Homework for Week Two

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1 Statistics

Working through the problem set Andrew posted on Slack.

1a was completed (and is on my RSTUDIO server).

1b:

Let $X_{ij} \sim \mathcal{N}(\mu, \sigma^2)$ be i.i.d. Then, find the distribution of:

$$W = \sum_{j=1}^{k} \sum_{i=1}^{n_j} (X_{ij} - \overline{X}_j)^2 \tag{1}$$

where:

$$\overline{X}_j = \frac{1}{n_j} \sum_{i=1}^{n_j} X_{ij} \tag{2}$$

We can define the random variable Z_{ij} as:

$$Z_{ij} = X_{ij} - \overline{X}_j = \left(1 - \frac{1}{n_j}\right) X_{ij} - \frac{1}{n_j} \sum_{\substack{i=1 \\ i \neq j}}^{j} X_{ij} \tag{3}$$

As the X_{ij} here are all independent normals, we can then show that $Z_{ij} \sim \mathcal{N}\left(0, \frac{\sigma^2}{n_j}\right)$. We then want to add a bunch of Z_{ij}^2 together (each Z_{ij}^2 will *individually* be χ_1^2), but the Z_{ij} will *not* be independent generically.

We then might wonder about:

$$\begin{aligned} \mathrm{Cov}(Z_{ij}^2, Z_{k\ell}^2) &= \mathrm{Cov}\left(\left((1 - n_j^{-1}) X_{ij} - n_j^{-1} \sum_{\substack{i=1 \\ i \neq j}}^{j} X_{ij}\right)^2, \left((1 - n_\ell^{-1}) X_{k\ell} - n_\ell^{-1} \sum_{\substack{k=1 \\ k \neq \ell}}^{j} X_{k\ell}\right)^2\right) \\ &= \mathrm{Cov}\big((1 - n_j^{-1})^2 X_{ij}^2\big) \end{aligned}$$

Note that when X^2, Y^2 are both mean zero (which is the case here), we get that:

$$\mathrm{Cov}(X^2, Y^2) = \mathbb{E}[X^2 Y^2] = \mathbb{E}[(XY)^2] \tag{4}$$

We therefore want to find (in terms of X_{ij}) what $X_{i_1 j_1} X_{i_2 j_2}$ looks like.

2 Laplace Plus ChiSquare

We have that the MGF of $X \sim \mathrm{Lap}(\mu, b)$ is:

$$\frac{\exp(\mu t)}{1 - b^2 t^2}, \quad |t| < 1/b \tag{5}$$

The MGF of $Y \sim \chi^2_{df}$ is:

$$(1-2t)^{-df/2}, \quad t < 1/2 \tag{6}$$

It follows that:

$$m_{X+Y}(t) = \frac{1}{\sqrt{1-2t}^{df}} \frac{e^{\mu t}}{1-b^2t^2} \tag{7}$$