ST1100 SOLUTIONS TO TEST 3

[WITH THE QUESTIONS]

1. A lecturer is performing a statistical test concerning the effectiveness of a new teaching technique. If he committed a Type I error by erroneously concluding that the technique is effective, **what** were the alternatives he was testing?

(A) H_0 : the technique is not effective,
H_1 : the technique is effective,

(B) H_0 : the technique is effective,
H_1 : the technique is not effective.

SOLUTION: The answer is **(A)**.

Details: This is *Problem 2* on *page 35* of the *Statistics Lecture Notes* and was discussed in the lectures. See also the solution to *Question 1* on *Problem Set 3*.

2. Assume that the population of heights of people has standard deviation $\sigma = 6$ cm. If a random sample of $n = 36$ people showed a mean height of 150 cm., **what** is an approximate 95% confidence interval for the population mean height μ?

(A) $149.68 < \mu < 150.33$
(B) $138.24 < \mu < 161.96$
(C) $148.04 < \mu < 151.96$
(D) $149.00 < \mu < 151.00$

SOLUTION: The answer is **(C)**.

Details: See also the *Example (Weights)* on *page 30* of the *Statistics Lecture Notes.*
A $100(1-\alpha)\%$ confidence interval for μ is $\bar{x} \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$. *Here this is* $150 \pm z_{0.05/2}\frac{6}{\sqrt{36}}$, *i.e.* $150 \pm z_{0.025}$,
i.e. 150 ± 1.96, or $148.04 < \mu < 151.96$ with 95% confidence.

3. Assume that the starting salaries (in €) of Irish university graduates has a normal distribution with unknown mean μ and standard deviation $\sigma = 1000.0$. **What** is the minimum number of graduates that should be sampled at random so that with 95% confidence the sample mean salary $\bar{x}$ will not differ from μ by more than ±98 ?

(A) 16 **(B)** 100 **(C)** 400 **(D)** 601 **(E)** 1068 **(F)** 1600 .

SOLUTION: The answer is **(C)**.

Details: This is identical to *Question 3* on *Problem Set 3* except that the number 49 of that problem is replaced by 98 here. Of course the solutions to the Problem Sheets are given but I will reproduce the details for this example here:
A $100(1-\alpha)\%$ confidence interval for μ is $\bar{x} \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$. You're thus being asked to find n so that $z_{\alpha/2}\frac{\sigma}{\sqrt{n}} = 98$. This is $z_{0.025}\frac{\sigma}{\sqrt{n}} = 98$ i.e. $(1.96)\frac{1000}{\sqrt{n}} = 98$ or $\frac{1960}{\sqrt{n}} = 98$.
Hence $\sqrt{n} = \frac{1960}{98} = 20$ and so $n = 400$

4. A random sample of $n = 100$ Jetta SportWagen cars sold showed that 64 were fueled by diesel. **What** is an approximate 95% confidence interval for the proportion of all Jetta SportWagen cars sold that are of the diesel model?

A) 0.5 ± 0.98 **B)** 0.64 ± 0.094 **C)** 0.64 ± 0.048

D) 0.64 ± 0.98 **E)** 0.36 ± 0.094 **F)** 0.36 ± 0.096

SOLUTION: The answer is **(B)**.

Details: This problem is solved in the example on *pages 31-32* of the *Statistics Lecture Notes*, with "voters favouring a certain political candidate" in that example replaced here by "Jetta SportWagen running on diesel"

5. A researcher conducted a large sample two-sided test of the null hypothesis that $\mu = 100$. She reports a *p*-value of 0.04. Which one of the following is correct concerning the z-test and the corresponding confidence interval for μ?

A) The null hypothesis is rejected at $\alpha = 0.01$
B) The null hypothesis is not rejected at $\alpha = 0.05$
C) The 95% confidence interval for μ would not contain 100
D) The 99% confidence interval for μ would not contain 100 .

SOLUTION: The answer is **(C)**.
Details: See also *Question 8* of *Problem Set 3.*
Recall from *page 34* of the *Statistics Lecture Notes* that we reject the null hypothesis if and only if the p-value is less than α. Hence A) and (D) are both false.
You get ideas relating to (and essentially the answers concerning the falsity or veracity of) each of (C) and (D) of the present question by looking at the solution to *Question 7* on *Problem Sheet 3* (see also *Question 6* below).

6. Suppose that based on a random sample of size 15 and the formula $\bar{x} \pm t_{n-1\ \alpha/2}\frac{s}{\sqrt{n}}$, a 95% confidence interval for the mean μ of a population was found to be $45 < \mu < 55$.

Consider the following four statements (1), (2), (3) and (4).
(1) the probability is 0.95 that μ lies in the interval $(45, 55)$
(2) of all possible samples of size 15 that could be taken from the population, 95% of the intervals that would be obtained (using the same formula as above) would contain μ
(3) 95% of the means of all possible samples of size 15 that could be taken from the population would lie in the interval $(45, 55)$
(4) if the appropriate *t*-test of the alternatives $H_0 : \mu = 56$ versus $H_1 : \mu \neq 56$ was conducted using a level of significance $\alpha = 0.05$, H_0 would be rejected.

Then it is true that
(A) None of statements (1)–(4) is true
(B) Exactly one of statements (1)–(4) are true
(C) Exactly two of statements (1)–(4) are true
(D) Exactly three of statements (1)–(4) are true
(E) All four of statements (1)–(4) are true

SOLUTION: The answer is **(C)**.

Details: See also the solution to *Question 7* on *Problem Sheet 3.* The problem here is identical except that there (4) had $H_0 : \mu = 52$ versus $H_1 : \mu \neq 52$ instead of what we have here, $H_0 : \mu = 56$ versus $H_1 : \mu \neq 56$. In the present problem, both (2) and (4) are correct and (1) and (3) are false.

Alert students will notice that in general if a $100(1-\alpha)\%$ confidence interval does not contain the null hypothesis value of a parameter (in the present example, the 95% confidence interval $45 < \mu < 55$ does not contain $\mu = 56$), then we would reject the null hypothesis at level of significance α (here $\alpha = 0.05$). So here we reject $H_0 : \mu = 56$.
The idea is of course that a confidence interval puts μ within $\pm t_{n-1\ \alpha/2}\frac{s}{\sqrt{n}}$ while the test rejects H_0 if $\bar{x}$ lies more that $\pm z t_{n-1\ \alpha/2}\frac{s}{\sqrt{n}}$ from the null hypothesis value $\mu = 56$.

Questions 7–10 below concern the following problem: Assume that the daily profits X (ignore units) of a certain company have a normal distribution with unknown mean μ and standard deviation $\sigma = 8$, and that we wish to test the null hypothesis $H_0 : \mu = 100$ versus the alternative $H_1 : \mu < 100$. Of course we will use a z-test (see e.g. *Statistics Lecture Notes* A1 on *page 37*, *Example 1* on *page 38-39*, *Example 2* on *Page 39*; *see also B1 on Problem Sheet 3*, and the many examples and problems on z-tests in the *e-book*)

7. Suppose that the profits over a random sample of $n = 64$ days will be observed by an analyst and that he will reject H_0 if the sample mean daily profits satisfies $\bar{x} < 98$. **What** is the power of the test if in fact $\mu = 99$?

(A) 0.0228 **(B)** 0.4772 **(C)** 0.1587 **(D)** 0.3413 **(E)** 0.9772 **(F)** 0.5.

SOLUTION: The answer is **(C)**.

Details: See also the solution to *Problem 5* on *Problem Set 3.*
Power at $\mu = 99$ is $P(\text{reject } H_0 \mid \mu = 99) = P(\bar{X} < 98 \mid \mu = 99)$. Standardizing as usual (see e.g. *Statistics Lecture Notes page 25* or the *solution to Problem B1(c) or of Problem 5* of *Problem Set 3)* this probability is $= P\left(\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} < \frac{98-99}{8/\sqrt{64}}\right) = P(Z < -1) \underset{\text{by symmetry}}{=} P(Z > 1) \underset{\text{from z-tables}}{=} 0.1587$

8. The decision rule for the test is (in non-standardized form) "reject H_0 if $\bar{x} < c$". If the level of significance of the test is to be $\alpha = 0.0228$, **what** should c be? (Assume that the analyst has $n = 64$.)

(A) 96 **(B)** 97 **(C)** 98 **(D)** 99 **(E)** 100 **(F)** 101.

SOLUTION: The answer is **(C)**.

Details: We require c so that $P(\text{Type I error}) = \alpha = 0.0228$. That is, $P(\text{reject } H_0 \mid H_0 \text{ true}) = 0.0028$, that is,
$P(\bar{X} < c \mid \mu = 100) = 0.0228$
But $P(\bar{X} < c \mid \mu = 100) = P\left(Z < \frac{c-100}{8/\sqrt{64}}\right) = P(Z < c - 100)$. Therefore
$P(Z < c - 100) = 0.0228$. Using the z-tables, we know that $P(Z > 2) = 0.0228$. Hence $c - 100 = -2$, so $c = 98$.

9. **What** values of n and c should the analyst use if it is desired that the rule "reject H_0 if $\bar{x} < c$" will have $P(\text{reject } H_0 \text{ when } \mu = 100) = 0.0228$ *and* $P(\text{reject } H_0 \text{ when } \mu = 98) = 0.9772$?

(A) $n = 100$ and $c = 98$ **(B)** $n = 100$ and $c = 100$ **(C)** $n = 256$ and $c = 99$

(D) $n = 200$ and $c = 98$ **(E)** $n = 64$ and $c = 99$ **(F)** $n = 36$ and $c = 98$.

SOLUTION: The answer is **(C)**.

Details: The statements $P(\text{reject } H_0 \text{ when } \mu = 100) = 0.0228$ *and* $P(\text{reject } H_0 \text{ when } \mu = 98) = 0.9772$.
are equivalent to
$P(\bar{X} < c \mid \mu = 100) = 0.0228$ *and* $P(\bar{X} < c \mid \mu = 98) = 0.9772$

Standardizing each of these we have
$P\left(Z < \dfrac{c-100}{\frac{8}{\sqrt{n}}}\right) = 0.0228$ and $P\left(Z < \dfrac{c-98}{\frac{8}{\sqrt{n}}}\right) = 0.9772$

Using the z-tables twice, we thus have $\dfrac{c-100}{\frac{8}{\sqrt{n}}} = -2$ and $\dfrac{c-98}{\frac{8}{\sqrt{n}}} = 2$
We complete the solution by solving these two simultaneous equations for n and c.
The equations are $c - 100 = -2\frac{8}{\sqrt{n}}$ and $c - 98 = 2\frac{8}{\sqrt{n}}$
Subtracting the first equation from the second gives $2 = 4\frac{8}{\sqrt{n}}$, i.e. $2 = \frac{32}{\sqrt{n}}$ i.e. $\frac{1}{16} = \frac{1}{\sqrt{n}}$ so $n = 256$.
Substituting $n = 256$ into either of the equations $c - 100 = -2\frac{8}{\sqrt{n}}$ or $c - 98 = 2\frac{8}{\sqrt{n}}$, we have $c - 100 = -2\frac{8}{\sqrt{256}}$, so $c = 100 - 1 = 99$.
So we have shown that $n = 256$ and $c = 99$

Note: The importance of Question 9 of this test is in *design and analysis of experiments*: we often want to choose the sample size and decision rule to control both the probability of a Type I error and the power of the test at desirable levels!

10. Consider the following two statements (i) and (ii):

(i) For any fixed sample size n, and any level of significance α, the power of the z-test is larger at $\mu = 97$ than at $\mu = 98$.

(ii) For any fixed sample size n, and any two fixed numbers α_1 and α_2 satisfying $0 < \alpha_1 < \alpha_2 < 1$, the power at $\mu = 97$ of the z-test is smaller if a level of significance α_1 is used than if a level of significance α_2 is used.

Choose one answer below regarding the veracity of these statements.
Hint: The z-test with level of significance α based on n observations rejects H_0 if $\bar{x} < c$ with $c = 100 - z_\alpha \frac{\sigma}{\sqrt{n}}$.

(A) Both of statements (i) and (ii) are true
(B) Statement (i) is true but statement (ii) is false
(C) Statement (i) is false but statement (ii) is true
(D) Both of statements (i) and (ii) are false.

SOLUTION: The answer is **(A)**.
Details: To see that (i) is true, draw graphs of the two normal distributions of $\bar{X}$ centered at 97 and 98 and shade the areas corresponding to the powers. Alternatively, proceed algebraically in a manner similar to the particular power calculation we did in the solution to *Question A7* above.
Of course you knew that statement (ii) is true. In fact from *Statistics Lecture Notes page 28*, we see "It can be shown that for a fixed sample size and design, it is not possible to have both the probabilities of Type I and Type II errors as small as we please." If we reduce the Type I error probability, we increase the Type II error probability, and vice versa. Since power at any $\mu < 100$ is $1 - P$(Type II error at that μ) we see that the smaller we make α, the smaller will be the power also.