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ECE 3793 – Project 2

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Introduction

In this project, I will be using given transmission signals ($x_{TX}(t)$), received signals($x_{RX}(t)$), and a desired audio waveform ($x_a(t)$) to find a low-pass filter, such that the audio waveform $x_a(t)$ can be output cleanly after transmission.

Part I(a): Derivation for the Transmitter

Given a signal with a frequency of 500 Hz,

$$x(t) = \cos(2\pi \times 500t)$$

and an arbitrary carrier frequency (F_c), to which the carrier signal is

$$x_c(t) = \cos(2\pi F_c t)$$

Calculation 1: $x_{TX}(t)$ as a Sum of Cosines

$$x_{TX}(t) = x(t)x_c(t)$$

Using the known signals, $x(t)$ and $x_c(t)$, we can substitute these functions into $x_{TX}(t)$.

$$x_{TX}(t) = (\cos(2\pi \times 500t))(\cos(2\pi F_c t))$$

Using the trigonometric identity

$$\cos(u)\cos(v) = \frac{1}{2}[\cos(u+v) + \cos(u-v)]$$

Substituting in $x(t)$ for the cos(u) term, and $x_c(t)$ for the cos(v) term,

$$\cos(2\pi \times 500)\cos(2\pi F_c t) = \frac{1}{2}[\cos(2\pi \times 500t + 2\pi F_c t) + \cos(2\pi \times 500t - 2\pi F_c t)]$$

Resulting in the equation

$$x_{TX}(t) = \frac{1}{2}\Big[\cos(2\pi \times 500t + 2\pi F_c t) + \cos(2\pi \times 500t - 2\pi F_c t)\Big]$$

Then distributing the 1/2 to each cosine

$$x_{TX}(t) = \frac{1}{2}\cos(2\pi \times 500t + 2\pi F_c t) + \frac{1}{2}\cos(2\pi \times 500t - 2\pi F_c t)$$

and factoring out 2π from each cosine term

$$x_{TX}(t) = \frac{1}{2}\cos(2\pi(500t + F_c t)) + \frac{1}{2}\cos(2\pi(500t - F_c t))$$

And finally, factoring out t from each term.

$$x_{TX}(t) = \frac{1}{2}\cos(2\pi t(500 + F_c)) + \frac{1}{2}\cos(2\pi t(500 - F_c))$$

Calculation 1 Results:

As shown through using trigonometic identities, and basic algebraic techniques, $x_{TX}(t)$ can be expressed as a sum of cosines as follows.

$$\boxed{x_{TX}(t) = \frac{1}{2}\cos(2\pi t(500 + F_c)) + \frac{1}{2}\cos(2\pi t(500 - F_c))}$$

Calculation 2: The Fourier Transform of $x_{TX}(t)$

The Fourier Transform for a continuous signal is defined as

$$X(F) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi Ft}dt$$

Thankfully, the function $x_{TX}(t)$ has a known Fourier Transform pair. The known pairing is

$$x(t) = \cos(2\pi F_0 t) \xleftrightarrow{\mathcal{F}} X(F) = \frac{1}{2}[\delta(F - F_0) + \delta(F + F_0)]$$

Since a property of the Fourier Transform is linearity, this means that we can split the function,

$$x_{TX}(t) = \frac{1}{2}\cos(2\pi t(500 + F_c)) + \frac{1}{2}\cos(2\pi t(500 - F_c))$$

into two parts and apply the transform to each part. The first cosine term is

$$\begin{aligned}
\frac{1}{2}\cos(2\pi t(500 + F_c)) \xleftrightarrow{\mathcal{F}} \frac{1}{2}\left[\frac{1}{2}[\delta(F - 500 + F_c) + \delta(F + 500 + F_c)]\right] \\
= \frac{1}{4}\Big[\delta(F - 500 + F_c) + \delta(F + 500 + F_c)\Big] \\
= \frac{1}{4}\delta(F - 500 + F_c) + \frac{1}{4}\delta(F + 500 + F_c)
\end{aligned}$$

Then the second cosine term,

$$\begin{aligned}
\frac{1}{2}\cos(2\pi t(500 - F_c)) \xleftrightarrow{\mathcal{F}} \frac{1}{2}\left[\frac{1}{2}[\delta(F - 500 - F_c) + \delta(F + 500 - F_c)]\right] \\
= \frac{1}{4}\Big[\delta(F - 500 - F_c) + \delta(F + 500 - F_c)\Big] \\
= \frac{1}{4}\delta(F - 500 - F_c) + \frac{1}{4}\delta(F + 500 - F_c)
\end{aligned}$$

Then, referencing back to the linearity property, we are able to add these two transformed terms back together to have the final Fourier Transform, $X_{TX}(F)$.

$$X_{TX}(F) = \frac{1}{4}\delta(F - 500 + F_c) + \frac{1}{4}\delta(F + 500 + F_c) + \frac{1}{4}\delta(F - 500 - F_c) + \frac{1}{4}\delta(F + 500 - F_c)$$

Finally, we can factor out the 1/4 term from each cosine, giving us our final answer.

$$\boxed{X_{TX}(F) = \frac{1}{4}\left[\delta(F - 500 + F_c) + \delta(F + 500 + F_c) + \delta(F - 500 - F_c) + \delta(F + 500 - F_c)\right]}$$

Calculation 3:

It is important to verify that the pairs and properties of the Fourier Transform were applied properly. In order to do this, the Fourier Transforms of both $x(t)$ and $x_c(t)$ will be performed. Beginning with $x_t(t)$, we can note that the signal can be used with the same pair used in finding the Fourier Transform of $X_{TX}(F)$ being,

$$x(t) = \cos(2\pi F_0 t) \xleftrightarrow{\mathcal{F}} X(F) = \frac{1}{2}[\delta(F - F_0) + \delta(F + F_0)]$$

Substituting in our $x(t)$ gives

$$x(t) = \cos(2\pi \times 500t) \xleftrightarrow{\mathcal{F}} X(F) = \frac{1}{2}[\delta(F - 500) + \delta(F + 500)]$$

$$x(t) = \cos(2\pi \times 500t) \xleftrightarrow{\mathcal{F}} X(F) = \frac{1}{2}\delta(F - 500) + \frac{1}{2}\delta(F + 500)$$

and doing the same for $x_c(t)$

$$x_c(t) = \cos(2\pi F_c t) \xleftrightarrow{\mathcal{F}} X_c(F) = \frac{1}{2}[\delta(F - Fc) + \delta(F + Fc)]$$

$$x_c(t) = \cos(2\pi F_c t) \xleftrightarrow{\mathcal{F}} X_c(F) = \frac{1}{2}\delta(F - Fc) + \frac{1}{2}\delta(F + Fc)$$

This results in our final calculations for $X(F)$ and $X_c(F)$ to be

$$\boxed{X(F) = \frac{1}{2}\delta(F - 500) + \frac{1}{2}\delta(F + 500)}$$

$$\boxed{X_c(F) = \frac{1}{2}\delta(F - Fc) + \frac{1}{2}\delta(F + Fc)}$$

Calculation 4:

With these two Fourier Transforms, and the multiplication property,

$$x(t)y(t) \xleftrightarrow{\mathcal{F}} X(F) * Y(F)$$

we can define the Fourier Transform of $x_{TX}(t)$ as

$$X_{TX}(F) = X(F) * X_C(F)$$

Substituing the values found above for $X(F)$ and $X_c(F)$, we have the new function

$$X_{TX}(F) = \frac{1}{2}\delta(F-500) + \frac{1}{2}\delta(F+500) * \frac{1}{2}\delta(F-Fc) + \frac{1}{2}\delta(F+Fc)$$

To simplify the terms of this $X_{TX}(F)$, the function must be convolved. To do this, the shifting and linearity properties of convolution must be applied.

$$x(t) * \delta(t-t_0) = x(t-t_0)$$

For ease of viewing, each term in the convolution has been labeled as follows

$$X_{TX}(F) = \overset{①}{\frac{1}{2}\delta(F-500)} + \overset{②}{\frac{1}{2}\delta(F+500)} * \overset{③}{\frac{1}{2}\delta(F-F_c)} + \overset{④}{\frac{1}{2}\delta(F+F_c)} \tag{1}$$

each term on the right hand side, ③ and ④, must be applied to each term on the left hand side, ① and ②.

$$\overset{①}{\frac{1}{2}\delta(F-500)} * \overset{③}{\frac{1}{2}\delta(F-F_c)} = \frac{1}{4}\delta(F-500-F_C)$$

$$\overset{①}{\frac{1}{2}\delta(F-500)} * \overset{④}{\frac{1}{2}\delta(F+F_c)} = \frac{1}{4}\delta(F-500+F_C)$$

$$\overset{②}{\frac{1}{2}\delta(F+500)} * \overset{③}{\frac{1}{2}\delta(F-F_c)} = \frac{1}{4}\delta(F+500-F_C)$$

$$\overset{②}{\frac{1}{2}\delta(F+500)} * \overset{④}{\frac{1}{2}\delta(F+F_c)} = \frac{1}{4}\delta(F+500+F_C)$$

Then, referencing back to the linearity property, we are able to add these terms back together to have the final Fourier Transform, $X_{TX}(F)$.

$$X_{TX}(F) = \frac{1}{4}\delta(F-500+F_c) + \frac{1}{4}\delta(F+500+F_c) + \frac{1}{4}\delta(F-500-F_c) + \frac{1}{4}\delta(F+500-F_c)$$

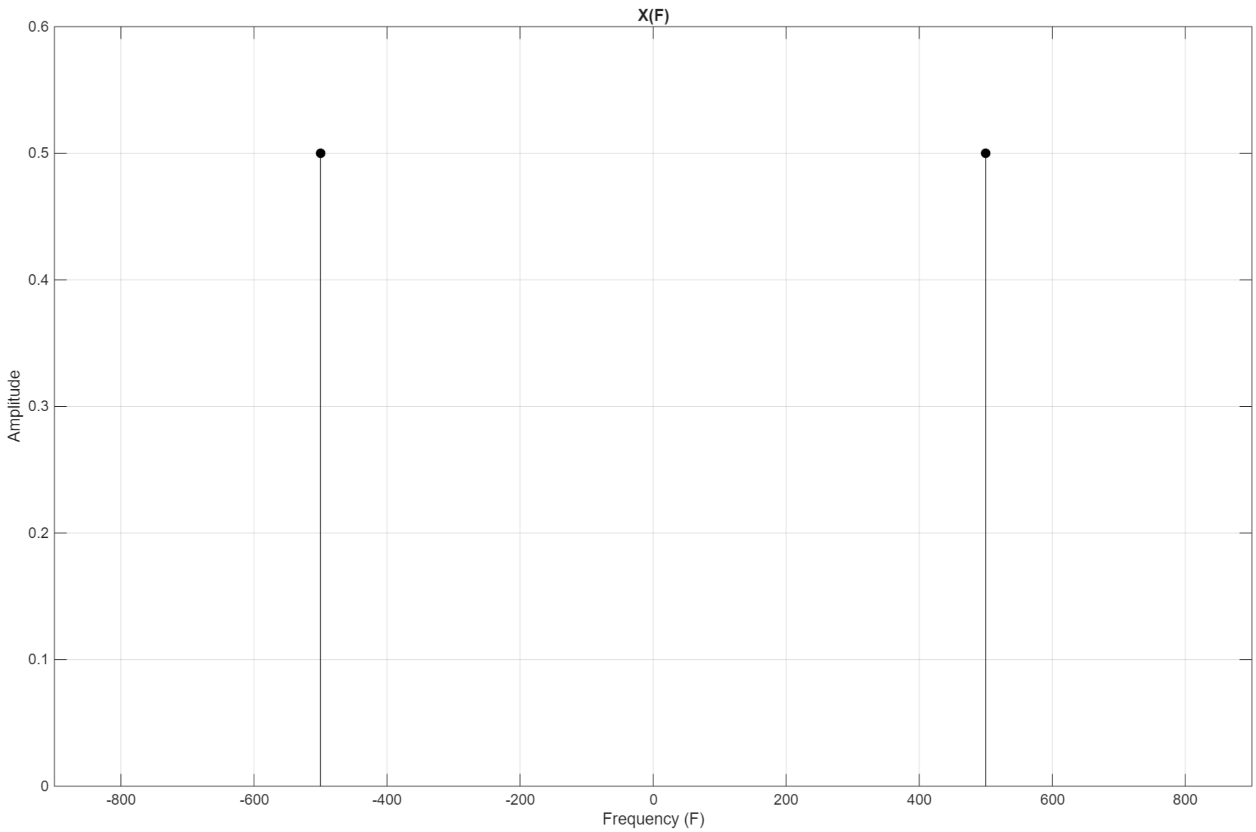
And we can factor out the 1/4 term from each cosine, giving us our final answer.

$$\boxed{X_{TX}(F) = \frac{1}{4}\left[\delta(F-500+F_c) + \delta(F+500+F_c) + \delta(F-500-F_c) + \delta(F+500-F_c)\right]} \tag{2}$$

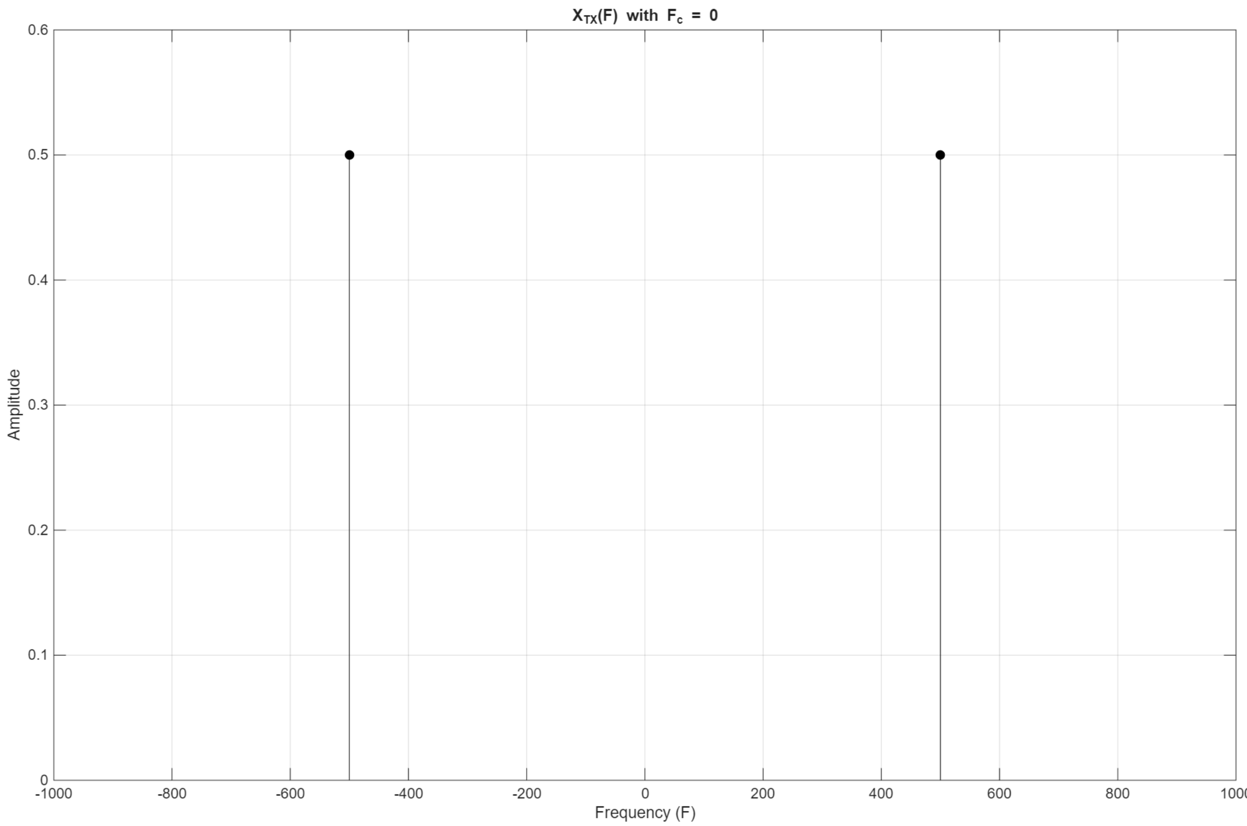
Comparing this to the $X_{TX}(F)$ found in **Calculation 2**, we can see that both of these ways result in the same answer, verifying the linearity and convolution properties of the Fourier Transform.

Question 1: How is $X_{TX}(F)$ related to $X(F)$?

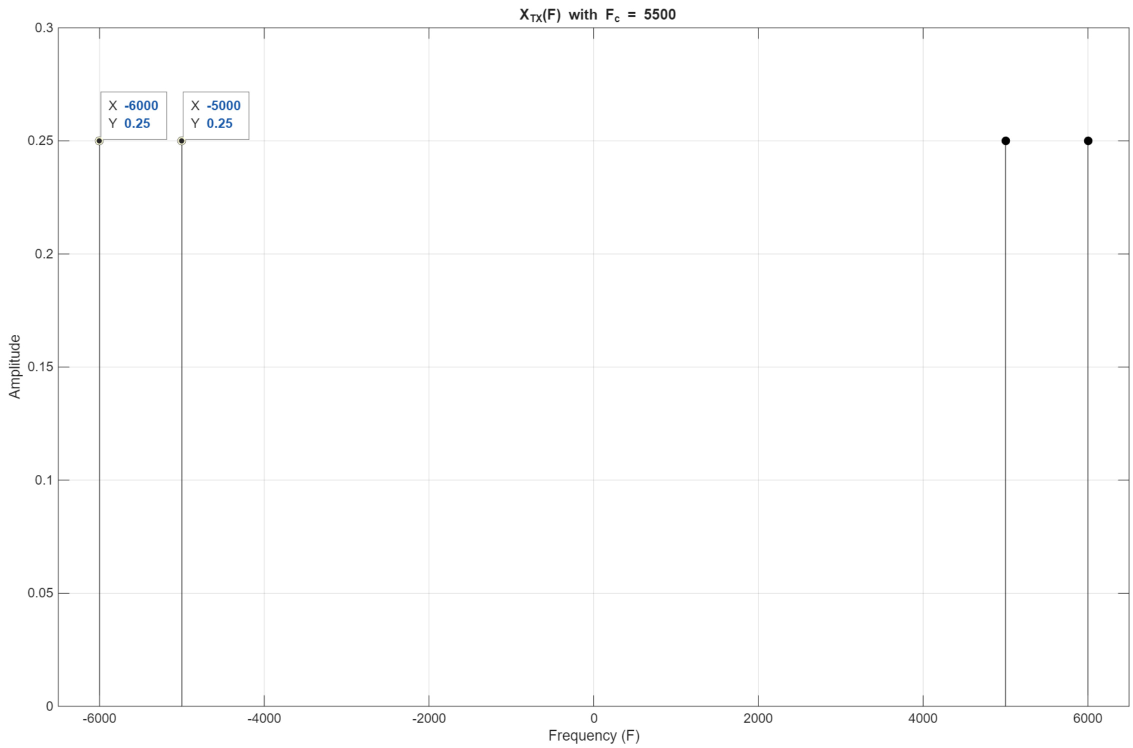
Let's begin by looking at a graph of $X(F)$.



Above we can see two stems, located at -500 and +500, with a height of 1/2. With the signal being centered around 0. Now let us look at a graph for $X_{TX}(F)$, with an F_c of 0.



We can see that this is the same graph. Looking at a graph where $F_c = 5500$, we can see a change in the graph.



Looking at this graph, we can see that when the signal is shifted by F_c, it is bound by $X(F)$. This tells us that **the relation between $X_{TX}(F)$ and $X(F)$ is that $X(F)$ bounds $X_{TX}(F)$ into 500 Hz frequency bands, which are then shifted by F_c.**

Now, should we have been given an audio waveform $x_a(t)$, given only the knowledge that it has a Fourier transform $X_a(F)$ which only has non-zero values for $|F| < 22$kHz and it is a real signal, meaning that it is conjugate symmetric.

We shall now redefine our transmission signal by

$$x_{TX}(t) = x_a(t)x_c(t)$$

Calculation 5: $X_{TX}(F)$ in terms of $X_a(F)$

Despite not knowing the signal $x_a(t)$, we will still take the Fourier transform to find our new, transformed signal, $X_{TX}(F)$. For now, we will define the Fourier transform of $x_a(t)$ as $X_a(F)$.

$$x_{TX}(t) = x_a(t)x_c(t) \xleftrightarrow{\mathcal{F}} X_{TX}(F) = X_a(F) * X_c(F)$$

Since we have previously defined $X_c(F)$, we can easily substitute it in.

$$X_{TX}(F) = X_a(F) * \left(\frac{1}{2}\delta(F - Fc) + \frac{1}{2}\delta(F + Fc)\right)$$

Now, using the shifting property of convolution, $x(t) * \delta(t - t_0) = x(t - t_0)$.

$$\boxed{X_{TX}(F) = \frac{1}{2}X_a(F - F_c) + \frac{1}{2}X_a(F + F_c)}$$

Question 2: How is $X_{TX}(F)$ related to $X_a(F)$?

Based on the above definition of $X_{TX}(F)$ we can determine that $X_{TX}(F)$ **is two scaled and shifted $X_a(F)$ signals.**

Question 3: What is the bandwidth of $X_a(F)$ if we only consider the positive side of the F-Axis?

With the hypothetical that the minimum frequency of $X_a(F)$ is F = 0 Hz, and the maximum frequency is F = 22kHz, the bandwith is the difference between the largest and smallest frequency. The difference between the two frequencies, 0 Hz and 22k Hz, is 22k Hz. With that we can say that **the bandwidth of $X_a(F)$ is 22k Hz**

Question 4: What is the bandwidth of $X_{TX}(F)$ if we only consider the positive side of the F-Axis?

Based on the definition of $X_{TX}(F)$,

$$X_{TX}(F) = \frac{1}{2}X_a(F - F_c) + \frac{1}{2}X_a(F + F_c)$$

This tells us that we have placed $X_a(F)$ on both the negative and positive side of F_c. Due to the bandwidth of $X_a(F)$ being 22k Hz, we can determine that $X_{TX}(F)$ will span from $F_c - 22$k Hz to $F_c + 22$k Hz. The difference between $F_c - 22$k Hz and $F_c + 22$k Hz is 44k Hz, so **the bandwidth of $X_{TX}(F)$ is 44k Hz**

Question 5: How does the bandwidth of $X_{TX}(F)$ compare with the bandwidth of $X_a(F)$

With the bandwidth of $X_{TX}(F)$ being 44k Hz, and the bandwidth of $X_a(F)$ being 22k Hz, we can deduce that **the bandwidth of $X_{TX}(F)$ is twice that of $X_a(F)$**.

Part I(b): Derivation for the Receiver

A received signal, $x_{RX}(t)$ is defined as

$$x_{RX}(t) = x_{TX}(t) + x_I(t)$$

where $x_I(t)$ denotes interfering signals. We know nothing about $x_I(t)$ other than it has a Fourier transform, $X_I(F)$, and $X_I(F) = 0$ for all values of F where $X_{TX}(F)$ is non-zero. (out-of-band interferers.)

Calculation 6: Write an expression for the Fourier transform $X_{RX}(F)$ in terms of $X_{TX}(F)$ and $X_I(F)$

$$x_{RX}(t) = x_{TX}(t) + x_I(t) \xleftrightarrow{\mathcal{F}} \boxed{X_{RX}(F) = X_{TX}(F) + X_I(F)}$$

Calculation 7: Next, replace the expression for $X_{TX}(F)$ with its expression in terms of $X_a(F)$

$X_{TX}(F)$ is defined as

$$X_{TX}(F) = \frac{1}{2}X_a(F - F_c) + \frac{1}{2}X_a(F + F_c)$$

now substituing this into the equation for $X_{RX}(F)$

$$\boxed{X_{RX}(F) = \frac{1}{2}X_a(F - F_c) + \frac{1}{2}X_a(F + F_c) + X_I(F)}$$

Calculation 8: The Fourier Transform of $x_d(t)$

Now, the signal $x_{RX}(t)$ is down-converted from the carrier frequency (F_c) to the audio frequency by multiplying it by $x_c(t)$. Let $x_d(t)$ be the down-converted signal, given by

$$x_d(t) = x_{RX}(t)x_c(t)$$

Using the multiplication property of the Fourier transform,

$$x(t)y(t) \xleftrightarrow{\mathcal{F}} X(F) * Y(F)$$

We can define $X_d(F)$ as

$$X_d(F) = X_{RX}(F) * X_c(F)$$

$X_{RX}(F)$ is defined as

$$X_{RX}(F) = \frac{1}{2}X_a(F - F_c) + \frac{1}{2}X_a(F + F_c) + X_I(F)$$

Allowing us to substitute into the equation for $X_d(F)$

$$X_d(F) = \left[\frac{1}{2}X_a(F - F_c) + \frac{1}{2}X_a(F + F_c) + X_I(F)\right] * X_c(F)$$

and $X_c(F)$ is defined as

$$X_c(F) = \frac{1}{2}\delta(F - Fc) + \frac{1}{2}\delta(F + Fc)$$

Allowing another substitution.

$$X_d(F) = \left[\frac{1}{2}X_a(F - F_c) + \frac{1}{2}X_a(F + F_c) + X_I(F)\right] * \left[\frac{1}{2}\delta(F - Fc) + \frac{1}{2}\delta(F + Fc)\right]$$

Using the shifting property of convolution,

$$X_d(F) = \frac{1}{4}X_a(F-F_c-F_c)+\frac{1}{4}X_a(F+F_c-F_c)+\frac{1}{2}X_I(F-F_c)+\frac{1}{4}X_a(F-F_c+F_c)+\frac{1}{4}X_a(F+F_c+F_c)+\frac{1}{2}X_I(F+F_c)$$

Simplifying this gives

$$X_d(F) = \frac{1}{4}X_a(F - 2F_c) + \frac{1}{4}X_a(F) + \frac{1}{2}X_I(F - F_c) + \frac{1}{4}X_a(F) + \frac{1}{4}X_a(F + 2F_c) + \frac{1}{2}X_I(F + F_c)$$

Further simplifying and rearranging terms,

$$\boxed{X_d(F) = \frac{1}{4}X_a(F - 2F_c) + \frac{1}{4}X_a(F + 2F_c) + \frac{1}{2}X_a(F) + \frac{1}{2}X_I(F + F_c) + \frac{1}{2}X_I(F - F_c)}$$

Our final answer, and expression for $X_d(F)$.

Calculation 8: The Inverse Fourier Transform of $X_d(F)$

Using the Fourier pairing

$$x(t) = \cos(2\pi F_0 t) \xleftrightarrow{\mathcal{F}} X(F) = \frac{1}{2}\left[\delta(F - F_0) + \delta(F + F_0)\right]$$

we are able to simplify the $X_a(F \pm 2F_c)$ and $X_I(F \pm F_c)$ terms. Firstly, the $X_a(F)$ terms.

$$\frac{1}{4}X_a(F - 2F_c) + \frac{1}{4}X_a(F + 2F_c)$$

$$\frac{1}{4}\Big[X_a(F - 2F_c) + X_a(F + 2F_c)\Big]$$

$$\frac{1}{2}\left[\frac{1}{2}\Big[X_a(F - 2F_c) + X_a(F + 2F_c)\Big]\right]$$

$$\frac{1}{2}\left[\frac{1}{2}\Big[X_a(F - 2F_c) + X_a(F + 2F_c)\Big]\right] \xrightarrow{\mathcal{F}^{-1}} \frac{1}{2}x_a(t)\cos(2\pi(2F_c)t)$$

$$X_a(F \pm 2F_c) \xrightarrow{\mathcal{F}^{-1}} \frac{1}{2}x_a(t)\cos(4\pi F_c t)$$

Now, the $X_I(F \pm F_c)$ terms.

$$\frac{1}{2}X_I(F + F_c) + \frac{1}{2}X_I(F - F_c)$$

$$\frac{1}{2}\Big[X_I(F + F_c) + X_I(F - F_c)\Big]$$

$$\frac{1}{2}\Big[X_I(F+F_c)+X_I(F-F_c)\Big] \xrightarrow{\mathcal{F}^{-1}} x_I(t)\cos(2\pi F_c t)$$

$$X_I(F \pm F_0) \xrightarrow{\mathcal{F}^{-1}} x_I(t)\cos(2\pi F_c t)$$

The final transform which has no shifts, is simple enough

$$\frac{1}{2}X_a(F) \xrightarrow{\mathcal{F}^{-1}} \frac{1}{2}x_a(t)$$

Now, with the pieces that we have found, we can reconstruct the signal using the linearity property of the Fourier transform.

$$\boxed{x_d(t) = \frac{1}{2}x_a(t) + \frac{1}{2}x_a(t)\cos(4\pi F_c t) + x_I(t)\cos(2\pi F_c t)}$$

Calculation 10: The Fourier transform for the ideal filter $H(F)$

To fully recieve our signal, we must now apply a filter to remove the unwanted content, so we are only left with $x_a(t)$. Knowing that our transformed signal, $X_a(F)$ does not have content above $F = 22$kHz. The ideal low pass filter is a rect function, centered at $F = 0$Hz, with a width of $2F_{LPF}$ (Low-Pass Filter), where F_{LPF} is the cutoff frequency of the filter. In our case, $F_{LPF} = 22$kHz Knowing that we are centered at F = 0, and have the width of $2F_{LPF}$, with the known value of $F_{LPF} = 22$kHz we can derive the piecewise function for the shape we want.

$$H(F) = \begin{cases} 1 & \text{if } |F| \leq 22 \text{ kHz} \\ 0 & \text{if } |F| > 22 \text{ kHz} \end{cases}$$

This works because if F is in the range -22 kHz $\leq F \leq 22$ Hz, we will allow the signal through, but if we are outside the range, $F > 22$ kHz or $F < -22$ kHz nothing will pass through.
Using the piecewise function we created for H(F), we can define H(F) formally as

$$\boxed{H(F) = \text{rect}\Big(\frac{F}{44 \text{ kHz}}\Big)}$$

Calculation 11: Determine the Impulse Response h(t)

The rect function has a known Fourier pairing,

$$X(F) = \text{rect}\Big(\frac{F}{W}\Big) \xleftrightarrow{\mathcal{F}} x(t) = W\text{sinc}(Wt)$$

Using our defined H(F)

$$H(F) = \text{rect}\Big(\frac{F}{44 \text{ kHz}}\Big) \xleftrightarrow{\mathcal{F}} h(t) = 44\text{k sinc}(44\text{k } t)$$

Otherwise expressed as,

$$\boxed{h(t) = 44000 \text{ sinc}(44000t)}$$

Calculation 12: The Fourier Transform $X_{out}(F)$

$x_{out}(t)$ is defined as

$$x_{out}(t) = x_d(t) * h(t)$$

Making the assumption that $X_I(F)$ does not overlap in frequency with $X_a(F)$, we can redefine $X_d(F)$ as

$$X_d(F) = \frac{1}{4}X_a(F)(F - 2F_c) + \frac{1}{4}X_a(F)(F + 2F_c) + \frac{1}{2}X_a(F)$$

using the Fourier transform property of convolution,

$$x(t) * y(t) \xleftrightarrow{\mathcal{F}^{-1}} X(F)Y(F)$$

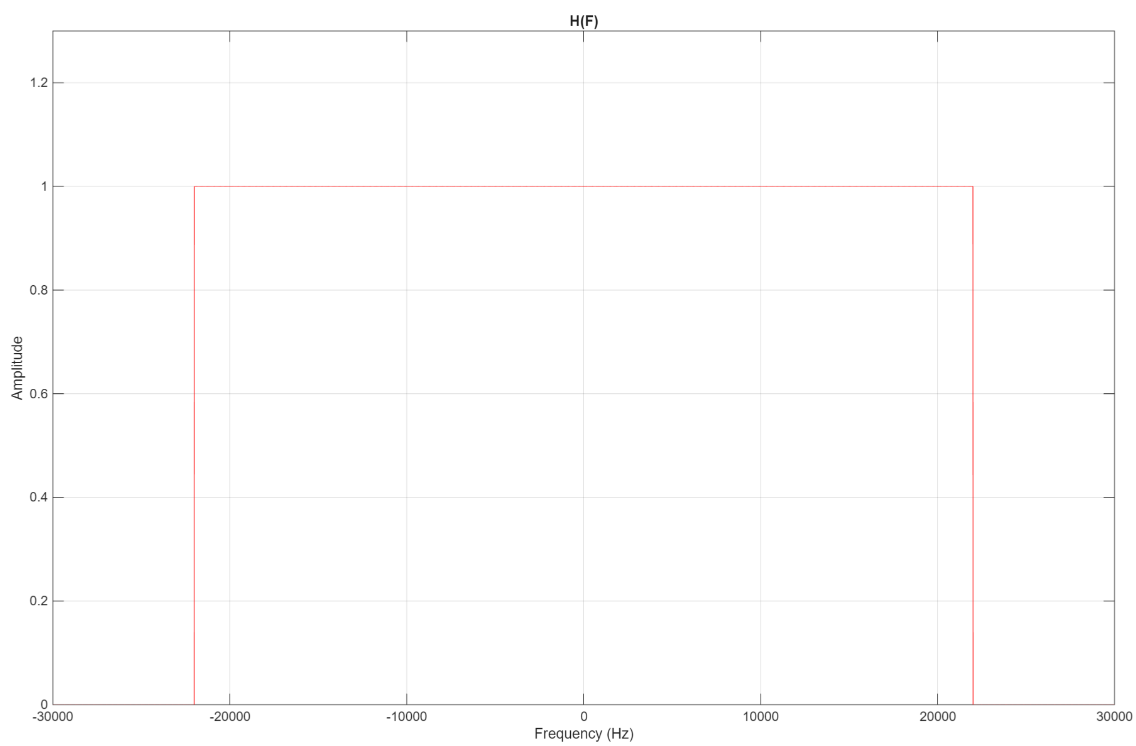
We can defined the Fourier transform of $x_{out}(t)$ as

$$X_{out}(F) = X_d(F)H(F)$$

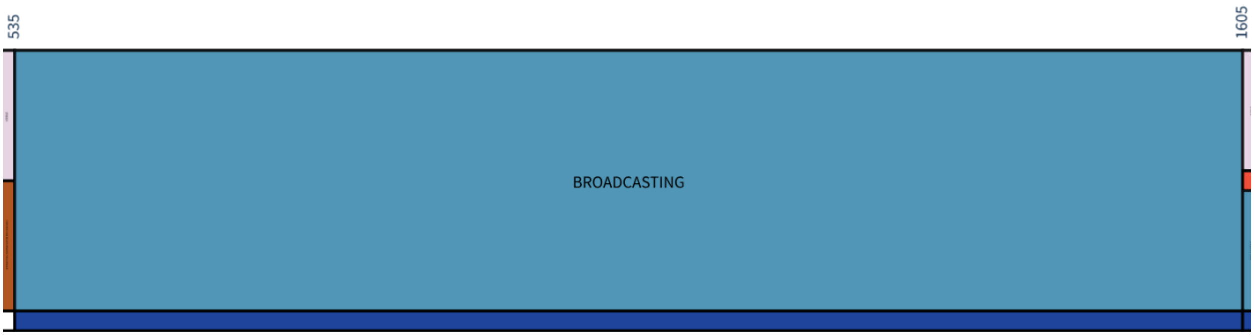
With the knowledge that $X_a(F)$ has a bandwith of 22 kHz, and the piecewise function for $H(F)$,

$$H(F) = \begin{cases} 1 & \text{if } |F| \leq 22 \text{ kHz} \\ 0 & \text{if } |F| > 22 \text{ kHz} \end{cases}$$

Firstly, let us visualize H(F)



We can see that just like the piecewise function describes, H(F) is a box centered about 0, extending from -22 kHz to 22 kHz. Looking at $X_d(F)$, we can see the term $\frac{1}{4}X_a(F)(F \pm 2F_c)$. Ideally, our carrier frequency would be very large. Given that $x_a(t)$ is an audio waveform, we can look at the United States Frequency Allocations Radio Spectrum and more specifically the lowest frequency "Broadcasting" section.



The figure above shows that the lowest frequency one can use in the United States to broadcast is 535 kHz. This means that the term $\frac{1}{4}X_a(F)(F \pm 2F_c)$ will be outside of our H(F), and thus be zero.
This leaves one remaining term, and thus the expression for $X_{out}(F)$,

$$\boxed{X_{out}(F) = \frac{1}{2}X_a(F)}$$

Calculation 13: The Final Expression for $x_{out}(t)$

After all that, we are able to perform a simple inverse Fourier transform on $X_{out}(F)$ as follows,

$$X_{out}(F) = \frac{1}{2}X_a(F) \xrightarrow{\mathcal{F}^{-1}} x_{out}(t) = \frac{1}{2}x_a(t)$$

$$\boxed{x_{out}(t) = \frac{1}{2}x_a(t)}$$