Improved Differentially Private Analysis of Variance

Marika Swanberg   Ira Globus-Harris   Iris Griffith
Anna Ritz   Andrew Bray   Adam Groce

Observed Data, $n = 30$

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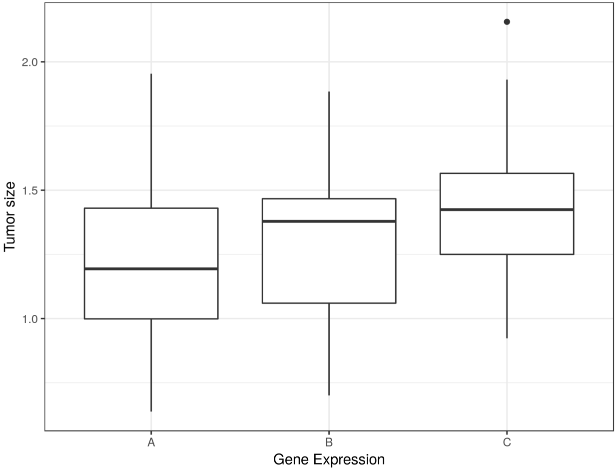

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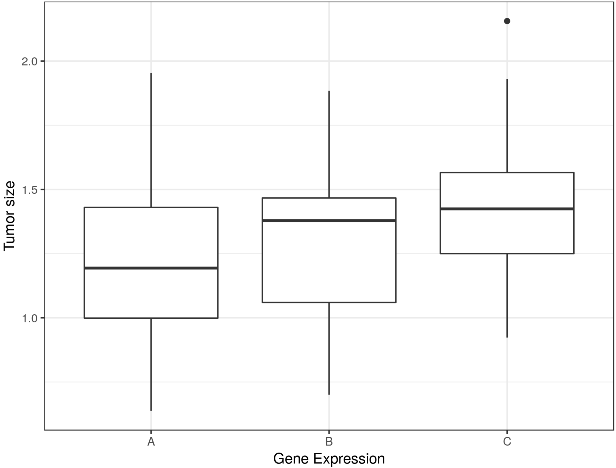

Random variation or real dependency?

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The metric:

$$SSA(D) = \sum_{j=1}^{k} n_j(\bar{y}_j - \bar{y})^2$$

$$SSE(D) = \sum_{i=1}^{n} (y_i - \bar{y}_{c_i})^2$$

$$F = \frac{SSA/(k-1)}{SSE/(n-k)}$$

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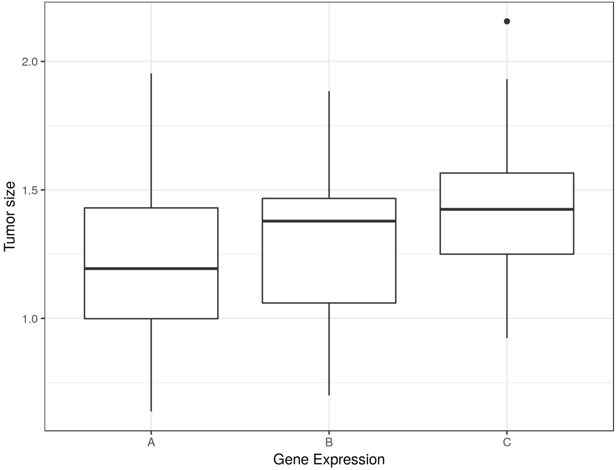

$$F = 4.988$$

Simulate Null Data

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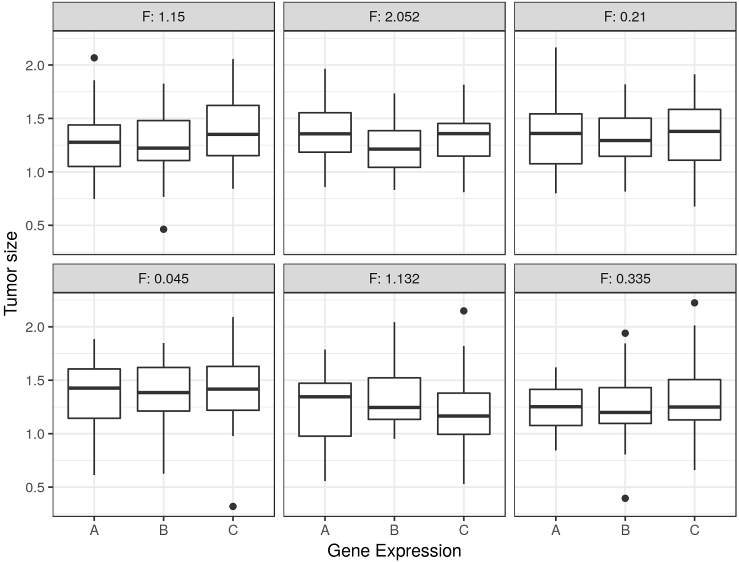

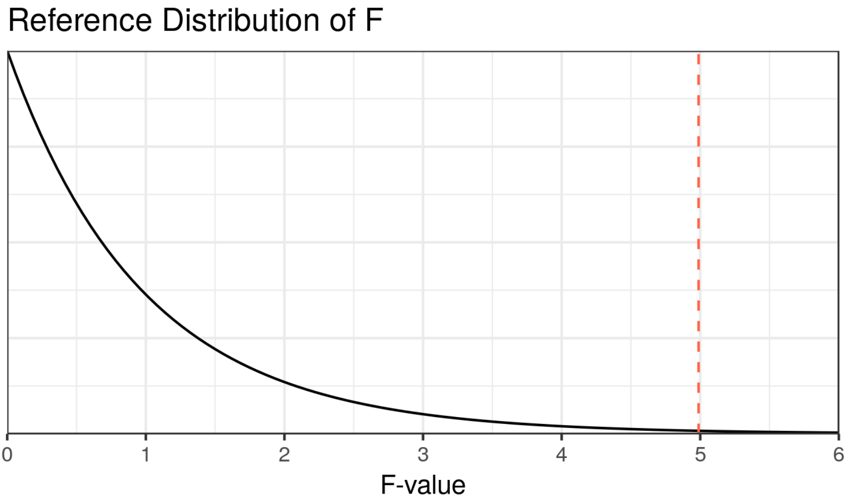
Reference Distribution of F
0
1
2
3
4
5
6
F-value

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Goal of any test statistic is achieving high power*.

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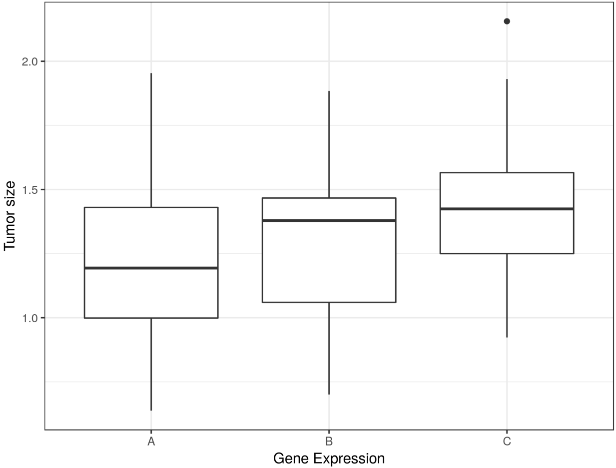

What if we want to keep this data private?

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A query f is ε-**differentially private** if for all neighboring databases D, D' and all output sets S

$$\Pr[f(D) \in S] \leq e^{\varepsilon} \Pr[f(D') \in S].$$

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If f is ε-differentially private then for any (randomized) function g, then if $h(D) = g(f(D))$, h is also ε-differentially private.

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Theorem (Composition)

If f is ε_1-differentially private and g is ε_2-differentially private then if $h(D) = (g(D), f(D))$, h is $(\varepsilon_1 + \varepsilon_2)$-differentially private.

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Theorem (Laplace Mechanism)

Given any deterministic, real-valued function f on databases, define $\widehat{f}$ as

$$\widehat{f}(D) = f(D) + Y,$$

where $Y \leftarrow \mathrm{Lap}(\Delta f/\varepsilon)$. The Laplace mechanism is ε-differentially private.

Related Works

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$$\widehat{SSE}(D) = SSE(D) + \text{Lap}\left(\frac{7}{\varepsilon/2}\right)$$

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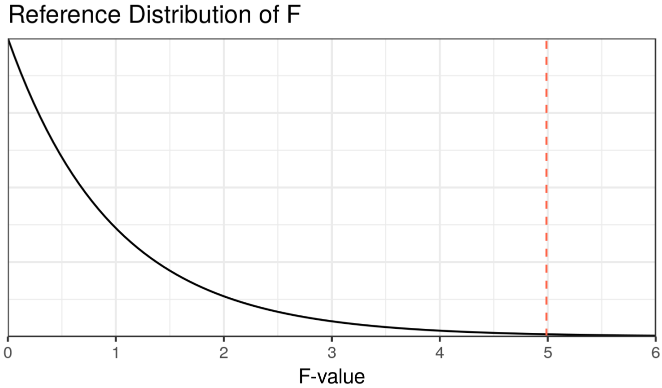

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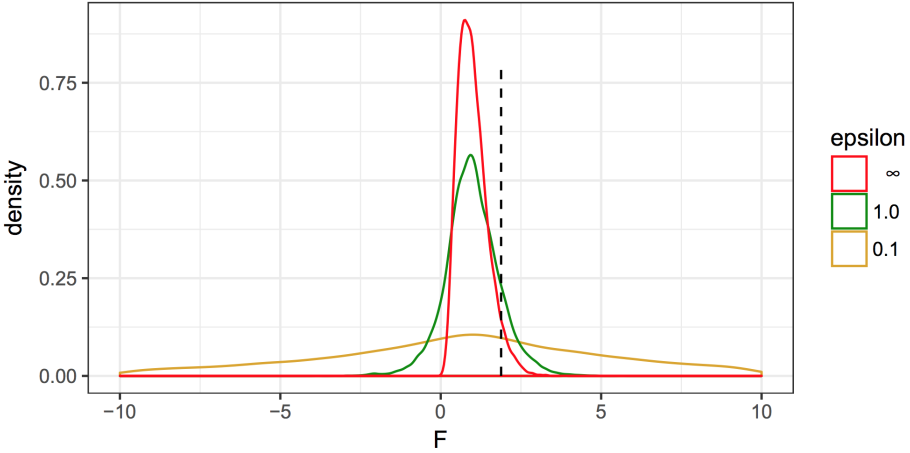

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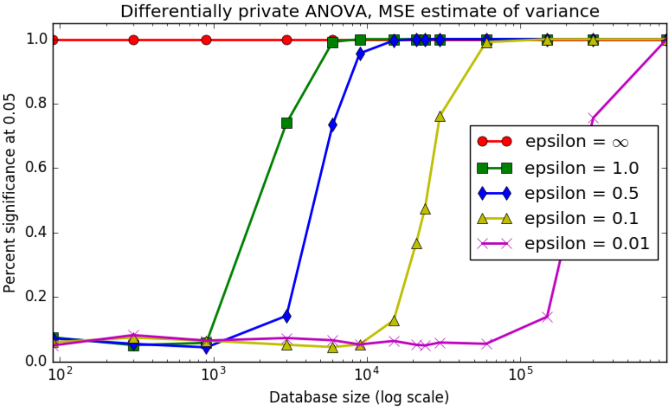

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- Problem: reference distribution depends on σ

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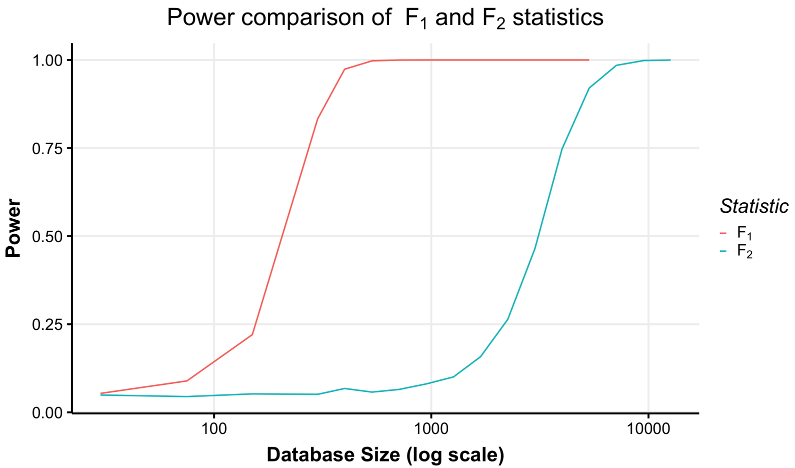
Proceed with simulation

Empirically verified to have valid p-values.



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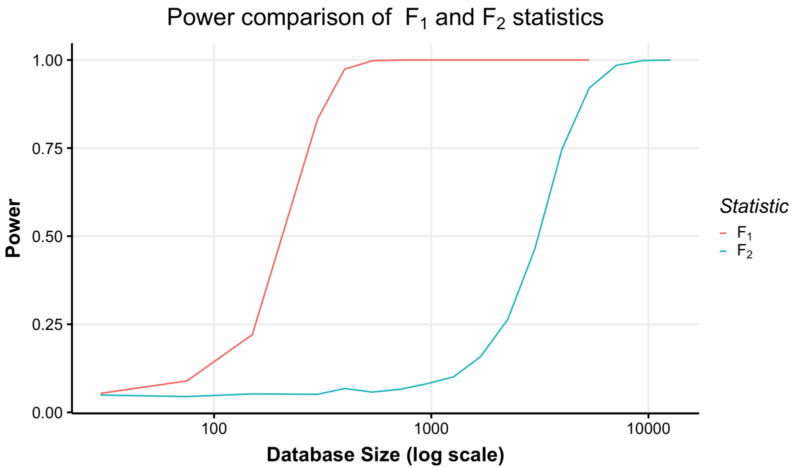

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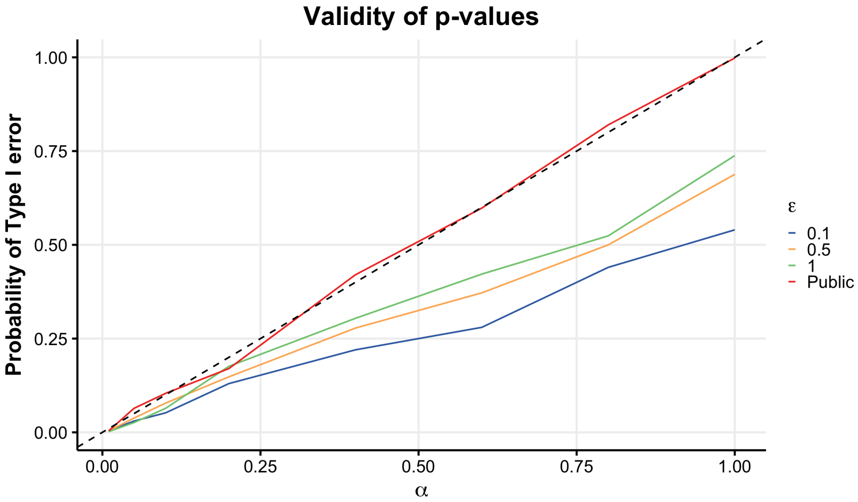
Validity of p-values
Probability of Type I error
1.00
0.75
0.50
0.25
0.00
0.00
0.25
0.50
0.75
1.00
α
ε
0.1
0.5
1
Public