Improved Differentially Private Analysis of Variance

Marika Swanberg Ira Globus-Harris Iris Griffith
Anna Ritz Andrew Bray Adam Groce

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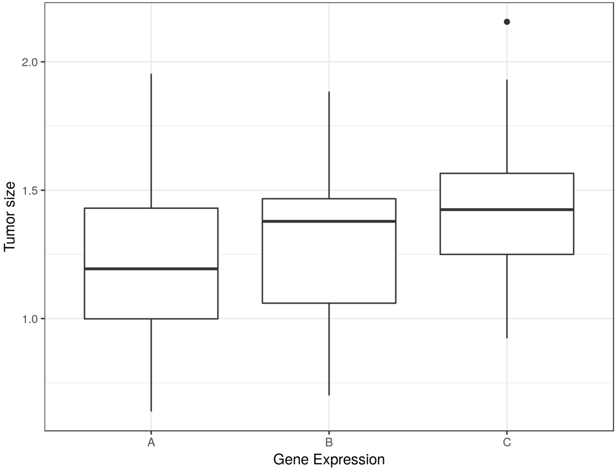

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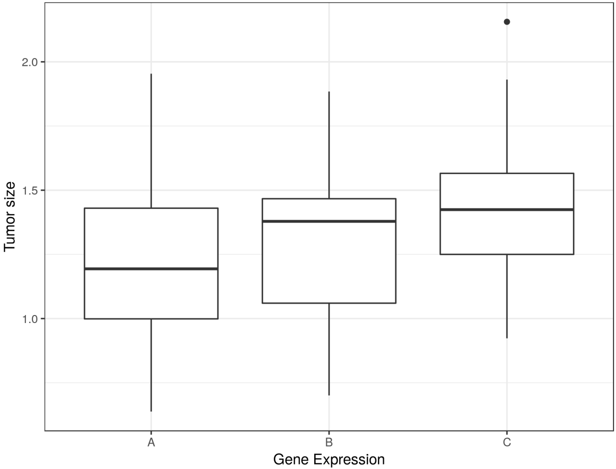

Random variation or real dependency?

Analysis of Variance

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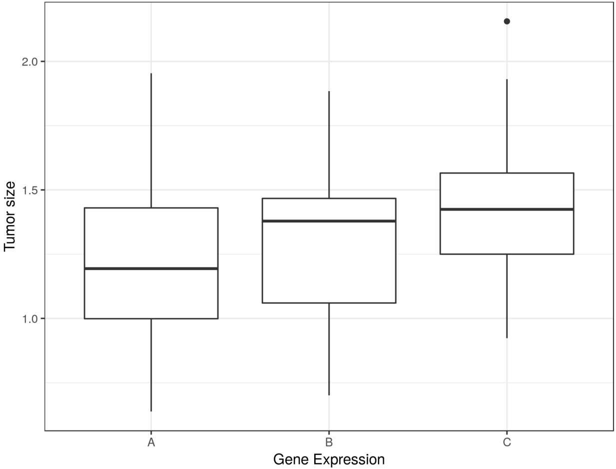
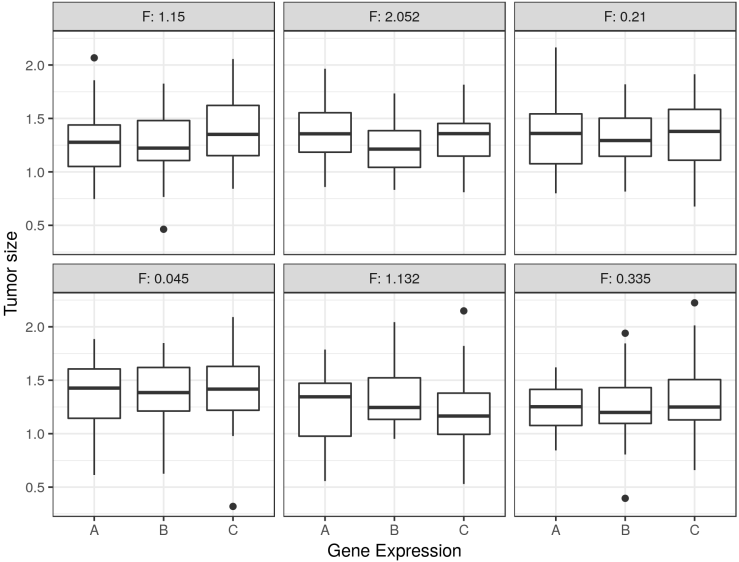
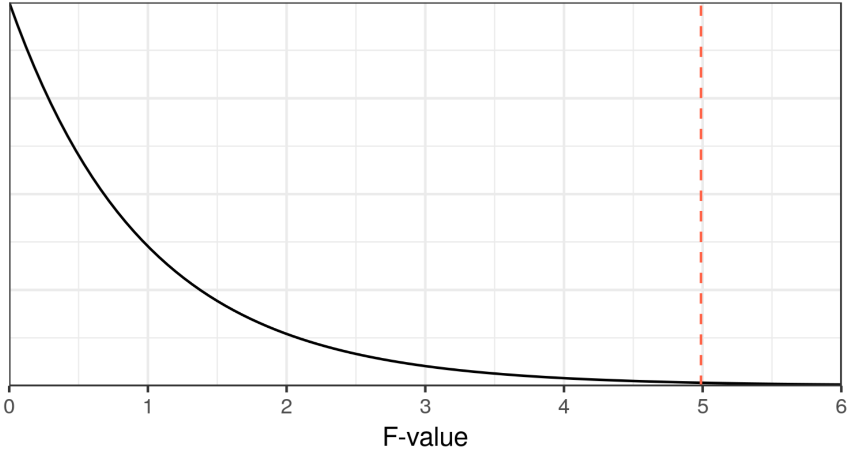


$$F = 4.988$$

Simulate Null Data



Reference Distribution of F



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Goal of any test statistic is achieving high power*.

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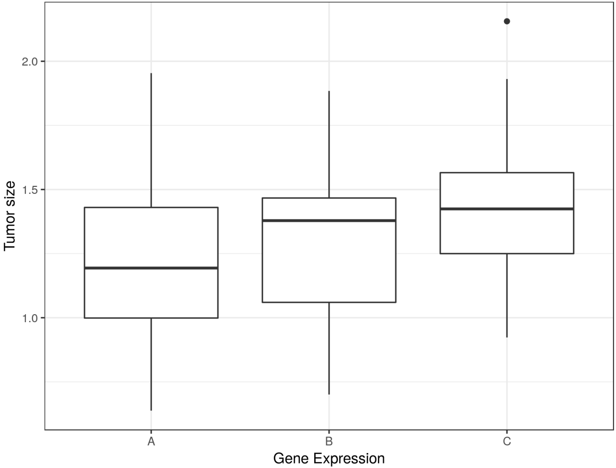

What if we want to keep this data private?

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A query f is ε-**differentially private** if for all neighboring databases D, D' and all output sets S

$$\Pr[f(D) \in S] \leq e^{\varepsilon} \Pr[f(D') \in S].$$

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Theorem (Post-processing)

If f is ε-differentially private then for any (randomized) function g, then if $h(D) = g(f(D))$, h is also ε-differentially private.

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Theorem (Composition)

If f is ε_1-differentially private and g is ε_2-differentially private then if $h(D) = (g(D), f(D))$, h is $(\varepsilon_1 + \varepsilon_2)$-differentially private.

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Theorem (Laplace Mechanism)

Given any deterministic, real-valued function f on databases, define $\widehat{f}$ as

$$\widehat{f}(D) = f(D) + Y,$$

where $Y \leftarrow \mathrm{Lap}(\Delta f/\varepsilon)$. The Laplace mechanism is ε-differentially private.

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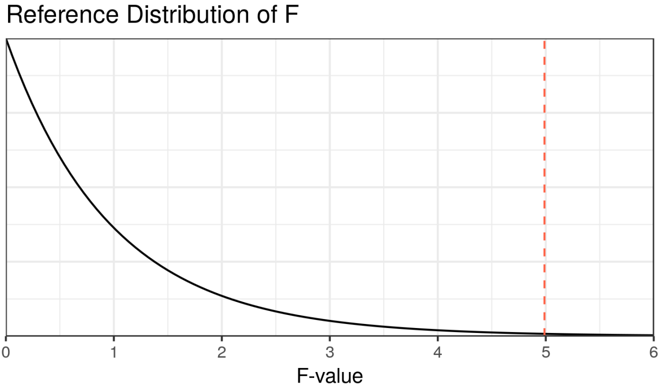

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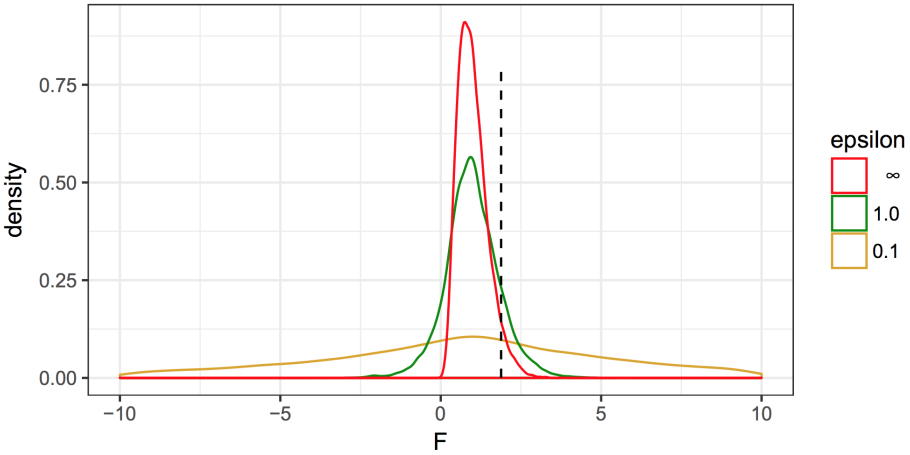

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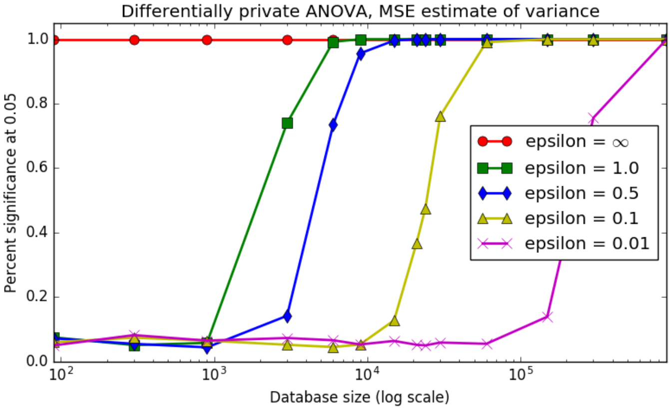

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- Problem: reference distribution depends on σ

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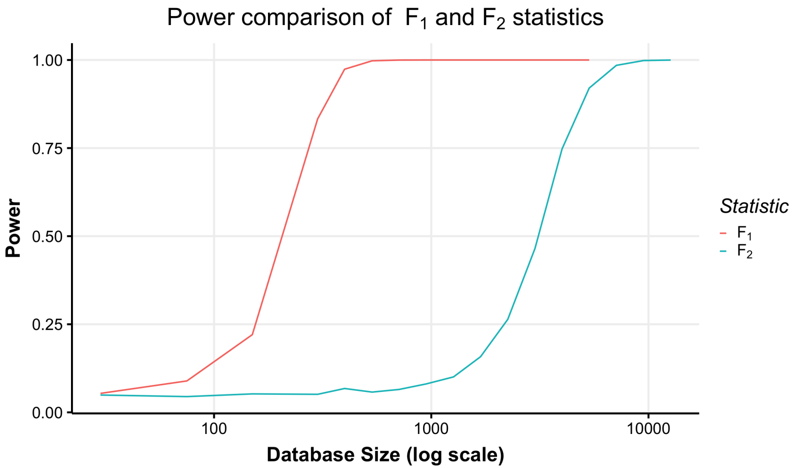
Proceed with simulation

Empirically verified to have valid p-values.



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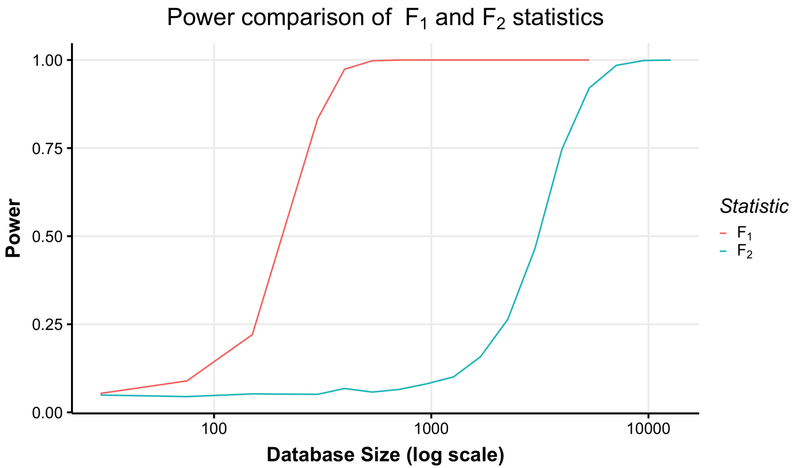

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