Discussion 3

1. SVD

Suppose we have a matrix $A \in \mathbb{R}^{m\times n}$ with rank r.
We define the compact SVD as follows:

$$\underbrace{A}_{m\times n} = \underbrace{U_r}_{m\times r}\underbrace{\Sigma_r}_{r\times r}\underbrace{V_r^\top}_{r\times n}.$$

Here, $\Sigma_r \in \mathbb{R}^{r\times r}$ is a diagonal matrix containing non-zero singular values of A.

$$\Sigma_r = \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{bmatrix},$$

with $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r$.
Next, $U_r \in \mathbb{R}^{m\times r}$ is given by,

$$U_r = \left[\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_r\right],$$

where u_i is a left singular vector corresponding to non-zero singular value, σ_i, for $i = 1, 2, \ldots, r$. The columns of U_r are orthonormal and together they span the columnspace of A.
Finally, $V_r^\top \in \mathbb{R}^{r\times n}$ is given by,

$$V_r^\top = \begin{bmatrix} \vec{v}_1^\top \\ \vec{v}_2^\top \\ \vdots \\ \vec{v}_r^\top \end{bmatrix},$$

where $\vec{v}_j$ is a right singular vector corresponding to non-zero singular value, σ_j for $j = 1, 2, \ldots, r$. The rows of $V_r^\top$ are orthonormal and span the rowspace of A. Equivalently the columns of V_r span the column space of $A^\top$.

The matrix A can be expressed as,

$$A = \sigma_1\vec{u}_1\vec{v}_1^\top + \sigma_2\vec{u}_2\vec{v}_2^\top + \ldots + \sigma_r\vec{u}_r\vec{v}_r^\top.$$

Assume now that $m \geq n$.

Another type of SVD which might be more familiar is the full SVD of A which is defined as follows:

$$\underbrace{A}_{m\times n} = \underbrace{U}_{m\times m}\underbrace{\Sigma}_{m\times n}\underbrace{V^\top}_{n\times n}.$$

Here, $\Sigma \in \mathbb{R}^{m\times n}$ has non-diagonal entries as zero. The diagonal entries of Σ contain the singular values and we can write Σ in terms of Σ_r as,

$$\Sigma = \left[\begin{array}{c|c} \Sigma_r & 0_{r\times(n-r)} \\ \hline 0_{(m-r)\times r} & 0_{(m-r)\times(n-r)} \end{array}\right]$$

Next, $U \in \mathbb{R}^{m\times m}$ is an orthogonal matrix. U can be expressed in terms of U_r as,

$$U = \begin{bmatrix} \underbrace{U_r}_{m\times r} & \underbrace{\vec{u}_{r+1} \quad \dots \quad \vec{u}_m}_{m\times(m-r)} \end{bmatrix}$$

The columns $\vec{u}_{r+1}, \vec{u}_{r+2}, \dots, \vec{u}_n$ are left singular vectors corresponding to singular value 0, and together span the nullspace of $A^\top$.

Finally, $V^\top$ is an orthogonal matrix and can be expressed in terms of $V_r^\top$ as,

$$V^\top = \begin{bmatrix} V_r^\top \\ \vec{v}_{r+1}^\top \\ \vdots \\ \vec{v}_n^\top \end{bmatrix} \begin{matrix} \} \quad r\times n \\ \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} \quad (n-r)\times n \end{matrix}$$

The rows $\vec{v}_{r+1}^\top, \vec{v}_{r+2}^\top, \dots, \vec{v}_n^\top$ when transposed are the right singular vectors corresponding to singular value of 0 and together span the nullspace of A.

(a) For this problem assume that $m > n > r$. Which of the following are True:

(a) $UU^\top = I$

(b) $U^\top U = I$

(c) $V^\top V = I$

(d) $VV^\top = I$

(e) $U_r^\top U_r = I$

(f) $U_r U_r^\top = I$

(g) $V_r V_r^\top = I$

(h) $V_r^\top V_r = I$

(b) Going from the full SVD to compact SVD. Find the compact SVD of A which has the full SVD:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(c) Going from compact SVD to full SVD: Find the full SVD of A which has the compact SVD:

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(d) (**Optional**) For a given matrix $A \in \mathbb{R}^{m \times n}$ with $\text{rank}(A) = r = \min\{m, n\}$. Prove the rank nullity theorem, i.e., $n = r + \dim(\mathcal{N}(A))$

2. SVD Part 2

Consider A to be the 4×3 matrix

$$A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{bmatrix} \tag{1}$$

where $\vec{a}_i$ for $i \in \{1, 2, 3\}$ form a set of *orthogonal* vectors satisfying $\|\vec{a}_1\|_2 = 3$, $\|\vec{a}_2\|_2 = 2$, $\|\vec{a}_3\|_2 = 1$.

(a) What is the SVD of A? Express it as $A = U\Sigma V^\top$, with Σ the diagonal matrix of singular values ordered in decreasing fashion, and explicitly describe U and V.

(b) What is the dimension of the null space, $\dim(\mathcal{N}(A))$?

(c) What is the rank of A, $\text{rank}(A)$? Provide an orthonormal basis for the range of A.

(d) Let I_3 denote the 3×3 identity matrix. Consider the matrix $\tilde{A} = \begin{bmatrix} A \\ I_3 \end{bmatrix} \in \mathbb{R}^{7 \times 3}$. What are the singular values of $\tilde{A}$ (in terms of the singular values of A)?

(e) (**Optional**) Find an SVD of the matrix $\tilde{A}$.

