32. Spectral methods for boundary-value problems

Last time

- Recurrence relation for Chebyshev polynomials
- Clenshaw–Curtis integration

Goals for today

- Spectral methods
- Boundary-value problems for ODEs
- Eigenvalue problems
- Time evolution

(Pseudo)-spectral methods for boundary-value problems

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- Notation: derivative $u_x := u' := \frac{du}{dx}$
- u_{xx} = 2nd derivative

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- Try to reduce to a linear algebra problem
- Will follow Trefethen, *Spectral Methods in MATLAB* (Chaps. 6 + 7)

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- Suppose that equation satisfed by function u (unknown)
- Look at collection of $N + 1$ nodes t_j
- With *unknown* values $u_j := u(t_j)$ at t_j

Collocation methods II

- Let $p(x)$ be degree-N polynomial interpolating u_j at t_j
- Note that u_j are the *unknowns*!
- Then we *impose*

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- So

$$\mathsf{M}\mathbf{u} = \mathbf{f}$$

  where $\mathsf{M} := \mathsf{D}_N^2$
- D_N is differentiation matrix

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- ODE is imposed at interior points

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- Additional advantage: **interpolant** is output from method

Summary

- Chebyshev spectral methods
- Collocation: Assume ODE is satisfied at interior points