Improved Differentially Private Analysis of Variance

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Reed College

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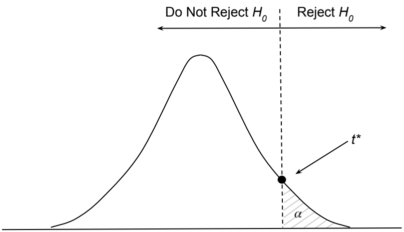
3. Compare p to a preset value α (usually .05). Reject H_0 if $p < \alpha$.



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Definition (Power)

The **power** of a hypothesis test is the probability it rejects H_0. It depends on the alternate distribution H_A and n.

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A query f is ε-**differentially private** if for all neighboring databases D, D' and all output sets S

$$\Pr[f(D) \in S] \leq e^{\varepsilon} \Pr[f(D') \in S].$$

Properties of differential privacy [DMNS06]

Theorem (Post-processing)

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Theorem (Composition)

If f is ε_1-differentially private and g is ε_2-differentially private then if $h(D) = (g(D), f(D))$, h is $(\varepsilon_1 + \varepsilon_2)$-differentially private.

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Theorem (Laplace Mechanism)

Given any deterministic, real-valued function f on databases, define $\widehat{f}$ as

$$\widehat{f}(D) = f(D) + Y,$$

where $Y \leftarrow \mathrm{Lap}(\Delta f/\varepsilon)$. The Laplace mechanism is ε-differentially private.

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  - Main point of improvement

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- Empirically checked: good enough, type 1 error rate bounded by α

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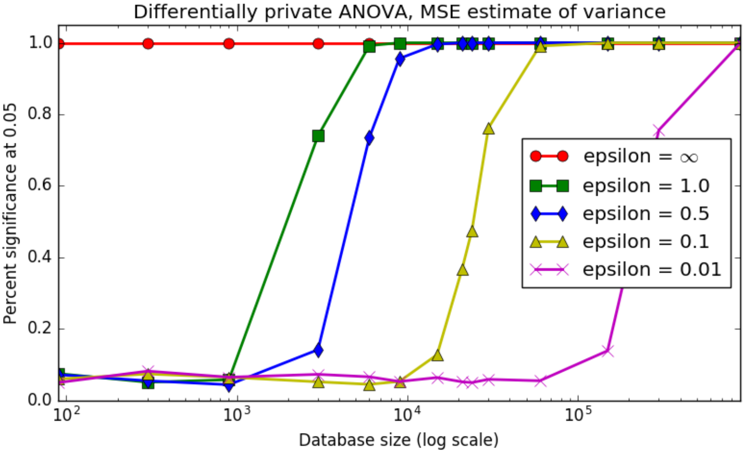

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- Much higher typical value

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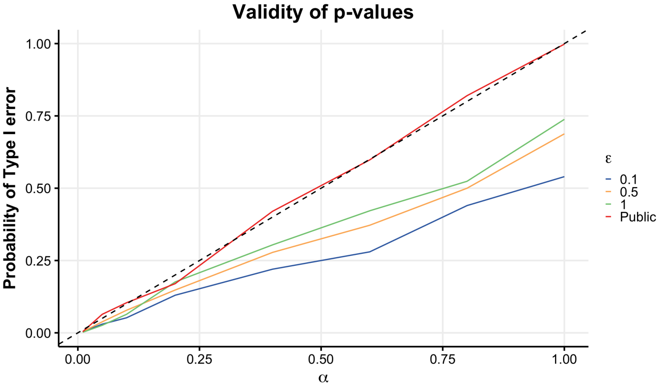
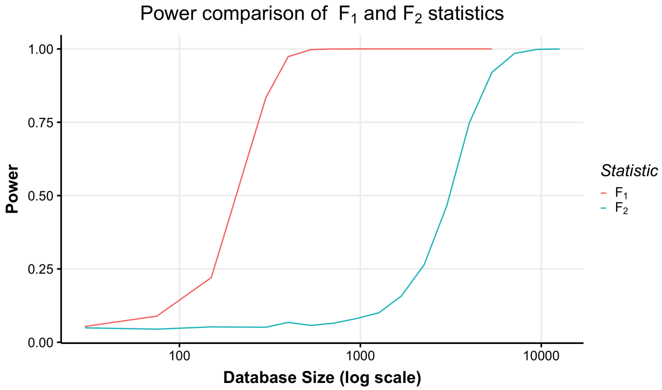
$$\hat{\sigma} = \sqrt{\pi/2} \cdot \frac{\widehat{SE}}{(N-k)}$$

Validity of p-values



Power of F_1 test



Power comparision at $\varepsilon = 1$. F achieves 80% power with 4500 observations. F_1 requires 300.

Further Optimization

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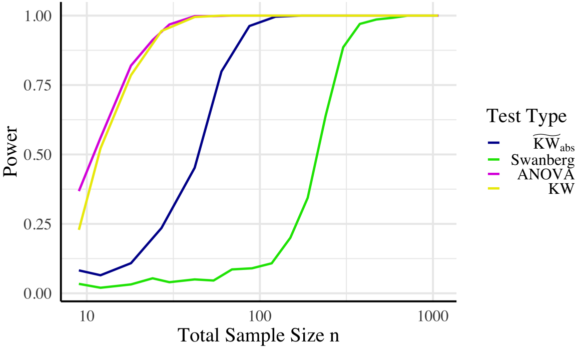

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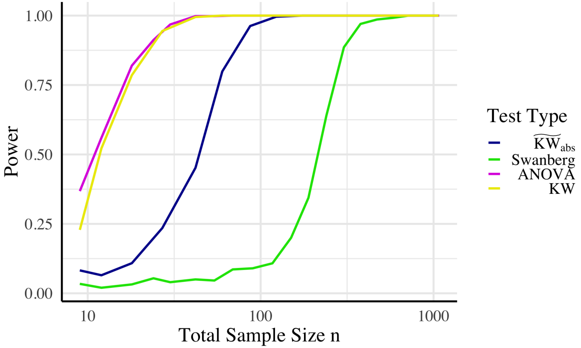

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Thank you