Lecture #3. Exercise

Amplitude Damping Master Equation:

$$\frac{d\rho}{dt} = -i[H,\rho] + \Gamma(\bar{N}+1)\mathcal{D}[\sigma_-]\rho + \Gamma\bar{N}\mathcal{D}[\sigma_+]\rho$$

In our case $H = \frac{\omega}{2}\sigma_z = \frac{\omega}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$\rho(t) = \begin{pmatrix} p(t) & \alpha(t) \\ \alpha^*(t) & 1-p(t) \end{pmatrix}$$

Recall $\mathcal{D}[L]\rho = L\rho L^+ - \frac{1}{2}\{L^+L, \rho\}$

and $\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

Lets perform matrix multiplication separately for each term.

- $-i[H,\rho] = -iH\rho + i\rho H =$

$$= -i\frac{\omega}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} p & \alpha \\ \alpha^* & 1-p \end{pmatrix} + i\frac{\omega}{2}\begin{pmatrix} p & \alpha \\ \alpha^* & 1-p \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} =$$

$$= -i\frac{\omega}{2}\begin{pmatrix} p & \alpha \\ -\alpha^* & p-1 \end{pmatrix} + i\frac{\omega}{2}\begin{pmatrix} p & -\alpha \\ \alpha^* & p-1 \end{pmatrix} = -i\frac{\omega}{2}\begin{pmatrix} 0 & 2\alpha \\ -2\alpha^* & 0 \end{pmatrix} =$$

$$= -i\omega\begin{pmatrix} 0 & -\alpha \\ \alpha^* & 0 \end{pmatrix}$$

- $\mathcal{D}[\sigma_-]\rho = \sigma_-\rho\sigma_-^+ - \frac{1}{2}\{\sigma_-^+\sigma_-, \rho\} = \left\{ \sigma_-^+\sigma_- = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\} =$

$$= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} p & \alpha \\ \alpha^* & 1-p \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} p & \alpha \\ \alpha^* & p \end{pmatrix} - \frac{1}{2}\begin{pmatrix} p & \alpha \\ \alpha^* & p \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 0 \\ p & \alpha \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} p & \alpha \\ 0 & 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} p & 0 \\ \alpha^* & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & p \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 2p & \alpha \\ \alpha^* & 0 \end{pmatrix} = \begin{pmatrix} -p & -\alpha/2 \\ -\alpha^*/2 & p \end{pmatrix}$$

$$\bullet\ \mathcal{D}[\sigma_+]\rho = \sigma_+\rho\sigma_+^\dagger - \frac{1}{2}\{\sigma_+^\dagger\sigma_+, \rho\} = \left\{\sigma_+^\dagger\sigma_+ = \begin{pmatrix}0&0\\1&0\end{pmatrix}\begin{pmatrix}0&1\\0&0\end{pmatrix} = \begin{pmatrix}0&0\\0&1\end{pmatrix}\right\} =$$

$$= \begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}p&\alpha\\\alpha^*&1-p\end{pmatrix}\begin{pmatrix}0&0\\1&0\end{pmatrix} - \frac{1}{2}\begin{pmatrix}0&0\\0&1\end{pmatrix}\begin{pmatrix}p&\alpha\\\alpha^*&1-p\end{pmatrix} - \frac{1}{2}\begin{pmatrix}p&\alpha\\\alpha^*&1-p\end{pmatrix}\begin{pmatrix}0&0\\0&1\end{pmatrix} =$$

$$= \begin{pmatrix}\alpha^*&1-p\\0&0\end{pmatrix}\begin{pmatrix}0&0\\1&0\end{pmatrix} - \frac{1}{2}\begin{pmatrix}0&0\\\alpha^*&1-p\end{pmatrix} - \frac{1}{2}\begin{pmatrix}0&\alpha\\0&1-p\end{pmatrix} =$$

$$= \begin{pmatrix}1-p&0\\0&0\end{pmatrix} - \begin{pmatrix}0&\alpha/2\\\alpha^*/2&1-p\end{pmatrix} = \begin{pmatrix}1-p&-\alpha/2\\-\alpha^*/2&p-1\end{pmatrix}$$

Combining everything together,

$$\frac{d\rho}{dt} = \begin{pmatrix}0&-i\omega\alpha\\i\omega\alpha^*&0\end{pmatrix} + \Gamma(\bar{N}+1)\begin{pmatrix}-p&-\alpha/2\\-\alpha^*/2&p\end{pmatrix} + \Gamma\bar{N}\begin{pmatrix}1-p&-\alpha/2\\-\alpha^*/2&p-1\end{pmatrix}$$

$$\frac{d\rho}{dt} = \begin{pmatrix}dp/dt & d\alpha/dt\\ d\alpha^*/dt & -dp/dt\end{pmatrix}$$

So, $\frac{dp}{dt} = -\Gamma(\bar{N}+1)p + \Gamma\bar{N}(1-p) = -\Gamma\bar{N}p - \Gamma p + \Gamma\bar{N} - \Gamma\bar{N}p =$
$= \Gamma\bar{N} - \Gamma(2\bar{N}+1)p$

$$\frac{d\alpha}{dt} = -i\omega\alpha - \Gamma(\bar{N}+1)\frac{\alpha}{2} - \Gamma\bar{N}\frac{\alpha}{2} =$$
$$= -i\omega\alpha - \frac{\Gamma}{2}(2\bar{N}+1)\alpha$$

$$\boxed{\begin{aligned}\frac{dp}{dt} &= \Gamma\bar{N} - \Gamma(2\bar{N}+1)p \\ \frac{d\alpha}{dt} &= -i\omega\alpha - \frac{\Gamma}{2}(2\bar{N}+1)\alpha\end{aligned}} \quad \begin{aligned}&(1)\\&(2)\end{aligned}$$

The solution for (2) is $\alpha(t) = e^{-[i\omega + \frac{\Gamma}{2}(2\bar{N}+1)]t}$

Let $\gamma = \Gamma(2\bar{N}+1)$, then $\alpha(t) = e^{-[i\omega + \gamma/2]t}$

(1) takes the form $\frac{dp}{dt} = \Gamma\bar{N} - \gamma p$

The solution for (1) is $p(t) = e^{-\gamma t} p(0) + \int_0^t dt' e^{-\gamma(t-t')} \Gamma \bar{N}$

After integration $p(t) = e^{-\gamma t} p(0) + \frac{\bar{N}}{2\bar{N}+1}(1 - e^{-\gamma t})$

So,

$$\boxed{\begin{aligned} p(t) &= e^{-\gamma t} p(0) + \frac{\bar{N}}{2\bar{N}+1}(1-e^{-\gamma t}) \\ \alpha(t) &= e^{-[i\omega + \gamma/2]t} \end{aligned}}$$

- $T_1 = \frac{1}{\gamma} = \frac{1}{\Gamma(2\bar{N}+1)}$ for $T \neq 0K$