QUANTUM COMPUTING

Learning Objective

Understand how to read quantum circuits

Student notes are added as annotations throughout

States

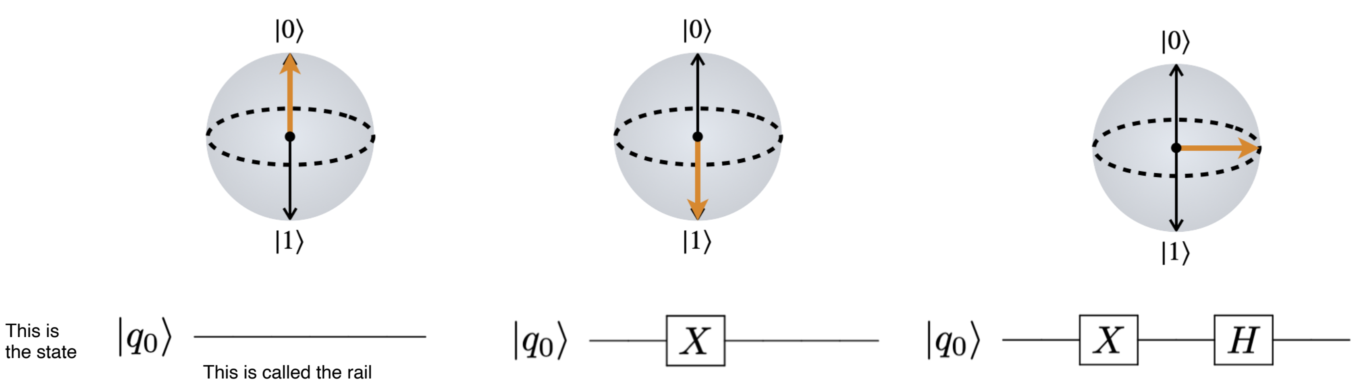
- A single-qubit state can be represented on a Bloch sphere and written as,

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle \tag{1}$$

- This picture works for a static, single qubit picture but if we want to look at gates acting on one or more qubits, often we turn to circuit diagrams

Gates supported on a single qubit

- **Unitary gates** are how we operate on qubits in gate-based quantum computing. A unitary gate in a circuit is equivalent to a unitary matrix or operator in traditional mathematics or quantum mechanics
- An example of a few **single-qubit gates** acting on a qubit state $|q_0\rangle = |0\rangle$:



$|q_0\rangle = |0\rangle$ $\qquad$ $|q_0\rangle = X|0\rangle$ $\qquad$ $|q_0\rangle = HX|0\rangle$

- Note here that if you have multiple qubits, you will have multiple qubit rails and the state will be given by a tensor product of qubits which we can write as

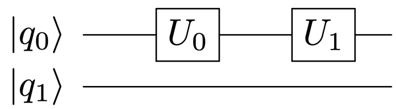
$$|\psi\rangle = |q_0\rangle \otimes |q_1\rangle \tag{2}$$

for two qubits or as,

$$|\psi\rangle = \bigotimes_i |q_i\rangle \tag{3}$$

for more qubits

- As an example circuit with two qubits, we have:



Mathematically we can write this as a produce of each operator,

$$\begin{aligned}|\psi\rangle &= (U_1 \otimes \mathbb{1})(U_0 \otimes \mathbb{1})(|q_0\rangle \otimes |q_1\rangle) \\ &= U_1 U_0 |q_0\rangle \otimes \mathbb{1}\mathbb{1} |q_1\rangle\end{aligned} \tag{4}$$

- In the above we use the **mixed product property** of the Kronecker product,

$$(A \otimes B)(C \otimes D) = AC \otimes BD \tag{5}$$

- ***Exercise 1***
- If a state can be written as a single Kronecker product, then we refer to the state as **separable**. Separable gates are also referred to as **gates supported on one qubit**
- The **Pauli gates** are a common gate set,

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{6}$$

- Single-qubit **rotation gates** are also constructed from exponentiating the Pauli operators,

$$\begin{aligned} R_X(\theta) &= e^{-\frac{iX\theta}{2}} = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) & -i\sin\left(\frac{\theta}{2}\right) \\ -\sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right) \end{pmatrix} \\ R_Y(\theta) &= e^{-\frac{iY\theta}{2}} = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) & -\sin\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right) \end{pmatrix} \\ R_Z(\theta) &= e^{-\frac{iY\theta}{2}} = \begin{pmatrix} e^{-\frac{i\theta}{2}} & 0 \\ 0 & e^{\frac{i\theta}{2}} \end{pmatrix} \end{aligned} \tag{7}$$

The bottom left element should have an i

Note that these are Unitary but NOT Hermitian

- The **Hadamard** gate is another very common gate,

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{8}$$

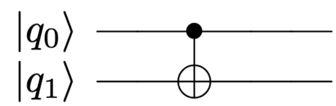
Gates supported on multiple qubits

- Also referred to as 2-qubit or k-qubit gates
- The most common 2-qubit gate is a CNOT (controlled-not) gate,

Sometimes called a CX gate

$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{9}$$

if we order our 2-qubit basis $\{00, 01, 10, 11\}$ and is represented in a circuit as:



The closed dot is called the control and the open is the target

- If we act this CNOT gate on the state where both qubits are initially in their ground state then,

$$\begin{aligned} |\psi\rangle &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} |0\rangle \otimes |0\rangle \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \end{aligned} \tag{10}$$

Physically, we are saying that the second qubit state depends on the first. If the first qubit is in state $|0\rangle$ then nothing will happen to the second qubit, which is exactly what we see here.

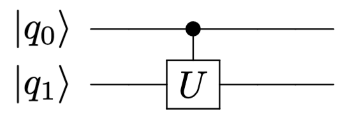
- Note in the above that one needs to be very careful about the basis for multiple qubits – different software will order basis states differently.
- ***Exercise 2***

- A more general 2-qubit control gate is given by,

$$CU = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & u_{00} & u_{01} \\ 0 & 0 & u_{10} & u_{11} \end{pmatrix} \tag{11}$$

These gates are also called entangling gates because the state can no longer be written as a kronecker product

and is represented in the following circuit:

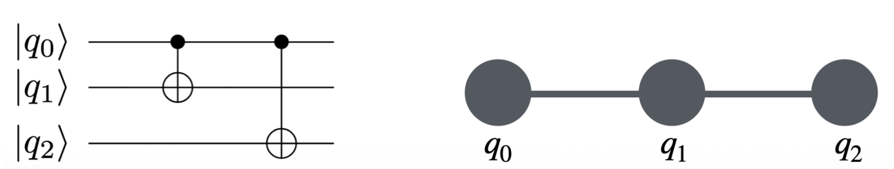


If the **control qubit** is in state $|0\rangle$ then nothing happens to the second qubit (the **target qubit**), if the control qubit is in state $|1\rangle$ then the operator U acts on the target qubit

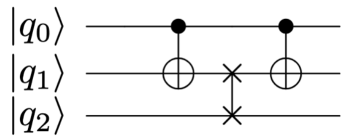
- Sometimes control qubits are only used to alter the target qubits that are more relevant for our algorithm. Qubits that are used to facilitate a computation but do not contain necessary information are often referred to as **ancilla** or **ancillary qubits**

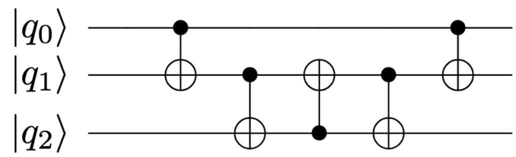
Circuit scaling and things to keep in mind

- We are currently in the **noisy-intermediate scale quantum (NISQ) era** which is defined by small qubit numbers where each qubit is individually fairly prone to errors
- Often NISQ algorithms report complexity in terms of CNOT complexity because these errors are the limiting factor
- The circuit order is the opposite from the ordering of mathematical multiplication
- **Circuit depth:** The number of gates that occur (length of circuit)
- **Circuit volume:** Combination of number of qubits and number of gates
- Connectivity of the device is relevant for circuit depth
- **Native gate set:** which gates are physically realizable on a given hardware platform; relevant for circuit depth
- **Transpilation:** Breaking a unitary down into native gates
- An example of where connectivity matters where the left is a circuit and the right is the hypothetical connection of three qubits:



The way this is written, when the circuit is implemented, a SWAP gate will be necessary:





where the SWAP gate exchanges the states,

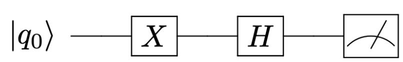
A SWAP gate is equivalent to 3 straight control gates

$$SWAP\big(\,|q_1\rangle \otimes |q_2\rangle\,\big) = |q_2\rangle \otimes |q_1\rangle \tag{12}$$

- ***Exercise 3***

Measurement

- The final step of a quantum circuit is referred to as **measurement** where the state needs to be collapsed to obtain information, which is depicted in the circuit symbol below:



- The process of measuring all the density matrix elements is referred to as **tomography**
- There are many more modern approaches to extract information that bypass full tomography
- Even with perfect qubits, this is still a statistical process and therefore many samples need to be taken to reproduce a quantum state (think flipping a coin, rolling dice)

Often times literature uses ‘sample’ or ‘shot’ to talk about the number of measurements needed to get the distribution