Improved Differentially Private Analysis of Variance

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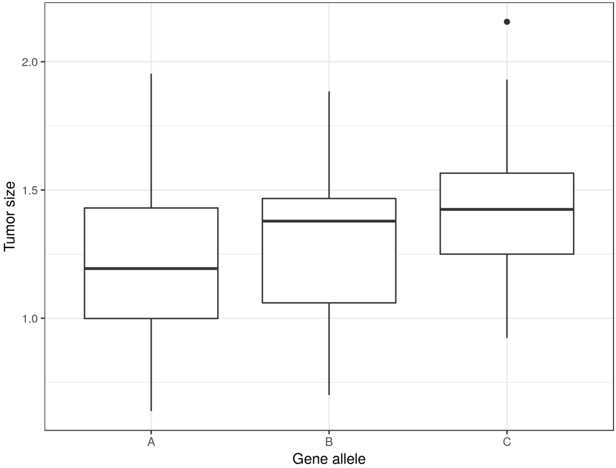
Reed College

Observed Data, $n = 30$



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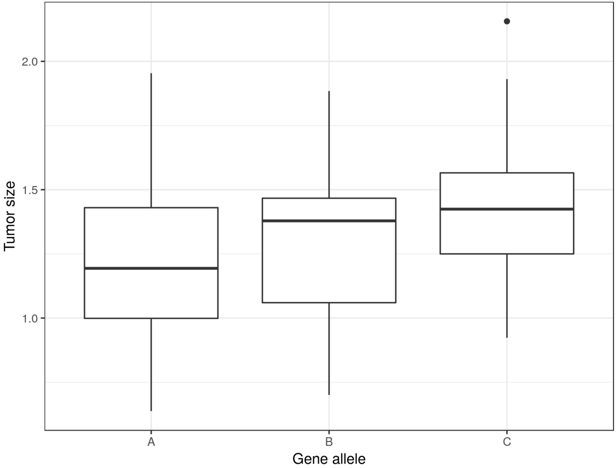

Are gene allele and tumor size dependent?

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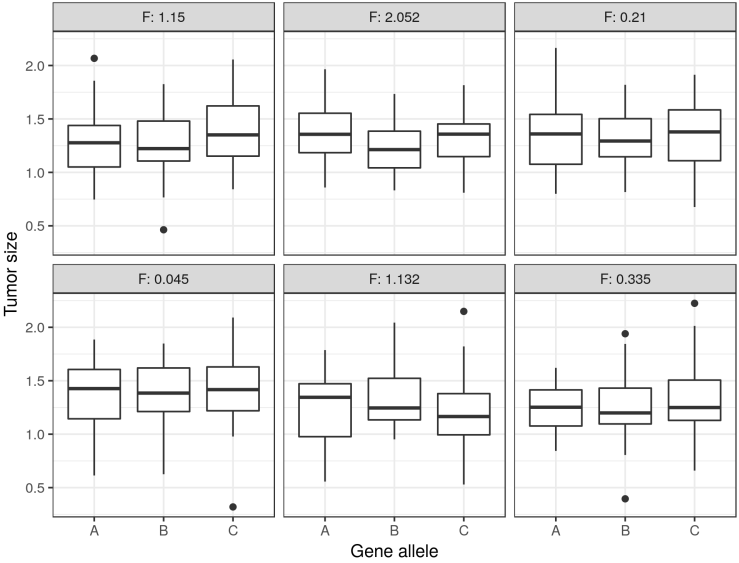
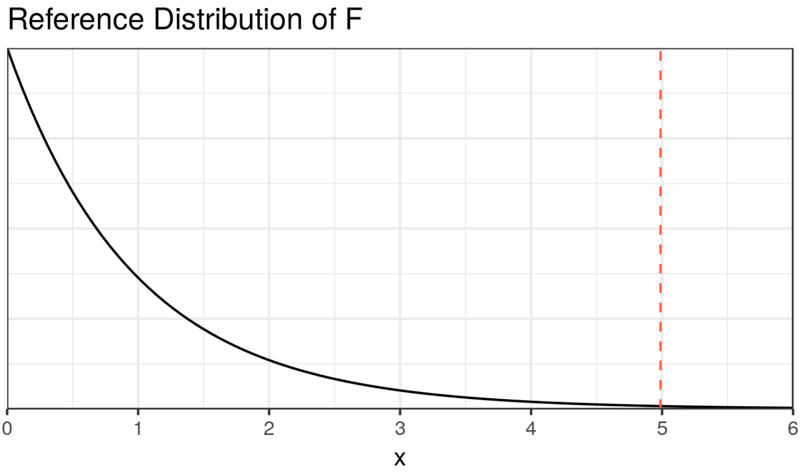
Simulate Random Data





Now do we think our data supports independence of gene allele and tumor size?

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Goal of any test statistic is achieving high power*.

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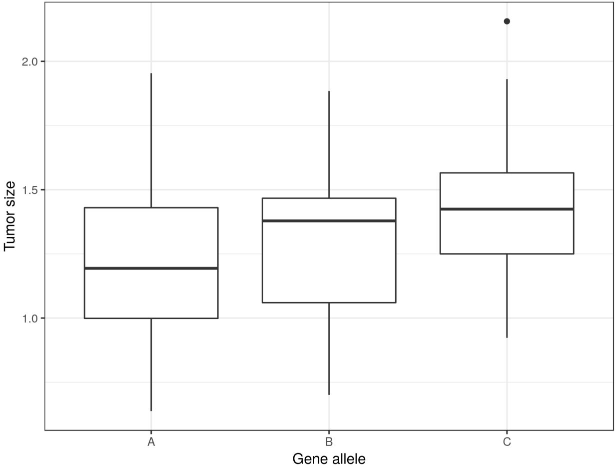

What if we want to keep this data private?

Differential privacy [DMNS06]

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A query f is ε-**differentially private** if for all neighboring databases D, D' and all output sets S

$$\Pr[f(D) \in S] \leq e^{\varepsilon} \Pr[f(D') \in S].$$

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Theorem (Post-processing)

If f is ε-differentially private then for any (randomized) function g, then if $h(D) = g(f(D)$, h is also ε-differentially private.

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Theorem (Composition)

If f is ε_1-differentially private and g is ε_2-differentially private then if $h(D) = (g(D), f(D))$, h is $(\varepsilon_1 + \varepsilon_2)$-differentially private.

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Theorem (Laplace Mechanism)

Given any deterministic, real-valued function f on databases, define $\widehat{f}$ as

$$\widehat{f}(D) = f(D) + Y,$$

where $Y \leftarrow \mathrm{Lap}(\Delta f/\varepsilon)$. The Laplace mechanism is ε-differentially private.

Related Works

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Problem: What is the reference distribution now?

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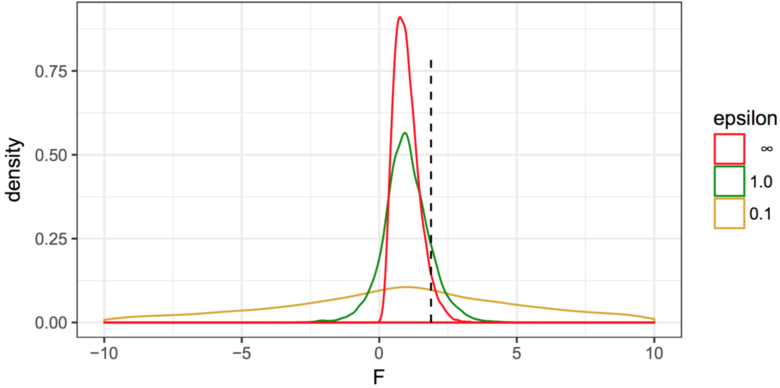

Public reference distribution gives inaccurate p-values.

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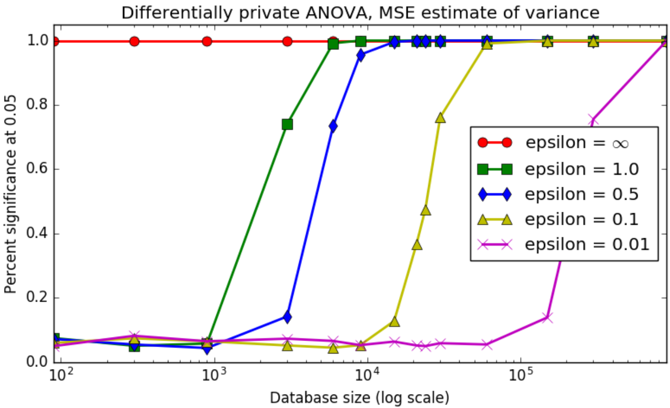

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- Much higher typical value

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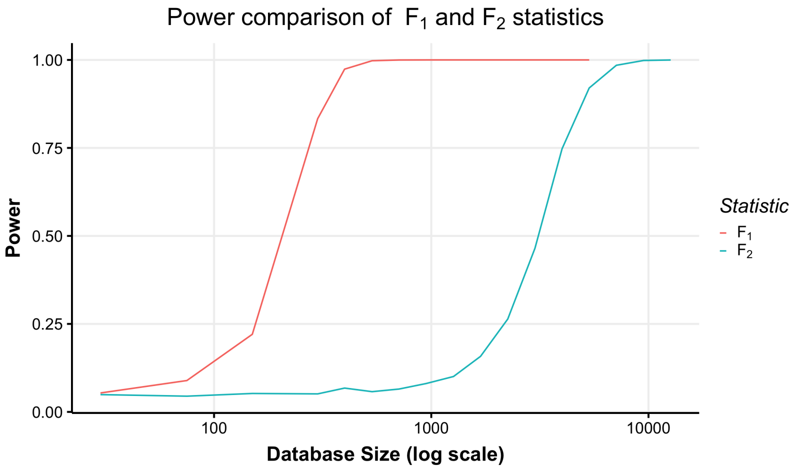
Proceed with simulation

Empirically verified to have valid p-values.


Power comparison of F_1 and F_2 statistics
1.00
0.75
0.50
0.25
0.00
Power
100
1000
10000
Database Size (log scale)
Statistic
F_1
F_2

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Perhaps a different exponent? Turns out 1 is optimal.

Conclusion

In the private framework . . .

- Not limited to closed-form test statistics
- Room for massive power gains
- Optimal public test $\not\Rightarrow$ optimal private test

. . . all of statistics is fair game.

New developments

[CKSBG19] gained another order of magnitude improvement. See *Differentially Private Nonparametric Hypothesis Testing* at CCS '19.

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Thank you