Homework 2

This homework is due at 11 PM on September 13th, 2023.

Submission Format: Your homework submission should consist of a single PDF file that contains all of your answers (any handwritten answers should be scanned).

1. **Norms**

(a) Show that the following inequalities hold for any vector $\vec{x} \in \mathbb{R}^n$:

$$\frac{1}{\sqrt{n}} \left\|\vec{x}\right\|_2 \le \left\|\vec{x}\right\|_\infty \le \left\|\vec{x}\right\|_2 \le \left\|\vec{x}\right\|_1 \le \sqrt{n} \left\|\vec{x}\right\|_2 \le n \left\|\vec{x}\right\|_\infty . \tag{1}$$

As an aside: note that we can interpret different norms as different ways of computing distance between two points $\vec{x}, \vec{y} \in \mathbb{R}^2$. The ℓ_2 norm is the distance as the crow flies (i.e. point-to-point distance), the ℓ_1 norm, also known as the Manhattan distance is the distance you would have to cover if you were to navigate from $\vec{x}$ to $\vec{y}$ via a rectangular street grid, and the ℓ_∞ norm is the maximum distance that you have to travel in either the north-south or the east-west direction.

(b) We define the *cardinality* of the vector $\vec{x}$ as the number of non-zero elements in $\vec{x}$. This is also commonly known as the ℓ_0 norm of the vector $\vec{x}$, denoted by $\left\|\vec{x}\right\|_0$. Show that for any non-zero vector x,

$$\left\|\vec{x}\right\|_0 \ge \frac{\left\|\vec{x}\right\|_1^2}{\left\|\vec{x}\right\|_2^2}. \tag{2}$$

Find all vectors $\vec{x}$ for which the lower bound is attained.

2. Distinct Eigenvalues, Orthogonal Eigenspaces

Let $A \in \mathbb{S}^n$ (i.e. the set of $n \times n$ symmetric matrices) and $(\lambda_1, \vec{u}_1), (\lambda_2, \vec{u}_2), \lambda_1 \neq \lambda_2$ be distinct eigen-pairs of A. Show that $\langle \vec{u}_1,\ \vec{u}_2 \rangle = 0$, i.e eigenspaces corresponding to distinct eigenvalues are mutually orthogonal.

Note: This exercise is part of the proof of the spectral theorem.

3. Eigenvectors of a Symmetric Matrix

Let $\vec{p}, \vec{q} \in \mathbb{R}^n$ be two linearly independent vectors, with unit norm ($\|\vec{p}\|_2 = \|\vec{q}\|_2 = 1$). Define the symmetric matrix $A := \vec{p}\vec{q}^\top + \vec{q}\vec{p}^\top$. In your derivations, it may be useful to use the notation $c := \vec{p}^\top \vec{q}$.

(a) Show that A is symmetric.

(b) Show that $\vec{p} + \vec{q}$ and $\vec{p} - \vec{q}$ are eigenvectors of A, and determine the corresponding eigenvalues.

(c) Determine the nullspace and rank of A.

(d) Find an eigenvalue decomposition of A, in terms of $\vec{p}, \vec{q}$. *HINT: Use the previous two parts.*

(e) **(OPTIONAL)** Now consider general vectors $\vec{p}_{\text{new}}, \vec{q}_{\text{new}}$ that are scaled versions of $\vec{p}, \vec{q}$. Note that $\vec{p}_{\text{new}}, \vec{q}_{\text{new}}$ are not necessarily norm 1. Define the matrix $A_{\text{new}} := \vec{p}_{\text{new}}\vec{q}_{\text{new}}^\top + \vec{q}_{\text{new}}\vec{p}_{\text{new}}^\top$.

Write A_{new} as a function of $\vec{p}, \vec{q}$ and the norms of $\vec{p}_{\text{new}}, \vec{q}_{\text{new}}$, and the eigenvalues of matrix A_{new} as a function of $\vec{p}, \vec{q}$ and the norms of $\vec{p}_{\text{new}}, \vec{q}_{\text{new}}$.

4. PSD Matrices

In this problem, we will analyze properties of positive semidefinite (PSD) matrices. A matrix M is a PSD matrix if all its eigenvalues are non-negative, and we denote that as $M \succeq 0$.
Assume $A \in \mathbb{R}^{n\times n}$ is a symmetric matrix.

(a) Show that $\forall \vec{x} \in \mathbb{R}^n, \vec{x}^\top A\vec{x} \geq 0 \iff$ all eigenvalues of A are non-negative.

Now we will show that a symmetric matrix A is positive semidefinite if and only if there exists a symmetric matrix $P \in \mathbb{R}^{n\times n}$ such that $A = P^\top P$.

(b) First, show that A having non-negative eigenvalues allows us to decompose $A = P^\top P$ where $P \succeq 0$.

(c) Now, show that any matrix of the form $A = P^\top P$ is positive semidefinite, i.e. $A \succeq 0$.

(d) Show that if $A \succeq 0$ then all diagonal entries of A are non-negative, $A_{ii} \geq 0$.

5. SVD Transformation

In this problem we will interpret the linear map corresponding to a matrix $A \in \mathbb{R}^{n\times n}$ by looking at its singular value decomposition, $A = UDV^\top$. Recall that here $U, D, V \in \mathbb{R}^{n\times n}$ and U, V are orthonormal matrices while D is a diagonal matrix. We will first look at how $V^\top$, D and U each separately transform the unit circle $C = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ and then look at their effect as a whole. This problem has an associated jupyter notebook, `svd_transformation.ipynb` that contains several parts (b, c, d, e) of the problem. These sub-parts can be answered in the notebook itself in the space provided and can be submitted as an attachment to this PDF using the "Download as PDF" feature that Jupyter notebook supports.

(a) Show that $V^\top\vec{x}$ represents $\vec{x}$ in the basis defined by the columns of V. Recall: $V^\top V = I$.

For the rest of the problem we restrict ourselves to the case where $A \in \mathbb{R}^{2\times 2}$ and move to the Jupyter notebook.

6. Homework Process

With whom did you work on this homework? List the names and SIDs of your group members.

NOTE: If you didn't work with anyone, you can put "none" as your answer.