31. Clenshaw–Curtis integration

Last time

- Chebyshev expansions
- How to choose number of interpolation points
- Differentiation
- Recurrence relation for Chebyshev polynomials

Goals for today

- Recurrence relation for Chebyshev polynomials
- Clenshaw–Curtis integration
- Aliasing

Recurrence relation

- 3-term **recurrence relation** relating T_k to T_{k-1} and T_{k_2}:

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- Where does this recurrence relation come from?

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- $\alpha_{k,j} \propto (xT_k, T_j)$, i.e. equal up to normalization

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- Change variables using $x = \cos(\theta)$:

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- In fact, *any* set of orthogonal polynomials have similar 3-term recurrence

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- Clenshaw & Curtis (1960)

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- Double N until last few α_k are of order ϵ_{mach}

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- E.g. exponential (spectral) convergence if f is analytic

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- Find $I_k = \frac{(-1)^k+1}{1-k^2}$

Clenshaw–Curtis integration IV

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- **Riesz representation theorem** for linear functional

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- Integrate Lagrange interpolant – explicit formulae
- But leads to messier results – effectively Discrete Cosine Transform of above

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- Consider sampling Cheb. polynomial T_k at Cheb. points t_j
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- $T_m, T_{2n-m}, T_{2n+m}, \dots$

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- $T_m, T_{2n-m}, T_{2n+m}, \dots$
- Gives relationship between coefficients of *truncation* of f and *interpolation* of f
- See Trefethen, *Approx. Theory*, Chap. 4

Summary

- 3-term recurrence relation for Chebyshev polynomials
- Clenshaw–Curtis integration
- Spectrally accurate for analytic functions