线性代数 (Linear Algebra)

第四章 Vector Spaces

§ 4.4 Coordinate Systems

坐标系统

衡益

2021 年 11 月 25 日，中山大学南校区

坐标系统

坐标系统

定理（唯一表示定理）

令B={ b_1， …， b_n }是向量空间 V的一个基，
则对 V中每个向量x, 存在唯一的一组数 $c_1, \cdots, c_n$,
使得：$x = c_1 b_1 + \cdots + c_n b_n$

坐标系统

Proof : Since β spans *V*, there exist scalars such that
$\mathbf{x} = c_1\mathbf{b_1}+\ldots+c_n\mathbf{b_n}$. Suppose x also has the representation

$$\mathbf{x} = d_1\mathbf{b_1}+\ldots+ d_n\mathbf{b_n}$$

for scalars d1,…,dn. Then, subtracting, we have

$$0 = \mathbf{x}-\mathbf{x} = (c_1-d_1)\mathbf{b_1}+\ldots+(c_n-d_n)\mathbf{b_n} \qquad (2)$$

Since β is linearly independent, the weights in (2) must
be all zero. That is , $c_j = d_j$ for $1 \le j \le n$.

坐标系统

定义　假设集合B={ b_1，…，b_n }是 V的一个基，x在 V中，x相对于基B的坐标(或x的B-坐标)是使得$x = c_1 b_1 + \cdots + c_n b_n$的权$c_1, \cdots, c_n$.

若$c_1, \cdots, c_n$是x的B-坐标，则$\mathbb{R}^n$中的向量

$$[x]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$ 是x(相对于B)的坐标向量，

映射$x \to [x]_B$ 称为（由B确定的）坐标映射.

坐标系统

- Example: The entries in the vector $x = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$ are the coordinates of x relative to the standard basis $\varepsilon = (e_1 \quad e_2)$, since

$$\begin{bmatrix} 1 \\ 6 \end{bmatrix} = 1 \bullet \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 6 \bullet \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1 \bullet e_1 + 6 \bullet e_2$$

$\varepsilon = \{e_1 \quad e_2\}$，则$[x]_\varepsilon = x$

坐标的几何意义

坐标的几何意义

举例：

标准向量$\varepsilon=\begin{pmatrix} e_1 & e_2 \end{pmatrix} \rightarrow \begin{pmatrix} b_1 & b_2 \end{pmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$

向量$x = \begin{bmatrix} 1 \\ 6 \end{bmatrix} \Rightarrow \begin{bmatrix} x \end{bmatrix}_B = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$

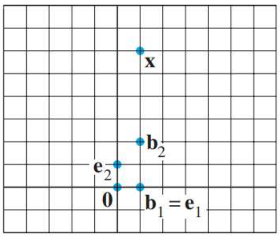


FIGURE 1 Standard graph paper.

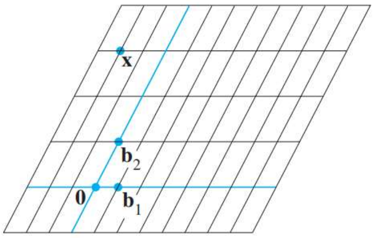


FIGURE 2 $\mathcal{B}$-graph paper.

$\mathbb{R}^n$ 中的坐标

$\mathbb{R}^n$ 中的坐标

- Example : Let $b_1=\begin{bmatrix}2\\1\end{bmatrix}, b_2=\begin{bmatrix}-1\\1\end{bmatrix}, x=\begin{bmatrix}4\\5\end{bmatrix}, \mathcal{B}=\{b_1,b_2\}$

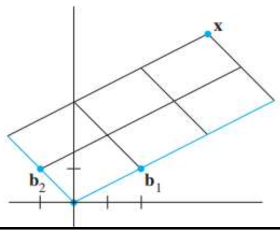
  Find the coordinate vector $\boxed{[\mathbf{x}]_\beta \text{ of } \mathbf{x} \text{ relative to } \boldsymbol{\beta}}$

- Solution： The **β**-coordinate c_1, c_2 of **x** satisfy

$$\underset{b_1}{c_1\begin{bmatrix}2\\1\end{bmatrix}}+\underset{b_2}{c_2\begin{bmatrix}-1\\1\end{bmatrix}}=\underset{x}{\begin{bmatrix}4\\5\end{bmatrix}} \quad \text{or} \quad \begin{array}{cc}\begin{bmatrix}2 & -1\\1 & 1\end{bmatrix} \\ b_1 \;\; b_2\end{array}\begin{bmatrix}c_1\\c_2\end{bmatrix}=\underset{x}{\begin{bmatrix}4\\5\end{bmatrix}}$$

So $c_1=3, c_2=2$

$x=3b_1+2b_2$ $\quad [x]_{\mathcal{B}}=\begin{bmatrix}c_1\\c_2\end{bmatrix}=\begin{bmatrix}3\\2\end{bmatrix}$



$\mathbb{R}^n$ 中的坐标

For a basis $\beta = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, let

$$P_\beta = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \end{bmatrix} \text{ and } [\mathbf{x}]_\beta = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

Then

$$\boxed{\mathbf{x} = P_\beta[\mathbf{x}]_\beta.}$$

We call P_β the **change-of-coordinates matrix** from β to the standard basis in $\mathbf{R}^n$. Then

回到标准坐标

$$[\mathbf{x}]_\beta = P_\beta^{-1}\mathbf{x}$$

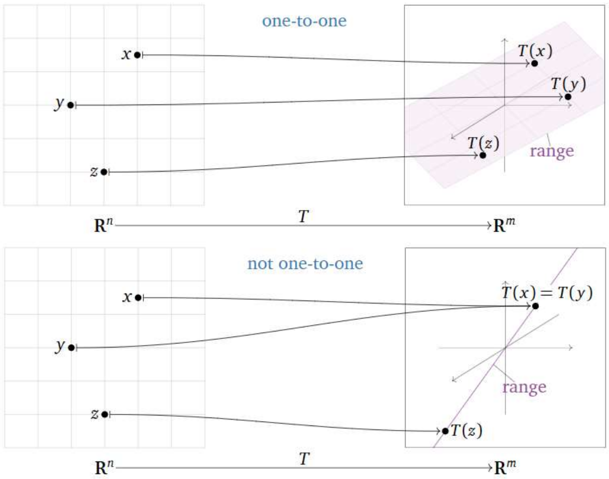
and therefore P_β^{-1} is a **change-of-coordinates matrix** from the standard basis in $\mathbf{R}^n$ to the basis β.

坐标映射

One-to-one Transformations 单射(也译作injective)

Definition (One-to-one transformations). A transformation $T: \mathbf{R}^n \to \mathbf{R}^m$ is ***one-to-one*** if, for every vector b in $\mathbf{R}^m$, the equation $T(x) = b$ has *at most one* solution x in $\mathbf{R}^n$.

定义：映射T：$\mathbb{R}^n \to \mathbb{R}^m$是单射，则对于任意$\mathbf{b} \in \mathbb{R}^m$，方程$T(\mathbf{x}) = \mathbf{b}$在$\mathbf{x} \in \mathbb{R}^n$中至多有一个解。



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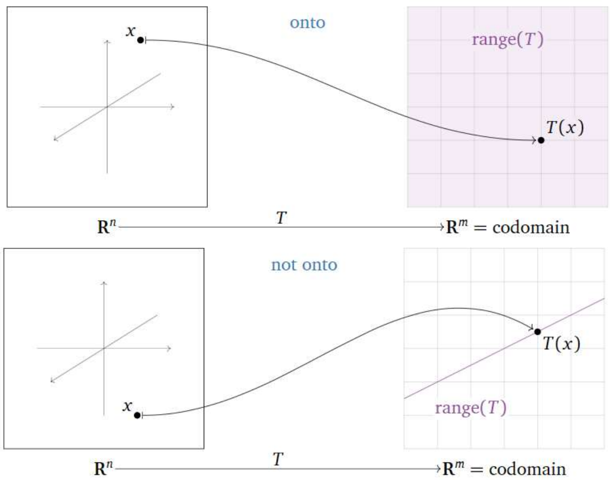
Onto Transformations 满射(也译作surjective)

Definition (Onto transformations). A transformation $T: \mathbf{R}^n \to \mathbf{R}^m$ is ***onto*** if, for every vector b in $\mathbf{R}^m$, the equation $T(x) = b$ has *at least one* solution x in $\mathbf{R}^n$.

定义：映射T：$\mathbb{R}^n \to \mathbb{R}^m$是满射，则对于任意$\mathbf{b} \in \mathbb{R}^m$，方程$T(\mathbf{x}) = \mathbf{b}$在$\mathbf{x} \in \mathbb{R}^n$中至少有一个解。



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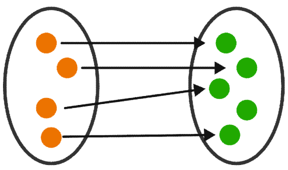
Bijective (双射)

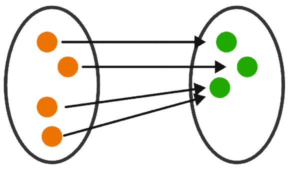
Definition: A transformation T: $\mathbb{R}^n \to \mathbb{R}^m$ is bijective if it is injective and surjective; that is, every element $\mathbf{b} \in \mathbb{R}^m$ is the image of exactly one element $\mathbf{x} \in \mathbb{R}^n$.

定义：映射T：$\mathbb{R}^n \to \mathbb{R}^m$是双射，则映射$T$既为单射也为满射，即对于任意$\mathbf{b} \in \mathbb{R}^m$，方程$T(\mathbf{x}) = \mathbf{b}$在$\mathbf{x} \in \mathbb{R}^n$中有唯一解。

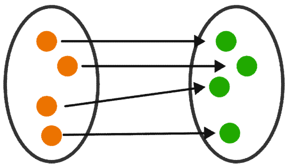
Injection (One-to-One)



Surjection (Onto)



Bijection (One-to-One and Onto)



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Comparison

The above expositions of one-to-one and onto transformations were written to mirror each other. However, “one-to-one” and “onto” are complementary notions: neither one implies the other. Below we have provided a chart for comparing the two. In the chart, A is an $m \times n$ matrix, and $T: \mathrm{R}^n \to \mathrm{R}^m$ is the matrix transformation $T(x) = Ax$.

T is one-to-one	T is onto
$T(\mathbf{x}) = \mathbf{b}$ has at most one solution for every $\mathbf{b}$. 对每个$\mathbf{b}$,方程$T(\mathbf{x}) = \mathbf{b}$在至多有一个解.	$T(\mathbf{x}) = \mathbf{b}$ has at least one solution for every $\mathbf{b}$. 对每个$\mathbf{b}$,方程$T(\mathbf{x}) = \mathbf{b}$至少有一个解.
The columns of A are linearly independent. A的列向量线性独立.	The columns of A span $\mathbb{R}^m$. A的列向量张成$\mathbb{R}^m$空间.
A has a pivot in every column. A的每列都有主元.	A has a pivot in every row. A的每行都有主元.
The range of T has dimension n. T的值域是n维的.	The range of T has dimension m. T的值域是m维的.

Note that in general, a transformation T is bijective if it is injective and surjective; that is, every element $\mathbf{b} \in \mathbb{R}^m$ is the image of exactly one element $\mathbf{x} \in \mathbb{R}^n$.
注意：映射T是双射，则映射T既为单射也为满射，即对于任意$\mathbf{b} \in \mathbb{R}^m$，方程$T(\mathbf{x}) = \mathbf{b}$在$\mathbf{x} \in \mathbb{R}^n$中有唯一解。

回顾

回顾 → 变换

符号：

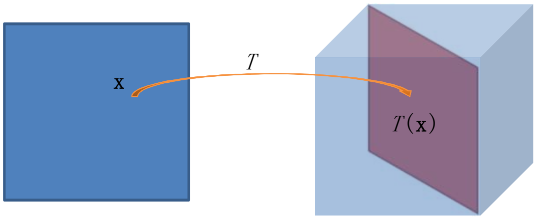
$\mathbb{R}^n$ ：定义域（domain of T）

$\mathbb{R}^m$ ：余定义域、陪域（codomain of T）

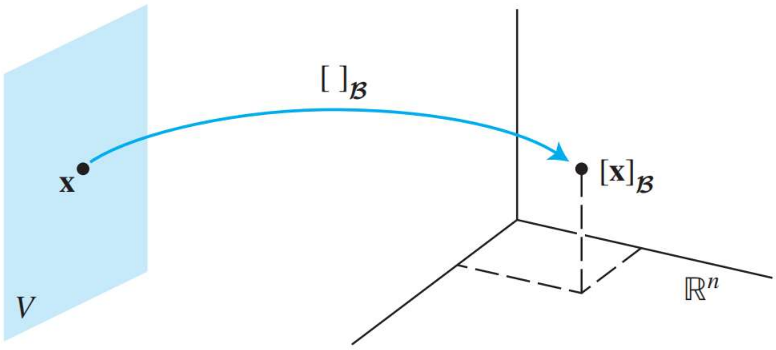
T: $\mathbb{R}^n \to \mathbb{R}^m$的变换

$T(\mathrm{x})$: x在$\mathbb{R}^m$的像（Image of x）

所有$T(\mathrm{x})$的集合：值域（Range of T）



坐标映射



把不熟悉的向量空间转换为熟悉的向量空间

坐标映射

Theorem 8

Let $\boldsymbol{\beta} = \{b_1, \ldots, b_n\}$ be a basis for a vector space V. Then the coordinate mapping $\boldsymbol{x} \to [\boldsymbol{x}]_{\boldsymbol{B}}$ is a **one-to-one linear transformation** from V onto $\mathbb{R}^n$

Proof : Take two typical vectors in V

$$\mathrm{u} = c_1\mathrm{b}_1 + \cdots + c_n\mathrm{b}_\mathrm{n},\ \mathrm{w} = d_1\mathrm{b}_1 + \cdots + d_n\mathrm{b}_\mathrm{n}$$

Then $\mathrm{u} + \mathrm{w} = (c_1 + d_1)\mathrm{b}_1 + \cdots + (c_n + d_n)\mathrm{b}_\mathrm{n}$

It follows that

$$[\mathrm{u} + \mathrm{w}]_\mathrm{B} = \begin{bmatrix} c_1 + d_1 \\ \vdots \\ c_n + d_n \end{bmatrix} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} + \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix} = [\mathrm{u}]_\mathrm{B} + [\mathrm{w}]_\mathrm{B}$$

Thus the coordinate mapping preserves addition.

坐标映射

If **r** is any scalar, then $r\mathrm{u} = r(c_1\mathrm{b}_1 + \cdots + c_n\mathrm{b}_n) = (rc_1)\mathrm{b}_1 + \cdots + (rc_n)\mathrm{b}_n$

So

$$[r\mathrm{u}]_\mathrm{B} = \begin{bmatrix} rc_1 \\ \vdots \\ rc_n \end{bmatrix} = r\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = r[\mathrm{u}]_\mathrm{B}$$

Thus the coordinate mapping also preserves scalar multiplication.

坐标映射

Standard basis for $\mathbf{P}_2$: $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\} = \{1, t, t^2\}$

Polynomials in $\mathbf{P}_2$ behave like vectors in $\mathbf{R}^3$. Since
$a + bt + ct^2 = \underline{\quad a \quad}\mathbf{p}_1 + \underline{\quad b \quad}\mathbf{p}_2 + \underline{\quad c \quad}\mathbf{p}_3$,

$$[a + bt + ct^2]_\beta = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

We say that the vector space $\mathbb{R}^3$ is *isomorphic* to $\mathbf{P}_2$.

Isomorphic:同构的
Isomorphism:同构

坐标映射

EXAMPLE: Parallel Worlds of $\mathbb{R}^3$ and $\mathbf{P}_2$.

Vector Space $\mathbb{R}^3$	Vector Space P_2
Vector Form: $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$	Vector Form: $a + bt + bt^2$
Vector Addition Example $\begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix}$	*Vector Addition Example* $(-1 + 2t - 3t^2) + (2 + 3t + 5t^2) = 1 + 5t + 2t^2$

Informally, we say that vector space V is **isomorphic** to W if *every vector space calculation in V is accurately reproduced in W*, and vice versa.

坐标映射

EXAMPLE: Use coordinate vectors to determine if $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ is a linearly independent set, where $\mathbf{p}_1 = 1 - t$, $\mathbf{p}_2 = 2 - t + t^2$, and $\mathbf{p}_3 = 2t + 3t^2$.

把不熟悉的向量空间有关的问题转换到熟悉的向量空间上来

Solution: The standard basis set for $\mathbf{P}_2$ is $\beta = \{1, t, t^2\}$. So

$$[\mathbf{p}_1]_\beta = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, [\mathbf{p}_2]_\beta = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, [\mathbf{p}_3]_\beta = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$$

Then

$$\begin{bmatrix} 1 & 2 & 0 \\ -1 & -1 & 2 \\ 0 & 1 & 3 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

By the IMT, $\{[\mathbf{p}_1]_\beta, [\mathbf{p}_2]_\beta, [\mathbf{p}_3]_\beta\}$ is linearly independent and therefore

$\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ is linearly independent

Q & A