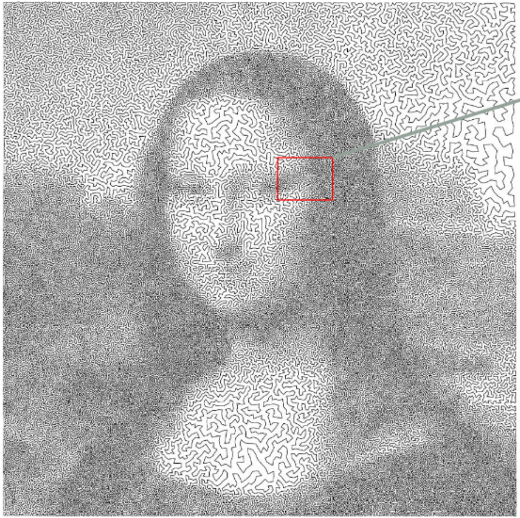
Traveling Salesman Problem (TSP)

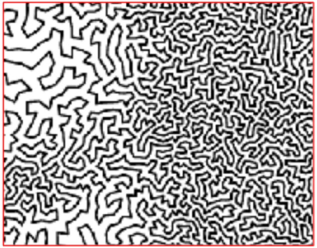
- TSP introduction
- NP-hardness
- Approximation algorithm

Traveling Salesman Problem

- Given a set of cities along with the cost of travel between them, find the cheapest route visiting all cities and returning to starting city
- Given a complete graph $G = (V, E)$ with integer cost $c(u, v)$ for each edge (u, v), find a min-cost H cycle
- Metric TSP: costs satisfy triangle inequality: $c(u, w) \leq c(u, v) + c(v, w)$
- Euclidean TSP: cities are points in the Euclidean space, costs are distances

Mona Lisa TSP





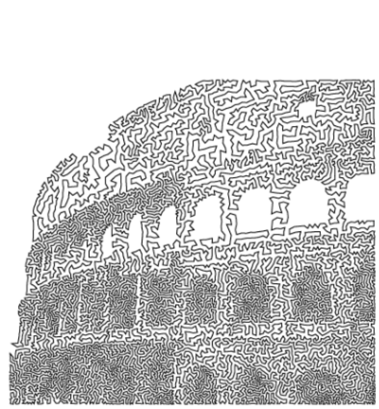
The current best known results for the Mona Lisa TSP are:

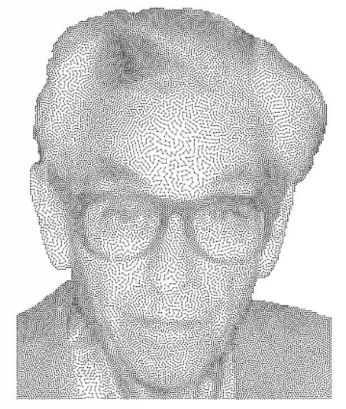
Tour: 5,757,191
Lower Bound: 5,757,084
Gap: 107 (0.0019%)

$1,000 prize to the first person to find a tour shorter than 5,757,191.

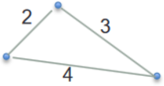
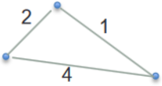
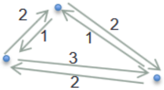
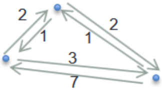
Source:
http://www.math.uwaterloo.ca/tsp/data/ml/monalisa.html

Mona Lisa TSP





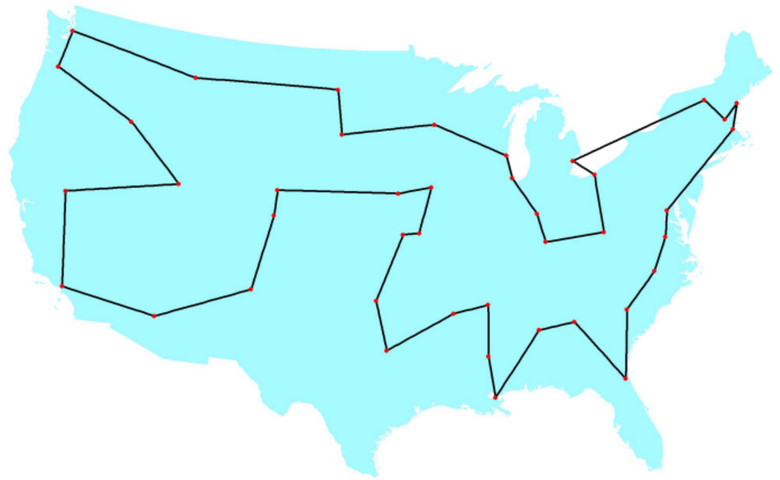
TSP variants

TSP	Metric (Δ-inequality)	Non-metric
Symmetric	 1.5-approx.	 No α-approx.
Asymmetric	 O(log n) -approx.	 No α-approx.

There is a PTAS for the Euclidean TSP

History of TSP

- Dantzig, Fulkerson and Johnson found an optimal tour through 42 cities



http://www.math.uwaterloo.ca/tsp/history/img/dantzig_big.html

DFJ solution

- Create a linear program: variable $x(u, v) = 1$ iff tour goes between u, v
- Solve the LP
- If the solution is integral and forms a tour, done
- Otherwise, find a new constraint to add (cutting plane)

Hardness of TSP

Theorem

$\forall \rho > 1$ finding a ρ-optimal TSP tour is NP-hard

Proof

Idea: reduce Hamiltonian Cycle problem to ρ-TSP

- Let $G = (V, E)$ be an instance of the hamiltonian-cycle problem
- Let $G' = (V, E')$ be a complete graph with costs $c_e = 1$ if $e \in E$, $c_e = \rho|V| + 1$ otherwise
- If G has a H-cycle, then G' contains a tour of cost $|V|$
- Otherwise, any tour T must use some edge $\notin E$
- $c(T) \geq (\rho|V| + 1) + (|V| - 1) = (\rho + 1)|V| > \rho|V|$
- ρ-optimal TSP tour in G' computes H-cycle in G (if one exists)

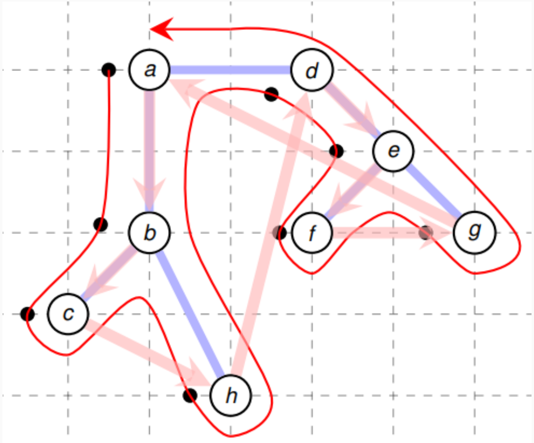
Metric TSP: algorithm

Idea: compute an min spanning tree (MST), create a tour based on it

```
APPROX-TSP-TOUR(G, c)
select a vertex r ∈ G.V to be a "root" vertex
compute a minimum spanning tree T for G from root r
        using MST-PRIM(G, c, r)
let H be a list of vertices, ordered according to when they are first visited
        in a preorder tree walk of T
return the hamiltonian cycle H
```

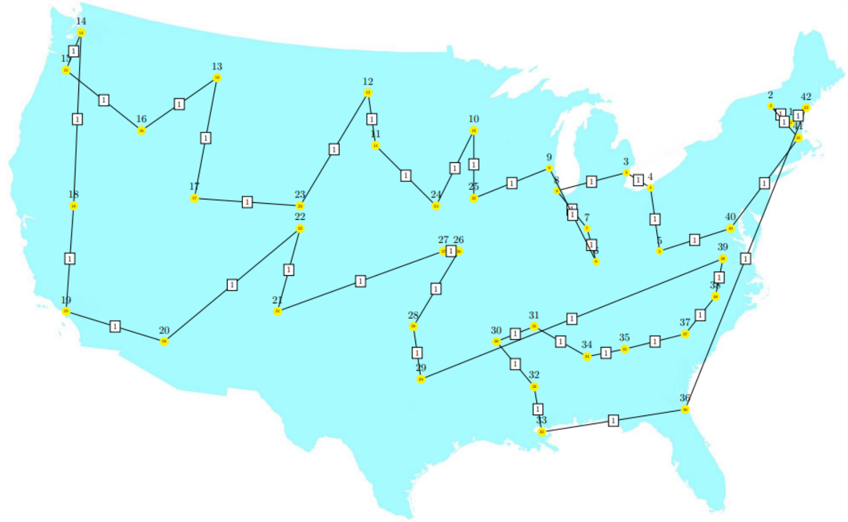
Metric TSP: algorithm

- Compute MST
- Perform preorder walk on MST
- Return list of vertices according to the preorder tree walk



Practical performance

Cost=921, min-cost=699



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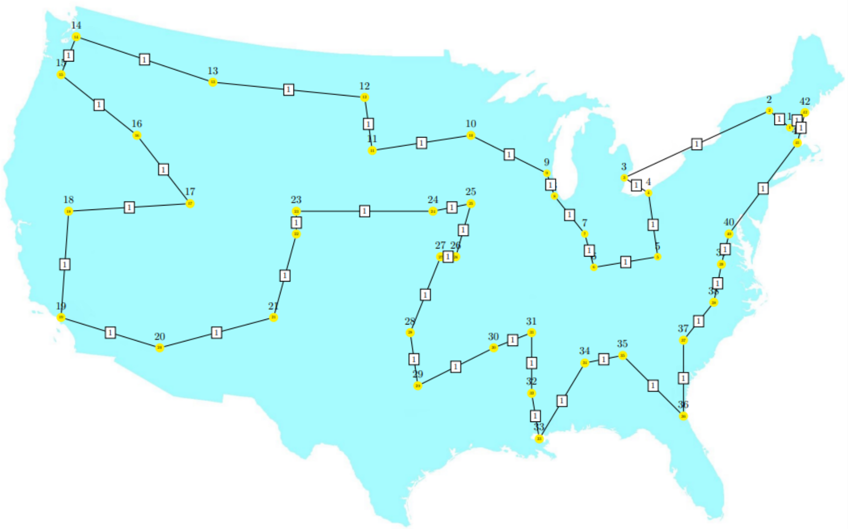


Proof of the Approximation Ratio

theorem

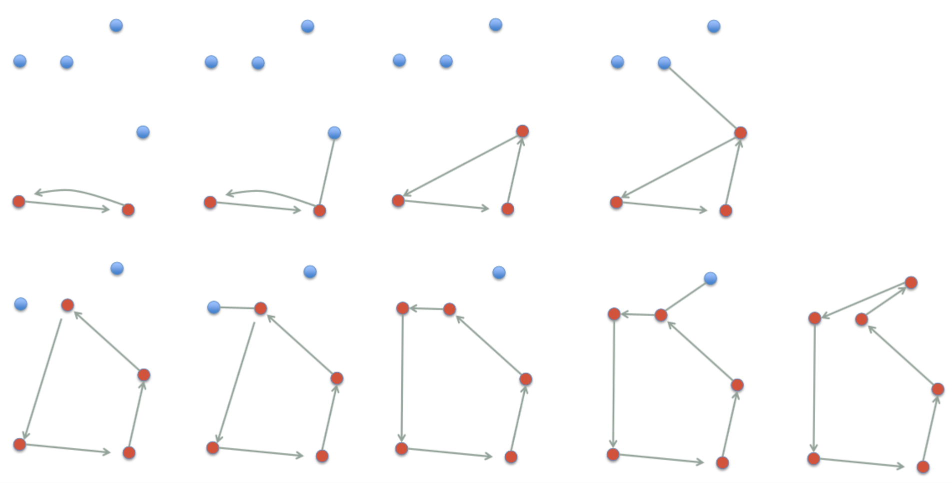
The algorithm produces a 2-optimal TSP tour for metric TSP

proof

- Consider the optimal tour H^* and remove an arbitrary edge
- yields a spanning tree T: $c(T) \leq c(H^*)$
- Let W be the walk of T_{min}: $c(W) \leq 2c(T_{min}) \leq 2c(T) \leq 2c(H^*)$
- Deleting duplicate vertices from W yields a tour H with smaller cost, by triangle inequality: $c(H) \leq c(W) \leq 2c(H^*)$

Another algorithm: nearest neighbor

- Start with any vertex
- Keep adding the nearest vertex



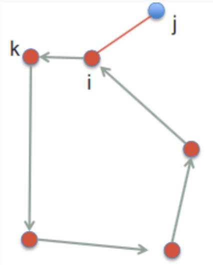
Nearest neighbor: approximation ratio

theorem

The algorithm produces a 2-optimal TSP tour

proof

- By triangle inequality: $c_{jk} \leq c_{ji} + c_{ik}$, hence $c_{jk} - c_{ik} \leq c_{ji}$
- Cost in this step: $c_{ij} + c_{jk} - c_{ik} \leq 2c_{ij}$
- Total cost ≤ 2cost(MST) $\leq$ 2OPT

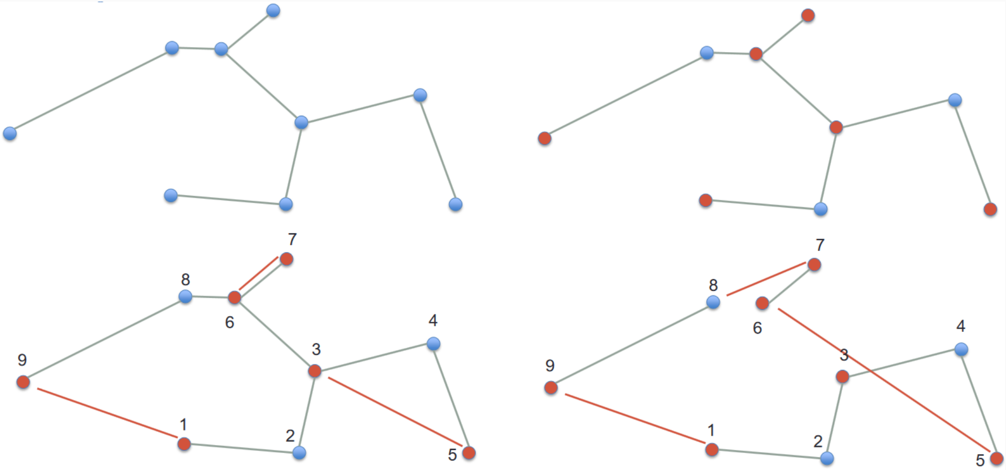


Algorithm with a better approximation ratio

Christofides algorithm

- Find an MST T
- Find a minimum matching M for the odd-degree vertices in T
- Add M to T
- Find an Euler tour
- Cut short

Christofides algorithm



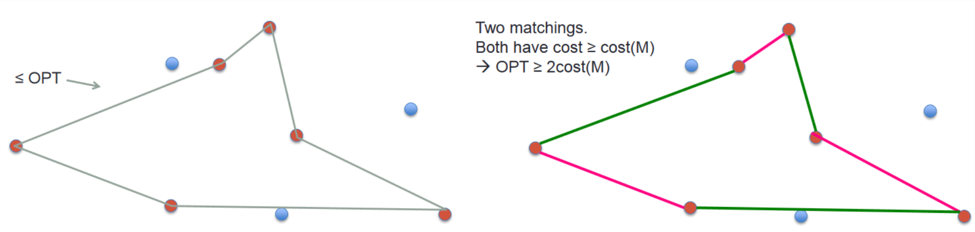
Christofides algorithm: approximation ratio

Theorem

Christofides algorithm produces a 1.5-optimal TSP tour

Proof

- $c(T) < OPT$, we only need to prove $c(M) \leq OPT/2$



TSP

ALGORITHMS

Computer Scientists Break Traveling Salesperson Record

22

After 44 years, there's finally a better way to find approximate solutions to the notoriously difficult traveling salesperson problem.

Now Karlin, Klein and Oveis Gharan have proved that an algorithm devised a decade ago beats Christofides' 50% factor, though they were only able to subtract 0.2 billionth of a trillionth of a trillionth of a percent. Yet this minuscule improvement breaks through both a theoretical logjam and a psychological one. Researchers hope that it will open the floodgates to further improvements.

Nathan Klein (left), a graduate student at the University of Washington, and his advisers, Anna Karlin and Shayan Oveis Gharan.

—

Flora Hollifield; from "Embracing Frustration," with permission from Microsoft; courtesy of Shayan Gharan.

"This is a result I have wanted all my career," said David Williamson of Cornell University, who has been studying the traveling salesperson problem since the 1980s.

Christos Papadimitriou: The TSP is not a problem. It's an addiction.

Excercise

Solve the multiple-salesmen variant of TSP