Discussion 2

1. Symmetric Matrices

(a) Show that any symmetric matrix $A \in \mathbb{R}^{n \times n}$ is positive semidefinite if and only if there exists a symmetric matrix $C \in \mathbb{R}^{n \times n}$ such that $A = C^\top C$.

(b) Draw the region $\left\{ \vec{x} \in \mathbb{R}^2 \middle| \vec{x}^\top \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \vec{x} \leq 1 \right\}$.

(c) Draw the region $\left\{ \vec{x} \in \mathbb{R}^2 \middle| \vec{x}^\top \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \vec{x} \leq 1 \right\}$.

(d) (**Optional**) Why is the region in part (b) bounded, whereas the region in part (c) is unbounded?

2. Eigenvalues

(a) Let $A \in \mathbb{R}^{n \times n}$ be a matrix with eigenvalues and corresponding eigenvectors $\lambda_1, \ldots, \lambda_n$ and $\vec{v}_1, \ldots, \vec{v}_n$ respectively. Now consider $B = A + cI_n$ where $c \in \mathbb{R}$ and I_n is the $n \times n$ identity matrix. What are the eigenvalues and eigenvectors of B in terms of c and $\lambda_i, \vec{v}_i$ for $i = 1, \ldots, n$?

(b) Let A be a $d \times n$ matrix. Prove that the non-zero eigenvalues of $AA^\top$ are the same as the non-zero eigenvalues of $A^\top A$.

(c) Given a matrix $A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$ find its eigenvalues and eigenvectors without using the characteristic polynomial. (The reason to not use the characteric polynomial method for certain matrices is that computing determinants can be expensive in general. But when we have matrices with more structure, we can simplify the problem.)

HINT: Use the fact that eigenvectors with eigenvalue 0 *span the null space. Also notice that the eigenvectors of a symmetric matrix corresponding to different eigenvalues are orthogonal to each other.*

(d) (**Optional**) Let Q be an orthonormal matrix, i.e., $Q^\top Q = I$ of size $n \times n$. Let $\lambda \in \mathbb{R}$ be a scalar, $\vec{v} \in \mathbb{R}^n$ be a vector, and $A \in \mathbb{R}^{n \times n}$ be a matrix. Prove that if we have

$$A\vec{v} = \lambda\vec{v}, \tag{1}$$

i.e., $\vec{v}, \lambda$ is an eigenpair of A, then we have

$$(QAQ^\top)(Q\vec{v}) = \lambda(Q\vec{v}). \tag{2}$$