Improved Differentially Private Analysis of Variance
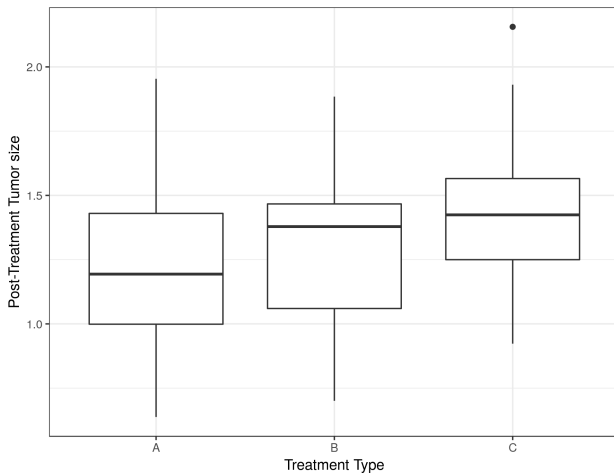
Marika Swanberg   Ira Globus-Harris   Iris Griffith
Anna Ritz   Andrew Bray   Adam Groce
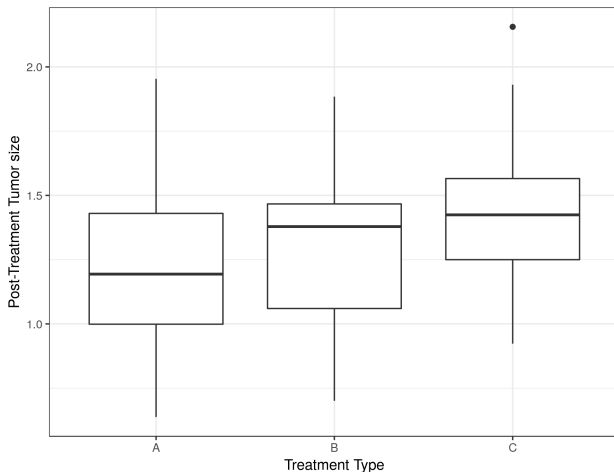
Observed Data, $n = 30$

Are treatment type and post-treatment tumor size dependent?

Metric for dependency

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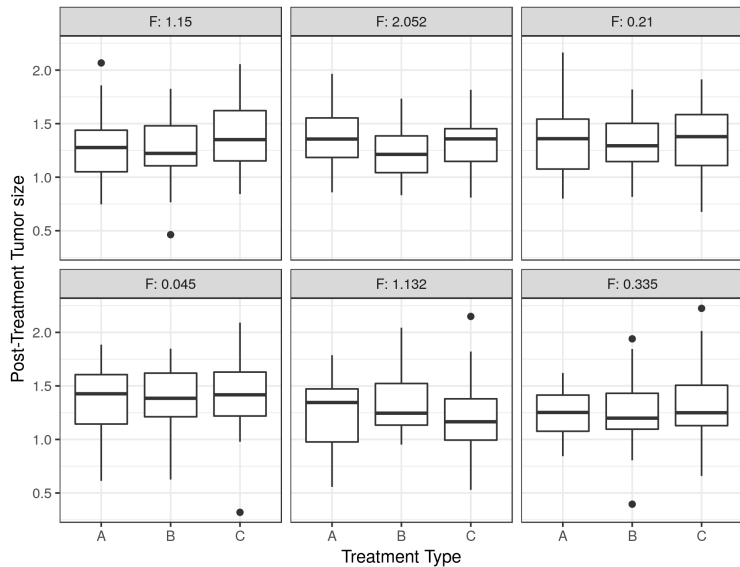$$F\text{-test} = \frac{\text{Variation between groups}}{\text{In-group variation}}$$

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Simulate Null Data

Reference Distribution of F
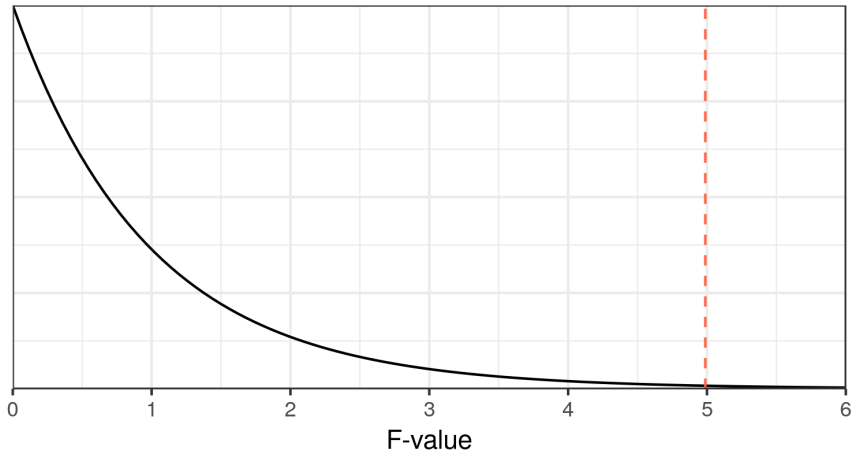
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Goal of any test statistic is achieving high power*.

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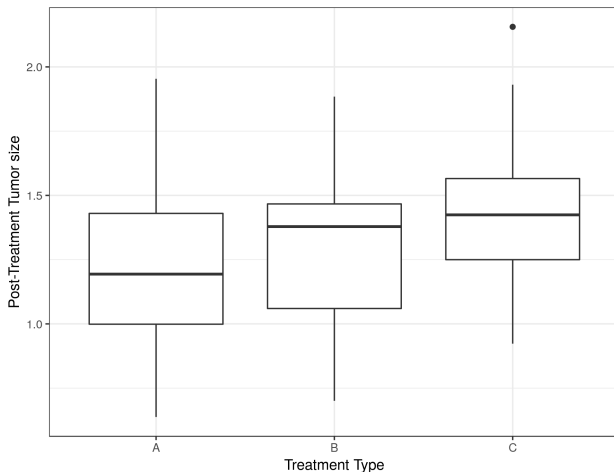
What if we want to keep this data private?

Differential privacy [DMNS06]

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A query f is ε-**differentially private** if for all neighboring databases D, D' and all output sets S

$$\Pr[f(D) \in S] \leq e^{\varepsilon} \Pr[f(D') \in S].$$

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Theorem (Composition)

If f is ε_1-differentially private and g is ε_2-differentially private then if $h(D) = (g(D), f(D))$, h is $(\varepsilon_1 + \varepsilon_2)$-differentially private.

Laplace mechanism

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Theorem (Laplace Mechanism)

Given any deterministic, real-valued function f on databases, define $\widehat{f}$ as

$$\widehat{f}(D) = f(D) + Y,$$

where $Y \leftarrow \mathrm{Lap}(\Delta f / \varepsilon)$. The Laplace mechanism is ε-differentially private.

Related Works

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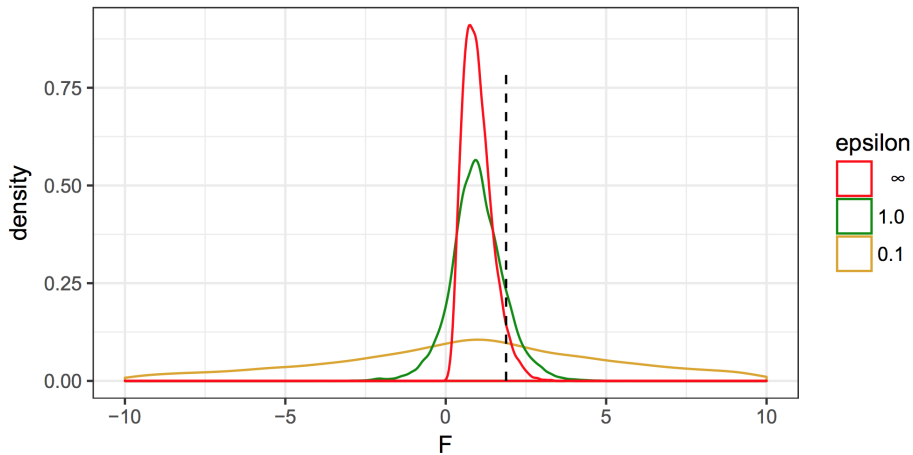
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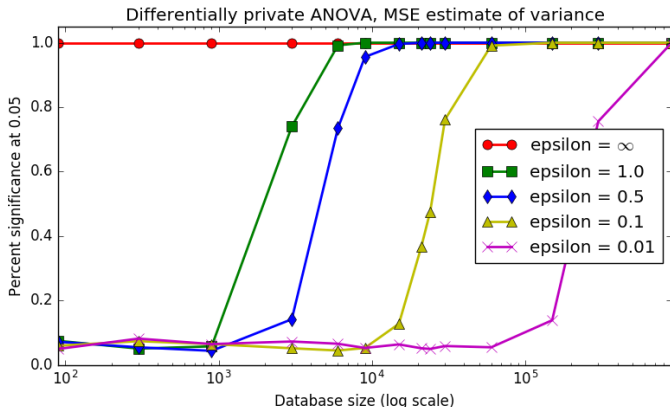
Private ANOVA [CBRG18]

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Problem: What is the reference distribution now?

Private ANOVA [CBRG18]

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Differentially private ANOVA, MSE estimate of variance

Legend:
- epsilon = ∞
- epsilon = 1.0
- epsilon = 0.5
- epsilon = 0.1
- epsilon = 0.01

x-axis: Database size (log scale)
y-axis: Percent significance at 0.05

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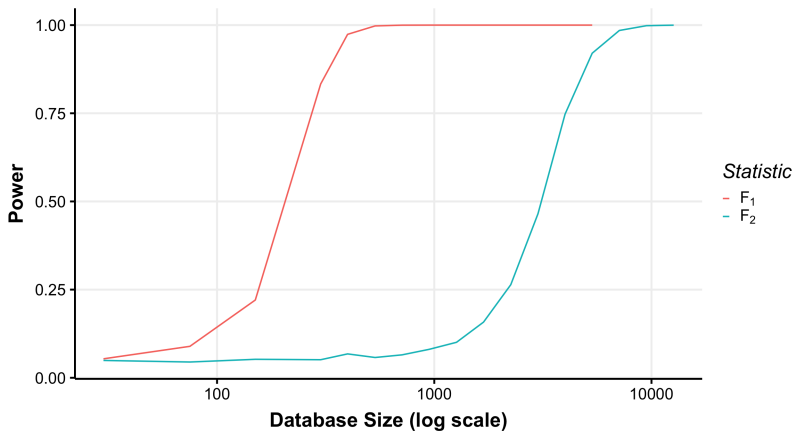
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Proceed with simulation

Empirically verified to have valid *p*-values.

Power comparison of F_1 and F_2 statistics

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Perhaps a different exponent? Turns out 1 is optimal.

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