EXERCISE 1: FIND THE ISOMETRY THAT WE CAN USE TO OBTAIN THE KRAUS OPERATORS FOR THE AMPLITUDE-DAMPING CHANNEL

The phenomenological process description of this process is:

$$|0\rangle_S \otimes |0\rangle_E \longrightarrow |0\rangle_S \otimes |0\rangle_E \tag{1}$$

$$|1\rangle_S \otimes |0\rangle_E \longrightarrow \sqrt{p}\,|0\rangle_S \otimes |1\rangle_E + \sqrt{1-p}\,|1\rangle_S \otimes |0\rangle_E \tag{2}$$

We can define an **isometry** V, which is a distance preserving,

$$V^\dagger V = \mathbb{1}, \tag{3}$$

such that,

$$V : \mathbb{C}^2 \longrightarrow \mathbb{C}^4. \tag{4}$$

The isometry that we are looking for will produce the following,

$$V\,|0\rangle_S = |0\rangle_S \otimes |0\rangle_E \tag{5}$$

and

$$V\,|1\rangle_S = \sqrt{p}\,|0\rangle_S \otimes |1\rangle_E + \sqrt{1-p}\,|1\rangle_S \otimes |0\rangle_E\,, \tag{6}$$

where the action of the isometry is on the system space and results in a state in the system and environment space.

We can construct this isometry in matrix form by considering the columns to be system states $|0\rangle_S$ and $|1\rangle_S$, and the rows to be composite system-environment states $|0\rangle_S \otimes |0\rangle_E$, $|0\rangle_S \otimes |1\rangle_E$, $|1\rangle_S \otimes |0\rangle_E$, and $|1\rangle_S \otimes |1\rangle_E$,

$$V = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{p} \\ 0 & \sqrt{1-p} \\ 0 & 0 \end{pmatrix}. \tag{7}$$

EXERCISE 2: VERIFY THAT THIS ISOMETRY IS DISTANCE PRESERVING

$$\begin{aligned}
V^\dagger V &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{p} & \sqrt{1-p} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{p} \\ 0 & \sqrt{1-p} \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & p+1-p \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \mathbb{1}_S
\end{aligned} \tag{8}$$

EXERCISE 3: DERIVE THE KRAUS OPERATORS FROM THE ISOMETRY ABOVE

$$M_k = (\mathbb{1} \otimes \langle k|)V, \tag{9}$$

where $\langle k|$ are the bath states.

For the amplitude damping channel, we will have 2 Kraus operators:

$$\begin{aligned}
M_0 &= (\mathbb{1} \otimes \langle 0|) \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{p} \\ 0 & \sqrt{1-p} \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{p} \\ 0 & \sqrt{1-p} \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}
\end{aligned} \tag{10}$$

and

$$M_1 = (\mathbb{1} \otimes \langle 1|) \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{p} \\ 0 & \sqrt{1-p} \\ 0 & 0 \end{pmatrix} \tag{11}$$
$$= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{p} \\ 0 & \sqrt{1-p} \\ 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix}.$$

EXERCISE 4: FIND THE ISOMETRY THAT WE CAN USE TO OBTAIN THE KRAUS OPERATORS FOR THE PHASE-DAMPING CHANNEL

A phenomenological process description is:

$$|0\rangle_S \longrightarrow \sqrt{1-p}\,|0\rangle_S \otimes |0\rangle_E + \sqrt{p}\,|0\rangle_S \otimes |1\rangle_E \tag{12}$$
$$|1\rangle_S \longrightarrow \sqrt{1-p}\,|1\rangle_S \otimes |0\rangle_E + \sqrt{p}\,|1\rangle_S \otimes |2\rangle_E \tag{13}$$

We can define an **isometry** V such that,

$$V : \mathbb{C}^2 \longrightarrow \mathbb{C}^6. \tag{14}$$

The isometry that we are looking for will produce the following,

$$V\,|0\rangle_S = \sqrt{1-p}\,|0\rangle_S \otimes |0\rangle_E + \sqrt{p}\,|0\rangle_S \otimes |1\rangle_E \tag{15}$$

and

$$V\,|1\rangle_S = \sqrt{1-p}\,|1\rangle_S \otimes |0\rangle_E + \sqrt{p}\,|1\rangle_S \otimes |2\rangle_E\,, \tag{16}$$

where the action of the isometry is on the system space and results in a state in the system and environment space. Noting here that the system is energetically unchanged but the environment scatters.

We can construct V such that we have two columns, representing the system states $|0\rangle$ and $|1\rangle$ respectively, and then six rows representing the combined system-environment states $|0\rangle \otimes |0\rangle$, $|0\rangle \otimes$

$|1\rangle, |0\rangle \otimes |2\rangle, |1\rangle \otimes |0\rangle, |1\rangle \otimes |1\rangle$, and $|1\rangle \otimes |2\rangle$ respectively. With this basis and the maps defined above we have,

$$V = \begin{pmatrix} \sqrt{1-p} & 0 \\ \sqrt{p} & 0 \\ 0 & 0 \\ 0 & \sqrt{1-p} \\ 0 & 0 \\ 0 & \sqrt{p} \end{pmatrix}.$$

EXERCISE 5: VERIFY THAT THIS ISOMETRY IS DISTANCE PRESERVING

$$\begin{aligned} V^\dagger V &= \begin{pmatrix} sqrt 1-p & \sqrt{p} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{1-p} & 0 & \sqrt{p} \end{pmatrix} \begin{pmatrix} \sqrt{1-p} & 0 \\ \sqrt{p} & 0 \\ 0 & 0 \\ 0 & \sqrt{1-p} \\ 0 & 0 \\ 0 & \sqrt{p} \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{1-p}\sqrt{1-p} + \sqrt{p}\sqrt{p} & 0 \\ 0 & \sqrt{1-p}\sqrt{1-p} + \sqrt{p}\sqrt{p} \end{pmatrix} \\ &= \begin{pmatrix} 1-p+p & 0 \\ 0 & 1-p+p \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & = \mathbb{1}_S \end{aligned}$$

EXERCISE 6: DERIVE THE KRAUS OPERATORS FROM THE ISOMETRY ABOVE

$$(\mathbb{1}_S \otimes \langle k|)V,$$

for all $|k\rangle = |0\rangle, |1\rangle, |2\rangle$ for the environment.

Starting with $|0\rangle$ for the environment,

$$
\begin{aligned}
M_0 &= (\mathbb{1}_S \otimes \langle 0|)V \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{1-p} & 0 \\ \sqrt{p} & 0 \\ 0 & 0 \\ 0 & \sqrt{1-p} \\ 0 & 0 \\ 0 & \sqrt{p} \end{pmatrix} \\
&= \begin{pmatrix} \sqrt{1-p} & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}
\end{aligned}
$$

Next using $|1\rangle$ for the environment,

$$
\begin{aligned}
M_1 &= (\mathbb{1}_S \otimes \langle 1|)V \\
&= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{1-p} & 0 \\ \sqrt{p} & 0 \\ 0 & 0 \\ 0 & \sqrt{1-p} \\ 0 & 0 \\ 0 & \sqrt{p} \end{pmatrix} \\
&= \begin{pmatrix} \sqrt{p} & 0 \\ 0 & 0 \end{pmatrix}
\end{aligned}
$$

And finally using $|2\rangle$ for the environment,

$$
\begin{aligned}
M_2 &= (\mathbb{1}_S \otimes \langle 2|)V \\
&= \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{1-p} & 0 \\ \sqrt{p} & 0 \\ 0 & 0 \\ 0 & \sqrt{1-p} \\ 0 & 0 \\ 0 & \sqrt{p} \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{p} \end{pmatrix}.
\end{aligned}
$$