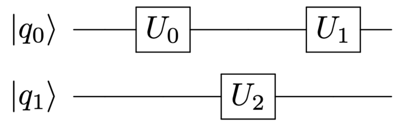
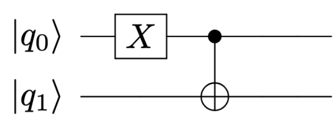
EXERCISE 1: WRITE OUT THE MATRIX MULTIPLICATION REPRESENTED BY THE CIRCUIT DIAGRAM:



Solution:

$$
\begin{aligned}
|\psi\rangle &= (U_1 \otimes \mathbb{1})(\mathbb{1} \otimes U_2)(U_0 \otimes \mathbb{1})(|q_0\rangle \otimes |q_1\rangle) \\
&= U_1 U_0 |q_0\rangle \otimes U_2 |q_1\rangle
\end{aligned}
$$

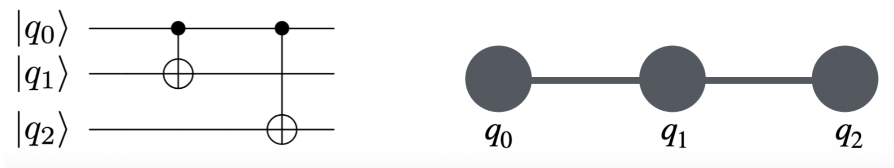
EXERCISE 2: WRITE OUT THE MATRIX MULTIPLICATION REPRESENTED BY THE CIRCUIT DIAGRAM:



Solution:

$$
\begin{aligned}
|\psi\rangle &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \Big(X|0\rangle \otimes |0\rangle \Big) \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \Big(|1\rangle \otimes |0\rangle \Big) \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}
\end{aligned}
$$

EXERCISE 3: CONSIDER THE FOLLOWING CIRCUIT AND QUBIT LAYOUT:



A natural mapping would be to say that the qubit furthest to the left is $|q_0\rangle$, the middle one is $|q_1\rangle$, and the one on the right is $|q_2\rangle$. Why would this be a bad mapping?

Solution: In the circuit, $|q_0\rangle$ needs to interact directly with both $|q_1\rangle$ and $|q_2\rangle$. It would be easier to have this qubit be in the center so it has direct connectivity for the required gates.

EXERCISE 4: WHY IS THE KRAUS OPERATOR HARDER TO MAP INTO QUANTUM GATES THAN THE HAMILTONIAN?

Solution: M_k is not necessarily unitary! If there is only one M_k then $M_k^\dagger M_k = \mathbb{1}$; however that is the exception and not the norm. Generally, $\sum_k M_k M_k^\dagger = \mathbb{1}$ but each individual operator is not unitary.

EXERCISE 5: CHECK THAT S IS SYMMETRIC AND A IS ANTISYMMETRIC

Solution:

$$S^\dagger = \left(\frac{M + M^\dagger}{2}\right)^\dagger = \frac{M^\dagger + M}{2} = S$$

$$A^\dagger = \left(\frac{M - M^\dagger}{2}\right)^\dagger = \frac{M^\dagger - M}{2} = -A$$

EXERCISE 6: VERIFY THE TAYLOR EXPANSION VERSIONS OF S AND A

Solution:

For S:

$$e^{-i\epsilon S} = 1 - i\epsilon S + \mathcal{O}(\epsilon^2)$$
$$e^{i\epsilon S} = 1 + i\epsilon S + \mathcal{O}(\epsilon^2)$$

then take the difference and reshuffle,

$$
\begin{aligned}
e^{-i\epsilon S} - e^{i\epsilon S} &= 1 - i\epsilon S + \mathcal{O}(\epsilon^2) - (1 + i\epsilon S + \mathcal{O}(\epsilon^2)) \\
e^{-i\epsilon S} - e^{i\epsilon S} &= -2i\epsilon S \\
S &= \frac{1}{-2i\epsilon}\left(e^{-i\epsilon S} - e^{i\epsilon S}\right) \\
S &= \frac{i}{2\epsilon}\left(e^{-i\epsilon S} - e^{i\epsilon S}\right)
\end{aligned}
$$

For A:

$$
\begin{aligned}
e^{\epsilon A} &= 1 + \epsilon A + \mathcal{O}(\epsilon^2) \\
e^{-\epsilon A} &= 1 - \epsilon A + \mathcal{O}(\epsilon^2)
\end{aligned}
$$

then take the difference and reshuffle,

$$
\begin{aligned}
e^{\epsilon A} - e^{-\epsilon A} &= 1 + \epsilon A + \mathcal{O}(\epsilon^2) - (1 - \epsilon A + \mathcal{O}(\epsilon^2)) \\
e^{\epsilon A} - e^{\epsilon A} &= 2\epsilon A \\
A &= \frac{1}{2\epsilon}\left(e^{\epsilon A} - e^{-\epsilon A}\right)
\end{aligned}
$$

EXERCISE 7: VERIFY THESE FOUR OPREATORS ARE UNITARY.

Solution:

$$
\begin{aligned}
(ie^{-i\epsilon S})^\dagger ie^{-i\epsilon S} &= -ie^{i\epsilon S^\dagger} ie^{-i\epsilon S} = e^{i\epsilon S}e^{-i\epsilon S} = \mathbb{1} \\
(-ie^{i\epsilon S})^\dagger (-i)e^{i\epsilon S} &= ie^{-i\epsilon S^\dagger}(-i)e^{i\epsilon S} = e^{-i\epsilon S}e^{i\epsilon S} = \mathbb{1} \\
(e^{\epsilon A})^\dagger e^{\epsilon A} &= (e^{\epsilon A^\dagger})e^{\epsilon A} = e^{-\epsilon A}e^{\epsilon A} = \mathbb{1} \\
(e^{-\epsilon A})^\dagger e^{-\epsilon A} &= (e^{-\epsilon A^\dagger})e^{-\epsilon A} = e^{\epsilon A}e^{-\epsilon A} = \mathbb{1}
\end{aligned}
$$

EXERCISE 8: LINEAR COMBINATION OF UNITARIES

Consider a simpler problem,

$$\begin{pmatrix} U_0 & 0 \\ 0 & U_1 \end{pmatrix} \begin{pmatrix} |\psi\rangle \\ |\psi\rangle \end{pmatrix} = \begin{pmatrix} U_0 |\psi\rangle \\ U_1 |\psi\rangle \end{pmatrix} \tag{1}$$

The probability of measuring the ancilla in the state $|0\rangle$ is $P(0) = \langle\psi| U_0^\dagger U_0 |\psi\rangle$ and in the state $|1\rangle$, $P(1) = \langle\psi| U_1^\dagger U_1 |\psi\rangle$. These are not the same thing as $\langle\psi| M^\dagger M |\psi\rangle$ because this requires a sum of the unitaries. So what can we do differently?

Solution: Rotation!

$$\begin{pmatrix} \mathbb{1} & \mathbb{1} \\ \mathbb{1} & -\mathbb{1} \end{pmatrix} \begin{pmatrix} U_0 & 0 \\ 0 & U_1 \end{pmatrix} \begin{pmatrix} |\psi\rangle \\ |\psi\rangle \end{pmatrix} = \begin{pmatrix} \mathbb{1} & \mathbb{1} \\ \mathbb{1} & -\mathbb{1} \end{pmatrix} \begin{pmatrix} U_0 |\psi\rangle \\ U_1 |\psi\rangle \end{pmatrix} \tag{2}$$

$$= \begin{pmatrix} (U_0 + U_1) |\psi\rangle \\ (U_0 - U_1) |\psi\rangle \end{pmatrix} \tag{3}$$