Homework 0

This homework is *required* and mostly consists of review problems.

Self grades are due at 11 PM on September 6, 2023.

1. **Course Setup**

   Please complete the following steps to get access to all course resources.

   (a) Visit the course website at `http://www.eecs127.github.io/` and familiarize yourself with the syllabus.

   (b) Verify that you can access the class Ed site at `https://edstem.org/us/courses/42325`.

   (c) Make sure you can access the Gradescope at `https://www.gradescope.com/courses/566395`. If not, make a private Ed post to contact us.

   (d) When are self grades due for this homework? In general, when are self grades due? Where are the self-grade assignments?

   Solution: Self grades for this homework are due September 6th at 11 PM PST. In general, self grades are due one week following initial homework submission. Self-grade assignments are on Gradescope and are also accessible using the course website.

   (e) How many homework drops do you get? Are there exceptions?

   Solution: Your lowest two homework grades will be dropped. There are no exceptions, and late work will not be accepted, though you may ask for extensions using the form on the course website.

2. What Prerequisites Have You Taken?

The prerequisites for this course are

- EECS 16A & 16B (Designing Information Devices and Systems I & II) **OR** MATH 54 (Linear Algebra & Differential Equations),
- CS 70 (Discrete Mathematics & Probability Theory), and
- MATH 53 (Multivariable Calculus).

Please list which of these courses you have taken. If you have taken equivalent courses at a separate institution, please list them here. If you are unsure of course material overlap, please refer to the EECS 16A, EECS 16B, and CS 70 websites (`https://www.eecs16a.org/`, `https://www.eecs16b.org/`, and `http://www.sp22.eecs70.org/`, respectively) and the MATH 53 textbook (*Multivariable Calculus* by James Stewart).

The course material this semester will rely on knowledge from these prerequisite courses. If you feel shaky on this material, please use the first week to reacquaint yourself with it.

3. Determinants

Consider a unit box $\mathcal{B}$ in $\mathbb{R}^2$ — i.e., the square with corners $\begin{bmatrix}0\\0\end{bmatrix}$, $\begin{bmatrix}0\\1\end{bmatrix}$, $\begin{bmatrix}1\\0\end{bmatrix}$, and $\begin{bmatrix}1\\1\end{bmatrix}$. Define $A(\mathcal{B})$ as the parallelogram generated by applying matrix A to every point in $\mathcal{B}$.

(a) For $A = \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}$, calculate the location of each corner of $A(\mathcal{B})$.

Solution: Directly applying matrix A to each corner point, we have

$$A\begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}.$$

$$A\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}2\\4\end{bmatrix}.$$

$$A\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\3\end{bmatrix}.$$

$$A\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}3\\7\end{bmatrix}.$$

(b) Write the area of $A(\mathcal{B})$ as a function of $\det(A)$.

HINT: How are the basis vectors $\begin{bmatrix}0\\1\end{bmatrix}$ *and* $\begin{bmatrix}1\\0\end{bmatrix}$ *transformed by the matrix multiplication?*

Solution: The area of the parallelogram $A(\mathcal{B})$ can be computed geometrically from its corner points and is equal to $|\det(A)|$. A full explanation of this relation can be found in Calafiore & El Ghaoui section 3.3.2.

(c) Calculate the area of $A(\mathcal{B})$ for each of the following values of A.

i. $A = \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}$

ii. $A = \begin{bmatrix}2 & 1\\4 & 3\end{bmatrix}$

iii. $A = \begin{bmatrix}1 & 2\\2 & 4\end{bmatrix}$

iv. $A = \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}$

Solution: We use the result from part (b) and use the determinant of A to calculate the area.

i. $\det(A) = \det\left(\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\right) = -2 \implies \text{area}[A(\mathcal{B})] = |\det(A)| = 2.$

ii. $\det(A) = \det\left(\begin{bmatrix}2 & 1\\4 & 3\end{bmatrix}\right) = 2 \implies \text{area}[A(\mathcal{B})] = |\det(A)| = 2.$

iii. $\det(A) = \det\left(\begin{bmatrix}1 & 2\\2 & 4\end{bmatrix}\right) = 0 \implies \text{area}[A(\mathcal{B})] = |\det(A)| = 0$ (Singular matrix!).

iv. $\det(A) = \det\left(\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}\right) = 1 \implies \text{area}[A(\mathcal{B})] = |\det(A)| = 1$ (Rotation matrix!).

4. Subspaces and Dimensions

Consider the set $\mathcal{S}$ of points $(x_1, x_2, x_3) \in \mathbb{R}^3$ such that

$$x_1 + 2x_2 + 3x_3 = 0, \quad 3x_1 + 2x_2 + x_3 = 0. \tag{1}$$

(a) Find a 2×3 matrix A for which $\mathcal{S}$ is exactly the null space of A.

Solution: Recall the definition of the null space of a matrix A as the set of all vectors $\vec{x}$ such that $A\vec{x} = \vec{0}$. The equations

$$x_1 + 2x_2 + 3x_3 = 0 \tag{2}$$

$$3x_1 + 2x_2 + x_3 = 0 \tag{3}$$

can be written in matrix-vector form as

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{4}$$

The set of $\vec{x} = \begin{bmatrix} x_1, x_2, x_3 \end{bmatrix}^\top$ which satisfy this equation form the null space of $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$. This is the matrix we are looking for.

(b) Determine the dimension of $\mathcal{S}$ and find a basis for it.

Solution: Recall the definitions of a basis and the dimension of a subspace, which are related. A basis for a space is a set of linearly independent vectors that span the space. The dimension of this space is then the number of vectors in the basis.

To find the dimension, we solve the equation and find that any solution to the equations is of the form $x_1 = x_3$, $x_2 = -2x_3$, where x_3 is free. Thus, the solutions are of the form $\begin{bmatrix} 1, -2, 1 \end{bmatrix}^\top u$ for $u \in \mathbb{R}$, and so $\mathcal{S} = \operatorname{span}\left(\begin{bmatrix} 1, -2, 1 \end{bmatrix}^\top\right)$. Hence, the dimension of $\mathcal{S}$ is 1, and a basis for $\mathcal{S}$ is the vector $\begin{bmatrix} 1, -2, 1 \end{bmatrix}^\top$.

5. Orthogonality

Let $\vec{x}, \vec{y} \in \mathbb{R}^n$ be two linearly independent unit-norm vectors; that is, $\|\vec{x}\|_2 = \|\vec{y}\|_2 = 1$.

(a) Show that the vectors $\vec{u} = \vec{x} - \vec{y}$ and $\vec{v} = \vec{x} + \vec{y}$ are orthogonal.

Solution: Orthogonal means dot product is 0. When x, y are both unit-norm, we have

$$(\vec{x} - \vec{y})^\top (\vec{x} + \vec{y}) = \vec{x}^\top \vec{x} - \vec{y}^\top \vec{y} - \vec{y}^\top \vec{x} + \vec{x}^\top \vec{y} = \vec{x}^\top \vec{x} - \vec{y}^\top \vec{y} = 0, \tag{5}$$

(b) Find an orthonormal basis for $\operatorname{span}(\vec{x}, \vec{y})$, the subspace spanned by $\vec{x}$ and $\vec{y}$.

Solution: Since $\vec{x}$ and $\vec{y}$ are linearly independent, we have $\vec{x} \neq \vec{y}$ and $\vec{x} \neq -\vec{y}$ (since they are both unit norm). Thus $\vec{u}$ and $\vec{v}$ are nonzero.

Motivated by the first part that asks us to show $\vec{u}$ and $\vec{v}$ are orthogonal, we first see that $\vec{u}, \vec{v}$ form an orthogonal basis for $\operatorname{span}(\vec{u}, \vec{v})$.

The subspace spanned by $\vec{x}, \vec{y}$ is $S_1 = \operatorname{span}(\vec{x}, \vec{y})$, We want to check if $S_2 = \operatorname{span}(\vec{u}, \vec{v})$ is the same set as S_1.

If this is true then $\vec{u}$ and $\vec{v}$ are orthogonal basis vectors for $\operatorname{span}(\vec{x}, \vec{y})$.

First we show $S_1 \subseteq S_2$ by checking that $\vec{z} \in S_1 \implies \vec{z} \in S_2$.

We can express any vector $\vec{z} \in \operatorname{span}(\vec{x}, \vec{y})$ as $\vec{z} = \lambda\vec{x} + \mu\vec{y}$, for some $\lambda, \mu \in \mathbb{R}$. We have $\vec{z} = \alpha\vec{u} + \beta\vec{v}$, where

$$\alpha = \frac{\lambda - \mu}{2}, \quad \beta = \frac{\lambda + \mu}{2}. \tag{6}$$

Hence $\vec{z} \in \operatorname{span}(\vec{u}, \vec{v})$. The converse is also true for similar reasons.

We can find orthonormal basis vectors by dividing each orthogonal basis vector by its norm. The desired orthonormal basis is thus given by $((\vec{x} - \vec{y}) / \|\vec{x} - \vec{y}\|_2, (\vec{x} + \vec{y}) / \|\vec{x} + \vec{y}\|_2)$.

We could have also gotten that $\operatorname{span}(\vec{x}, \vec{y}) = \operatorname{span}(\vec{u}, \vec{v})$ in a slightly faster way by noting that

$$\vec{u} = \vec{x} - \vec{y}, \quad \vec{v} = \vec{x} + \vec{y}, \quad \vec{x} = \frac{\vec{u} + \vec{v}}{2}, \quad \vec{y} = \frac{\vec{v} - \vec{u}}{2} \tag{7}$$

so linear combinations of $\vec{u}$ and $\vec{v}$ are linear combinations of $\vec{x}$ and $\vec{y}$, and vice versa. Thus the spans are the same, e.g., $\operatorname{span}(\vec{x}, \vec{y}) = \operatorname{span}(\vec{u}, \vec{v})$. And the solution proceeds from there in the same way.

6. Homework Process

With whom did you work on this homework? List the names and SIDs of your group members.

NOTE: If you didn't work with anyone, you can put "none" as your answer.