ST1100 SOLUTIONS TO TEST 1

[WITH THE QUESTIONS]

1. In a class, there are 100 students of whom 50 are males and 50 are females. If two students will be drawn at random from the class what is the probability that the second student drawn is male?
Note: In future we do not need to say that sampling is without replacement; that is, the first selected person is not replaced to the room before the second student is selected; sampling is always without replacement unless otherwise stated!

(A) $\frac{1}{5}$ **(B)** $\frac{3}{10}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{1}{2} \times \frac{49}{99}$ **(E)** $\frac{51}{99}$ **(F)** $\frac{49}{99}$

SOLUTION: The answer is **(C)**

Details: This is similar to *Question 5(a)* on page 1 of *Problem Set 1* [i.e. *SOME PROBABILITY PROBLEMS (with Solutions and Complements)*].

By partitioning, P(second is male) $=$
P(first is male $\cap$ second is male) $+ P$(first is female $\cap$second is male)
(see also *formula 15* on *Useful Probability Formulae* in the file *Some Probability Lecture Notes.* See also the *solution to Question 3* below.).
Now using *formula 10* on *Useful Probability formulae* twice, we get
P(second is male) $=$
P(first is male)P(second is male | first is male) $+ P$(first is female)P(second is male| first is female) $=$

$$\frac{1}{2} \times \frac{49}{99} + \frac{1}{2} \times \frac{50}{99} = \frac{1}{2} \times \left(\frac{49}{99} + \frac{50}{99}\right) = \frac{1}{2} \times \frac{99}{99} = \frac{1}{2}$$

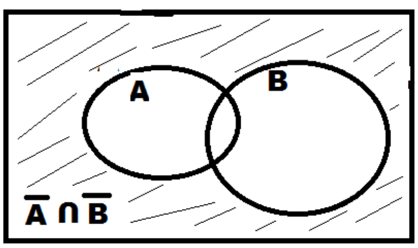
Note: This answer should be obvious! In the long run (that is, over a great many trials) the second-selected person should be male about 50% of the time! Why should it be more likely to be a male than a female, or vice versa? Note also that the number $\frac{1}{2} \times \frac{49}{99} + \frac{1}{2} \times \frac{50}{99}$ is a weighted average of the numbers $\frac{49}{99}$ and $\frac{50}{99}$, with weights, $\frac{1}{2}$, given by the proportions of males and females in the room

2. A and B are events satisfying $P(A) = P(B) = 0.7$ and $P(A \cap B) = 0.6$. What is $P(\overline{A} \cap \overline{B})$?
Note: The complement $\overline{C}$ of an event is the set of all elements of the sample space that are not in C.

(A) 0.8 **(B)** 0.7 **(C)** 0.5 **(D)** 0.4 **(E)** 0.2 **(F)** none of these

SOLUTION: The answer is **(E)**

Details: By drawing a Venn diagram or using one of De Morgan's laws, we see that the event $\overline{A} \cap \overline{B}$ is the complement of the event $A \cup B$ (see diagram below where the rectangle represents the entire sample space Ω, the shaded portion is the set $\overline{A} \cap \overline{B}$, i.e. all the elements that are outside of A and outside of B; and $A \cup B$ is the portion inside one or both circles that represent the events A and B)



Hence $P(\overline{A} \cap \overline{B}) = 1 - P(A \cup B)$.
But by the addition formula (*formula 6* on *Useful Probability Formulae* in *Some Probability Lecture Notes)*
$P(A \cup B) = P(A) + P(B) - P(A \cap B)$. Accordingly we get

$$P(\overline{A} \cap \overline{B}) = 1 - [P(A) + P(B) - P(A \cap B)] = 1 - [0.7 + 0.7 - 0.6] = 0.2$$

[Note that *Question 2* on the *Sample Quiz* had asked for $P(A \cup B)$!]

3. Refer to Question 2 above. What is $P(\bar{A} \cap B)$?

(A) 0.2 **(B)** 0.3 **(C)** 0.4 **(D)** 0.5 **(E)** 0.6 **(F)** 0.1

SOLUTION: The answer is **(F)**

Details: By partitioning (see also Formula 15 on *Useful Probability Formulae* in *Some Probability Lecture Notes)* we see that $B = (A \cap B) \cup (\bar{A} \cap B)$. [Draw a Venn diagram to see how obvious this is!]
Since the two bracketed sets on the right are disjoint, we have, by (3) on *Useful Probability Formulae* in *Some Probability Lecture Notes*, $P(B) = P(A \cap B) + P(\bar{A} \cap B)$.
Thus $P(\bar{A} \cap B) = P(B) - P(A \cap B)$. Hence $P(\bar{A} \cap B) = 0.7 - 0.6 = 0.1$

4. A card will be taken at random from a pack of 52 cards. Let *A* be the event that the card is a Jack, and let *B* be the event that the card is a picture card (that is, a Jack, a Queen or a King). Then it is true that

(A) *A* and *B* are disjoint [i.e. mutually exclusive] and independent, and $P(AUB) = \frac{12}{52}$

(B) *A* and *B* disjoint but not independent, and $P(AUB) = \frac{10}{52}$

(C) *A* and *B* not disjoint but are independent, and $P(AUB) = \frac{8}{52}$

(D) *A* and *B* neither disjoint nor independent, and $P(AUB) = \frac{12}{52}$

(E) *A* and *B* neither disjoint nor independent, and $P(AUB) = \frac{10}{52}$

(F) *A* and *B* neither disjoint nor independent, and $P(AUB) = \frac{8}{52}$

SOLUTION: The answer is **(D)**

Details: This is somewhat similar to e.g. *Question 5* on the *Sample Quiz*.
We have $P(A \mid B) = \frac{4}{12}$ (since there are 4 jacks among the 12 picture cards).
Since 4/12 = 0.3333 is *not* 0, we have by (*) that *A* and *B* are *not disjoint* by one of the formulae in *7* on *Useful Probability Formulae* in *Some Probability Lecture Notes*. (Of course you knew that! For clearly it is possible to get a card which is both a Jack and a picture card at the same time, namely, the Jack of Spades!)

Concerning whether or not the events *A* and *B* are independent, look at (***) above. Note that from above, $P(A \mid B) =$ [the probability that the card is a Jack given that the card will be a picture card] equals 4/12. But $P(A) =$ [Probability of getting a Jack] equals 4/52. Thus since 4/12 is not equal to 4/52, the events *A* and *B* are *not independent*. Of course this was intuitively obvious: if you know you are going to get a picture card, it surely increases your chances of getting a Jack; that is, we have a bigger chance of getting a Jack from among the picture cards than a Jack from the whole pack. This is precisely what lack of independence means ... and it is a crucial concept in probability and statistics; after all, what is the point of adding, for example, a catalyst to a chemical process if this catalyst does not affect the response? Or what would be the point of tracking a certain multi-national's share prices if they do not affect your investment portfolio?! Or what would be the value of taking seismic soundings if the type of structure they show does not give any evidence whether or not there is oil underground!

Finally, using *formula 6* and *formula 10* on *Useful Probability Formulae* in *Some Probability Lecture Notes*, we have

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = P(A) + P(B) - P(A)P(B \mid A) = \frac{4}{52} + \frac{12}{52} - \frac{4}{52} = \frac{12}{52}.$$

Note: Of course, in getting $P(A \cap B)$, you could just count that there are 4 of the 52 cards which are both jacks and picture cards (the set of Jacks is actually a subset of the set of picture cards, so actually $P(A) = P(A \cap B)$ in this example). Then incidentally, it is obvious that *A* and *B* are neither disjoint nor independent! Or of course you could use the formula $P(A \cap B) = P(A)P(B \mid A)$.
Note: Of course there are many other ways to answer the present problem. See the *Useful Probability Formulae* for the different ways of checking for disjointness and the various ways for checking if events are independent.

5. There are 2 radar units, operating independently, at an airport. Each of them has a 90% chance of detecting an incoming plane. What is the probability that *neither* of 2 planes, arriving independently, will be detected?
Hint: Think first *only* of one plane arriving and work out the probability that it will not be detected.
(A) 0.1 **(B)** 0.9 **(C)** 0.01 **(D)** 0.09 **(E)** 0.0001 **(F)** 0.99

SOLUTION: The answer is **(E)**

Details: $P(\text{one random plane detected}) = P(\text{at least one of the two units detects it}) =$
$1 - P(\text{neither unit detects it}) = 1 - P(\text{unit 1 does not detect it} \cap \text{unit 2 does not detect it})$
$\underset{\text{by independence}}{=} 1 - P(\text{unit 1 does not detect it})P(\text{unit 2 does not detect it})$
$= 1 - (0.1)(0.1) = 1 - 0.01 = 0.99.$
Then $P(\text{neither of two planes is detected}) = P(\text{plane 1is not detected} \cap \text{plane 2 is not detected})$
$\underset{\text{by independence}}{=} P(\text{plane 1is not detected})P(\text{plane 2 is not detected}) =$
$(1 - 0.99)(1 - 0.99) = 0.01 \times 0.01 = 0.0001$
Note: Of course one can use the binomial distribution instead by letting X = number of planes detected among the $n = 2$ detected. Then $P(\text{neither plane detected}) =$
$\binom{2}{0}(0.99)^0(1 - 0.99)^{2-0} = (0.01)^2 = 0.0001,$ as above.

6. Consider the following five statements (1), (2), (3), (4) and (5).
(1) Probability is concerned with making inferences about samples from knowledge of the populations from which the samples will be taken
(2) Statistics is primarily concerned with making inferences about populations based on samples taken from these populations
(3) Probability is used to solve statistical inference problems
(4) If X = height of one randomly selected Irish person, then X is equivalent to the population of heights of Irish people
(5) If X = height of one randomly selected Irish person, then X and its distribution are equivalent to the population of heights of Irish people

Then
(A) All of statements (1)–(5) are true
(B) Exactly four of statements (1)–(5) are true
(C) Exactly three of statements (1)–(5) are true
(D) Exactly two of statements (1)–(5) are true
(E) Exactly one of statements (1)–(5) is true
(F) None of statements (1)–(5) is true

SOLUTION: The answer is **(B)**

Some details: Note that the points made in this question have been discussed *ad infinitum* in class. See also the important boxed comment on *page 10* of *Some Probability Lecture Notes* and *page 9* of *Lecture Notes Statistics*. Only statement 4 is false!

7. Eight men and eight women can be seated in a row in 16! ways. How many of these arrangements will have the eight men next to each other?
(A) $\frac{16!}{4}$ **(B)** $\frac{16!}{2}$ **(C)** $\binom{16}{8}$ **(D)** 2 **(E)** 8!9!2! **(F)** 8!9!

SOLUTION: The answer is **(F)**

Details: There are 6!6!2! See Example 1 on page 6 of *Problem Sheet 1* and *Example 3 on page 7* of *Some Probability Lecture Notes*. The 8 men can be arranged in 8! ways. If we regard the 8 men (stuck together) as one unit, this unit and the 8 women can be arranged in 9! ways.
By the Fundamental Counting formula (*C1* on *page 5* of *Some Probability Lecture Notes)* we get the answer by multiplying these two numbers 8! and 9!

8. Based on his past record of a weakness for Taytos, the police figure that there is an 80% chance that Mr. Perri was the thief who stole a bag of Taytos from Jerome's office today. If he did not steal them, the chance that he'd be eating a bag of them right now would be only 20%, while if he did steal them, the chance that he would be eating them now would be 70%. Mr. Perri is seen eating a bag of Taytos right now. What is the chance that he stole them?

A) $\frac{14}{15}$ **B)** $\frac{7}{8}$ **C)** $\frac{4}{5}$ **D)** $\frac{3}{4}$ **E)** $\frac{14}{21}$ **F)** none of these

SOLUTION: The answer is **(A)**

Details: This is *question 6* on *page 2* of *Problem Set 1* (with different numbers). By Bayes' formula (draw a tree diagram or see *formula 16* on *Useful Probability Formulae* in *Some Probability Lecture Notes*, or see me, Jerome!) the answer is $\frac{0.8\times 0.7}{0.8\times 0.7+0.2\times 0.2} = \frac{0.56}{0.56+0.04} = \frac{0.56}{0.60} = \frac{56}{60} = \frac{14}{15}$

9. Suppose you have one line of 6 numbers for the next draw of the UK Lotto $\frac{6}{49}$, in which six numbers are drawn at random from the set of numbers $\{1,2,3,\ldots,49\}$, and then a seventh number (called the bonus ball) is drawn. What is the probability that you match "five and the bonus"? That is, what is the probability that your line of six numbers has exactly five of the first six numbers drawn from the 49 numbers and also has the seventh bonus ball?

(A) $\frac{\binom{6}{5}\binom{43}{1}}{\binom{49}{6}}\times\frac{1}{43}$ **(B)** $\frac{\binom{6}{5}\binom{43}{1}}{\binom{49}{6}}\times\frac{3}{43}$ **(C)** $\frac{\binom{6}{5}\binom{43}{1}}{\binom{49}{6}}\times\frac{5}{43}$

(D) $\frac{\binom{6}{3}\binom{43}{3}}{\binom{49}{6}}\times\frac{1}{43}$ **(E)** $\frac{\binom{6}{3}\binom{36}{3}}{\binom{42}{6}}\times\frac{3}{36}$ **(F)** $\frac{\binom{6}{3}\binom{36}{3}}{\binom{42}{6}}\times\frac{1}{36}$

SOLUTION: The answer is **(A)**

Details: See also the solution to *problem 10* (Lotto) on *page 3* of *Problem Set 1*, where we had 42 numbers in the drum (instead of the 49 in the present problem).
In the present problem we want P(match 5 of the first 6 drawn AND match bonus ball). By the multiplication formula (*formula 10* on *Useful Probability Formulae* in the *Lecture Notes on Probability*), this is
P(match 5 of the first 6 drawn AND match bonus ball) =

$$P(\text{match 5 of the first 6 drawn})P(\text{match bonus ball} \mid \text{match 5 of first 6}) = \frac{\binom{6}{5}\binom{43}{1}}{\binom{49}{6}}\times\frac{1}{43}$$

Here we used the hypergeometric distribution and we note also that if we have matched 3 of the first six numbers drawn, then there are still three of our numbers among the 43 numbers still in the drum, so the chance that we get the seventh ball drawn is 3/43. For interest, note that the above numerical expression simplifies to

$$\frac{\binom{6}{5}\binom{43}{1}}{\binom{49}{6}}\times\frac{1}{43} = \frac{6\times 43}{13983816}\times\frac{1}{43} = \frac{6}{13983816} = \frac{1}{2330636}$$

(see also e.g. http://www.murderousmaths.co.uk/books/bkmm6xlo.htm where this probability is written as approximately 1 in 2.3 million)

10. In a factory, 1% of screws are too short for purpose (defective). If 10 screws will be selected at random, what is a numerical expression for the probability that zero will be defective given that at most one will be defective?

(A) $\dfrac{(0.99)^{10}}{(0.99)^{10}+(0.1)(0.99)^{9}}$ **(B)** $\dfrac{(0.1)^{1}(0.99)^{9}}{(0.99)^{10}+(0.1)(0.99)^{9}}$ **(C)** $\dfrac{(0.99)^{10}+(0.1)(0.99)^{9}}{(0.1)^{1}(0.99)^{9}}$

(D) $1-\dfrac{(0.01)(0.99)^{10}}{\binom{10}{1}(0.99)^{10}+\binom{10}{2}(0.99)^{9}}$ **(E)** $\frac{1}{2}$ **(F)** $\frac{1}{5}$.

SOLUTION: The answer is **(A)**

Details: This is similar to a number of examples/problems in various sources, e.g. *Example 3(d)* on *page 12* of *Some Probability Lecture Notes*, or *Problem 11(c)* on *pages 4-5* of *Problem Set 1*, *Problem 4(b)* on *pages 5-6* of that problem sheet, *Problem 10* of the *Sample Test*, etc!

We want $P(X = 0 \mid X \le 1)$

Using *Formula 11* on *Useful Formulae in Probability* on *page 3* of *Some Probability Lecture Notes* we have

$P(X = 0 \mid X \le 1) = \dfrac{P(X = 0 \cap X \le 1)}{P(X \le 1)}$.

But since the event "$X = 0$" is a subset of the event "$X \le 1$" (that is, if 0 defective occur, we must have at most 2 defectives occurring) we have $P(X = 0 \cap X \le 1) = P(X = 0\,)$. Thus

$P(X = 5 \mid X \le 1) = \dfrac{P(X = 0\,)}{P(X \le 1)}$.

(Students should try to see the last formula above as being obvious!)

We now complete the solution by applying the binomial distribution (not the hypergeometric [Why?]) to get

$$\frac{P(X = 0\,)}{P(X \le 1)} = \frac{P(X = 0\,)}{P(X = 0) + P(X = 1)} =$$

$$\frac{\binom{10}{0}(0.01)^{0}(1-0.01)^{10}}{\binom{10}{0}(0.01)^{0}(1-0.01)^{10}+\binom{10}{1}(0.01)^{1}(1-0.01)^{9}} = \frac{(0.99)^{10}}{(0.99)^{10}+10(0.01)(0.99)^{9}} = \frac{(0.99)^{10}}{(0.99)^{10}+(0.1)(0.99)^{9}}$$

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