28. Chebyshev methods II

Last time

- Convergence rate for trapezium rule
- Orthogonal polynomials

Goals for today

- Chebyshev polynomials

Global approximation on an interval

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- With orthogonal basis functions ϕ_n

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- $x := \mathrm{Re}(z)$
- So $x = \cos(\theta)$

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- From $f(x)$ construct function $g(\theta) := f(x) = f(\cos(\theta))$ that is periodic!

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- Thus

$$g(\theta) = \sum_{n=0}^{\infty} g_n \cos(n\theta)$$

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- This does not look like a pleasant set of functions!
- What are the functions $\phi_n(x) := \cos(n \arccos(x))$?

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Global approximation on an interval VI

- Idea: Recall: $e^{in\theta} = (e^{i\theta})^n$
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- So can express $\cos(n\theta)$ in terms of $\cos(\theta)$
- What is the relationship?

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- Recall: $\sin(\theta)^2 + \cos(\theta)^2 = 1$
- So $\cos(2\theta) = 2\cos(\theta)^2 - 1 = 2x^2 - 1$
- In general: $\cos(n\theta)$ is a *polynomial* in x !
- Gives **Chebyshev polynomials** (of first kind):

$$T_n(x) := \cos(n \arccos(x))$$

Chebyshev polynomials

- Have shown that function f on $[-1,+1]$ can be written as

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- With T_n the Chebyshev polynomials

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- Are they orthogonal?

Chebyshev polynomials II

- If we calculate $\int_{-1}^{1} T_1(x)T_2(x)dx$ we do not get 0

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- If we calculate $\int_{-1}^{1} T_1(x) T_2(x) dx$ we do not get 0

- But inner product can include a **weight function** w:

$$(f, g) := \int f(x)\, g(x)\, w(x)\, dx$$

- w must be *positive* function

Chebyshev polynomials III

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- Factor is due to change of variables $x = \cos(\theta)$ in integral:

$$\int_{-1}^{+1} f(x) g(x) w(x) dx = \frac{1}{2} \int_{0}^{2\pi} f(\cos(\theta)) g(\cos(\theta)) d\theta$$

- Consequence of fact that Chebyshev polynomials satisfy differential equation of Sturm–Liouville type

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- In Fourier case we found alternative solution:
- *Interpolation* using finite linear combinations of basis functions
- Can we do the same here?

Chebyshev interpolation II

- We want to interpolate $f : [-1, +1] \to \mathbb{R}$
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- Interpolate using $g_N = \sum_{n=0}^{N-1} \alpha_n T_n$
- Since *polynomial* interpolation, we *must not* use equally-spaced points!
- Solve following for α_k:

$$\sum_{k=0}^{N-1} \alpha_k T_k(t_j) = f_j := f(t_j)$$

- How choose nodes t_j?

Chebyshev interpolation III

- Due to `arccos` in expression for T_n:
- Choose $t_j = \cos(\frac{j\pi}{N})$
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- **Discrete Cosine Transform**
- Calculated in same way as FFT but 1/4 of the work

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- Revisit Lagrange interpolation

Barycentric Lagrange interpolation II

- Recall Lagrange interpolation for polynomial p of degree n:

$$p(x) = \sum_{j=0}^{n} \alpha_j \ell_j(x)$$

- Where

$$\ell_j(x) := \frac{\prod_{k \neq j} (x - t_k)}{\prod_{k \neq j} (t_j - t_k)}$$

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- Evaluating requires $\mathcal{O}(n^2)$ operations
- If add new node, must recalculate
- Numerically unstable

Barycentric Lagrange interpolation III

- Rewrite using product

$$\ell(x) := (x - t_0)(x - t_1) \cdots (x - t_n)$$

- Define **barycentric weights**

$$w_j := \frac{1}{\prod_{k \neq j}(t_j - t_k)}$$

- Then $w_j = \frac{1}{\ell'(t_j)}$

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- Then $w_j = \frac{1}{\ell'(t_j)}$
- And $\ell_j(x) = \ell(x) \frac{w_j}{x - t_j}$
- So $p(x) = \ell(x) \sum_{j=0}^{n} \frac{w_j}{x - t_j} f_j$

Barycentric Lagrange interpolation IV

- Now suppose we interpolate the constant function $\mathbf{1}(x) := 1 \quad \forall x$
- Get following for all x:

$$1 = \ell(x) \sum_{j=0}^{n} \frac{w_j}{x - t_j}$$

. . .

- Divide $p(x)$ by this:

$$p(x) = \frac{\sum_{j=0}^{n} \frac{w_j}{x-t_j} f_j}{\sum_{j=0}^{n} \frac{w_j}{x-t_j}}$$

Barycentric Lagrange interpolation V

- For given set of nodes t_j:
- Calculate weights w_j only *once*, with $\mathcal{O}(N^2)$ operations
- Or known analytically, e.g. for Chebyshev points have $w_j = (-1)^j \delta_j$,

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- Evaluate interpolant $p(x)$ at each x with $\mathcal{O}(N)$ operations
- Numerically stable (despite divisions)

Summary

- Spectral convergence of trapezium rule

- “Transplanting” Fourier analysis gives orthogonal expansion of $f : [-1, +1] \to \mathbb{R}$
- Basis functions are **Chebyshev polynomials**
- Orthogonal with respect to a particular inner product
- Barycentric Lagrange interpolation reconstructs interpolant rapidly