27. Global approximation on an interval

Last time

- Interpolation with trigonometric functions
- Fast Fourier transform

Goals for today

- Spectral convergence of the trapezium rule for smooth, periodic functions

- Global approximation of functions on an interval
- Chebyshev polynomials

Trapezium rule

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- So if $\hat{f}_n$ decay exponentially fast then so does $|I - S_N|$

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- Take $f : [-1, +1] \to \mathbb{R}$
- Other intervals via linear transformation

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- Not sinusoids, since not periodic

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- $x := \mathrm{Re}(z)$
- So $x = \cos(\theta)$

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- From $f(x)$ construct function $g(\theta) := f(x) = f(\cos(\theta))$ that is periodic!

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- This does not look like a pleasant set of functions!

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- So can express $\cos(n\theta)$ in terms of $\cos(\theta)$
- What is the relationship?

Global approximation on an interval VI

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- So $\cos(2\theta) = 2\cos(\theta)^2 - 1 = 2x^2 - 1$
- In general: $\cos(n\theta)$ is a *polynomial* in x !
- **Chebysev polynomials** (of the first kind), $T_n(x)$

Chebyshev polynomials

- Have shown that function f on $[-1, +1]$ can be written as

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- Are they orthogonal?

Chebyshev polynomials II

- If we calculate $\int_{-1}^{1} T_1(x)T_2(x)dx$ we do not get 0

Chebyshev polynomials II

- If we calculate $\int_{-1}^{1} T_1(x)T_2(x)dx$ we do not get 0
- But inner product can include a **weight function** w:

$$(f, g) := \int f(x)\, g(x)\, w(x)\, dx$$

- w must be *positive* function

Chebyshev polynomials III

- T_n are orthogonal with respect to weight function

$$w(x) = \frac{1}{\sqrt{1-x^2}}$$

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- This is a consequence of the fact that Chebyshev polynomials satisfy a differential equation of Sturm–Liouville type

Summary

- Spectral convergence of trapezium rule

- By “transplanting” Fourier analysis, found orthogonal expansion of $f : [-1, +1] \to \mathbb{R}$
- Basis functions are **Chebyshev polynomials**
- Orthogonal with respect to a particular inner product