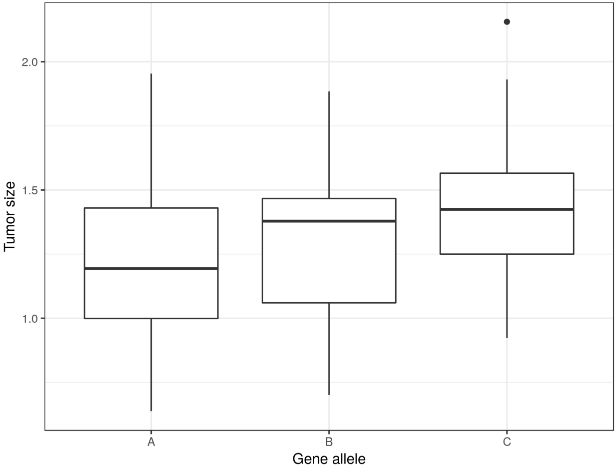
Improved Differentially Private Analysis of Variance

Marika Swanberg Ira Globus-Harris Iris Griffith
Anna Ritz Andrew Bray Adam Groce

REED COLLEGE

BOSTON UNIVERSITY

Observed Data, $n = 30$



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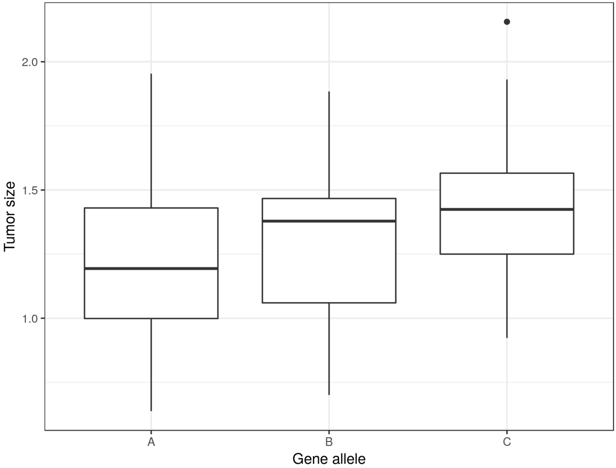

Are gene allele and tumor size dependent?

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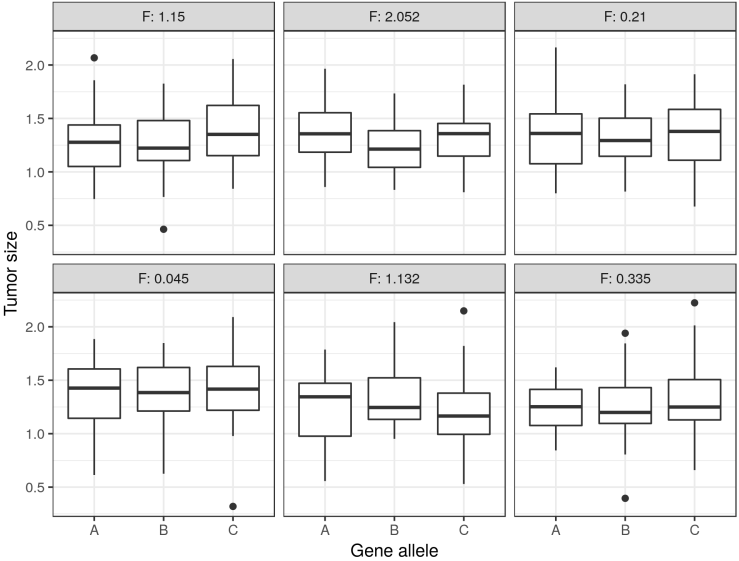
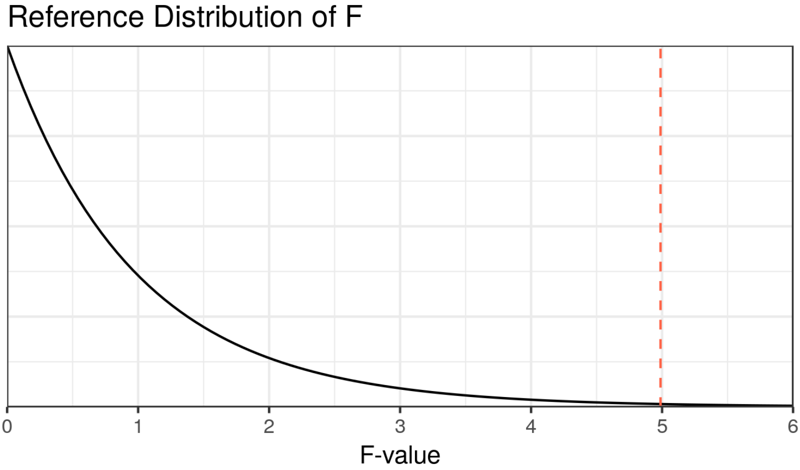
Simulate Random Data





Now do we think our data supports independence of gene allele and tumor size?

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Goal of any test statistic is achieving high power*.

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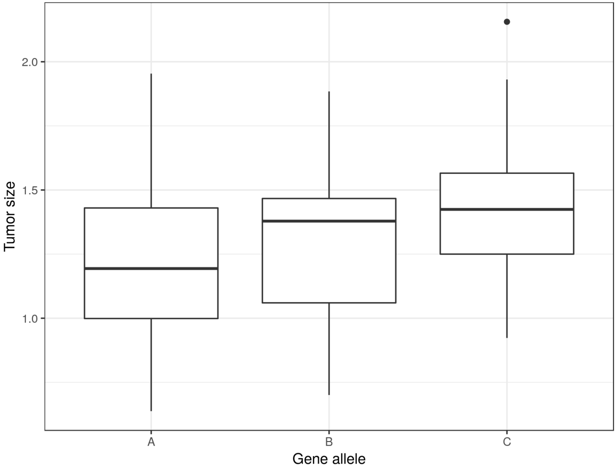

What if we want to keep this data private?

Differential privacy [DMNS06]

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A query f is ε-**differentially private** if for all neighboring databases D, D' and all output sets S

$$\Pr[f(D) \in S] \leq e^{\varepsilon} \Pr[f(D') \in S].$$

Properties of differential privacy [DMNS06]

Theorem (Post-processing)

If f is ε-differentially private then for any (randomized) function g, then if $h(D) = g(f(D)$, h is also ε-differentially private.

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Theorem (Composition)

If f is ε_1-differentially private and g is ε_2-differentially private then if $h(D) = (g(D), f(D))$, h is $(\varepsilon_1 + \varepsilon_2)$-differentially private.

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Theorem (Laplace Mechanism)

Given any deterministic, real-valued function f on databases, define $\widehat{f}$ as

$$\widehat{f}(D) = f(D) + Y,$$

where $Y \leftarrow \mathrm{Lap}(\Delta f/\varepsilon)$. The Laplace mechanism is ε-differentially private.

Related Works

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Problem: What is the reference distribution now?

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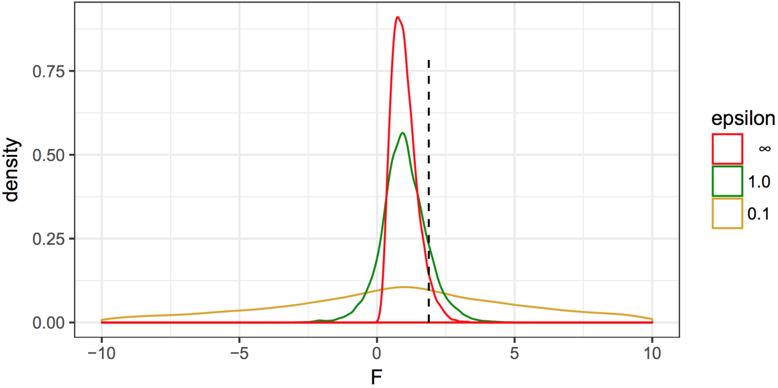

Public reference distribution gives inaccurate p-values.

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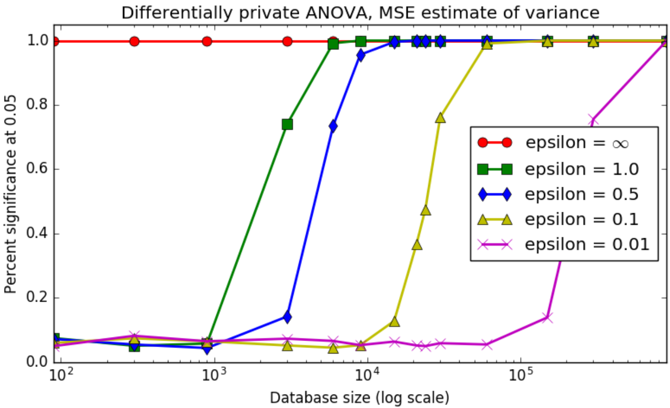

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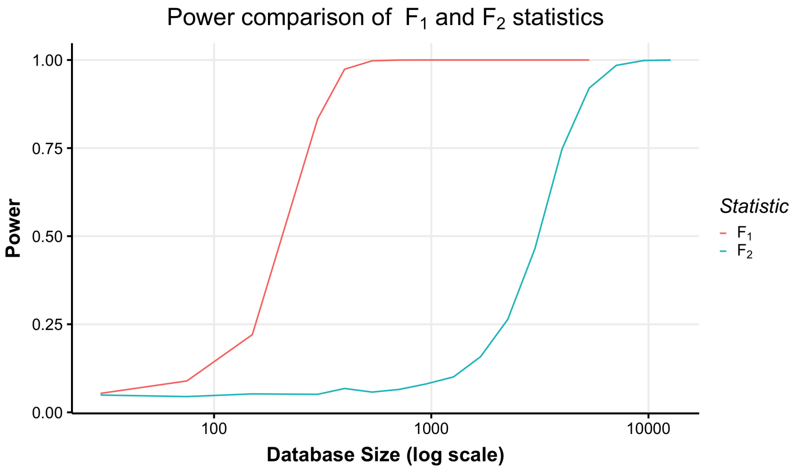
Proceed with simulation

Empirically verified to have valid p-values.


Power comparison of F_1 and F_2 statistics
Power
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0.75
0.50
0.25
0.00
100
1000
10000
Database Size (log scale)
Statistic
F_1
F_2

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Perhaps a different exponent? Turns out 1 is optimal.

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