Exercises: Quantum Information IV

Texas Winter Quantum School

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Exercise 1 (Quantum mutual information)**.** *The* mutual information *measures correlations, both quantum and classical, in a bipartite state ρ_{AB}, and is defined as*

$$I(A:B) = \min_{\sigma_A,\sigma_B} D(\rho_{AB}\|\sigma_A \otimes \sigma_B); \tag{1}$$

it measures how far ρ_{AB} is from a product state.

Show that

(a) $I(A:B) = D(\rho_{AB}\|\rho_A \otimes \rho_B)$*;*

(b) $I(A:B) = S(\rho_A) + S(\rho_B) - S(\rho_{AB})$*.*

Solution. (a) Recall that the relative entropy

$$D(\rho\|\sigma) = \mathrm{Tr}\,[\rho \log \rho - \rho \log \sigma], \tag{2}$$

so

$$I(A:B) = \min_{\sigma_A,\sigma_B} \mathrm{Tr}\,(\rho_{AB} \log \rho_{AB} - \rho_{AB} \log \sigma_A \otimes \sigma_B). \tag{3}$$

We can add and subtract the same thing:

$$\begin{aligned} I(A:B) &= \min_{\sigma_A,\sigma_B} \mathrm{Tr}\,[\rho_{AB} \log \rho_{AB} - \rho_{AB} \log (\sigma_A \otimes \sigma_B)] \\ &\qquad + \min_{\sigma_A,\sigma_B} \mathrm{Tr}\,[\rho_{AB} \log (\rho_A \otimes \rho_B) - \rho_{AB} \log (\rho_A \otimes \rho_B)] \\ &= \mathrm{Tr}\,[\rho_{AB} \log \rho_{AB} - \rho_{AB} \log (\rho_A \otimes \rho_B)] \\ &\qquad + \min_{\sigma_A,\sigma_B} \mathrm{Tr}\,[\rho_{AB} \log (\rho_A \otimes \rho_B) - \rho_{AB} \log (\sigma_A \otimes \sigma_B)]. \end{aligned} \tag{4}$$

Consider the terms in the minimization. We make use of the identity

$$\mathrm{Tr}\,[A_{AB} \log (B_A \otimes B_B)] = \mathrm{Tr}\,[A_{AB} \log B_A] + \mathrm{Tr}\,[A_{AB} \log B_B] \tag{5}$$

to observe that

$$\begin{aligned} &\min_{\sigma_A,\sigma_B} \mathrm{Tr}\,[\rho_{AB} \log (\rho_A \otimes \rho_B) - \rho_{AB} \log (\sigma_A \otimes \sigma_B)] = \\ &\qquad \min_{\sigma_A} \mathrm{Tr}\,[\rho_{AB} \log \rho_A - \rho_{AB} \log \sigma_A] + \\ &\qquad\qquad \min_{\sigma_B} \mathrm{Tr}\,[\rho_{AB} \log \rho_A - \rho_{AB} \log \sigma_B]; \end{aligned} \tag{6}$$

next, we note that we may take partial traces in sequence,

$$\operatorname{Tr} A_{AB} = \operatorname{Tr}_A [\operatorname{Tr}_B A_{AB}] = \operatorname{Tr}_B [\operatorname{Tr}_A A_{AB}], \tag{7}$$

and that $\operatorname{Tr}_A \rho_{AB} = \rho_B$, $\operatorname{Tr}_B \rho_{AB} = \rho_A$. This yields

$$\begin{aligned} &\min_{\sigma_A} \operatorname{Tr} [\rho_{AB} \log \rho_A - \rho_{AB} \log \sigma_A] = \\ &\qquad \min_{\sigma_A} \operatorname{Tr}_A [\operatorname{Tr}_B (\rho_{AB} \log \rho_A - \rho_{AB} \log \sigma_A)] = \\ &\qquad\qquad \min_{\sigma_A} \operatorname{Tr}_A [\rho_A \log \rho_A - \rho_A \log \sigma_A]. \end{aligned} \tag{8}$$

This is the minimum over σ_A of the relative entropy $D(\rho_A \| \sigma_A) \geq 0$; it attains its minimum when $\sigma_A = \rho_A$, which is contained in the set of allowable σ_A, so this term vanishes.

By the same logic, the minimization over σ_B also yields zero, and we are left with

$$I(A:B) = \operatorname{Tr} [\rho_{AB} \log \rho_{AB} - \rho_{AB} \log (\rho_A \otimes \rho_B)], \tag{9}$$

which we recognize as $D(\rho_{AB} \| \rho_A \otimes \rho_B)$.

(b) We use the same trace identity to rewrite

$$I(A:B) = \operatorname{Tr} [\rho_{AB} \log \rho_{AB}] - \operatorname{Tr} [\rho_{AB} \log \rho_A] - \operatorname{Tr} [\rho_{AB} \log \rho_B]. \tag{10}$$

Recalling that the entropy is defined as $S(\rho) = \operatorname{Tr} [\rho \log \rho]$, we immediately recognize the first term as $-S(\rho_{AB})$. For the second term, we perform the partial trace over B first to see that

$$\begin{aligned} -\operatorname{Tr} [\rho_{AB} \log \rho_A] &= -\operatorname{Tr}_A [\operatorname{Tr}_B (\rho_{AB}) \log \rho_A] = -\operatorname{Tr}_A [\rho_A \log \rho_A] \\ &= +S(\rho_A); \end{aligned} \tag{11}$$

by an exactly analogous argument, $-\operatorname{Tr} [\rho_{AB} \log \rho_B] = S(\rho_B)$. So we obtain

$$I(A:B) = S(\rho_A) + S(\rho_B) - S(\rho_{AB}), \tag{12}$$

as desired. $\square$