Improved Differentially Private Analysis of Variance

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Anna Ritz   Andrew Bray   Adam Groce

Reed College   Boston University

Observed Data, $n = 30$

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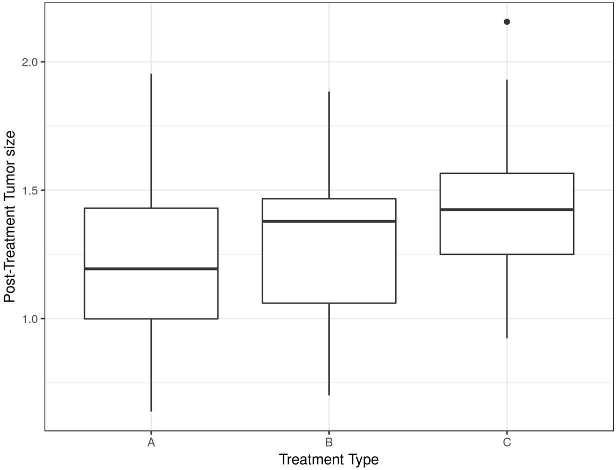

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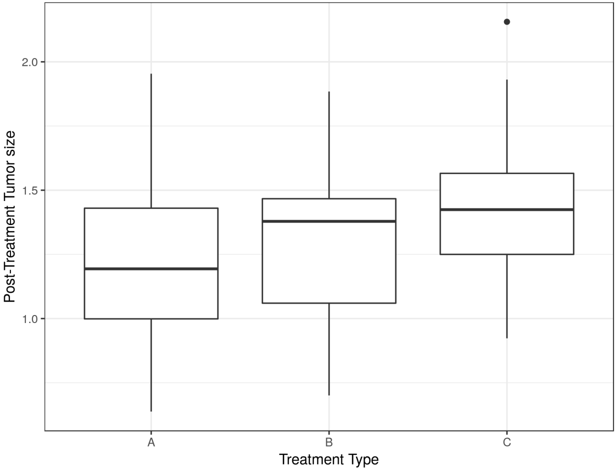

Are treatment type and post-treatment tumor size dependent?

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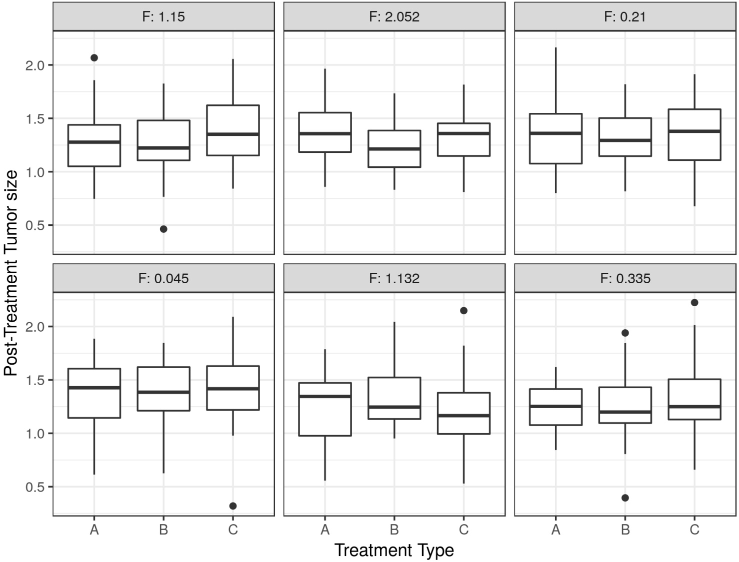
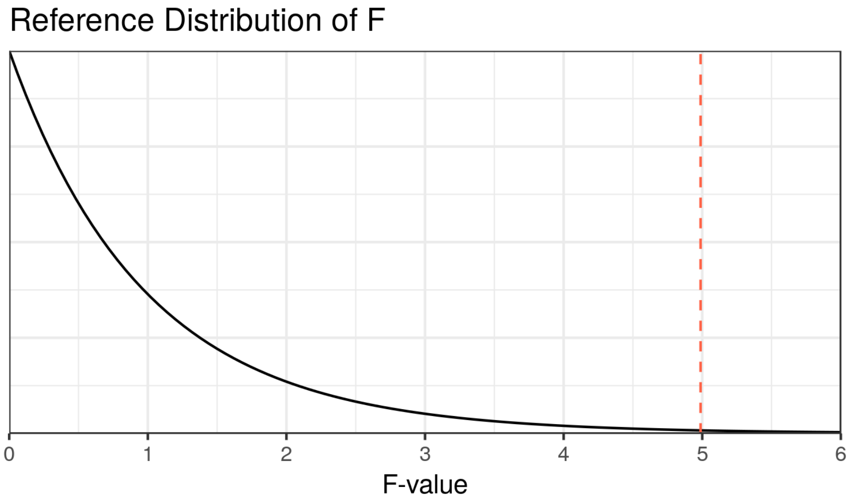
Simulate Null Data




Reference Distribution of F
0
1
2
3
4
5
6
F-value

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Goal of any test statistic is achieving high power*.

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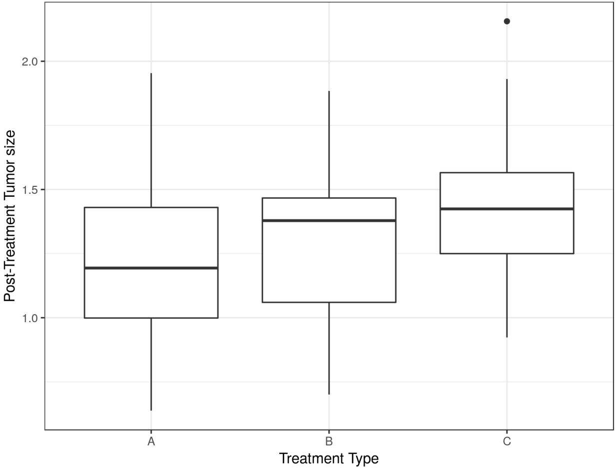

What if we want to keep this data private?

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A query f is ε-**differentially private** if for all neighboring databases D, D' and all output sets S

$$\Pr[f(D) \in S] \leq e^{\varepsilon} \Pr[f(D') \in S].$$

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If f is ε-differentially private then for any (randomized) function g, then if $h(D) = g(f(D))$, h is also ε-differentially private.

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Theorem (Composition)

If f is ε_1-differentially private and g is ε_2-differentially private then if $h(D) = (g(D), f(D))$, h is $(\varepsilon_1 + \varepsilon_2)$-differentially private.

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Theorem (Laplace Mechanism)

Given any deterministic, real-valued function f on databases, define $\widehat{f}$ as

$$\widehat{f}(D) = f(D) + Y,$$

where $Y \leftarrow \mathrm{Lap}(\Delta f/\varepsilon)$. The Laplace mechanism is ε-differentially private.

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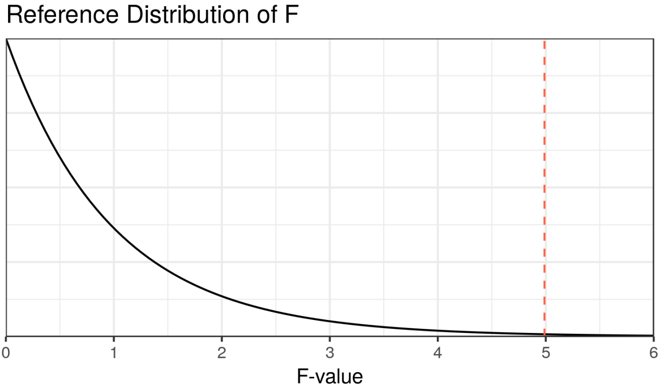

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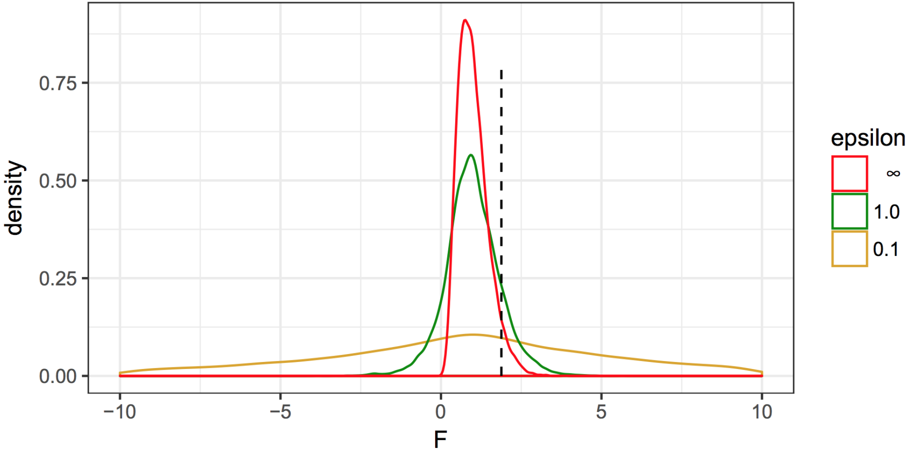

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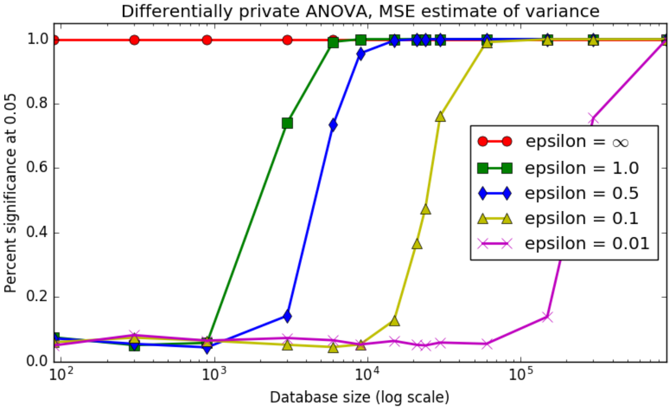

Fixed effect size: $\mu = [0.35, 0.5, 0.65], \sigma = 0.15.$

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- Problem: reference distribution depends on σ

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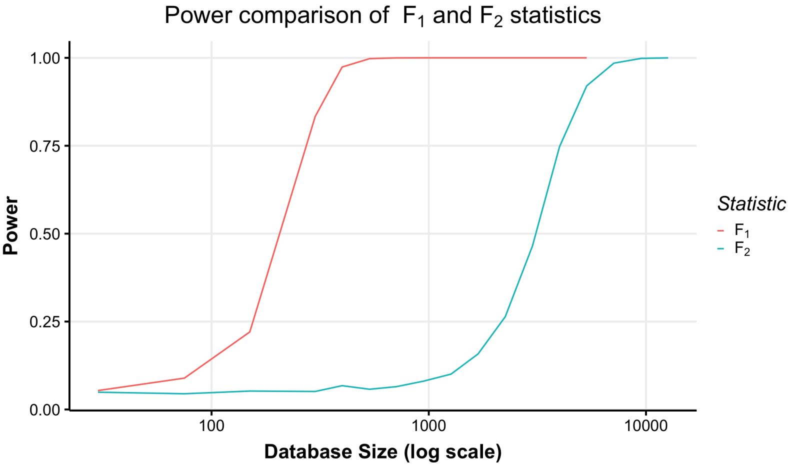
Proceed with simulation

Empirically verified to have valid p-values.



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