On the Distribution of $Z \sim \chi^2_{df} + \mathrm{Lap}(b)$

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When discussion a differentially private form of ANOVA, we looked into the following random variable:

$$\widehat{F} \sim \frac{U_1 + \mathrm{Lap}(b_1)}{U_2 + \mathrm{Lap}(b_2)} \tag{1}$$

Here, U_1 and U_2 are both χ^2_{df} for some parameter df. The first step to understanding this random variable would be understanding the distribution of $U + \mathrm{Lap}(b)$ for $U \sim \chi^2_{df}$. Does this potentially have a "nice" distribution?

The answer to this is both yes and no — it has an *explicitly computable* density function (without resorting to things like various special functions such as the hypergeometric function, Meijer G function, or Fox H function). We can easily[1] *algorithmically* compute this distribution, which will be detailed below.

1 Sums of independent variables via Moment Generating Functions

We've so far seen multiple ways to compute to compute the pdf of a random variable. One of the "most powerful" is that of the *moment generating function*, which is defined as:

$$m_X(t) = \mathbb{E}[e^{tX}] = \int_{x=-\infty}^{\infty} e^{tx}\rho_X(x)\mathrm{d}x \tag{2}$$

Much of the power of this method lies in its close connection to the *Laplace Transform*. This shares many desirable theoretical properties with the *Fourier Transform*[2], so one can think of them as roughly equivalent if you want to. The two main properties we've seen so far are:

[1]Given access to a tool that's good at Symbolic Integration, such as *Mathematica*.

[2]In fact, the *characteristic function* $\varphi_X(t) = \mathbb{E}[e^{itX}]$ is defined precisely in terms of the Fourier transform. This is a complex-valued function though, so is at least a *little* worse to work with numerically. The benefit is that it always exists, unlike the mgf.

- The Fourier/Laplace Transform take the *convolution product* to the standard product (multiplication). Specifically, if we define:

$$(f * g)(x) = \int_{-\infty}^{\infty} f(y)g(x-y)\mathrm{d}x \tag{3}$$

  Then it follows that:

$$\mathcal{F}[f * g](x) = \mathcal{F}[f](x)\mathcal{F}[g](x) \tag{4}$$

  where $\mathcal{F}[f](x)$ is the Fourier transform (or Laplace transform) of f evaluated at x.

- There is a unique correspondence between $f(x)$ and $\mathcal{F}[f](x)$. In terms of probability theory, either of the characteristic function, or the mgf *uniquely* determine the pdf[3].

One property we *haven't* discussed is known as *Fourier Inversion*. This is just the fact than Fourier-like transforms have easily computable *inverse* transforms. Unfortunately, this "easy computation" can require a certain amount of complex analysis to define, so will be omitted from this writeup.

Fortunately, *Mathematica* has a function:

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InverseLaplaceTransform[f(s), s, t]
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This *essentially* computes precisely what we need, and allows us to "invert" mgfs. As a result, we can look up the (known) mgfs for $X \sim \mathrm{Lap}(b)$ and $Y \sim \chi^2_{df}$. It's then quite easy to compute $m_{X+Y}(t) = m_X(t)m_Y(t)$, which we can then invert. This can easily be done in *Mathematica*, which the .nb file in this directory includes code for.

This allows us to describe the distribution of $\mathrm{Lap}(b) + \chi^2_{df}$, which Mathematica outputs as:

$$\frac{1}{2\,b\,\mathrm{Gamma}\left[\frac{df}{2}\right]} e^{-\frac{x}{b}} (-x)^{df/2} \left(e^{\frac{2x}{b}} \left(\left(-1+\frac{2}{b}\right) x\right)^{-df/2} \left(-\mathrm{Gamma}\left[\frac{df}{2}\right] + \mathrm{Gamma}\left[\frac{df}{2}, \left(-\frac{1}{2}+\frac{1}{b}\right) x\right]\right) + \left(-\frac{(2+b)\,x}{b}\right)^{-df/2} \left(\mathrm{Gamma}\left[\frac{df}{2}\right] - \mathrm{Gamma}\left[\frac{df}{2}, -\frac{(2+b)\,x}{2\,b}\right]\right)\right)$$

Note here that this includes both $\Gamma(a)$, and $\Gamma(a,x)$. Here, $\Gamma(a,x)$ is the *incomplete* Gamma function, defined as:

$$\Gamma(a,x) = \int_x^{\infty} t^{s-1}e^{-t}\mathrm{d}t \tag{5}$$

Note that $\Gamma(a,0) = \Gamma(a)$, so the traditional gamma function is a special case of this.

[3]Provided they exist, which the mgf doesn't always

We can transcribe this into LaTeX, to get that:

$$\begin{aligned}\rho_{X+Y}(x) = &\frac{1}{2b\Gamma(df/2)} e^{-x/b}(-x)^{df/2} \\ &\times \Bigg[e^{2x/b}\left(\left(-1+\frac{2}{b}\right)x\right)^{-df/2}(-\Gamma(df/2)+\Gamma(df/2,(-(1/2)+(1/b))x)) \\ &+\left(-\frac{(2+b)x}{b}\right)^{-df/2}\left(\Gamma(df/2)-\Gamma(df/2,-\frac{(2+b)x}{2b})\right)\Bigg]\end{aligned}$$

We can now try to clean this up some. To do this, we'll note that it's basic structure has three parts:

$$\rho_{X+Y}(x) = A(B+C) \tag{6}$$

where:

$$\begin{aligned}A &= \frac{1}{2b\Gamma(df/2)} e^{-x/b}(-x)^{df/2} \\ B &= e^{2x/b}\left(\left(-1+\frac{2}{b}\right)x\right)^{-df/2}(-\Gamma(df/2)+\Gamma(df/2,(-(1/2)+(1/b))x)) \\ C &= \left(-\frac{(2+b)x}{b}\right)^{-df/2}\left(\Gamma(df/2)-\Gamma(df/2,-\frac{(2+b)x}{2b})\right)\end{aligned}$$

Note that:

$$\left(\left(-1+\frac{2}{b}\right)x\right)^{-df/2} = \left(-\frac{b-2}{b}x\right)^{-df/2} \tag{7}$$

We additionally have that:

$$\left(-\frac{(2+b)x}{b}\right)^{-df/2} = \left(-\frac{b+2}{b}x\right)^{-df/2} \tag{8}$$

We can next look at the (other) factor of each term, which is of the form:

$$\pm(\Gamma(df/2)-\Gamma(df/2,a)) \tag{9}$$

for some a. First, note that:

$$\Gamma(a)-\Gamma(a,x) = \int_0^\infty t^{a-1}e^{-t}\mathrm{d}t - \int_x^\infty t^{a-1}e^{-t}\mathrm{d}t = \int_0^x t^{a-1}e^{-t}\mathrm{d}t = \gamma(a,x) \tag{10}$$

Here, $\gamma(a,x)$ is the *lower* incomplete Gamma function.

We have that the first of these a values will be:

$$a = \left(-\frac{1}{2}+\frac{1}{b}\right)x = -\frac{b-2}{2b}x \tag{11}$$

It follows that:

$$\Gamma(df/2) - \Gamma(df/2, -\frac{b-2}{2b}x) = \gamma(df/2, -\frac{b-2}{2b}x) \tag{12}$$

The other value will be:

$$\gamma(df/2, -\frac{b+2}{2b}x) \tag{13}$$

We can therefore try to rewrite the entire pdf, to get that:

$$\begin{aligned}
\rho_{X+Y}(x) &= \frac{1}{2b\Gamma(df/2)} e^{-x/b}(-x)^{df/2}\left(-e^{2x/b}\left(\frac{b}{(b-2)(-x)}\right)^{df/2}\gamma(df/2, -\frac{b-2}{2b}x)\right.\\
&\left. + \left(\frac{b}{(2+b)(-x)}\right)^{df/2}\gamma(df/2, -\frac{(2+b)x}{2b})\right)\\
&= \frac{b^{(df/2)-1}}{2\Gamma(df/2)} e^{-x/b}\left(\frac{-e^{2x/b}}{(b-2)^{df/2}}\gamma(df/2, -\frac{b-2}{2b}x) + \frac{1}{(2+b)^{df/2}}\gamma(df/2, -\frac{(b+2)x}{2b})\right)\\
&= \frac{b^{(df/2)-1}}{2\Gamma(df/2)(b^2-4)^{df/2}} e^{-x/b}\left((b-2)^{df/2}\gamma\left(df/2, -\frac{(b+2)x}{2b}\right)\right.\\
&\left. - e^{2x/b}(b+2)^{df/2}\gamma\left(df/2, -\frac{b-2}{2b}x\right)\right)
\end{aligned}$$

At this point, it's unclear if further simplification is possible.