

Exercises: Stochastic methods for open quantum systems I

Texas Winter Quantum School

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Exercise 1 (Wide open system limit). *Consider a chain of four one-level quantum systems in the system-bath coupling limit: each site k is coupled to a bath with one Lindblad operator $L_k = |k\rangle\langle k|$. An arbitrary initial state is*

$$|\psi\rangle = \sum_{k=1}^4 c_k |k\rangle; \quad c_k \in \mathbb{C}, \quad \sum_k |c_k|^2 = 1. \quad (1)$$

Define the delocalization entropy

$$S(\psi) = - \sum_k |c_k|^2 \ln(|c_k|^2). \quad (2)$$

(a) Show that $S = 0$ for $|\psi\rangle = |k\rangle$;

(b) Compute $S(\psi)$ for the fully delocalized state

$$|\psi\rangle = \frac{1}{\sqrt{4}} (|1\rangle + |2\rangle + |3\rangle + |4\rangle). \quad (3)$$

Is it higher or lower than for the localized state?

(c) In a wide open system, the dispersion entropy theorem states that

$$\frac{d}{dt} \mathcal{M}_z[S(|\psi_z(t)\rangle)] = - \sum_k \frac{1 - |c_k|^2}{|c_k|^2} \langle \psi_z | L_k | \psi_z \rangle \leq 0. \quad (4)$$

What does this mean for the average delocalization of the wavefunction as a function of time? What should happen if the system starts in a totally delocalized state? A localized state?

(d) Returning to our original formulation of the problem, observe that $|\Psi\rangle$ is the total wave function for the system and bath. Explain what your result for (c) means in terms of this total wave function. What does the dispersion entropy theorem say in terms of the total wave function?

Solution.

- (a) For a state fully localized on site k , $c_k = 1$ and $c_i = 0$ ($i \neq k$). So

$$\begin{aligned}
S(|k\rangle) &= -|c_k|^2 \ln(|c_k|^2) - \sum_{i \neq k} |c_i|^2 \ln(|c_i|^2) \\
&= -(1 \ln 1) - \sum_{i \neq k} (0 \ln 0) \\
&= 0.
\end{aligned} \tag{5}$$

- (b) In the totally delocalized state, $c_k = 1/\sqrt{4} = 1/2$ for all k , and

$$S(|\psi\rangle) = -4 \left(\frac{1}{4} \ln \frac{1}{4} \right) = -\ln \frac{1}{4} = \ln 4. \tag{6}$$

(More generally, for d sites, $c_k = 1/\sqrt{d}$, and $S(|\psi\rangle) = \ln d$.) This is strictly positive, and in particular greater than the delocalization entropy of the localized state $|k\rangle$.

- (c) The entropy of any $|\psi_z\rangle$ decreases over time; thus, the delocalized state will evolve towards a point of minimal entropy (a fully localized state $|k\rangle$, which has the minimum $S = 0$). In $|k\rangle$, the entropy cannot decrease, so the system must stay there; since there are no couplings between sites, either in the system Hamiltonian or in the bath operators, it will stay in $|k\rangle$ for all time.
- (d) Despite the fact that the system is completely uncoupled, the stochastic evolution of the bath drives fluctuations in the amplitudes c_k . Eventually, one will become larger than the others, and the system will localize on that site, effectively transferring its delocalization entropy to the bath.

□

Exercise 2 (Limitations of non-Markovian quantum state diffusion). *Looking back over this method, what are the major limitations of the method presented? We spoke at the very beginning about the long timescale of method development, so there have been a lot of developments since this method was first developed - what are the key developments you would want to go look for in the literature if you were interested in this method?*

Solution. Limitations include

1. The functional derivative in the non-Markovian kernel is not at all amenable to easy calculation.
2. We used Bargmann states in our derivation, so this equation is propagating an unnormalized wavefunction. This will lead to difficulty interpreting amplitudes as connected to probabilities.
3. We evolve only pure states, not general density matrices.

In extensions of the method, I would look for solutions to some or all of these problems. □