25. Approximating functions: Fourier analysis

Last time

- Conditioning of solving linear systems
- Iterative methods for solving linear systems

Goals for today

- Approximating functions
- Fourier series for periodic functions
- Rate of decay of Fourier coefficients

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  - Addition: $g = f + f$,
    i.e. $g(x) = f(x) + f(x) \quad \forall x \in [a, b]$
  - Multiplication $g = \alpha f^2$, i.e. $g(x) = \alpha * f(x) * f(x)$
  - Derivative: f'
  - Definite integral: $\int_a^b f(x)\, dx$
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- Idea: Find **approximation** of f that can manipulate

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- So far we have not discussed **periodic** functions f:

$$f(t+T) = f(t) \quad \forall t$$

T is **period** of function

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- Will use “linear algebra with functions” – i.e. functional analysis

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Fourier series

- Consider a **periodic** function f with period 2π
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- Can show that f can be written as

$$f(x) = \sum_{n=-\infty}^{\infty} \hat{f}_n \, e^{inx}$$

- i.e. sum of **basis functions** ϕ_n with $\phi_n(x) := e^{inx}$
- Alternatively

$$f(x) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)]$$

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- Need to specify *of which space*

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- Eigenfunctions of Laplacian operator $\frac{\partial^2}{\partial x^2}$
- Sturm–Liouville theory

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- When have **orthogonal vectors**
- Need to be able to talk about orthogonality of *functions*

Inner product

- Inner product **generalizes** the dot product in Euclidean space
- Abstract the properties of the dot product
- Denote as

$$(f, g) := \int_0^{2\pi} \overline{f(x)}\, g(x)\, dx$$

- Where for $z = x + iy \in \mathbb{C}$, the complex conjugate $\overline{z}$ denotes **complex conjugate** is $\bar{z} := x - iy$
- Properties: Linear in g and $(f, f) \geq 0$
- Two functions are **orthogonal** if $(f, g) = 0$

Orthogonality

- Let's look at inner products of our basis functions:

$$(\phi_n, \phi_m) = \int_0^{2\pi} e^{-inx} e^{imx} dx$$

$$= \int_0^{2\pi} e^{i(m-n)x} dx$$

$$= \frac{1}{i(m-n)} \left[e^{i(m-n)x} \right]_{x=0}^{2\pi}$$

- So $(\phi_n, \phi_m) = 0$ for $m \neq n$ and $= 2\pi$ for $m = n$
- Hence basis functions (ϕ_n) are orthogonal

Fourier coefficients II

- We have $f = \sum_{-\infty}^{\infty} \hat{f}_n \phi_n$
- Take inner product with ϕ_m:
- Obtain

$$(\phi_m, f) = \sum_n \hat{f}_n (\phi_m, \phi_n)$$

$$= 2\pi \hat{f}(m)$$

- Thus

$$\hat{f}_m = \frac{1}{2\pi} \int_0^{2\pi} e^{-imx} f(x) dx$$

Representation on computer

- Natural idea to represent periodic function on computer:

  sum finite *number of terms*

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- What does this depend on?

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- Depends on how fast $\hat{f}(n)$ *decay*

Fourier coefficients of derivative

. . .

- Suppose f is periodic and differentiable
- f' is also periodic, so can expand in Fourier series:

$$f' = \sum \widehat{f'}_n \phi_n$$

where $\widehat{f'}_n = \frac{1}{2\pi} \int_0^{2\pi} e^{-inx} f'(x) dx$

- Integrate by parts to get

$$\widehat{f'}_n = in\hat{f}_n$$

Rate of decay of Fourier coefficients

- Can show: If f integrable, e.g. continuous, on interval, then

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- Riemann–Lebesgue lemma

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- If $f \in C^k$ then $\widehat{f^{(k)}}_n = (in)^k \hat{f}_n \to 0$
- Hence $\hat{f}_n = o(n^{-k})$

Rate of decay II

- If f is **smooth** (C^∞) then decays faster than any polynomial

- In fact, if analytic in a suitable region then

$$\hat{f}_n = \mathcal{O}(e^{-\alpha n})$$

  i.e. coefficients decay **exponentially** fast

- **Spectral convergence**

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- This suggests that we should calculate *enough* terms
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- Soon reach machine epsilon
- Stop!
- We now have an excellent approximation of our general periodic function to something like machine epsilon!

Summary

- Periodic functions may be expanded in Fourier series
- Expansion in (infinite) linear combination of orthogonal basis functions
- Fourier coefficients decay at a rate determined by smoothness of f