ENGS 20 Short Assignment 17 - Introduction to ODEs

In this short assignment we will be exploring the numerical solution to ordinary differential equations (or ODEs). It's perfectly okay if you have never heard of differential equations before. You will explore how to solve them numerically with Euler's method below, which should give you a good foundation for when you see them in future classes and in Long Assignment 3.

An ordinary differential equation is an equation that describes how a function $y(t)$ and its derivative $y'(t)$ are related to each other in the form

$$y' = f(t, y)$$

For example: For the function $y = e^t$, we know $y' = e^t$ and therefore we can say that $y' = y$
Another example: $y = e^{t^2}$ has the derivative $y' = 2t \cdot e^{t^2}$ so we could say $y' = 2t \cdot y$

Differential equations appear pretty much everywhere when describing the world around us and you will encounter them in higher level classes if you haven't already. An example is Newton's second law of motion:

$$F(y(x)) = m \cdot y''(x)$$

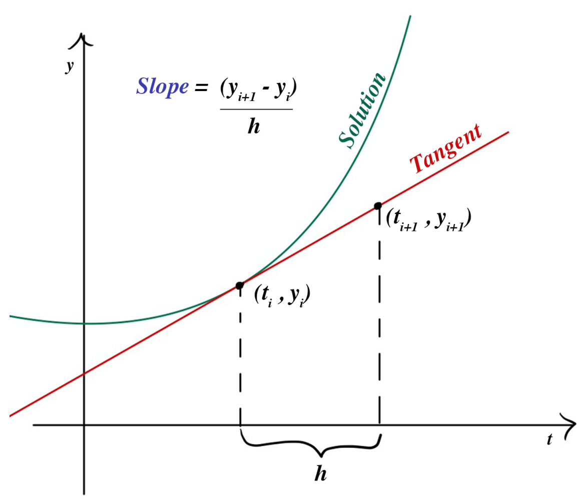
The force at position $y(x)$ equals mass times the acceleration $y''(x)$

Pretty much any mechanical system involves displacement $y(x)$, velocity $y'(x)$, and acceleration $y''(x)$. And electrical systems involve voltages, currents, and their time derivatives. The functions can vary with position x (like in Newton's law) or time t (like we will see in Long Assignment 3) or a manner of other things. These systems can often be expressed in terms of ODEs and are the subjects of courses like ENGS 22/23 and Math 23.

Oftentimes it is not simple, or even possible, to solve a differential equation analytically like we have above and find an equation for a solution. This is where Euler's method comes in. Euler's method is a numerical procedure that results in an approximation of the solution of an ODE. In other words, given an ODE in the form $y' = f(t, y)$ along with an initial condition $y(t_0) = y_0$, we can estimate values of the function $y(t)$ and therefore plot it numerically. Below we will implement the forward Euler method for a simple ODE and explore the resulting graphs to prepare for Long Assignment 3.

QUESTION 1

Let's remember from calculus that the derivative of a function at a point (t_i, y_i) is the slope of the tangent line at that point.
As you can see in the diagram to the right, given a second point (t_{i+1}, y_{i+1}) we can write the derivative as $y'(t_i) = \frac{(y_{i+1} - y_i)}{h}$ (where $h = t_{i+1} - t_i$ as seen in the picture)
A differential equation relates the derivative of a function to the function itself. If we are given a differential equation in the form $y'(t_i) = f(t_i, y_i)$, we now have two equations for $y'(t_i)$.
We can set these two equations equal to each other and rearrange to find y_{i+1}, which is and approximation for the value of the function at t_{i+1}. It is not exactly on the solution curve, but if our h value is small enough, this gives a good approximation to a point on the solution curve.
We've done that algebra for you, and rearranged to find:



$$y_{i+1} = y_i + h \cdot f(t_i, y_i)$$

Now, since we are given the initial condition (t_0, y_0) (meaning that we want the solution to go through that point), we can use the above equation to find y_1 by plugging in:

$$y_1 = y_0 + h \cdot f(t_0, y_0)$$

Hopefully you can see how we can use this process iteratively to find approximations for all the desired y values at $t_1, t_2, t_3, \ldots$. If we plot all of these y values with their corresponding t values, we now have a numerical approximation for our solution.

For question 1 we will start with a simple equation in the form $y' = ay + b$

Write a Matlab script that implements Euler's method for the equation:

$$y' = -2y + 9$$

Given the initial condition $t_0 = 0, y_0 = 5$, a final t value of 3, and using a step size of $h = 0.1$

QUESTION 2

Now let's explore what happens if you change the parameters a or b.

a. Plot 3 different versions of $y' = ay + b$ like you did in question 1, but change the values and/or signs of the parameter a. Use $a = -5, -2, 0.5$ but leave $b = 9$ for all. Plot all 3 curves on the same axis. You can leave the initial conditions, h, and final t value the same. Use the figure command to create a new figure (i.e. you should now have two figure windows on your screen. One for question 1 and now another for question 2).

b. Now do the same but with the parameter b. Use $b = -3, 3, 9$ but leave $a = -2$ for all. Use the subplot command. You should now have two subplots in this figure. One for different values of a and one for different values of b.

c. Observe the effect that changing the sign and magnitude of a and b has on the graphs. Note you have observed in a comments at the end of your code.

QUESTION 3

The differential equation $y' = -2y + 9$ with the initial condition $(0, 5)$ is quite easy to solve analytically. The analytical solution is $y = \frac{e^{-2t}}{2} - \frac{9}{2}$ (you can verify this by taking the derivative of y and comparing it to y'). Plot the analytical solution on the same graph as your Euler's method solution from question 1.

Comment on the differences you observe. This equation has a very simple analytical solution, but you could imagine a more complicated ODE where finding an analytical solution would be much harder if not mathematically impossible. We will encounter such a system in Long Assignment 3, and you will use Euler's method to solve the system.