Texas Quantum Winter School: Lecture 8 Notes

January 2026

I. Lecture 8

A. Non-Hermitian dynamics & Exceptional Points

Non-Hermitian Hamiltonians naturally appear in:

- **Open quantum systems** — where energy or probability leaks into an environment.
- **Optical systems with gain and loss** — e.g., coupled microresonators or waveguides.
- **Atomic and cavity QED systems** — effective descriptions under dissipation.

These systems often displays real-to-complex spectral transitions not possible in closed, Hermitian quantum mechanics.

Exceptional points (EPs) occur where two (or more) eigenvalues **and** their eigenvectors coalesce. They mark the boundary between qualitatively distinct dynamical regimes. Near EPs, systems exhibit **enhanced sensitivity**, **non-analytic response**, and **mode hybridization**. They provide a bridge between **open-system dynamics** and **topological phenomena**, and enable new forms of **control**, **sensing**, and **state engineering**. They also offer experimentally accessible models for testing quantum-classical transitions.

Conventional / Hermitian QM

$$\hat{H} = H^\dagger \ \& \ \hat{H}|\psi_n\rangle = E_n|\psi_n\rangle$$

- Eigenvalues real: $E_n \in \mathbb{R}$
- Orthonormal eigenvectors:

$$\langle\psi_m|\psi_n\rangle = \delta_{mn}$$

- Unitary time evolution:

$$U(t) = e^{-iHt/\hbar}$$

- Time-reversal is preserved.

Open / Non-Hermitian QM

$$\hat{H} \neq \hat{H}^\dagger$$

- Eigenvalues complex: $E_n = E'_n - i\Gamma_n/2$
  - $\Gamma_n > 0 \Rightarrow$ decay
  - $\Gamma_n < 0 \Rightarrow$ amplification
- Biorthogonal basis:

$$\left\langle\phi_m^L\middle|\psi_n^R\right\rangle = \delta_{mn}$$

- Non-unitary evolution:

$$|\psi(t)\rangle = e^{-iH_{\text{eff}}t/\hbar}\,|\psi(0)\rangle$$

- Time-reversal broken.

Consider a parity operator $\mathcal{P} : x \mapsto -x$ and a time-reversal operator $\mathcal{T} : t \mapsto -t$, i.e. $H = H^\dagger$. The $\mathcal{PT}$ symmetry is preserved if $[\mathcal{PT}, H] = 0$.

Bender & Boettcher [1] also showed it's possible to have a non-Hermitian H with real eigenvalues. Consider a 2×2 Hamilton matrix of the following form:

$$H_{\text{PT}} = \begin{pmatrix} \frac{\omega}{2} + i\gamma & g \\ g & -\frac{\omega}{2} - i\gamma \end{pmatrix}, \tag{1}$$

where g is the coupling between two modes and γ is a balanced gain/loss parameter. Under parity ($\mathcal{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$) and time reversal ($\mathcal{T}$: complex conjugation), we can verify $\mathcal{PT}H_{\text{PT}}(\mathcal{PT})^{-1} = H_{\text{PT}}$. It's eigenvalues $\lambda_\pm$ are:

$$\lambda_{\pm} = \pm\sqrt{g^2 - \left(\frac{\omega}{2}\right)^2 - \gamma^2}\,. \tag{2}$$

For $g > \sqrt{(\omega/2)^2 + \gamma^2}$, the eigenvalues are real and the PT symmetry is unbroken. For $g < \sqrt{(\omega/2)^2 + \gamma^2}$, they form complex-conjugate pairs and the PT symmetry is spontaneously broken. The exceptional point occurs at $g_{\mathrm{EP}} = \sqrt{(\omega/2)^2 + \gamma^2}$.

In the context of ordinary (Hermitian) degeneracy, when two or more eigenvalues coincide ($E_m = E_n$), the corresponding eigenvectors remain *linearly independent*. The Hamiltonian remains diagonalizable.

For non-hermitian matricies, when two (or more) eigenvalues coalesce, so does their **eigenvectors**. The Hamiltonian becomes *defective* (non-diagonalizable). A general example is the *Jordan block*:

$$H \sim \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}. \tag{3}$$

The exceptional points marks the critical boundary between PT-unbroken and PT-broken phases. At the EPs, matrices have only one eigenvector $|\psi_0\rangle$ and one generalized eigenvector $|\psi_1\rangle$ such that:

$$(H - \lambda_0 I)\,|\psi_0\rangle = 0, \qquad (H - \lambda_0 I)\,|\psi_1\rangle = |\psi_0\rangle\,. \tag{4}$$

Bittner & Tyagi [2] showed how EPs emerge from the eigenstructure of H_{eff} or the Liouvillian. Lindblad master equation:

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\frac{i}{\hbar}\left[\hat{H}, \rho\right] + \sum_j \left(F_j \rho F_j^\dagger - \frac{1}{2}\{F_j^\dagger F_j, \rho\}\right) \tag{5}$$

For quantum trajectories with no *quantum jumps*, the conditional (non-unitary) evolution is governed by the effective Hamiltonian:

$$\frac{\mathrm{d}\,|\psi(t)\rangle}{\mathrm{d}t} = -\frac{i}{\hbar} H_{\mathrm{eff}}\,|\psi(t)\rangle\,, \quad H_{\mathrm{eff}} = \underbrace{H}_{\text{Hermitian}} - \underbrace{\tfrac{i}{2}\sum_j F_j^\dagger F_j}_{\text{Anti-Hermitian}}\,, \tag{6}$$

which is explicitly non-Hermitian and encodes irreversible processes.

B. Itô and Stratonovich forms

The evolution given by H_{eff} describes a non-unitary evolution where the trace of ρ is not conserved. Introducing a suitable noise in the effective Hamiltonian restores the conservation of the trace. Let us consider a stochastic differential equation of the form:

$$dx = a(x)dt + b(x)dW(t) \tag{7}$$

where $W(t)$ describes Wiener processes with zero mean. The solution to this differential equation is ambiguous without specifying how the stochastic process is integrated. The two well-known stochastic integrals utilized to describe stochastic processes are of the kind known as Stratonovich and Itô stochastic integrals. Unlike the latter, the integrals of the former are easier to manipulate and are defined such that the chain rule of the ordinary integral holds. For the Itô integrals, one has to resort to the use of the more complicated Itô lemma to evaluate the ordinary integrals. These integrals can be defined in the limit of the Riemann sums. Therefore, in the Riemann sum representation of an integral, the Itô form evaluates the integrand at the left endpoint, while the Stratonovich form evaluates it at the midpoint.

Consider a stochastic integral of the Itô form:

$$\int_0^t b(X)\,dW. \tag{8}$$

The increment is evaluated at the *beginning* of the interval:

$$\int b(X)\, dW \;\equiv\; \lim_{\Delta t \to 0} \sum_n b(X(t_n))\; \Delta W_n. \tag{9}$$

In the Stratonovich form, the increment is evaluated at the *midpoint* of the interval:

$$\int b(X) \circ dW \;\equiv\; \lim_{\Delta t \to 0} \sum_n b\Big(\tfrac{X(t_n)+X(t_{n+1})}{2}\Big)\; \Delta W_n. \tag{10}$$

In the form Itô, the knowledge of $X(t_{n+1})$ is not needed for averaging, therefore it is required to modify the chain rule. The Stratonovich calculus preserves ordinary calculus rules, this property arises naturally from finite-correlation-time noise and it is appropriate for the Hamiltonian dynamics. Transformation bewteen the 2 forms is done by the general rule: [3]

$$A \circ \mathrm{d}W = A\, \mathrm{d}W + \frac{1}{2}\, \mathrm{d}A\, \mathrm{d}W\,. \tag{11}$$

Consider the quantum dynamics is generated by a Hamiltonian:

$$\dot{\rho}(t) = -\frac{i}{\hbar}[H(t), \rho(t)] \tag{12}$$

where the system Hamiltonian $H(t)$ contains noise,

$$H(t) = H_0 + \sum_\alpha \xi_\alpha(t) V_\alpha, \tag{13}$$

where V_α are a set of operators acting on the stochastic wave function. Then products like $H(t)\rho(t)$ must obey the usual product rule. The Stratonovich calculus preserves ordinary calculus rules and allow us to write the evolution of an stochastic Hamiltonian as:

$$d\rho = -\frac{i}{\hbar}[H_0, \rho]dt - \frac{i}{\hbar}\sum_\alpha [V_\alpha, \rho] \circ dW_\alpha \tag{14}$$

with

$$\langle \mathrm{d}W_\alpha \rangle = 0, \qquad \langle \mathrm{d}W_\alpha\, \mathrm{d}W_\beta \rangle = D_{\alpha\beta}\, \mathrm{d}t \tag{15}$$

We can convert to the Itô form using $A_\alpha \circ \mathrm{d}W_\alpha = A_\alpha\, \mathrm{d}W_\alpha + \frac{1}{2}\, \mathrm{d}A_\alpha\, \mathrm{d}W_\alpha$ where $A_\alpha = [V_\alpha, \rho]$

$$\mathrm{d}\rho_{\text{Itô}} = \mathrm{d}\rho_{\text{Strat}} - \frac{1}{2\hbar^2}\sum_{\alpha,\beta} D_{\alpha\beta}[V_\alpha, [V_\beta, \rho]]\, \mathrm{d}t$$

The Itô equation becomes:

$$\mathrm{d}\rho = -\frac{i}{\hbar}[H_0, \rho]\, \mathrm{d}t - \frac{i}{\hbar}\sum_\alpha [V_\alpha, \rho]\, \mathrm{d}W_\alpha - \frac{1}{2\hbar^2}\sum_{\alpha,\beta} D_{\alpha\beta}[V_\alpha, [V_\beta, \rho]]\, \mathrm{d}t$$

So far we assumed Hermitian noise operators $V_\alpha = V_\alpha^\dagger$. We now consider *non-Hermitian* operators to model Population relaxation:

$$L = \sigma_-, \qquad L^\dagger = \sigma_+.$$

Consider a stochastic Hamiltonian with complex noise:

$$H(t) = H_0 + \xi(t)\, L + \xi^*(t)\, L^\dagger, \tag{16}$$

with correlations

$$\langle \xi(t)\xi^*(t') \rangle = \Gamma\, \delta(t - t'). \tag{17}$$

The stocastic evolution is given by:

$$d\rho = -\frac{i}{\hbar}[H_0, \rho]dt - \frac{i}{\hbar}\big(L\rho \circ dW^* - \rho L^\dagger \circ dW\big).$$

Coverting into Ito form, usign the general rule, and keeping terms with $(dW)(dW^*) \sim dt$, the stochastic equation in Ito form lead to:

$$d\rho = -\frac{i}{\hbar}[H_0, \rho]dt + \Gamma\left(L\rho L^\dagger - \frac{1}{2}\{L^\dagger L, \rho\}\right)dt. \tag{18}$$

which is the Lindblad master equation.

References

[1] Carl M. Bender and Stefan Boettcher. Real spectra in non-hermitian hamiltonians having pt symmetry. *Phys. Rev. Lett.*, 80:5243–5246, Jun 1998.

[2] Eric R. Bittner and Bhavay Tyagi. Statistical control of relaxation and synchronization in open anyonic systems. *Scientific Reports*, 16(1):748, 2025.

[3] Pietro De Checchi, Federico Gallina, Barbara Fresch, and Giulio G Giusteri. On the noisy road to open quantum dynamics: The place of stochastic hamiltonians. *Annalen der Physik*, 538(1):e00482, 2026.