an economy that could borrow and save freely at rate $i^{\star}$. But there is an additional term, which captures the reason why the equilibrium consumption outcome lies strictly within the feasibility frontier. As stressed by Fanelli and Straub (2015), this term represents a loss: as the foreigners invest when the domestic interest is above the foreign one, they gain a profit which is a loss to the country. As can be seen, the losses are proportional to the amount of wealth invested by the foreign investors, $a^{\star}$, and the differential interest rate, $r - i^{\star}$. Note that differently from Fanelli and Straub (2015), in our environment, these losses may arise even absent a Central Bank intervention, if the foreign wealth is not large enough to take the economy to the first best allocation. When studying the zero lower bound environment, we will find it more useful to work with a version of equation (18), rather than with (19).

Figure 2: **The welfare costs of Central Bank interventions**



Bank. The movement from point A to the gray dot in the figure isolates the effect that operates through an increase in the domestic interest rate (which negatively affects the country, given that it is originally a borrower). The movement of the budget set from BC_1 to BC_2 captures the resource costs associated with the Central Bank interventions.

In this non-monetary world, Central Bank interventions would not be desirable (at best they have no effect). As a result, it would be optimal in this environment for the Central Bank to always set $F = 0$. We show below how, in a monetary environment, the Central Bank may be forced (because of its exchange rate objective and the zero lower bound) to engage on this type of costly interventions.

3.2 The implementation of an exchange rate policy

So far, we have seen that the Central Bank can generate a wedge between the domestic and the foreign interest rate by accumulating foreign reserves. In a non-monetary world, these interventions are never desirable because they entail welfare losses for the domestic household. The question we ask now is when, and under what conditions, the Central Bank will need to engage in these costly interventions in order to sustain a given exchange rate objective (s_1, s_2).

To explore this issue, we return to the monetary economy. From the household's optimization problem we know that in any monetary equilibrium given (s_1, s_2), the following conditions must hold

$$u'(c_1) = \beta(1+i)\frac{s_1}{s_2}u'(c_2) \tag{20}$$

$$(1+i)\frac{s_1}{s_2} \geq (1+i^\star) \tag{21}$$

$$h'\left(\frac{m}{s_1}\right) = \frac{i}{1+i}\frac{u'(c_1)}{s_1}, \tag{22}$$

and $f = 0$ if $(1+i)\frac{s_1}{s_2} > (1+i^\star)$.

Using the budget constraints of the households, together with market clearing condition in the money market, we get the following equation:

$$y_1 - c_1 + \frac{y_2 - c_2}{\frac{s_1}{s_2}(1+i)} - (f+F)\left[1 - \frac{s_2(1+i^\star)}{s_1(1+i)}\right] + \frac{i}{\frac{s_1}{s_2}(1+i)}\frac{m^\star}{s_2} = 0.$$

Note however that $f = 0$ if $1 - \frac{s_2(1+i^\star)}{s_1(1+i)} > 0$. Therefore, the above expression simplifies to

$$y_1 - c_1 + \frac{y_2 - c_2}{\frac{s_1}{s_2}(1+i)} - F\left[1 - \frac{s_2(1+i^\star)}{s_1(1+i)}\right] + \frac{i}{\frac{s_1}{s_2}(1+i)}\frac{m^\star}{s_2} = 0.$$

The first three terms in the above expression correspond to the inter-temporal resource constraint for the non-monetary economy, equation (18), as the domestic real interest rate in this monetary economy equals $(1+i)\frac{s_1}{s_2}$. The last term, which is peculiar to the monetary economy, captures the potential seigniorage collected from foreigners. Because foreigners do not receive liquidity services from holding money balances, they set $m^\star = 0$, unless the domestic nominal interest rate is 0, implying that $im^\star = 0$. As a result, the inter-temporal resource constraint further simplifies to

$$y_1 - c_1 + \frac{y_2 - c_2}{\frac{s_1}{s_2}(1+i)} - F\left[1 - \frac{s_2(1+i^\star)}{s_1(1+i)}\right] = 0 \tag{23}$$

The final equilibrium condition revolves around the Central Bank asset position. Recall from equation (10) that

$$c_1 - y_1 + F = \frac{m^\star + a^\star}{s_1} - f \leq \bar{w},$$

where the last inequality follows from $f \geq 0$ and $m^\star + a^\star \leq s_1\bar{w}$. In addition, if $\frac{1+i}{1+i^\star}\frac{s_1}{s_2} - 1 > 0$, then we know that $m^\star + a^\star = s_1\bar{w}$ and $f = 0$ (i.e., foreigners invest everything in the domestic assets, and households do no invest in the foreign asset). Therefore, in any monetary equilibrium we must have

$$c_1 \leq y_1 - F + \bar{w}; \text{ with equality if } \frac{1+i}{1+i^\star}\frac{s_1}{s_2} - 1 > 0 \tag{24}$$

In other words, the foreign wealth must finance the trade deficit plus the reserve accumulation of the Central Bank.

Note that equations (20), (21), (23), and (24) are the same equations that characterize a non-monetary equilibrium, equations (13), (14), (15), and (16), with $r = (1+i)\frac{s_1}{s_2} - 1$, $\tilde{y}_1 = y_1 - F$, and $\tilde{y}_2 = y_1 + (1+i^\star)F$. Thus, any monetary equilibrium must deliver an allocation consistent with a non-monetary equilibrium outcome. In addition, however, a monetary equilibrium imposes the restriction that the nominal interest rate must be non-negative (i.e., the zero lower bound), a key restriction that will play an important role in what follows.

As a result, there is potentially a continuum of monetary equilibria given the exchange rate objective (s_1, s_2). Each equilibrium differs for the level of foreign reserve F accumulated by the Central Bank and potentially for the level of the nominal interest rate i and for the consumption allocation.

For future reference, we denote by $\underline{r}$ the domestic real interest rate in the non-monetary equilibrium associated with $F = 0$. From Proposition 1 we know that $\underline{r} \geq i^\star$.

We can now study how the Central Bank can implement a given policy for the exchange rate (s_1, s_2) in the monetary economy. We will distinguish between two cases, depending on whether the zero lower bound constraint under the exchange rate policy binds or not.

3.2.1 Implementation when the zero lower bound constraint does not bind

We first consider the case in which $(1+\underline{r})\frac{s_2}{s_1} \geq 1$. We have the following result.

Proposition 2. *Suppose that $(1+\underline{r})\frac{s_2}{s_1} \geq 1$. Then, for all $F \in [0, y_1 + \bar{w})$, the non-monetary equilibrium given F constitutes a monetary equilibrium outcome. Household's welfare is maximized in the equilibrium with $F = 0$.*

The intuition behind this proposition is as follows. When the Central Bank does not accumulate foreign reserves, the real interest rate in the non-monetary economy will be equal to $\underline{r}$. This real rate, along with the exchange rate policy (s_1, s_2), does not violate the zero lower bound constraint because, by assumption, the domestic nominal interest rate would be such $i = (1+\underline{r})\frac{s_2}{s_1} - 1 \geq 0$. Therefore, the allocation $(c_1, c_2, \underline{r})$ for $F = 0$ constitutes a monetary equilibrium outcome. From Corollary 1, we know that the real interest rate is weakly increasing in F. Thus, all non-monetary equilibria given F, for $F > 0$, will not violate the zero lower bound constraint on nominal interest rates, and will also constitute a monetary equilibrium outcome.

Combining Proposition 1 and 2, we can see that the Central Bank can implement an exchange rate objective (s_1, s_2) in two distinct ways. First, the Central Bank could implement (s_1, s_2) by adjusting the nominal interest rate in order to guarantee that foreign investors are indifferent between holding domestic or foreign currency assets, i.e. that the interest rate parity condition in (21) holds with equality. This is case (i) in Proposition 1. In this first scenario, the accumulation of foreign reserves does not impact the equilibrium outcomes (locally), this mirroring the classic irrelevance result of Backus and Kehoe (1989).

There is, however, a second way to implement the exchange rate objective (s_1, s_2). This is described in case (ii) of Proposition 1: the Central Bank could achieve its desired exchange rate policy (s_1, s_2) by accumulating foreign reserves while setting a higher domestic interest rate than the one consistent with interest rate parity.

These results specialize the classic trilemma of international finance to an environment with limits to international arbitrage. The Central Bank can implement an exchange rate policy by adjusting the nominal interest rate and eliminate arbitrage opportunities in capital markets. In our environment, however, this is not the only option, and the Central Bank could follow an exchange rate policy (s_1, s_2) while maintaining some degrees of monetary independence. To do so, it will need to engage in the costly interventions described in Section 3.1.

In the model described here, though, this trade-off is not operating: given an exchange rate policy (s_1, s_2), the optimal Central Bank policy would be not to accumulate foreign reserves (a result that follows directly from Proposition 1).[10] However, there is a sense in which this is a stronger result. If a Central Bank has no fiscal support in the first period, then it may not be feasible for the Central Bank to engineer a deviation from interest parity:

Proposition 3. *Suppose that $(1 + \underline{r})\frac{s_2}{s_1} \geq 1$ and that assumption 1 holds. In addition, suppose that $c_1^{fb} - y_1 + \bar{x} \leq \bar{w}$. Then all monetary equilibria attain the first best consumption allocation, the same domestic welfare, and the interest rate parity condition (21) holds with equality.*

Proposition 3 tells us that a Central Bank that cannot issue interest rate paying liabilities and does not receive transfers from the fiscal authority is constrained in its ability to raise the

[10] There is an issue, related to the value of money balances, a consideration that, of course, does not appear in the non-monetary equilibria analysis. However, the equilibrium with $F = 0$ is the monetary equilibrium with the lowest possible nominal interest rate, given the exchange rate policy. And thus, it ends up maximizing total households' utility, inclusive of money balances. Indeed, there is no additional value of raising the domestic interest rate beyond what's necessary to support the exchange rate policy under no reserve accumulation.

domestic real rate above the foreign one. In order to understand why, suppose that the Central Bank tries to do so. This leads to an immediate inflow of foreign capital of size $\bar{w}$, which puts downward pressure on the domestic interest rate. To keep the interest rate from falling, the Central Bank must purchase a large amount of the inflow and accumulate foreign reserves. But the purchasing power of the Central Bank is limited by its balance sheet because, by assumption 1, the Central Bank's liabilities are bounded by the satiation point of money $\bar{x}$. If the external wealth is sufficiently high, the Central Bank will not be able to sustain a deviation from interest rate parity, and the domestic interest rate will need to adjust. Therefore, it could be challenging for the Central Bank to gain monetary independence while committing to an exchange rate policy when nominal interest rates are positive.

3.2.2 Implementation when the zero lower bound constraint binds

The second case we analyze is when $(1+\underline{r})\frac{s_2}{s_1} < 1$. In this case, the non-monetary equilibrium with $F = 0$ cannot arise as a monetary equilibrium outcome because it would lead to a domestic nominal interest rate that violates the zero lower bound constraint. As a result, the monetary equilibrium will necessarily feature a deviation from interest rate parity, and the domestic real interest rate will need to lie strictly above the foreign one.[11]

So, for there to be a monetary equilibrium, the Central Bank will need to intervene and accumulate reserves of a magnitude sufficient to increase the real interest rate above the level consistent with interest parity. Let $\bar{r}$ to be the highest possible real interest rate in the non-monetary economy (that is, the interest rate associated with the maximum possible intervention). We then have the following result:

Proposition 4. *Suppose that $1+\underline{r} < \frac{s_1}{s_2} < 1+\bar{r}$, then there exists an $\underline{F} > 0$ such that for all $F \in [\underline{F}, y_1 + \bar{w})$, the non-monetary equilibrium given F, (c_1, c_2, r), constitutes a monetary equilibrium outcome. In all these monetary equilibria, the interest rate parity condition (21) holds as a strict inequality. Household's welfare is maximized in the equilibrium with $F = \underline{F}$.*

Proposition 4 tells us that the Central Bank is able to sustain the exchange rate policy.

[11]This follows immediately from the following set of inequalities:

$$(1+i)\frac{s_1}{s_2} - 1 \geq \frac{s_1}{s_2} - 1 > \underline{r} \geq i^\star$$

where the first term is the domestic real rate, the first inequality follows from the zero lower bound constraint, the second defines the case of interest, and the last one is the restriction that appears in any non-monetary equilibrium.

However, because of the zero lower bound, it *has to* engage in the costly interventions described in Section 3.1.

It follows however that, given an exchange rate policy (s_1, s_2), the optimal Central Bank policy is to accumulate the minimum amount of foreign reserves necessary to deliver a monetary equilibrium. As a result, the best monetary equilibrium in this case will feature $i = 0$ and a violation of the interest parity condition. Differently from the situation in which the zero lower bound constraint is slack, the Central Bank can always sustain these exchange rate policies, even without the support of the fiscal authority:

Proposition 5. *Suppose that $(1 + \underline{r})\frac{s_2}{s_1} < 1$ and that assumption 1 holds. In addition, suppose that $c_1^{fb} - y_1 + \bar{x} \leq \bar{w}$. Then the unique monetary equilibrium outcome is the one where $F = \underline{F}$ and $i = 0$.*

Proposition 5 tells us that a Central Bank without fiscal support is able to raise the domestic real rate above the foreign one, as long as the nominal interest rate remains at zero. In this case, by sustaining the exchange rate path, the Central Bank is forced to issue currency to purchase the foreign assets necessary to maintain the domestic rate above the foreign one. The main difference from the case analyzed previously is that now, because of the zero nominal rate, bonds and money are perfect substitutes. Thus, the Central Bank can expand its balance sheet without limits.

4 The costs of foreign reserve accumulation

In the previous section we have seen that a Central Bank that wishes to implement an exchange rate path while its nominal interest rates are at zero needs to accumulate foreign reserves. We have also seen that these interventions are costly from the perspective of the small open economy. In this section we study in more details those costs, and discuss how they are affected by changes in the underlying economic environment.

We consider the effects of increases in foreign wealth, $\overline{w}$, and of reductions in the foreign interest rate $i^\star$ (when the country is a net borrower). Before moving to the ZLB environment, let us first argue that both of these changes will unambiguously improve welfare when the zero lower bound constraint does not bind, that is, when $(1 + \underline{r})\frac{s_2}{s_1} \geq 1$.

To see this, note that, away from the zero lower bound, the best monetary equilibrium given an exchange rate policy (s_1, s_2) sets $F = 0$. As a result, the welfare effects can be read by studying the effects of such changes in the budget constraint of domestic households,

$$y_1 - c_1 + \frac{y_2 - c_2}{1+r} \geq 0,$$

where r is the domestic equilibrium real rate. So, whether increases in $\overline{w}$ or decreases in $i^\star$ are welfare improving or not, depend on the effect of these changes on the equilibrium domestic real interest rate. The following helps in clarifying the effects:

Lemma 1. *Consider the non-monetary equilibrium given $F = 0$. Then,*

(i) if $c_1^{fb} > y_1 + \overline{w}$, a marginal increase in $\overline{w}$ strictly decreases the domestic real interest rate, while a marginal decrease in $i^\star$ has no effect.

(ii) if $c_1^{fb} < y_1 + \overline{w}$, a marginal increase in $\overline{w}$ has no effect on the domestic interest rate, while a marginal decrease in $i^\star$ strictly decreases it.

The results of this lemma follow from our characterization of the non-monetary equilibrium. When $F = 0$, if $c_1^{fb} < y_1 + \overline{w}$, then the economy achieves the first best consumption outcome, and the domestic real interest rate will equal $i^\star$. As a result, an increase in $\overline{w}$ would have no effect on the real interest rate in this region, but a reduction in $i^\star$ will reduce the domestic rate one to one, explaining part (ii) of the lemma. However, if $c_1^{fb} > y_1 + \overline{w}$, then the economy is constrained, and the domestic interest rate is the unique value r that solves the following equation

$$(1+r) = \frac{u'(y_1 + \overline{w})}{\beta u'(y_2 - (1+r)\overline{w})}$$

In this case, changes in the foreign interest rate have no effects on the equilibrium r. An increase in $\overline{w}$, however, strictly reduces r, a natural outcome of the increase in competition from foreign investors.[12]

It follows then an increase in $\overline{w}$ either has no effect on the domestic real rate, or reduces it when $c_1 = y_1 + \overline{w}$, that is, when the country is net borrower. From the households budget constraint, an increase in $\overline{w}$ then weakly increases welfare.

A reduction in $i^\star$ has no effect on the domestic real rate when the economy is constrained, and reduces the real rate when the economy is at its first best allocation, $c_1 = c_1^{fb}$. If $c_1^{fb} > y_1$, that

[12]To see this, we can differentiate the equation with respect to $\overline{w}$ and obtain that

$$\frac{dr}{d\overline{w}} = \frac{(1+r)^2 \beta u''(c_2) + u''(c_1)}{\beta u'(c_2) - (1+r)\beta u''(c_2)\overline{w}} < 0.$$

is, the economy is a net borrower at the first best consumption allocation, then the reduction in $i^\star$ will increase welfare.

We now proceed to show how these beneficial changes become welfare-reducing when the economy follows an exchange rate policy at the zero lower bound.

4.1 Changes in foreign wealth

We start analyzing how a change in foreign wealth $\bar{w}$ affects the costs of foreign reserve accumulation when the Central Bank is committed to the exchange rate path (s_1, s_2) and $(1+\underline{r})\frac{s_2}{s_1} < 1$. In this case, the best monetary equilibrium will set the nominal interest rate to zero.

We can characterize domestic welfare as follows

$$W \equiv \max_{(c_1,c_2)} u(c_1) + \beta u(c_2) + h(\overline{x}) \tag{25}$$

$$\text{subject to:}$$

$$y_1 - c_1 + \frac{y_2 - c_2}{s_1/s_2} - \underline{F}\left[1 - \frac{s_2(1+i^\star)}{s_1}\right] = 0,$$

where $\underline{F}$ is the minimum level of foreign reserves necessary for $i = 0$ given (s_2/s_1).[13]

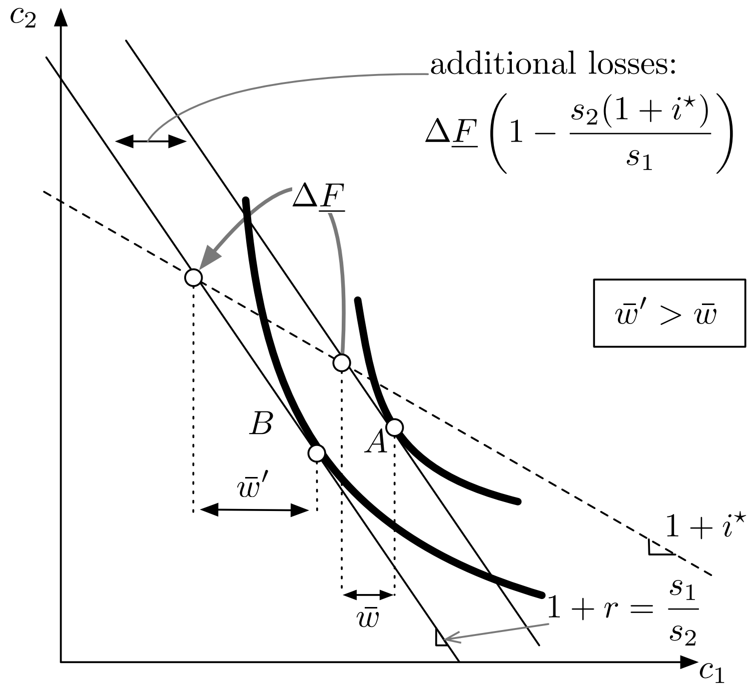
Note that at the zero lower bound, the domestic real interest rate equals the rate of appreciation of the currency, which is fixed under the Central Bank policy. Therefore, a change in the welfare of domestic household purely reflects changes in the arbitrage losses that the Central Bank sustain when accumulating foreign reserves, the term $\underline{F}\left[1 - \frac{s_2(1+i^\star)}{s_1}\right]$ in the resource constraint.

Let's now consider how an increase in the wealth of foreign investors affects this term. The term in the square bracket is independent on $\bar{w}$, so a change in $\bar{w}$ affects welfare only through its effect on $\underline{F}$: if higher $\bar{w}$ leads to higher $\underline{F}$, then welfare unambiguously declines. This is indeed what happens. When foreign wealth increases, the Central Bank is forced to accumulate more foreign reserves in order to sustain the exchange rate path (s_1, s_2). Because for every penny of foreign reserve accumulated the Central Bank makes a loss, the overall costs of the intervention increases with $\bar{w}$.

Figure 3 illustrates this graphically. Point A depicts the equilibrium consumption under the exchange rate policy for a given level of foreign wealth $\bar{w}$. Suppose now that foreign wealth

[13] The presence of $h(\overline{x})$ arises from the utility value of money balances. At $i = 0$, domestic households are satiated with respect to money balances, so that $m/s_1 \geq \overline{x}$ and $h(m/s_1) = h(\overline{x})$.

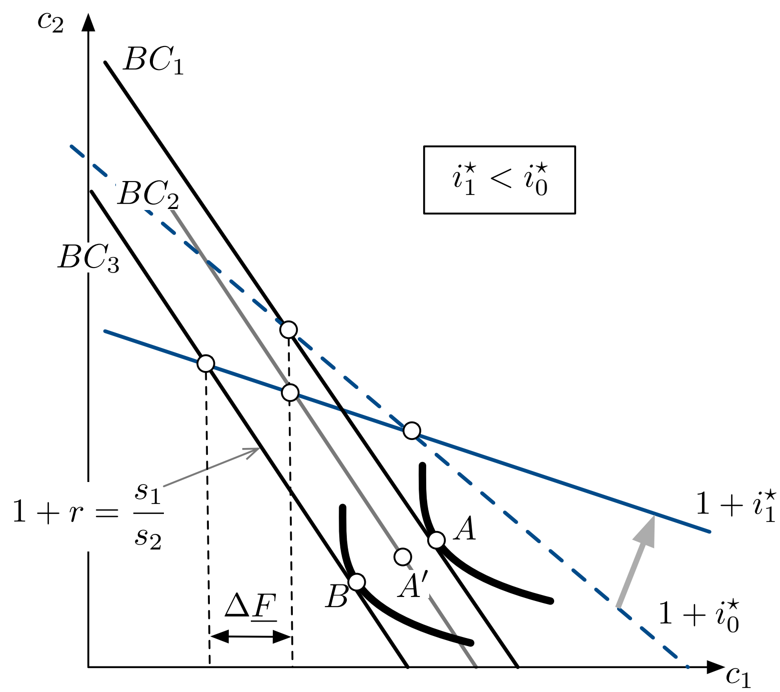
Figure 3: **Changes in $\bar{w}$ at the ZLB**



increases to $\bar{w}'$. Because the interest rate parity condition is violated under the policy, more foreign capital will fly toward the small open economy, putting downward pressure on the domestic real interest rate. In order to sustain the path for the exchange rate, the Central Bank will have to lean against these capital flows and purchase foreign assets, so $\underline{F}$ must increase. Point B in the figure represents the equilibrium that prevails when foreign wealth moves to $\bar{w}'$. The domestic real interest rate at equilibrium B is the same as at A, as the economy is at the ZLB in both points and domestic real rate is pinned down by $\frac{s_1}{s_2}$. Despite the fact the real rate has not changed, welfare at B is unambiguously lower than at A, as the higher $\bar{w}$ forces the Central Bank to intervene more ($\Delta \underline{F} > 0$). The losses generated by this bigger intervention are represented by the parallel shift in the budget lines from point A to point B.

This result shows that a higher degree of capital market integration makes more costly the pursue of an exchange rate objective when the economy is at the zero lower bound: the Central Bank has to accumulate more foreign reserves to sustain the path (s_1, s_2), and this accumulation leads to resource costs for the small open economy.

Figure 4: **Changes in $i^\star$ at the ZLB**



4.2 Changes in the foreign interest rate

A similar result occurs when $i^\star$ declines. Suppose, again, that the Central Bank is sustaining the path (s_1, s_2) at the zero lower bound, and the foreign interest rate decline. As it was for a change in foreign wealth, the impact of a decline in $i^\star$ on welfare depends on its impact on $\underline{F}\left[1 - \frac{s_2(1+i^\star)}{s_1}\right]$. However, there are now two effects to consider. First, for given a (s_1, s_2), the decline in the foreign interest rate implies a larger deviation from interest rate parity, and it leads to an increase in the arbitrage losses made by the Central Bank for a given $\underline{F}$. Second, the decline in $i^\star$ forces the Central Bank to accumulate more reserves, and $\underline{F}$ must increase. Both of these forces increase the costs of sustaining the exchange rate path for the Central Bank.

We illustrate this result in Figure 4. We are considering a situation where the SOE is already at the zero lower bound, and its consumption lies at point A. The dashed budget line represents the resource constraint using an initial foreign rate equal to $i_0^\star$. We then consider a reduction in the international rate to $i_1^\star < i_0^\star$. The first effect is isolated by the shift of the inter-temporal resource constraint from BC_1 to BC_2: the Central Bank intervention generates bigger resource costs for the small open economy because the interest parity deviations, for a given exchange rate path, are larger if the foreign interest rate is $i_1^\star$. However, A' is not an equilibrium. The domestic household now would like to save because his endowment in the second period is not as high as it used to be before the decrease in the foreign rate, which implies that the domestic asset market is not in equilibrium. As a result, the Central Bank must increase its foreign reserves, driving the economy to its equilibrium in point B, with an even higher reduction in welfare.[14]

5 Expectational mistakes

So far, we have assumed rational expectations of private agents. We now relax this assumption and ask what happens when private agents are mistaken in their expectations regarding the path of the exchange rate set by the Central Bank.

We show that when the Central Bank follows an exchange rate path that conflicts with the zero lower bound, incorrect beliefs of a future appreciation of the domestic currency necessarily induce an increase in the foreign reserve holdings of the Central Bank, and they end up increasing the welfare costs of sustaining the exchange rate path (s_1, s_2). This stands in contrast with the case when the zero lower bound constraint does not bind, as in this latter scenario a Central Bank can always exploit these expectational mistakes and strictly increase the welfare of the domestic household.

We introduce the possibility of expectational mistakes as follows. We continue to let (s_1, s_2) denote the actual exchange rate policy, and we maintain the assumption that the Central Bank will pursue it. Market participants (i.e., domestic households and foreign investors) see the value of s_1 in the first period, and believe that the exchange rate in the second period will be $\hat{s}_2$. Expectational mistakes arise when $\hat{s}_2 \neq s_2$. Keeping with our desire to maintain simplicity, we assume that the private agents do not learn or infer any information from the actions of the Central Bank. We define an equilibrium under potentially mistaken market beliefs as follows:

Definition 3. An *equilibrium given* (s_1, s_2) *and market beliefs* $\hat{s}_2$ consists of a domestic interest rate i, a consumption profile $(c_1, c_2, \hat{c}_2)$, asset positions for foreign investors $(a^\star, f^\star, m^\star)$, money M, investments by the monetary authority (A, F); transfers from the monetary authority to the fiscal, $(\tau_1, \tau_2, \hat{\tau}_2)$, and transfers from the fiscal authority to the households, $(T_1, T_2, \hat{T}_2)$ such that

(i) the allocation $(c_1, \hat{c}_2, a, f, m, a^\star, f^\star, m^\star, \tau_1, \hat{\tau}_2, A, F, T_1, \hat{T}_2)$ with nominal interest rate i constitutes a monetary equilibrium given the exchange rate $s_1, \hat{s}_2$.

[14] There is potentially another effect that we do not consider here. Suppose that the reduction in $i^\star$ allows foreigners to borrow more from the international financial markets. This is equivalent to a larger amount foreign wealth $\bar{w}$ available for investment in the SOE in the first period. The additional effects generated by this will be similar to the already discussed exogenous increase in $\bar{w}$: it will increase the foreign reserve holdings by the Central Bank, magnifying the welfare losses.

(ii) The second-period consumption and transfers, (c_2, τ_2, T_2) satisfy

$$\begin{aligned}
c_2 &= y_2 + T_2 + \frac{(1+i)a + m}{s_2} + (1+i^\star)f \\
\tau_2 &= (1+i^\star)F + (1+i)\frac{A}{s_2} - \frac{M}{s_2} \\
T_2 &= \tau_2 - (1+i)\frac{B}{s_2}
\end{aligned}$$

Note that in part (i) we use the beliefs to define the monetary equilibrium. However, in period 2, the realization of the exchange rate will be s_2, and the second period "true" allocations (c_2, τ_2, T_2) are calculated with respect to the true exchange rate (part (ii) of the definition). We will call $(c_1, \hat{c}_2)$ the *perceived* consumption allocation, and (c_1, c_2) the *true* consumption allocation. We will also say that $(1+i)\frac{s_1}{\hat{s}_2}$ is the *perceived* real interest rate, and we call $(1+i)\frac{s_1}{s_2}$ the *true* real rate interest rate. Clearly if $s_2 = \hat{s}_2$, the definition of equilibrium above is identical to our definition of a monetary equilibrium given the exchange rate policy (s_1, s_2).

As explained earlier, we will consider the case where $s_2 > \hat{s}_2$, that is, when private agents expect the currency to be more appreciated next period relative to the policy that is actually chosen by the Central Bank. We evaluate welfare by considering the household's utility under the true consumption allocation:

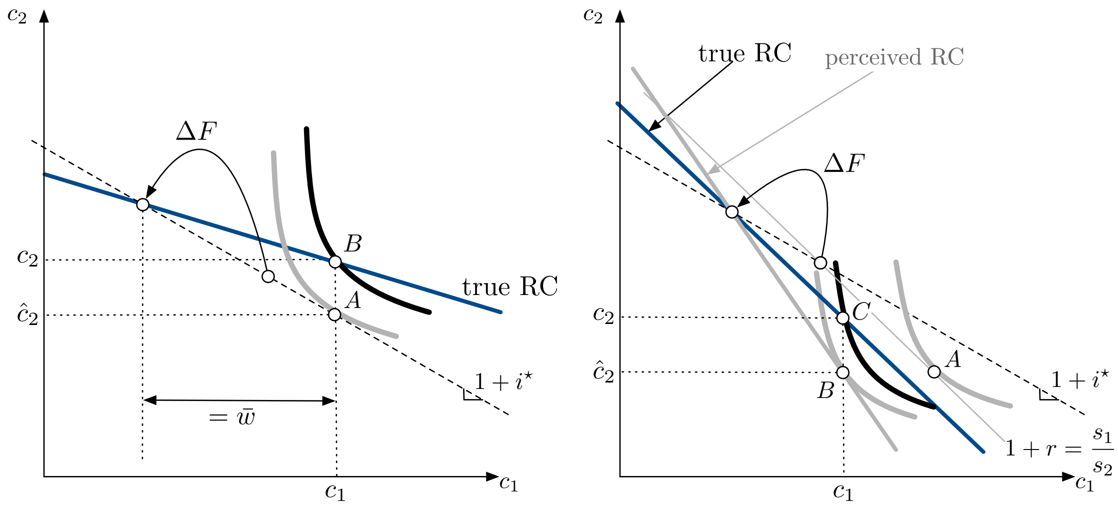
Proposition 6. *Suppose that $s_2 > \hat{s}_2$. Consider the equilibrium with market beliefs $\hat{s}_2$ that maximizes household's welfare. Then*

(i) if $(1+\underline{r})\frac{\hat{s}_2}{s_1} \geq 1$, household's welfare strictly decreases with $\hat{s}_2$;

(ii) if $(1+\underline{r})\frac{s_2}{s_1} < 1$, household's welfare strictly increases in $\hat{s}_2$.

In case (i) of the proposition, the Central Bank maximizes welfare by reducing the interest rate to that private agents perceive the interest rate parity condition to hold, and by accumulating foreign assets. Ex post, these foreign assets are worth more under the realized exchange rate in period 2 because $s_2 > \hat{s}_2$. These profits are rebated back to the households, and increase their effective second period consumption. As a result, household's welfare, as evaluated by the Central Bank, increases. An example of this is illustrated in Figure 5 panel (a). Point A represents the rational expectation case, $\hat{s}_2 = s_2$. In our example, the Central Bank is sustaining the path for the exchange rate with no interventions ($F = 0$). We then consider how the equilibrium changes when $\hat{s}_2 < s_2$. In this case, the Central Bank reduces the nominal interest rate, such that the "perceived" real rate of return remains the same. Moreover, it accumulates foreign reserves to

Figure 5: **Expectational mistakes**



(a) (b)

the point in which all foreign wealth enters the small open economy. These two changes do not affect the behavior of private agents, which believe the equilibrium will be at point A. However, ex-post, the exchange rate equals s_2, and the consumption of the domestic households will be at point B, rather than A, generating a strictly positive welfare gain.

The key reason why the Central Bank can exploit the mistaken beliefs in case (i) of Proposition 6 is based on its ability to lower the domestic nominal interest rate when the beliefs deviate from the true ones. At the ZLB, this is not possible. As a result, the expectation mistakes cannot be exploited and welfare cannot be increased. We show now an even more negative result: the mistaken beliefs at the ZLB unambiguously generate a reduction in welfare.

Figure 5 panel (b) illustrates case (ii) of the proposition. Point A represents the original equilibrium, where the ZLB binds, and $\hat{s}_2 = s_2$. We consider then how the equilibrium changes if $\hat{s}_2 < s_2$. In this case, the Central Bank cannot further reduce i, because the interest rate is at zero. As a result, the perceived domestic rate of return is necessarily above the foreign one. Thus, the Central Bank has to accumulate foreign reserves in order to maintain its desired path for the exchange rate. The perceived equilibrium consumption allocation shifts to point B. However, ex-post, the exchange rate remains at s_2, and the realized rate of return is identical to the original. The realized consumption allocation that is attained in equilibrium is given by point C, which dominated by A. Hence, welfare has been reduced.

The results of this section highlight that even if the Central Bank is committed to its exchange rate policy, if the markets do not believe it, then there will be costs associated to defending the policy when the economy operates at the zero bound. In addition, the larger the expectational mistake, the larger the required foreign exchange interventions by the Central Bank, and the larger the welfare losses. That is, at the ZLB, expectational mistakes are accompanied with costly balance sheet expansions by the Central Bank. Those expansions could, by themselves, trigger an abandonment of the exchange rate policy if the Central Bank finds it costly to maintain a large balance sheet.[15]

The results in this section open the door to the possibility of self-fulfilling "appreciation" runs at the ZLB. In particular, if the private agents erroneously anticipate an appreciation, when the Central Bank is at the ZLB, such mistaken expectations will be costly. This may force the Central Bank to abandon the defense of the current exchange rate regime, allowing the currency to appreciate, somewhat validating the mistaken expectations of the markets. The analysis in this section calls for a more detailed study of the game between the Central Bank and the private agents, something that we leave for future work.

6 Alternative Policies

This section studies how alternative policies can complement or substitute the use of foreign exchange intervention to implement an exchange rate policy at the ZLB.

6.1 Capital Controls

We analyze the role for capital controls and study how they interact with foreign exchange intervention. We consider both price and quantity instruments. We proceed by first characterizing these two types of controls, and then comparing the outcomes achieved in the two cases.

Taxes on inflows. We start by considering a tax, τ, on all capital inflows from foreign investors. We note that at the ZLB, it is crucial that those taxes apply not only to bond holdings, but also to money holdings by foreigners to have any effects on capital inflows. Because bonds and money are perfect substitutes at the ZLB, taxing only bonds would shift inflows to money without any effects on the total magnitude of inflows. With taxes on bond and money inflows, the budget

[15] This is something that we do not analyze here, but is a point we studied in Amador et al. (2016).

constraint of foreign investors become

$$f^\star + \frac{a^\star + m^\star}{s_1} = \bar{w}$$

$$c_2^\star = (1+i^\star)f^\star + (1+i)(1-\tau)\frac{a^\star}{s_2} + \frac{m^\star}{s_2}(1-\tau)$$

In this case, foreign investors find optimal to invest all their wealth in domestic assets if

$$(1+i)(1-\tau) \geq (1+i^\star)\frac{s_2}{s_1} \tag{26}$$

Consider now the central bank problem of implementing an exchange rate policy, when the central bank has access to capital controls at the zero lower bound.

$$\max_{\{c_1, c_2, \tau\}} u(c_1) + h(\bar{x}) + \beta u(c_2)$$

subject to:

$$(1+i^\star)c_1 + c_2 \leq (1+i^\star)y_1 + y_2 - \left(\frac{s_1}{s_2}(1-\tau) - (1+i^\star)\right)\bar{w} \tag{27}$$

$$1 = \frac{u'(c_1)}{\beta u'(c_2)}\frac{s_2}{s_1} \tag{28}$$

$$(1-\tau) \geq (1+i^\star)\frac{s_2}{s_1} \tag{29}$$

$$y_1 - c_1 + \bar{w} \geq 0 \tag{30}$$

We have the following proposition.

Proposition 7. *The optimal tax is such that* (29) *holds with equality.*

Proposition 7 indicates that the CB optimally uses taxes on inflows to eliminate all arbitrage profits. To see this, notice that the right hand side of (27) is increasing in τ (i.e., the government gets revenues from taxing foreign lenders). Since the inter-temporal resource constraint must hold with equality, if (29) were to hold with strict inequality, it would be feasible to increase the value for the CB by increasing τ. Intuitively, since the demand for domestic assets by foreign investors is inelastic as long as the IP condition holds with strict inequality, the CB can extract rents by taxing foreign investors.

Quantity Controls. An alternative policy is to place restrictions on the quantities of capital inflows in the economy, w.

$$W \equiv \max_{(c_1,c_2,w)} u(c_1) + \beta u(c_2) + h\left(\frac{m}{s_2}\right) \tag{31}$$

subject to:

$$y_1 - c_1 + \frac{y_2 - c_2}{1+i^*} - \left(\frac{\frac{s_1}{s_2} - (1+i^\star)}{1+i^*}\right) w \geq 0 \tag{32}$$

$$u'(c_1) = \beta \frac{s_1}{s_2} u'(c_2) \tag{33}$$

$$y_1 - c_1 + w \ \geq 0 \tag{34}$$

$$\bar{w} \geq w \geq 0 \tag{35}$$

Proposition 8. *The optimum w^* is such that $F^* = 0$.*

Proposition 8 indicates that the CB restrict capital inflows until the point that there is no need for reserve accumulation to implement the desired exchange rate policy. When the CB saves abroad, it pushes households to borrow at higher rates than the world interest rate. Since the CB can achieve the desired exchange rate policy by reducing $\bar{w}$ instead of accumulating foreign assets, it is strictly optimal to reduce $\bar{w}$ to the minimum level that guarantees the implementation of the exchange rate policy.

Price vs. Quantity Controls Capital controls do not allow the CB to achieve the efficient allocation. In fact, capital controls do not eliminate the interest rate distortion, i.e., we have $(1+i^*) < (1+i)s_1/s_2 = \beta^{-1}u'(c_1)/u'(c_2)$. It is important to notice that because capital controls do not entail other distortions in our simple setup, foreign exchange intervention becomes redundant when capital controls are chosen optimally. In this sense, capital controls and foreign exchange intervention are substitutes.[16]

Which type of controls achieve higher welfare? In a large class of models, these policies render equivalent welfare results (see e.g. Bianchi, 2011). This is not the case here, when the economy is a net borrower. When the economy is a net saver, it is easy to see that $\bar{w}^* = 0$, achieves

[16] For the case of quantity controls, we show in Section (7) that the relationship may be non-monotonic between F and $\bar{w}$ when there are benefits from depreciating the exchange rate resulting from nominal rigidities. When $\bar{w}$ is low, the CB sets $F = 0$ as it can achieve the exchange rate objective, as shown in Proposition 8. As $\bar{w}$ increases, the CB finds optimal to intervene to affect the exchange rate. Finally, when $\bar{w}$ becomes large enough, it is no longer desirable to intervene and depreciate the exchange rate.

the same degree of welfare as the one obtained with the optimal tax. If the economy is a net borrower, however, the CB sets $\bar{w}^{\star} = c_1 - y_1 > 0$. From the resource constraint (32), it follows that the CB does not avoid losses and the rents accrued to foreign investors. On the other hand, as we have shown in Proposition 7, the CB eliminates the losses with the optimal tax rate. Since real rates are the same across the two types of controls, the CB obtains strictly higher welfare with taxes.[17]

6.2 Negative Interest Rates

We consider next the implementation of negative rates using a tax on money holdings.[18] The first-order condition of households with respect to money is modified as follows:

$$h'\left(\frac{m}{s_1}\right) = (i + \tau^m)\frac{\lambda_2}{s_2}$$

It follows that by setting $i = -\tau^m = (1+i^*)\frac{s_2}{s_1} - 1$, the CB can implement the exchange rate policy with negative nominal interest rates, rather than by accumulating foreign assets and introducing deviations from interest parity. In the absence of taxes on money, the CB cannot lower nominal rates below zero, because households have an incentive to issue infinite amounts of bonds in exchange for money. Taxing money reduces the return on money and eliminates this incentive to arbitrage by households when the CB sets negative nominal rates.

7 Optimal Exchange Rate Policy

Until now we have studied how the economy achieves an equilibrium for a given exchange rate policy (s_1, s_2). We now study optimal exchange rate interventions in a model with nominal rigidities. To simplify the analysis, we assume that s_2 is given.[19]

We consider a model where first-period output is produced with labor, $y_1 = f(l_1)$, while in the second period we continue to assume that there is an endowment. Wages, are sticky in domestic

[17] This relates to the result in trade that tariffs dominate quotas when the economy is large.

[18] The policy prescription of using negative tax on money a to implement negative rates goes back to Gesell (1916).

[19] In the Appendix we address the joint determination of (s_1, s_2) and show that the results we derive in this section still apply. The only difference when s_2 is also endogenous is that the Central Bank can relax the zero lower bound constraint either by appreciating the exchange rate today or by depreciating it tomorrow. As long as there is a cost from depreciating in the second period, for example due to overheating or higher prices, the same trade-offs we uncover in this section apply in that version of the model.

currency, $\omega = \bar{\omega}$. Taking wages as given, firms solve

$$\Pi_1 \equiv \max_{l_1} s_1 f(l_1) - \bar{\omega} l_1,$$

which leads to a labor demand condition

$$s_1 f'(l_1) = \omega_1 \tag{36}$$

Households lifetime utility is now given by

$$u(c_1) + v(n_1) + \beta u(c_2)$$

where $v(n_1)$ represents the disutility from working. Households face essentially the same problem as in the previous version of the model with the difference households' income is composed of labor income and firms' profits. We assume that households are off their labor supply, and essentially supply as many hours as firms demand at the sticky wage (i.e., $n_1 = l_1$).[20] Hence, their problem is exactly as in Section 2, except that the first-period budget constraint is replaced by

$$s_1 c_1 + s_1 a^\star + a = \bar{\omega} n_1 + \Pi_1 + T_1$$

In this economy, the efficient level of employment solves $\max_{h_1} u(f(n_1)) - v(n_1)$ and implies a zero labor wedge $u_c(c_1^*) f'(n_1^*) = v_n(n_1^*)$. The flexible wage economy would achieve this level of employment with associated real wage, $\omega^* = f'(n_1^*)$. When ω is above (below) ω^*, we have unemployment (over-employment). From equation (36) we can see that the central bank can offset the nominal rigidity in wages by adjusting the level of the exchange rate. Following the standard prescription, the central bank can appreciate whenever there is over-employment, and depreciate whenever there is unemployment, to achieve the efficient level of employment. In the former case, the central bank can raise sufficiently the nominal interest rate to appreciate the exchange rate and achieve the efficient level of employment. The presence of the zero lower bound on nominal interest rates, however, imposes limits on the ability of the Central Bank to depreciate the exchange rate and reduce unemployment when real wages are too high. We study next the optimal central bank problem of jointly choosing exchange rate policy and foreign

[20]We could assume that wages are downward rigid, as inSchmitt-Grohé and Uribe (2016), but in this example, this does not play a role in period 1 since for the optimal exchange rate the real wage is above the equilibrium wage that would prevail with flexible wages

exchange rate intervention.

7.1 Central Bank's Problem

The problem of the Central Bank consists of choosing $\{s_1, i, F, T_1, T_2\}$ to maximize household lifetime utility choosing $\{s_1, F, T_1, T_2\}$, the Central Bank is subject to implementability constraints given by firms' employment decisions,households' optimization, foreign lenders' optimal demand for domestic assets, as well as the zero lower bound. Combining these conditions, the problem can be written as

$$\max_{\{c_1,c_2,s_1,s_2,i\}} u(c_1) - v\left(g\left(\frac{\bar{\omega}_1}{s_1}\right)\right) + \beta u(c_2)$$

subject to:

$$(1+i^\star)c_1 + c_2 \leq (1+i^\star)\left[f\left(g\left(\frac{\bar{\omega}_1}{s_1}\right)\right)\right] + y_2 - \left(\frac{s_1}{s_2} - (1+i^\star)\right)\bar{w} \tag{37}$$

$$1+i \geq 0 \tag{38}$$

$$1+i \geq (1+i^\star)\frac{s_2}{s_1} \tag{39}$$

$$1+i = \frac{u'(c_1)}{\beta u'(c_2)}\frac{s_2}{s_1} \tag{40}$$

$$f\left(g\left(\frac{\bar{\omega}_1}{s_1}\right)\right) - c_1 + \bar{x} \geq 0 \tag{41}$$

where we have denoted $g(x) = f'^{-1}(x)$, replaced the equilibrium level of employment using (36) and we have used that $\frac{u'(c_1)}{\beta u'(c_2)} = 1+i$ in (38).

Consider a relaxed problem where we ignore the ZLB constraint, constraint (38), $\frac{u'(c_1)}{\beta u'(c_2)} \geq \frac{s_1}{s_2}$. Let $c_1^\star, c_2^\star, s_1^\star$, denote its solution. To the extent that $\frac{u'(c_1^\star)}{\beta u'(c_2^\star)} \geq \frac{s_1^\star}{s_2}$, the solution to the planning problem coincides with the relaxed problem: the ZLB constraint is not binding and the economy achieves the efficient allocations by setting $1+i = (1+i^*)\frac{s_2^\star}{s_1^\star}$. If instead $\frac{u'(c_1^\star)}{\beta u'(c_2^\star)} < \frac{s_1^\star}{s_2}$, the solution to the relaxed problem cannot be attained in the planning problem and the economy is at the ZLB. The exchange rate is too appreciated relative to the efficient level (i.e., $s_1 < s_1^\star$).

At the ZLB, the central bank faces a trade-off. It can reduce the level of unemployment by raising s_1 above the interest parity condition (i.e, $s_1^\star > (1+i^*)s_2^\star$), but this exposes the CB to losses on the balance sheet. To inspect this trade-off, consider a state with deviation from IP condition and binding ZLB constraint. Let λ, ξ be respectively the Lagrange multiplier o a and

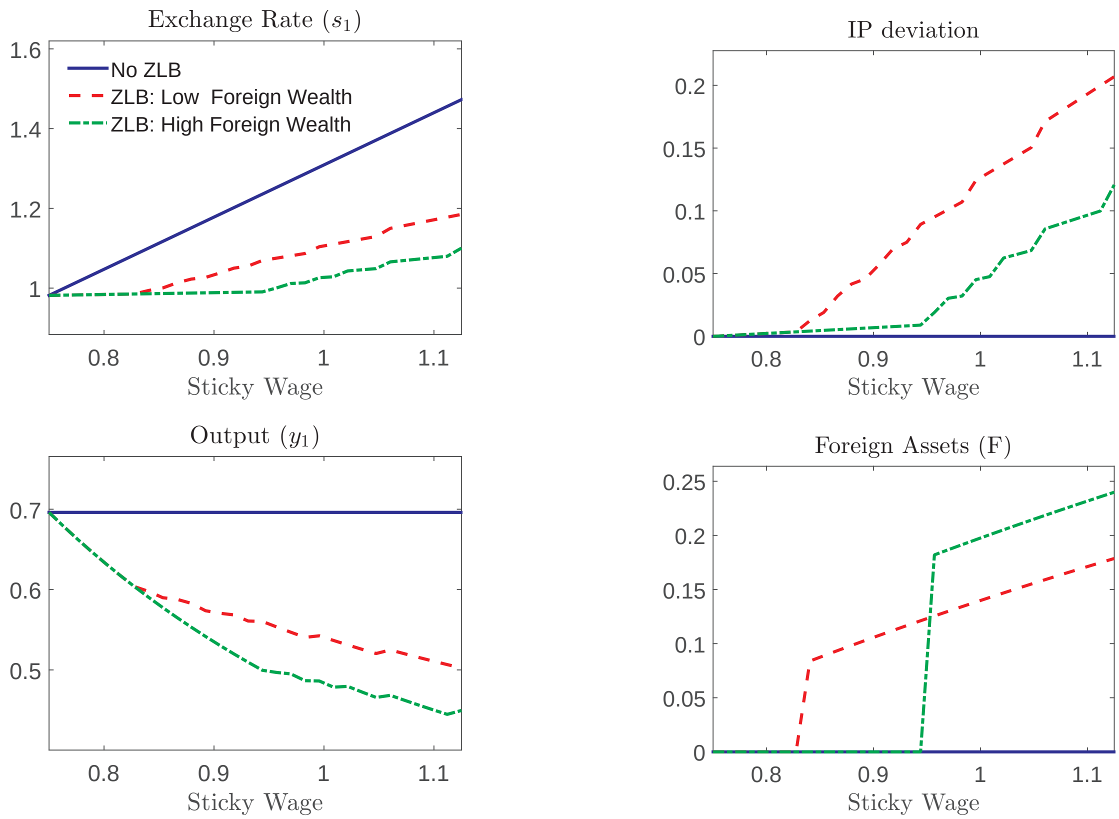


Figure 6: Optimal Exchange Rate Interventions

the ZLB constraint, the inter-temporal resource foreign wealth constraint. We have the following optimality condition with respect to s_1

$$-g'\left(\frac{\bar{\omega}_1}{s_1}\right)\frac{\bar{\omega}_1}{s_1^2}\left(\underbrace{(\lambda(1+i^*))f'(h)-v'(h_1)}_{\text{Labor Wedge}}\right)=\underbrace{\frac{\lambda}{s_2}}_{\text{Intervention Losses}}\bar{w}+\underbrace{\xi\beta u'(c_2)}_{\text{Interest Rate Distortion}} \tag{42}$$

Equation (42) illustrates the key trade-off in the model. The left-hand side indicates the benefits of depreciating the exchange rate. Since $g'<0$, this shows that there are positive benefits from depreciating when the labor wedge is positive. The right-hand side indicates the marginal costs from depreciating the exchange rate, which is composed of two objects we analyzed before. The first term represents the intervention losses, expressed in period 2 consumption goods, that arise when there is a strict violation from interest parity. As we have discussed before, these losses are proportional to the foreign wealth of investors. The second term is the loss due to the distortion in the consumption-saving decisions of domestic households. A rise in s_1 increases the real rate, and distorts consumption towards the second period.

Figure 6 shows how the optimal policy and allocations vary with the level of the sticky wage for three different cases. The straight blue line is the economy not subject to a ZLB constraint. In this case, the CB is able to fully stabilize output by depreciating the exchange rate as we increase the sticky wage. The red broken line represents the economy subject to a ZLB constraint, with a low degree of financial integration $\bar{w}$. For low levels of the sticky wage, the CB is able to achieve the efficient level of output because the ZLB is not binding. As the sticky wages increases, the ZLB becomes binding and the economy enters a recession. Notice that only when output drops sufficiently, the CB intervenes by accumulating foreign assets, opening up a gap in the CIP condition. Starting at the efficient allocations, an increase in the real wage induces a second order loss, whereas a deviation from IP generates a first-order loss proportional to $\bar{w}$. As a result, only when the ZLB is sufficiently binding, the CB induces a deviation from IP. Finally, the green line shows that when $\bar{w}$ is larger, the CB intervenes in fewer states and imposes lower deviations from CIP, consistent with our results that the costs of intervention are increasing at $\bar{w}$.In addition, to be able to affect the exchange rate, the CB has to intervene at a larger scale.[21]

8 Empirical Evidence

We now verify whether the main predictions of our theory find support in the data. In Section 8.1 we explore the links between foreign reserves, deviations from interest rate parity and nominal interest rate for a group of advanced economies. We document two main facts that are consistent with the basic mechanisms of our theory. First, foreign reserves held by monetary authorities are positively related, both across countries and over time, to deviations from the covered interest rate parity. Second, deviations from the covered interest rate parity arise mostly for currencies whose nominal interest rates gravitate around zero. In Section 8.2 we use our simple formula to provide an estimate of the costs of the recent foreign exchange interventions by the Swiss National Bank (SNB).

8.1 Foreign reserves, nominal interest rates and CIP gaps

In our model the monetary authority can increase the domestic real interest rate relative to the world real interest rate by accumulating foreign assets. While these interventions are costly from the point of view of a small open economy, we have seen that a Central Bank may optimally

[21] As discussed above, a decline in the world interest rate also leads to larger losses from interventions, and hence the model predicts a lower depreciation. Through the lens of our model, the ECB's QE policies offer an explanation of the Swiss decision to let their currency appreciate in January 2015.

implement them in order to temporarily depreciate its currency while at the zero lower bound. We now check whether these model predictions are consistent with basic facts about the relation between foreign reserves, nominal interest rates, and deviations between domestic and the world real interest rates.

To this end, we construct proxies for these three variables for a group of advanced economies over the 2000-2015 period.[22] We obtain yearly data on foreign reserves holdings from the IMF *International Financial Statistics*,[23] and we scale it by annual gross domestic product obtained from the OECD *National Accounts*. Both foreign reserves and gross domestic product are expressed in U.S. dollars at current prices. The ratio between foreign reserves and gross domestic product is our proxy for the size of foreign reserves.

The measurement of deviations between the domestic and the world real interest rates is more involved. In our model, these gaps generate arbitrage opportunities for foreign investors, and they have drastic effects on capital flows. Because our model is deterministic, we could proxy these gaps either with deviations from the covered interest rate parity (CIP) conditions, or using deviations from the uncovered one (UIP). However, it is well know that deviations from UIP may reflect compensation that risk averse lenders require for holding currency risk, and the literature has hardly interpreted their presence in the data as an indication of arbitrage opportunities (Engel, 2014). Because of that, we map the gap between a country's real interest rate and the U.S. real interest rate using deviations from the CIP condition. Specifically, letting $i^{\$}_{t,t+n}$ denote the nominal interest rate in US dollars between time t and time $t+n$, $i^{j}_{t,t+n}$ the corresponding interest rate in currency j, $s^{j,\$}_{t}$ the spot exchange rate of currency j per U.S. dollar, and $f^{j,\$}_{t,t+n}$ the n-periods ahead associated forward contract, we can express deviations from the CIP condition as

$$\text{cip gaps}^{j,\$}_{t,t+n} = i^{j}_{t,t+n} - i^{\$}_{t,t+n} + \frac{1}{n}[\log(s^{j,\$}_{t}) - \log(f^{j,\$}_{t,t+n})].$$

A positive value for this indicator is equivalent, in our model, to a positive gap between the real interest rate in country j and the world real interest rate.

We calculate deviations from the CIP condition at a three months horizon between major currencies and the U.S. dollar for the period 2000-2015. We map $i^{j}_{t,t+n}$ to the interest rate on an

[22] As we detail below, we proxy deviations between the domestic and the world real interest rates using gaps in the covered interest rate parity condition. Because of that, we restrict the analysis to a set of countries for which data on currency forwards are of sufficiently good quality. Our sample borrows mostly from Du et al. (2016) and it includes Switzerland, Japan, Denmark, Sweden, Canada, the U.K., Australia, New Zealand, Norway and Israel.

[23] Total reserves comprise holdings of monetary gold, special drawing rights, reserves of IMF members held by the IMF, and holdings of foreign exchange under the control of monetary authorities. The gold component of these reserves is valued at year-end (December 31) London prices.

overnight indexed swap (OIS) of three-month duration in currency j, while $i^{\$}_{t,t+n}$ is the respective OIS rate in U.S. dollars with the same duration.[24] The variable $f^{j,\$}_{t,t+n}$ is the three-months forward rate between currency j and the U.S. dollar. All these data are obtained at daily frequency from Bloomberg, and we use the mid-point between the bid and the ask quotes.

Figure x in the Appendix plots these three time series for each country in our group. The figure shows interesting patterns, both over time and across countries. First, there has been a sizable increase in the foreign reserve to GDP ratio for advanced economies over this period, which on average went from 9% in 2001 to 25.4% in 2015. This trend was more pronounced for certain economies than for others: Central Banks in Switzerland, Japan, and Denmark have substantially increased their foreign asset position during the sample, while Central Banks in Australia, Canada and the UK didn't. Second, nominal interest rates have been declining over time: by the end of the period, we have a group of countries with zero or even negative nominal interest rates (Denmark, Switzerland, Japan, and Sweden), and countries with clearly positive nominal rates (Australia, and New Zealand). The third panel in the figure reports yearly averages CIP gaps in our sample. Prior to 2007, CIP deviations were on average small and close to zero for all the advanced economies considered here, a fact that is well establish in the literature. During the global financial crisis of 2007-2009, we have observed major deviations from covered interest rate parity for all the currencies in our sample.[25] Interestingly, these deviations have persisted even after the financial crisis for a group of countries, most notably Switzerland, Denmark, Japan, and Sweden. Du et al. (2016) and Borio et al. (2016) discuss in details these failures of the CIP condition in the post 2008 period.

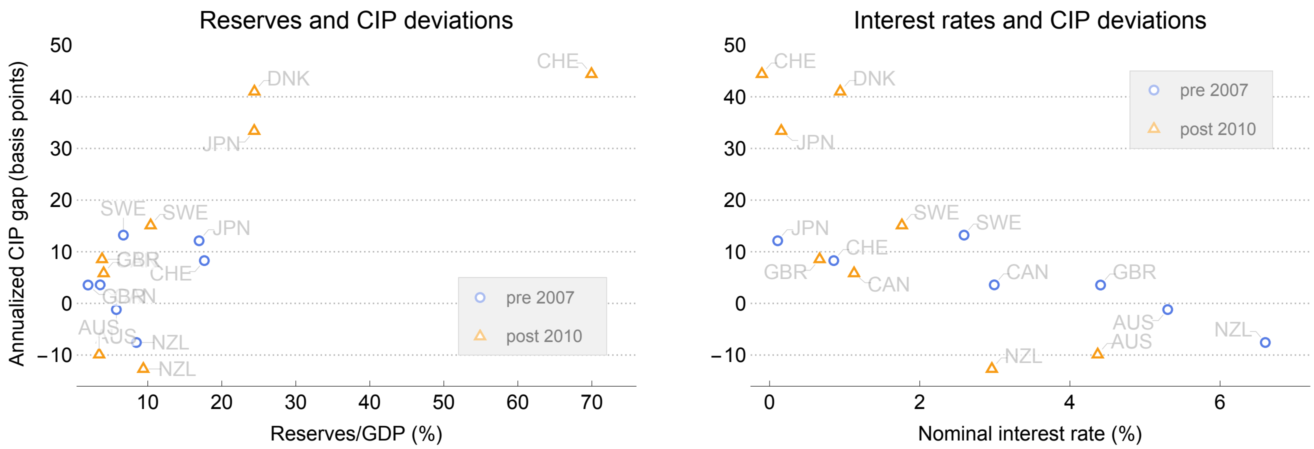
We exploit the variation in these three series, both across countries and over time, to verify two main predictions of the model. Because the deviations from CIP during 2007, 2008 and 2009 were extreme and probably due to the effects of the crisis, we exclude this period and split the time dimension of our sample in two subsets: before (2002-2006) and after (2010-2015) the financial crisis. The left panel of Figure 7 plots the average foreign reserve holdings to GDP ratio against the average CIP deviations in these two sub-samples for each country in our group. The plot shows a positive relationship, both across countries and over time, between the level of foreign reserves and the deviations from the CIP. This empirical finding, which to best of our knowledge has not been previously noted in the literature, is consistent with the mechanism at

[24] We also do the computations using Libor rates. See Appendix X for details.

[25] Baba and Packer (2009) and Ivashina et al. (2015) explain the deviations during the financial crisis through a combination of dollar funding shortages in foreign exchange markets and the inability of financial intermediaries to take advantage of these arbitrage opportunities because of funding constraints during the financial crisis. Our focus however, is in the period after the financial crisis.

the heart of our model, whereby the monetary authority increase the domestic real interest rate relative to the world real interest rate by accumulating a sufficiently large position in foreign assets.

Figure 7: **Foreign reserves, nominal interest rates and CIP deviations**



The right panel of Figure 7 plots the nominal interest rate against the average CIP deviations in these two subsamples. We can observe that the CIP gaps are positive for countries-time periods characterized by low nominal interest rates, while they tend to be small when nominal interest rates are positive. This negative relation between CIP gaps and nominal interest rates, originally uncovered by Du et al. (2016), lends support to our result that Central Banks find it optimal to engage in large foreign exchange interventions when the zero lower bound constraint on nominal interest rates binds.

8.2 The costs of the SNB foreign exchange interventions

In this subsection, we use the Swiss experience to obtain guidance on the size of the potential losses faced by Central Banks. Starting from 2010, the Swiss National Bank has intervened massively in foreign exchange markets, either to defend an explicit target for the exchange rate,[26] or, more informally, to fight appreciation pressures on the Swiss franc. Our theory provides a

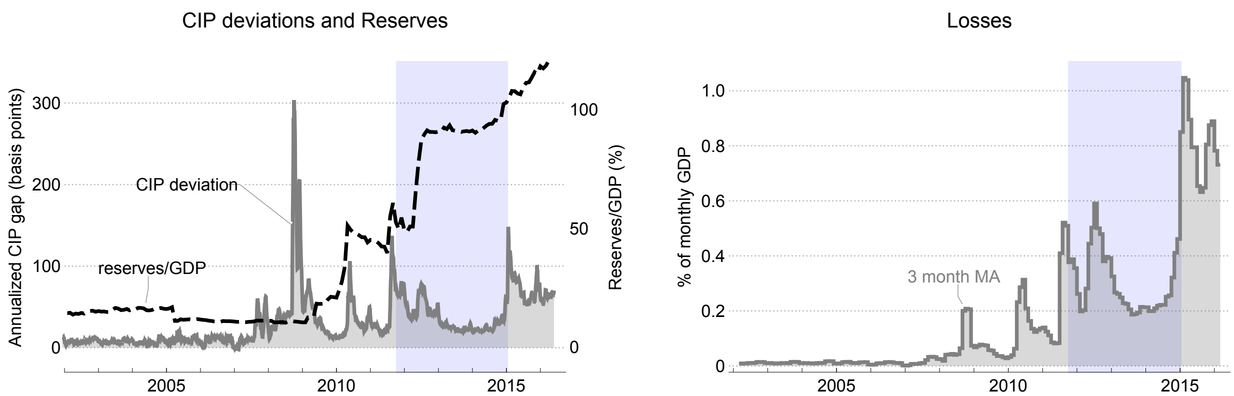
[26] Between 2011 and 2015, the Swiss National Bank successfully kept a floor of 1.2 Swiss franc per euro.

simple expression to measure the costs associated to these interventions,

$$\text{losses}_t = \left[\frac{(1+i_t)}{(1+i_t^\star)} \frac{s_t}{s_{t+1}} - 1 \right] \times F_t. \tag{43}$$

We can use our data on CIP deviations (on a horizon of three months) and on foreign reserves to provide an approximation for the costs of these foreign exchange interventions.[27]

Figure 8: **Foreign reserves, CIP deviations and losses**



In the left panel of Figure 8, we report the monthly three-month CIP deviations between the Swiss franc and the U.S. dollar along with a monthly series for the stock foreign reserves as a fraction of Swiss GDP. This plot confirms that the positive relation between foreign reserves and CIP deviations that we have uncovered earlier in our panel holds at a much higher frequency: after the U.S. financial crisis, spikes in the CIP gaps are associated to massive interventions of the SNB.

In the right panel of Figure 8, we report our corresponding measure of the monthly losses as a fraction of monthly Swiss GDP. The lightly shaded area represents the period where the SNB maintained an official floor on the franc. As can be seen, throughout this period, the losses were significant. They reach their highest point around January 2015, the month when the SNB decided to abandon the currency floor vis a vis the euro.

[27] See Appendix B.

9 Conclusions

[TO BE COMPLETED]

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A Proof of Proposition 7 and 8

Proof of Proposition 7

The first-order condition with respect to τ yields

$$\tau :: \lambda \frac{s_1}{s_2} - \zeta(1+i) = 0$$

where λ and η are respectively the non-negative Lagrange multipliers on (27) and (26). Since $\lambda > 0$, it follows that $\zeta > 0$, and by the complementary slackness condition, this implies that (29) holds with equality, yielding $\tau = 1 - (1+i^\star)\frac{s_2}{s_1}$. That is, the CB taxes domestic inflows, eliminating any rents made by foreign investors.

Proof of Proposition 8

An alternative policy is to place restrictions on the quantities of capital inflows in the economy, $\bar{w}$.

$$W \equiv \max_{(c_1, c_2 w \leq \bar{w})} u(c_1) + \beta u(c_2) + h\left(\frac{m}{s_2}\right) \tag{44}$$

$$\text{subject to:} \tag{45}$$

$$y_1 - c_1 + \frac{y_2 - c_2}{1+i^*} - \left(\frac{\frac{s_1}{s_2} - (1+i^\star)}{1+i^*}\right) w \geq 0 \tag{46}$$

$$u'(c_1) = \beta \frac{s_1}{s_2} u'(c_2) \tag{47}$$

$$y_1 - c_1 + w \geq 0 \tag{48}$$

$$w \geq 0 \tag{49}$$

The first order condition with respect to w:

$$\bar{w} :: \lambda \left(\frac{\frac{s_1}{s_2} - (1+i^\star)}{1+i^*}\right) = \zeta + \eta$$

where ζ denotes the Lagrange multiplier on (48) and η denotes. Since $\lambda > 0$, it follows that either $\zeta > 0$, or $\eta > 0$. and by the complementary slackness condition (48) must hold with equality, or $\eta > 0$ and $w = 0$. Using condition (), it follows that $F^* = 0$.

B Calculating the losses

Our formula in (43) is an approximation because the Swiss National Bank holds several assets in the form of foreign reserves that differ by maturity, currency of denomination, and underlying riskiness. The appropriate way to measure the losses would be that of computing CIP deviations for different currencies and at different horizons, and appropriately matching these gaps with the different asset purchases made by SNB.

The way we proceed is different. We will assume that all assets in the SNB balance sheet are three month zero coupon bonds, denominated in US dollars. We observe the monthly market value of the reserves portfolio from the SNB. Let us denote that series by S_t. Let n_t denote the amount of foreign denominated zero coupon bonds purchased by the SNB in period t.

The market value of the foreign reserve portfolio at the end of period t is:

$$S_t = q_t^3 n_t + q_t^2 n_{t-1} + q_t^1 n_{t-2},$$

where q_t^i is the international price at time t of a zero-coupon bonds that matures in i periods. The amount of foreign reserves purchased in period t is then

$$F_t = q_t^3 n_t.$$

And thus, we have that the market value of the stock can be written as

$$S_t = F_t + \frac{q_t^2}{q_{t-1}^3} F_{t-1} + \frac{q_t^1}{q_{t-2}^3} F_{t-2}$$

We know that $q_t^3 = (1+i_t^\star)^{-1}$, and we approximate the one period and the 2 period ahead interest rate to be $q_t^2 = (1+i_t^\star)^{-2/3}$ and $q_t^1 = (1+i_t^\star)^{-1/3}$.

Using the observed series for S_t, and the three month international interest rate, $i_t^\star$, we can then solve for the series F_t that solves

$$S_t = F_t + \frac{1+i_{t-1}^\star}{(1+i_t^\star)^{2/3}} F_{t-1} + \frac{1+i_{t-2}^\star}{(1+i_t^\star)^{-1/3}} F_{t-2},$$

for some starting points with $F_{t_0} = F_{t_0+1} = 0$.

Having computed F_t, we then apply the formula (43) to calculate the monthly losses of the Central Bank.

A Optimal Exchange Rate Policy (s_1, s_2)

In Section 7, we study the optimal exchange rate interventions in a model with endogenous exchange rate in period 1, s_1, taking as given the exchange rate in period 2, s_2. We consider now a model with endogenous production and disutility from working in both period 1 and period 2. We assume that the wage is expected to decline in period 2 $\bar{\omega}_1 > \bar{\omega}_2$. The basic motivation for this assumption is is that we want to consider a model where given a constant exchange rate, the recession is more severe in period 1 than in period 2.[28]

In this section, we study the optimal determination of the path for exchange rates (s_1, s_2). At the ZLB, the CB problem is:

$$\max_{\{c_1,c_2,s_1,s_2\}} u(c_1) - v\left(g\left(\frac{\bar{\omega}_1}{s_1}\right)\right) + \beta u(c_2) - \beta v\left(g\left(\frac{\bar{\omega}_2}{s_2}\right)\right)$$

subject to:

$$(1+i^\star)\left[f\left(g\left(\frac{\bar{\omega}_1}{s_1}\right)\right) - c_1\right] + \left[f\left(g\left(\frac{\bar{\omega}_2}{s_2}\right)\right) - c_2\right] - \left(\frac{s_1}{s_2} - (1+i^\star)\right)\bar{w} \geq 0 \tag{50}$$

$$s_1 \geq s_2(1+i^\star) \tag{51}$$

$$\frac{u'(c_1)}{\beta u'(c_2)} \geq \frac{s_1}{s_2} \tag{52}$$

$$f\left(g\left(\frac{\bar{\omega}_1}{s_1}\right)\right) - c_1 + \bar{w} \geq 0 \tag{53}$$

Let us write the Lagrangian of the CB problem

$$\begin{aligned}
L(c_1, c_2, s_1, s_2) = {} & u(c_1) - v\left(g\left(\frac{\bar{\omega}_1}{s_1}\right)\right) + \beta u(c_2) - \beta v\left(g\left(\frac{\bar{\omega}_2}{s_2}\right)\right) \\
& + \lambda\left[(1+i^\star)\left[f\left(g\left(\frac{\bar{\omega}_1}{s_1}\right)\right) - c_1\right] + \left[f\left(g\left(\frac{\bar{\omega}_2}{s_2}\right)\right) - c_2\right] - \left(\frac{s_1}{s_2} - (1+i^\star)\right)\bar{w}\right] \\
& + \zeta(s_1 - s_2(1+i^\star)) \\
& + \xi(u'(c_1)s_2 - s_1\beta u'(c_2)) \\
& + \eta(f\left(g\left(\frac{\bar{\omega}_1}{s_1}\right)\right) - c_1 + \bar{w})
\end{aligned}$$

[28] Considering a discount factor shock and adding a non-tradable sector would be an alternative way to make the ZLB binding in equilibrium.

The first order conditions are as follows:

$$c_1 :: u'(c_1) + u''(c_1)\xi s_2 - \eta = \lambda(1+i^*) \tag{54}$$

$$c_2 :: \beta u'(c_2) = \lambda + s_1\beta\xi u''(c_2) \tag{55}$$

$$s_1 :: v'(n_1)g'\left(\frac{\bar{\omega}_1}{s_1}\right)\frac{\bar{\omega}_1}{s_1^2} - (\lambda(1+i^*)+\eta)f'(l)g'\left(\frac{\bar{\omega}_1}{s_1}\right)\frac{\bar{\omega}_1}{s_1^2} + \zeta = \frac{\lambda}{s_2}\bar{w} + \xi\beta u'(c_2) \tag{56}$$

$$s_2 :: \beta v'(l_2)g'\left(\frac{\bar{\omega}_2}{s_2}\right)\frac{\bar{\omega}_2}{s_2^2} - \lambda f'(l_2)g'\left(\frac{\bar{\omega}_2}{s_2}\right)\frac{\bar{\omega}_2}{s_2^2} - \zeta(1+i^*) = -\frac{\lambda s_1}{s_2^2}\bar{w} - \xi u'(c_1) \tag{57}$$

Consider a state with deviation from IP ($\zeta = 0$) condition and binding ZLB ($\xi > 0$). We have the following optimality condition with respect to s_1

$$-g'\left(\frac{\bar{\omega}_1}{s_1}\right)\frac{\bar{\omega}_1}{s_1^2}\left(\underbrace{(\lambda(1+i^*))f'(h) - v'(h_1)}_{\text{Labor Wedge}}\right) = \underbrace{\frac{\lambda}{s_2}\bar{w}}_{\text{Intervention Losses}} + \underbrace{\xi\beta u'(c_2)}_{\text{Interest Rate Distortion}} \tag{58}$$

which is condition () in the text.

Similarly, we can re-express optimality condition (57) as:

$$s_2 :: -g'\left(\frac{\bar{\omega}_2}{s_2}\right)\frac{\bar{\omega}_2}{s_2^2}\underbrace{(\lambda f'(n_2) - \beta v'(n_2))}_{\text{Labor Wedge}} = \underbrace{-\frac{\lambda s_1}{s_2^2}\bar{w}}_{\downarrow\text{Sterilization Loss}} - \xi u'(c_1) \tag{59}$$

Equation (59) indicates that the CB at the optimum chooses to have a negative labor wedge in the second period. Since raising s_2 reduces sterilization losses and interest rate distortion, the CB would find strictly optimal to increase s_2 if the economy had a positive labor wedge.

Figure 9 shows how the period 2 exchange rates and output, in addition to the same variables shown in Figure 6, vary with the level of the first period sticky wage. As the wage increases, the CB increases $s_2$beyond the level that would lead to full employment. This allows to relax the ZLB and reduces unemployment in period 1.

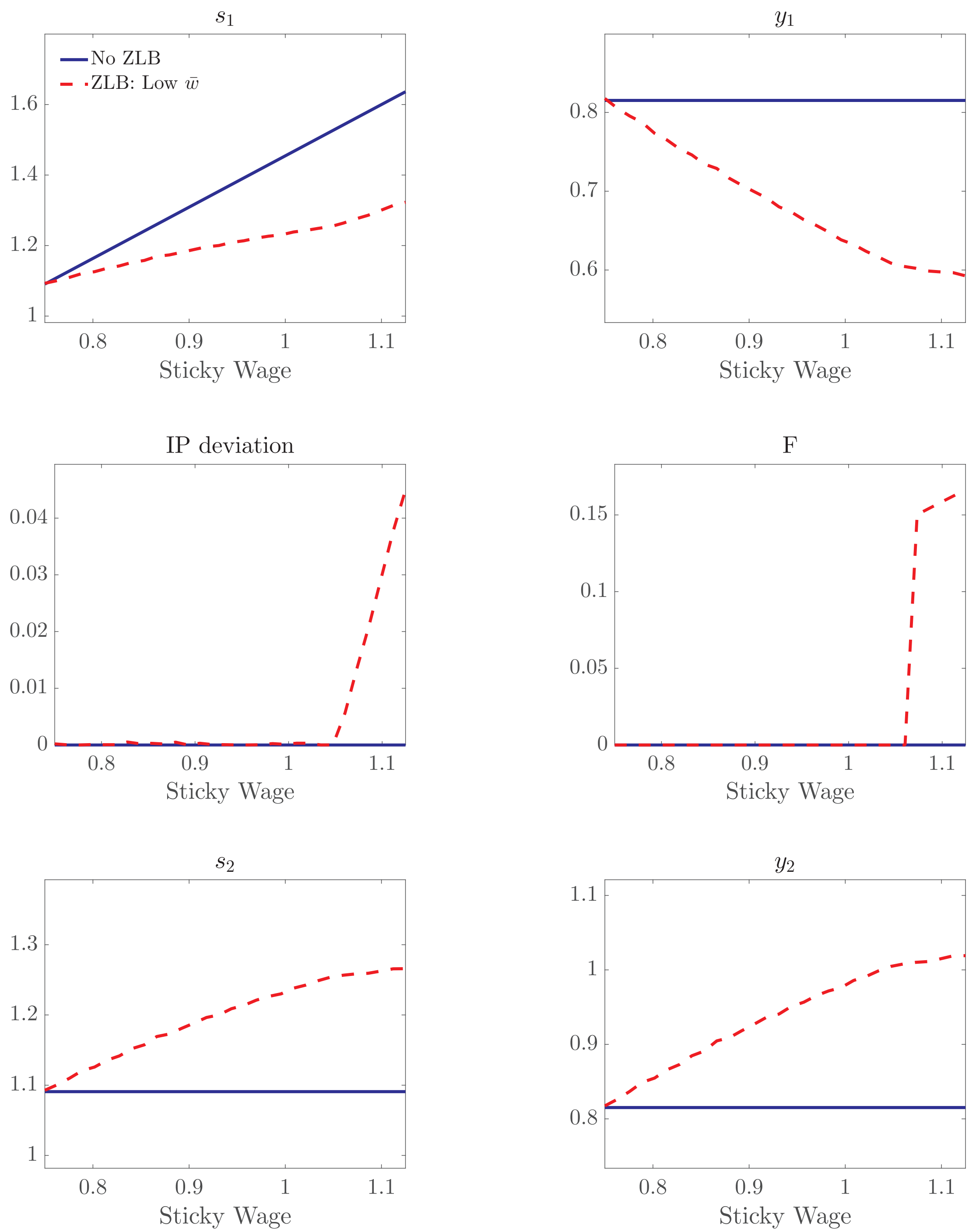

Figure 9: Optimal Exchange Rate Interventions with Endogenous (s_1, s_2)