Chapter 1

What is an Open Quantum System?

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1.1 Introduction

Most physical systems are not isolated but interact with an external environment. These interactions lead to phenomena such as dissipation, decoherence, relaxation, and irreversible decay, which cannot be captured by unitary dynamics alone. The formalism of open quantum systems provides the tools to describe and analyze such systems.

In this lecture, we examine two important approaches that provide insight into the emergence of irreversible behavior in quantum systems:

1. System–Bath partitioning.
2. The Wigner–Weisskopf approximation for decay into a continuum
3. The Feshbach projection operator method for deriving effective dynamics in a subspace

Both approaches highlight how system-environment couplings can lead to effective non-Hermitian behavior, energy shifts, and finite lifetimes of quantum states.

1.2 Wigner–Weisskopf Theory: Exponential Decay into a Continuum

We begin with a model consisting of a discrete state $|e\rangle$ of energy E_e coupled to a continuum of states $|k\rangle$ with energies E_k. The total Hamiltonian is

$$\hat{H} = \hat{H}_0 + \hat{V}, \tag{1.1}$$

where

$$\hat{H}_0 = E_e|e\rangle\langle e| + \int \mathrm{d}k\, E_k|k\rangle\langle k|, \tag{1.2}$$

$$\hat{V} = \int \mathrm{d}k\, \left[V_k|k\rangle\langle e| + V_k^*|e\rangle\langle k|\right]. \tag{1.3}$$

We assume the system is prepared in state $|\psi(0)\rangle = |e\rangle$ and evolves under the time-dependent Schrödinger equation:

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle. \tag{1.4}$$

We expand the state as

$$|\psi(t)\rangle = c_e(t)e^{-iE_e t/\hbar}|e\rangle + \int \mathrm{d}k\, c_k(t)e^{-iE_k t/\hbar}|k\rangle. \tag{1.5}$$

Substituting into the Schrödinger equation and projecting onto $\langle e|$ yields

$$\dot{c}_e(t) = -\frac{i}{\hbar}\int \mathrm{d}k\, V_k^* e^{-i(E_k-E_e)t/\hbar} c_k(t). \tag{1.6}$$

Similarly, projecting onto $\langle k|$ gives

$$\dot{c}_k(t) = -\frac{i}{\hbar} V_k e^{i(E_k-E_e)t/\hbar} c_e(t). \tag{1.7}$$

Integrating this and substituting into the equation for $\dot{c}_e(t)$, we obtain an integro-differential equation:

$$\dot{c}_e(t) = -\int_0^t \mathrm{d}t'\, \Sigma(t-t')c_e(t'), \tag{1.8}$$

with the self-energy kernel

$$\Sigma(t-t') = \frac{1}{\hbar^2}\int \mathrm{d}k\, |V_k|^2 e^{-i(E_k-E_e)(t-t')/\hbar}. \tag{1.9}$$

Assuming the coupling is weak and $|c_e(t)|$ varies slowly compared to the kernel, we apply the Markov approximation:

$$c_e(t') \approx c_e(t), \tag{1.10}$$

which gives

$$\dot{c}_e(t) = -c_e(t)\int_0^t \mathrm{d}\tau\, \Sigma(\tau). \tag{1.11}$$

We define the decay rate via

$$\Gamma = \frac{2\pi}{\hbar}|V(E_e)|^2 \rho(E_e), \tag{1.12}$$

where $\rho(E)$ is the density of states. Then, we find

$$c_e(t) = e^{-\Gamma t/2}. \tag{1.13}$$

The survival probability decays exponentially:

$$P_e(t) = |c_e(t)|^2 = e^{-\Gamma t}. \tag{1.14}$$

1.3 Feshbach Projection Operator Method

We now consider a more general approach where the Hilbert space is partitioned into two orthogonal subspaces via projection operators P and Q, such that

$$P + Q = 1, \quad P^2 = P, \quad Q^2 = Q, \quad PQ = QP = 0. \tag{1.15}$$

Let $|\Psi(t)\rangle$ be the total wavefunction evolving under $\hat{H}$. We apply the projections:

$$P|\Psi(t)\rangle = |\psi_P(t)\rangle, \tag{1.16}$$
$$Q|\Psi(t)\rangle = |\psi_Q(t)\rangle. \tag{1.17}$$

The Schrödinger equation becomes

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}P|\Psi(t)\rangle = P\hat{H}P|\psi_P\rangle + P\hat{H}Q|\psi_Q\rangle, \tag{1.18}$$
$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}Q|\Psi(t)\rangle = Q\hat{H}P|\psi_P\rangle + Q\hat{H}Q|\psi_Q\rangle. \tag{1.19}$$

The second equation can be formally solved:

$$|\psi_Q(t)\rangle = -\frac{i}{\hbar}\int_0^t \mathrm{d}t'\, e^{-\frac{i}{\hbar}Q\hat{H}Q(t-t')}Q\hat{H}P|\psi_P(t')\rangle. \tag{1.20}$$

Substituting into the first equation, we obtain a closed integro-differential equation for $|\psi_P(t)\rangle$:

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\psi_P(t)\rangle = P\hat{H}P|\psi_P(t)\rangle - \frac{1}{\hbar^2}\int_0^t \mathrm{d}t'\, P\hat{H}Qe^{-\frac{i}{\hbar}Q\hat{H}Q(t-t')}Q\hat{H}P|\psi_P(t')\rangle. \tag{1.21}$$

This equation describes non-Markovian dynamics of the system subspace, including memory effects and energy shifts due to coupling with the environment.

This framework lays the foundation for more sophisticated treatments, including master equations and the influence functional formalism. In the next lecture, we will connect these ideas to the density matrix formalism and Lindblad-type equations.

1.4 Time-Independent Embedding of a Discrete State into a Continuum

An alternative to the time-dependent Wigner–Weisskopf treatment is a time-independent scattering theory approach, where we consider how a discrete state becomes embedded in a continuum and acquires a complex energy (i.e., a resonance pole).

We consider a Hamiltonian of the form:

$$\hat{H} = \hat{H}_0 + \hat{V}, \tag{1.22}$$

where:

- $\hat{H}_0$ has a discrete bound state $|e\rangle$ with energy E_e
- A continuum of scattering states $|k\rangle$ with energies E_k

We treat the full Green's function:

$$G(z) = \frac{1}{z - \hat{H}}, \tag{1.23}$$

and define the self-energy via the Feshbach projection:

$$P = |e\rangle\langle e|, \quad Q = 1 - P. \tag{1.24}$$