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MATHEMATICS

MATHEMATICAL STUDY ON THERMODYNAMIC SECOND LAW FOR A GRAVITY – DRIVEN VISCOSITY LIQUID FILM USING HOMOTOPY ANALYSIS METHOD

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Abstract*:*

The present paper is concerned with the analysis of inherent irreversibility in a gravity-driven temperature-dependent variable viscosity thin liquid film along an inclined heated plate with convective cooling is investigated. In this study, both the isothermal and isoflux heating of the plate are considered. These dimensionless equations are solved analytically and graphically. The free surface of the liquid film is assumed to exchange heat with the surroundings following Newton's cooling law and the fluid viscosity model varies as an inverse linear function of the temperature. A convective boundary condition is employed which makes this study unique and the results are realistic and practically useful. The approximate analytical expressions of the dimensionless axial velocity and the dimensionless temperature fields are derived using the Homotopy analysis method. We can also drive analytically and graphically for the physical quantities of the problem like the entropy generation number and the Bejan number. The Homotopy analysis method can be easily extended to solve the other non-linear boundary value problems in MHD fluid flow problems.

Keywords: Inclined plate; Liquid film; Variale viscosity; Thermal stability; Convective cooling; Entropy analysis.

Mathematics Subject Classification (2010): 80A20, 76S05, 76A05.

1. Introduction

In the literature, the study of thin liquid films has a long history. A falling film is the gravity flow of a continuous liquid film down a solid tube having one free surface. Makinde studied the flow of liquid film with variable viscosity along an inclined heating plate. Apart from permitting a fundamental simplification of the viscous equations governing free surface flows, the tracking of such films has a significant impact on the manufacturing and final quality of the product. Flows occurring on inclined plates have been numerically investigated in the work of B.Vasu et al. [2]. They studied the free surface of a thin liquid film in the presence and absence of gravitational body force using a boundary-fitted coordinate system where the irregular free surface conformed to one of the flow boundaries.

The study of entropy generation in a falling variable viscosity liquid film along an inclined heated plate with convective cooling was carried out by Das et. al [3]. An analysis of laminar free convective flow and heat transfer about an inclined isothermal plate has been described by Bejan [8-9]. Bejan [14] have studied the natural convective effects on impulsively started inclined plate with heat and mass transfer. Analysis of thermal boundary layer flows of a thin liquid film on a heated inclined plate is extremely important in many industrial applications, especially in the area of handling and processing of such fluid. Sahin [18] employed the Hermite-Padé approximation technique to investigate the problem of heat transfer in steady flows of a liquid film with adiabatic free surface along an inclined heat plate.

Moreover, thermodynamic irreversibility in any fluid flow process can be quantified through entropy analysis. The first law of thermodynamics is simply an expression of the conservation of energy principle. The second law of thermodynamics is applied to investigate the irreversibility in terms of the entropy generation rate. As the generation of entropy destroys system energy, its minimization has been used as the optimal design criteria for thermal systems. The study of entropy generation in conductive and convective heat transfer processes has assumed considerable importance since the pioneering work of Makinde [21]. Many entropy generation studies on flows with heat transfer can be found in the literature. Makinde [22] analyzed the variation of entropy generation in the function of viscosity in forced flow.

In all of the abovementioned studies, the thermodynamic second law analysis for gravity-driven flow over both the isothermal- and isoflux-heated inclined plates with convective heat exchange at the free surface have not yet been investigated. Thus, in the present paper, we extend the work of Kierkus [24] on variable viscosity thin liquid films with a free surface flowing down a slightly inclined smooth solid substrate to include entropy generation analysis and the effect of convective cooling at the free surface.

2. Mathematical formulation of the problem

The configuration of the problem studied in this paper is depicted in Fig .1. We considered an isothermal- and isoflux-heated inclined plate surfaces placed in parallel streams of hydrodynamically and thermally-developed liquid films (Fig. 1). It has been assumed that the fluid is Newtonian with temperature dependent viscosity and all other fluid properties remaining constant.

$$\frac{d^2T}{d\bar{y}^2}+\frac{\bar{\mu}}{k}\left(\frac{d\bar{u}}{d\bar{y}}\right)^2=0, \quad \frac{d}{d\bar{y}}\left(\bar{\mu}\frac{d\bar{u}}{d\bar{y}}\right)+\rho g\sin(\phi)=0, \tag{1}$$

In order to investigate both the isothermal- and isoflux-heated inclined plate situations, we write the boundary conditions at the plate surface as:

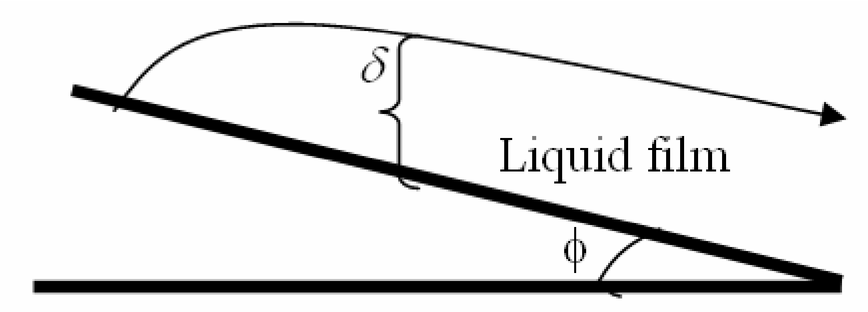


Fig. 1. Geometry of the problem

$$\bar{u}=0, \quad ak\frac{dT}{d\bar{y}}-\frac{bk}{\delta}(T-T_a)=-\frac{k}{\delta}(T_0-T_a), \quad on \quad \bar{y}=0 \tag{2}$$

Where $a=0, b=1$ for isothermal heating and $a=1, b=0$ for isoflux heating. The convective heat exchange with the ambient at the free surface is given by

$$\frac{d\bar{u}}{d\bar{y}}=0\ , \qquad k\frac{dT}{d\bar{y}}=-h(T-T_a) \qquad on \quad \bar{y}=\delta \tag{3}$$

where u is the axial velocity, T is the absolute temperature, T_0 is the plate reference temperature (i.e., uniform temperature for isothermal case or initial plate temperature for isoflux case), T_a is the ambient temperature, h is the heat transfer coefficient, k is the thermal conductivity, α is the liquid film thickness, Φ is the inclination angle, ρ is the fluid density, g is the gravitational acceleration, y is the vertical distance, a and b are the inclined plate heating parameters. The temperature dependency of dynamic viscosity (μ) can be expressed as:

$$\bar{\mu}=\frac{\mu_0}{1+m(T-T_a)} \tag{4}$$

where μ_0 is the fluid viscosity at ambient temperature T_a and the coefficient m determines the strength of dependency between viscosity and temperature. Eqn. (4) is most appropriate for liquid film since the viscosity of liquid decreases as the temperature increases eqn. [24]. The following dimensionless quantities are introduced into eqns. (1)-(4)

$$\theta=\frac{T-T_a}{T_0-T_a}\ ,\ y=\frac{\bar{y}}{\delta}, Br=\frac{\left(\delta^2\rho g\, sin(\phi)\right)^2}{\mu_0 k(T_0-T_a)}, u=\frac{\mu_0\bar{u}}{\delta^2\rho g\, sin(\phi)}, \alpha=m(T_0-T_a),$$
$$\alpha=m(T_0-T_a),\ \mu=\frac{\bar{\mu}}{\mu_0}, Bi=\frac{h\delta}{k} \tag{5}$$

Using the above dimensionless variables, we can obtain the following non-linear differential eqn. is as follows:

$$\frac{d^2\theta}{dy^2}+\mu Br\left(\frac{du}{dy}\right)^2=0\ ,\ \frac{d}{dy}\left(\mu\frac{du}{dy}\right)=-1 \tag{6}$$

with

$$u=0\ ,\ a\frac{d\theta}{dy}-b\theta=-1 \quad at \quad y=0 \tag{7}$$

and

$$\frac{du}{dr}=0\ ,\quad \frac{d\theta}{dy}=-Bi\,\theta\ ,\quad at \quad y=1 \tag{8}$$

Where $\mu=1/(1+\alpha\theta)$, Br is the Brinkmann number, Bi is the Biot number, and α is the variable viscosity parameter.
These eqns. can also be in the form of

$$\frac{d^2\theta}{dy^2}+Br(1-y)^2(1+\alpha\theta)=0 \tag{9}$$

$$\frac{du}{dy}=(1-y)(1+\alpha\theta) \tag{10}$$

The corresponding boundary conditions are as follows: $\frac{d\theta}{dy}(1)=-Bi\,\theta(1), a\frac{d\theta}{dy}(0)-b\theta(0)=-1, u(0)=0$ (11)

3. Solution of the problems using the Homotopy analysis method

This paper deals with a basic strong analytic tool for nonlinear problems, namely the Homotopy analysis method (HAM) which was generated by Liao [19], is employed to solve the nonlinear differential eqns. (9) - (12). The Homotopy analysis method is based on a basic concept in topology. Unlike perturbation techniques like [24], the Homotopy analysis method is independent of the small/large parameters. Unlike all other reported perturbation and non-perturbation techniques such as the artificial small parameter method[25], the δ-expansion method [26] and Adomian's decomposition method [27], the Homotopy analysis method provides us a simple way to adjust and control the convergence region and rate of approximation series. The Homotopy analysis method has been successfully applied to many nonlinear problems such as heat transfer [28], viscous flows [29], nonlinear oscillations [30], Thomas-Fermi's atom model [31], nonlinear water waves [32], etc. Such varied successful applications of the Homotopy analysis method confirm its validity for nonlinear problems in science and engineering. The Homotopy analysis method is a good technique when compared to other perturbation methods. The existence of the auxiliary parameter h in the Homotopy analysis method provides us with a simple way to adjust and control the convergence region of the solution series.

In this paper we have used the Homotopy analysis method for the non-linear boundary value problem which is expressed in the equations (9) and (10) with the boundary condition (11). And we have obtained the approximate analytical expression for the dimensionless axial velocity u(y) and dimensionless temperature θ(y).

$$u(y)=y-\frac{y^2}{2}+\alpha\left[\frac{c1y^2}{2}+c2y-\frac{c1y^3}{3}-\frac{c2y^2}{2}\right] \tag{12}$$

$$\theta(y)=-h\left(\begin{array}{l} Br\left(\frac{y^2}{2}-\frac{y^3}{3}+\frac{y^4}{12}\right)+C_1y+C_2-\frac{Br\alpha C_2y^2}{2}+E_1\frac{y^3}{6}+E_2\frac{y^4}{12}+E_3\frac{y^5}{20} \\ \qquad +Br^2\alpha\left(\frac{3y^6}{72}-\frac{y^7}{84}+\frac{y^8}{672}\right)+C_3y+C_4 \end{array}\right) \tag{13}$$

Where

$$C_1=\frac{\frac{bBr}{3}+\frac{bBiBr}{4}-Bi}{(b(1+Bi)+aBi)} \tag{14}$$

$$C_2=\frac{1}{Bi}\left[\frac{Br}{3}+\frac{BiBr}{4}-(1+Bi)C_1\right] \tag{15}$$

$$C_3 = \frac{\begin{pmatrix} Br\alpha C_2 - \frac{15Br^2\alpha}{84} + \frac{C_2 BiBr\alpha}{2} + \frac{21BiBr^2\alpha}{672} - \frac{E_1}{2} - \frac{E_2}{3} \\ -\frac{E_3}{4} - \frac{BiE_1}{6} - \frac{BiE_2}{12} - \frac{BiE_3}{20} \end{pmatrix}}{b(1+Bi) + a\,Bi} \quad (16)$$

$$C_4 = \frac{\left(\begin{pmatrix} Br\alpha C_2 - \frac{15Br^2\alpha}{84} + \frac{C_2 BiBr\alpha}{2} + \frac{21BiBr^2\alpha}{672} - \frac{E_1}{2} - \frac{E_2}{3} - \frac{E_3}{4} \\ -\frac{BiE_1}{6} - \frac{BiE_2}{12} - \frac{BiE_3}{20} \end{pmatrix} - C_3(1+Bi)\right)}{Bi} \quad (17)$$

$$E_1 = -\alpha BrC_1 + 2Br\alpha C_2 \quad (18)$$

$$E_2 = \frac{Br^2\alpha}{2} + 2\alpha BrC_1 - \alpha BrC_2 \quad (19)$$

$$E_3 = \frac{-4Br^2\alpha}{3} - \alpha BrC_1 \quad (20)$$

The formula for the entropy generation number is

$$Ns = \frac{\delta^2 T_0^{\ 2} S^m}{k(T_0 - T_a)^2} = \left(\frac{d\theta}{dy}\right)^2 + \frac{\mu Br}{\Omega}\left(\frac{du}{dy}\right)^2 \quad (21)$$

where Ω is the temperature difference parameter. In the eqn. (21), the first term can be assigned as N_1, while the second term due to viscous dissipation as N_2:

$$N_1 = \left(\frac{d\theta}{dy}\right)^2, \ N_2 = \frac{\mu Br}{\Omega}\left(\frac{du}{dy}\right)^2 \quad (22)$$

$$N_1 = \left(\frac{\begin{pmatrix} -Br\left[\frac{2y}{4} - \frac{3y^2}{9} + \frac{4y^3}{48}\right] + C_1 - \frac{Br\alpha C_2 2y}{4} + \frac{E_1 3y^2}{18} + \frac{E_2 4y^3}{48} + \frac{E_3 5y^4}{100} \\ + Br^2\alpha\left[\frac{18y^5}{432} - \frac{7y^6}{588} + \frac{8y^7}{5376}\right] + C_3 \end{pmatrix}}{dy}\right) \quad (23)$$

$$N_2 = \frac{\mu Br}{\Omega}\left(\frac{1 - \frac{2y}{4} + \alpha\left[\frac{C_1 2y}{4} + C_2 - \frac{C_1 3y^2}{9} - \frac{C_2 2y}{4}\right]}{dy}\right)^2 \quad (24)$$

The analytical expression of the entropy generation number using the eqn.(21) is given by

$$Ns = \left(\left(\frac{\left(-Br\left[\frac{2y}{4} - \frac{3y^2}{9} + \frac{4y^3}{48}\right] + C_1 - \frac{Br\alpha C_2 2y}{4} + \frac{E_1 3y^2}{18} + \frac{E_2 4y^3}{48} + \frac{E_3 5y^4}{100} + Br^2\alpha\left[\frac{18y^5}{432} - \frac{7y^6}{588} + \frac{8y^7}{5376}\right] + C_3 \right)}{dy} \right)^2 + \frac{\mu Br}{\Omega}\left(\frac{1 - \frac{2y}{4} + \alpha\left[\frac{C_1 2y}{4} + C_2 - \frac{C_1 3y^2}{9} - \frac{C_2 2y}{4}\right]}{dy} \right)^2 \right) \tag{25}$$

The formula for the Bejan number is given by

$$Be = \frac{N_1}{Ns} = \frac{1}{1+\Phi} \tag{26}$$

The analytical expressions of the Bejan number using the eqn. (26) is given by

$$Be = \frac{\left(\frac{\left(-Br\left[\frac{2y}{4} - \frac{3y^2}{9} + \frac{4y^3}{48}\right] + C_1 - \frac{Br\alpha C_2 2y}{4} + \frac{E_1 3y^2}{18} + \frac{E_2 4y^3}{48} + \frac{E_3 5y^4}{100} + Br^2\alpha\left[\frac{18y^5}{432} - \frac{7y^6}{588} + \frac{8y^7}{5376}\right] + C_3 \right)}{dy} \right)}{\left(\left(\frac{\left(-Br\left[\frac{2y}{4} - \frac{3y^2}{9} + \frac{4y^3}{48}\right] + C_1 - \frac{Br\alpha C_2 2y}{4} + \frac{E_1 3y^2}{18} + \frac{E_2 4y^3}{48} + \frac{E_3 5y^4}{100} + Br^2\alpha\left[\frac{18y^5}{432} - \frac{7y^6}{588} + \frac{8y^7}{5376}\right] + C_3 \right)}{dy} \right)^2 + \frac{\mu Br}{\Omega}\left(\frac{1 - \frac{2y}{4} + \alpha\left[\frac{C_1 2y}{4} + C_2 - \frac{C_1 3y^2}{9} - \frac{C_2 2y}{4}\right]}{dy} \right)^2 \right)} \tag{27}$$

5. Results and discussion

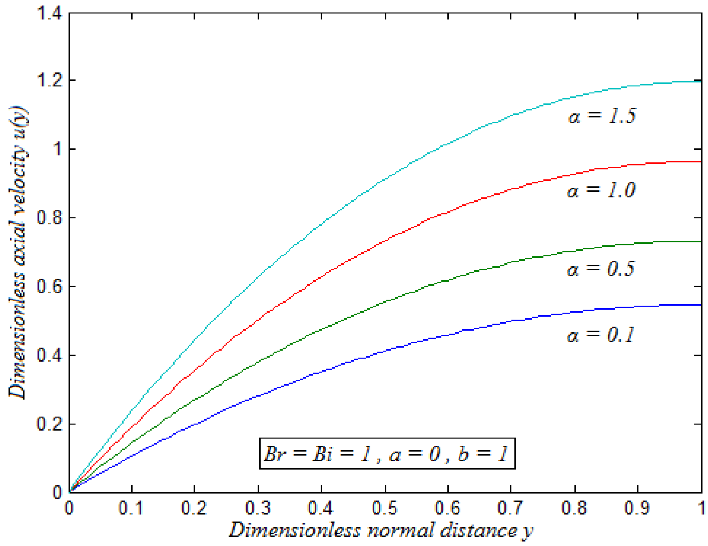
Figure 1 shows geometry of the problem. Figure 2 represents the dimensionless axial velocity $u(y)$ versus the dimensionless normal distance y. From Fig. 2(a) and 2(b), it is noted that in both the isothermal and isoflux case, when the variable viscosity parameter α increases the corresponding axial velocity profile also increases in some fixed values of the other parameters $Br, Bi, a, b.$ From Fig. 2(c) and 2(d), it is inferred that in both the isothermal and isoflux case, when the Biot number Bi increases the corresponding dimensionless axial velocity profile decreases in some fixed values of the other parameters $Br, \alpha, a, b.$ From Fig. 2(e) and 2(f), it is depict that in both the isothermal and isoflux case, when the Brinkmann number Br increases the corresponding axial velocity profile also increases in some fixed values of the other parameters $Br, Bi, a, b.$

Figure 3 represents the dimensionless temperature $\theta(y)$ versus the dimensionless normal distance y. From Fig. 3(a) and 3(b), it is noted that in both the isothermal and isoflux case, when the variable viscosity parameter α increases the corresponding dimensionless temperature increases in some fixed values of the other parameters $Br, Bi, a, b.$ From Fig. 3(c) and 3(d), it is depict that in both the isothermal and isoflux case, when the variable Biot number Bi increases the corresponding dimensionless temperature increases in some fixed values of the other parameters $Br, Bi, a, b.$ From Fig. 3(e) and 3(f), it is depict that when the Brinkmann number Br increasing in the isothermal and decreasing in the isoflux case. The corresponding dimensionless temperature profile also increases in some fixed values of the other parameters $Br, Bi, a, b.$

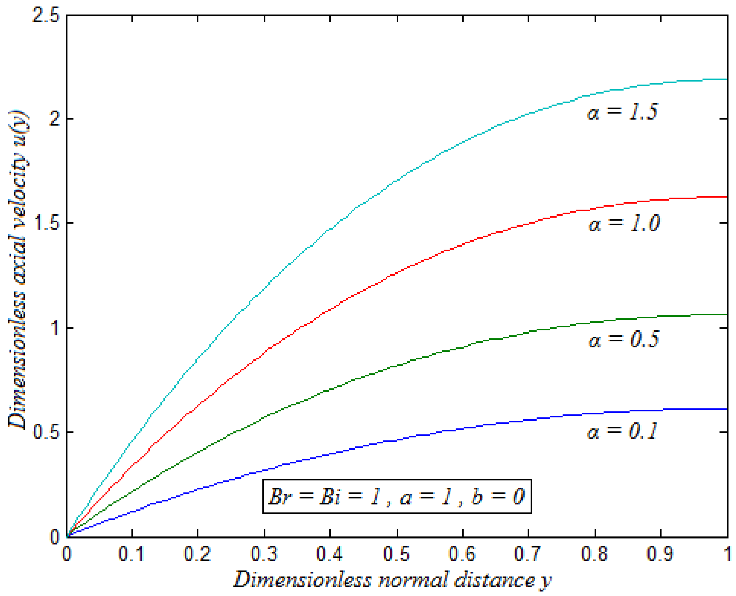
Figure 4 represents the Entropy generation number Ns versus the dimensionless normal distance y. From Fig. 4(a) and 4(b), it is noted that when the variable viscosity parameter α increasing in the isothermal case and decreasing in the isoflux case. The corresponding entropy generation number increases in some fixed values of the other parameters $Br\Omega^{-1}, Bi, a, b.$ From Fig. 4(c) and 4(d), it is depict that in both the isothermal and isoflux case, when the Brinkmann number Br decreases the corresponding entropy generation number increases in some fixed values of the other parameters $\alpha, Bi, a, b.$ From Fig. 4(e) and 4(f), it is inferred that when the Biot number Bi decreases in the both isothermal and isoflux case the corresponding entropy generation number increases in some fixed values of the other dimensionless parameters $Br\Omega^{-1}, \alpha, a, b.$

Figure 5 represents the Bejan number Be versus the dimensionless normal distance y From Fig. 5(a) and 5(b), it is noted that when the variable viscosity parameter α decreases in both the isothermal and isoflux case. The corresponding Bejan number also increases in some fixed values of the other dimensionless parameters $Br\Omega^{-1}, Bi, a, b.$ From Fig. 5(c) and 5(d), it is depict that when the Brinkmann number Br increases in both the isothermal and isoflux case. The corresponding Bejan number increases in some fixed values of the other dimensionless parameters $\alpha, Bi, a, b.$ From Fig. 5(e) and 5(f), it is inferred that when the Biot number Bi decreases in the isothermal and increases in the isoflux case. The corresponding Bejan number increases in some fixed values of the other dimensionless parameters.

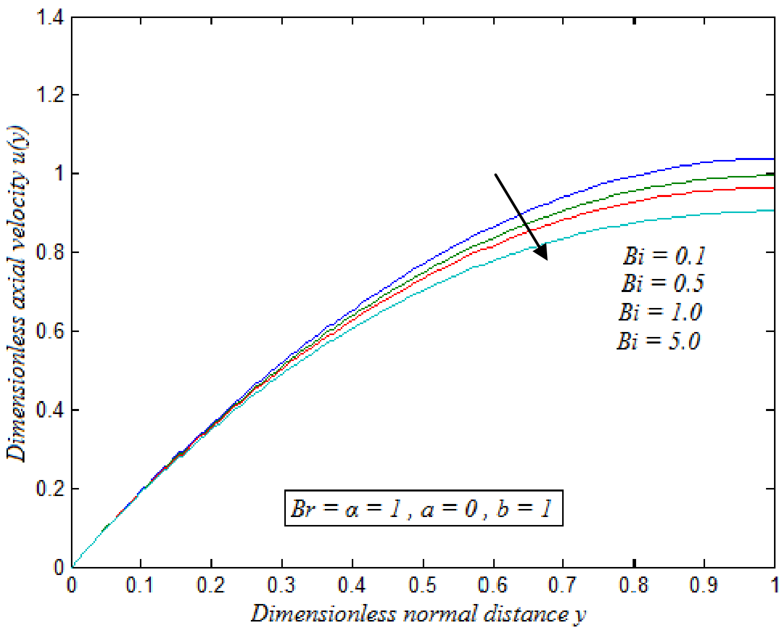
(a)



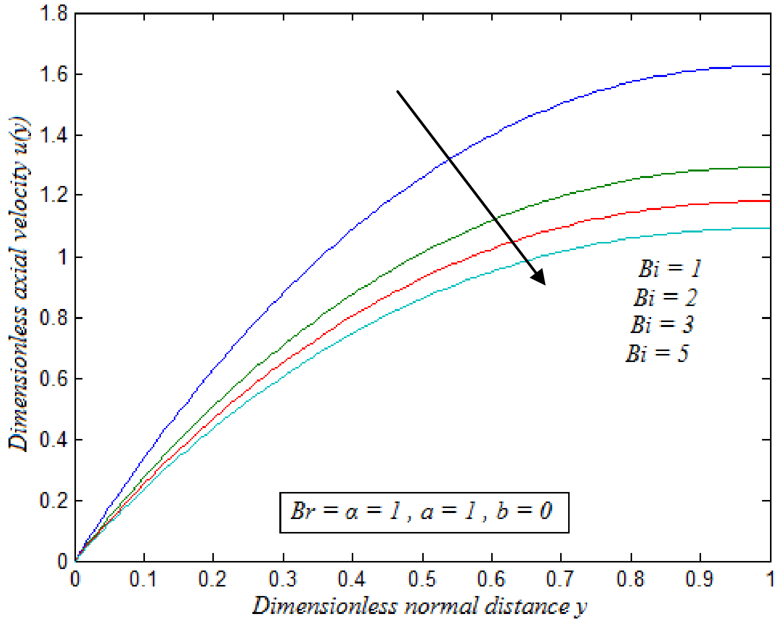
(b)


α = 1.5
α = 1.0
α = 0.5
α = 0.1
Br = Bi = 1 , a = 1 , b = 0
Dimensionless axial velocity u(y)
Dimensionless normal distance y

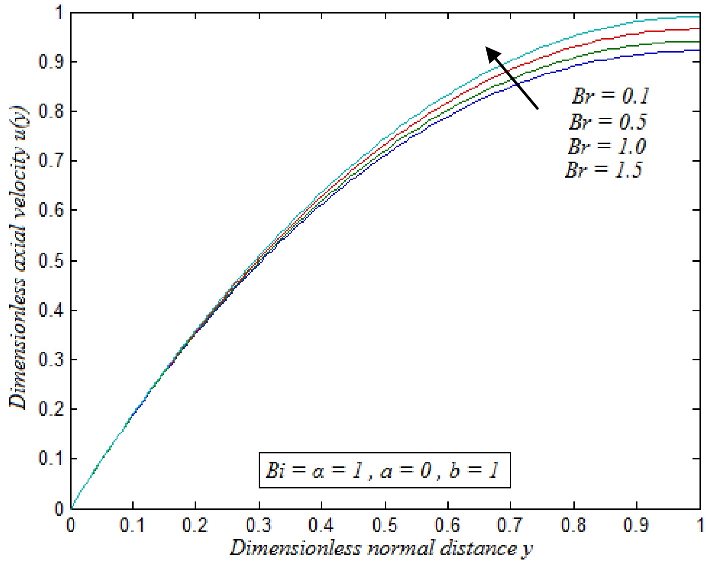
(c)


Bi = 0.1
Bi = 0.5
Bi = 1.0
Bi = 5.0
Br = α = 1 , a = 0 , b = 1
Dimensionless axial velocity u(y)
Dimensionless normal distance y

(d)


Dimensionless axial velocity u(y)
Dimensionless normal distance y
Bi = 1
Bi = 2
Bi = 3
Bi = 5
Br = α = 1 , a = 1 , b = 0
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
0.1
0.3
0.5
0.7
0.9

(e)


Dimensionless axial velocity u(y)
Dimensionless normal distance y
Br = 0.1
Br = 0.5
Br = 1.0
Br = 1.5
Bi = α = 1 , a = 0 , b = 1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

(f)

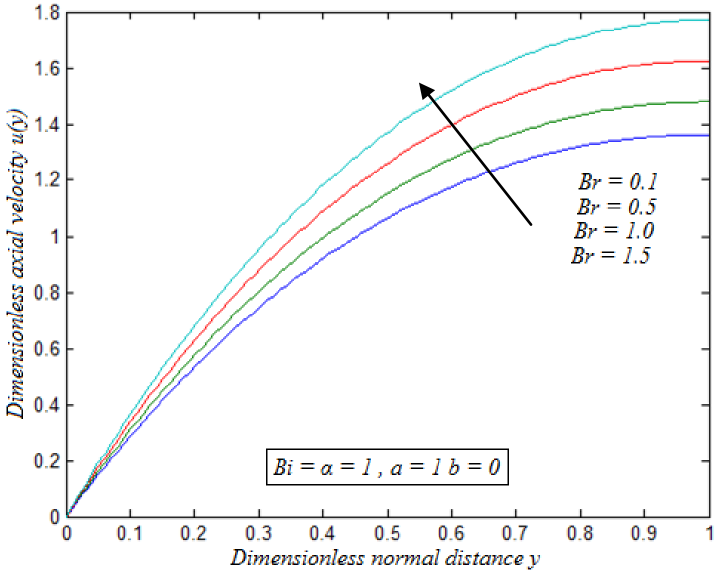
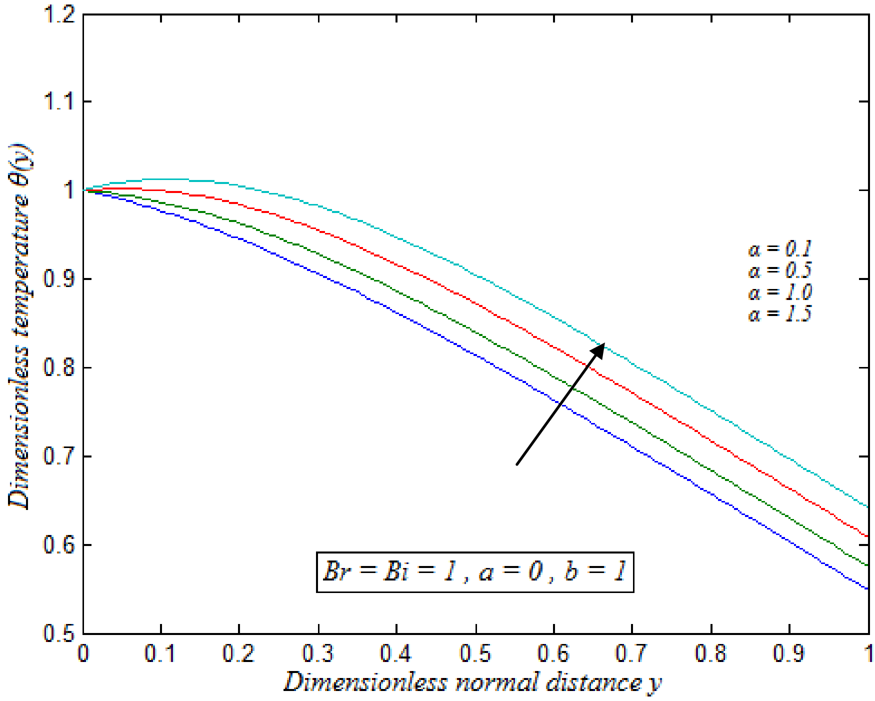
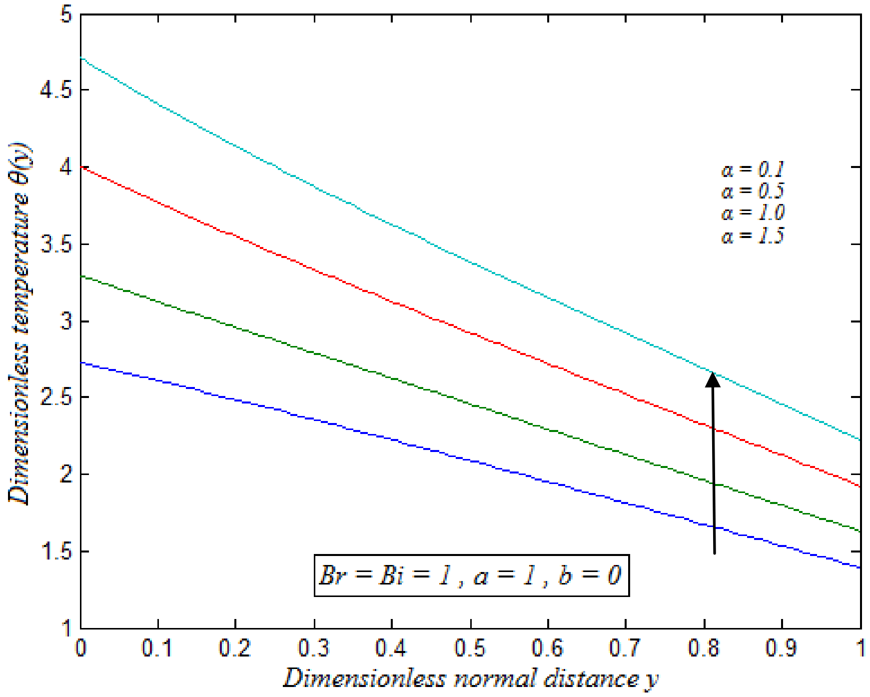


Fig. 2: Dimensionless axial velocity $u(y)$ versus the dimensionless normal distance y. The curves are plotted using the eqn.(12) for various values of the dimensionless parameters α, Bi, Br and in some fixed values of the other dimensionless parameters, when
(a) $Br = Bi = 1, a = 0, b = 1$; (b) $Br = Bi = 1, a = 1, b = 0$; (c) $Br = \alpha = 1, a = 0, b = 1$;
(d) $Br = \alpha = 1, a = 1, b = 0$; (e) $Bi = \alpha = 1, a = 0, b = 1$; (f) $Bi = \alpha = 1, a = 1, b = 0$.

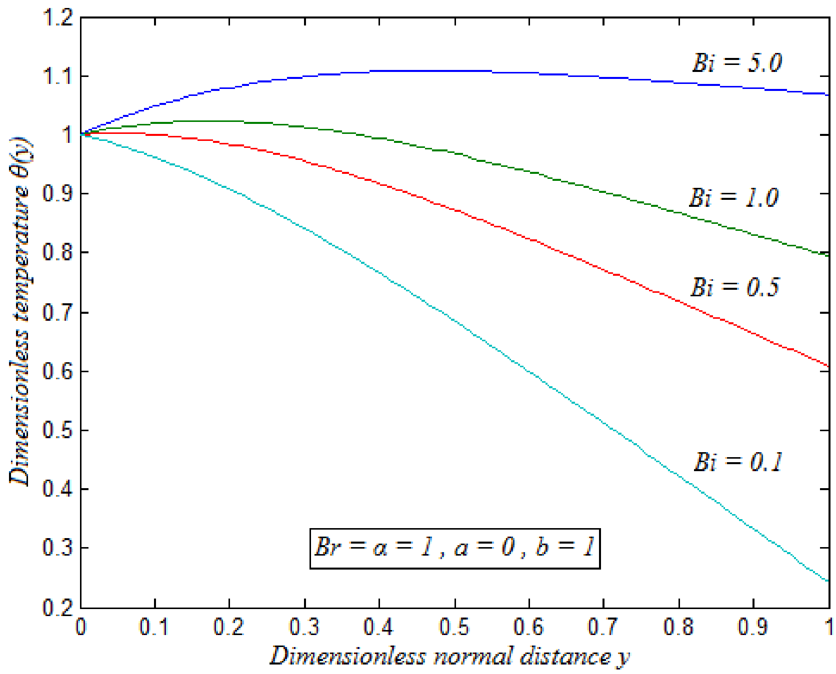
(a)



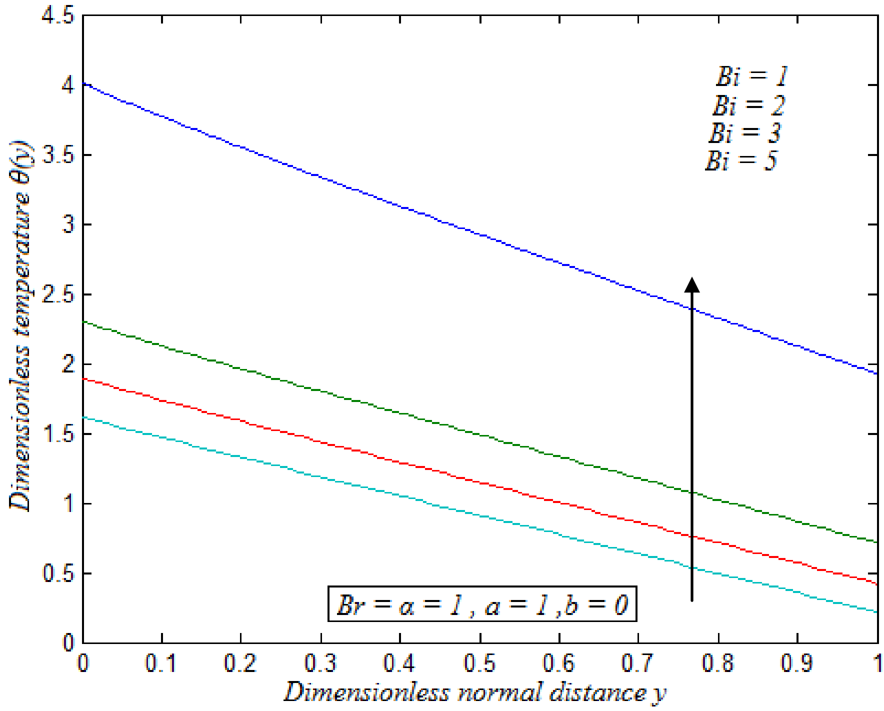
(b)


α = 0.1
α = 0.5
α = 1.0
α = 1.5
Br = Bi = 1 , a = 1 , b = 0
Dimensionless temperature θ(y)
Dimensionless normal distance y

(c)

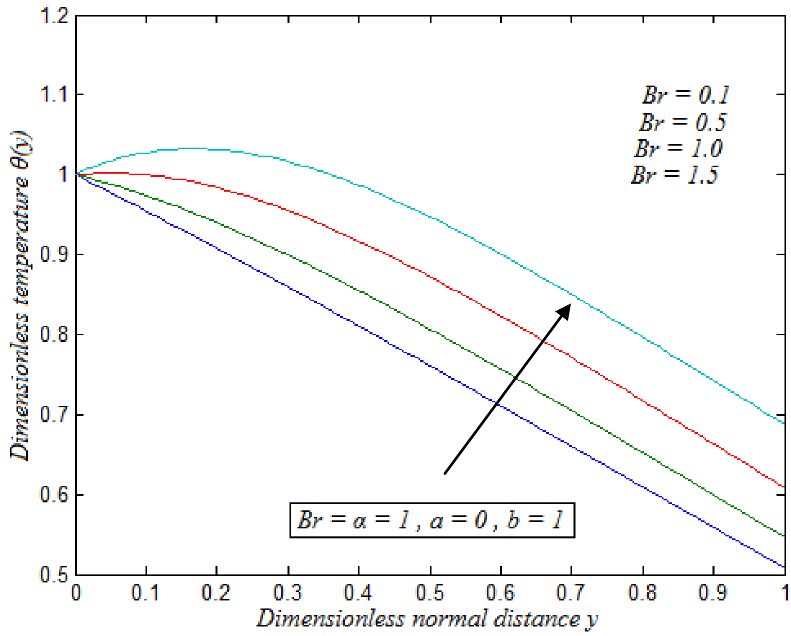

Bi = 5.0
Bi = 1.0
Bi = 0.5
Bi = 0.1
Br = α = 1 , a = 0 , b = 1
Dimensionless temperature θ(y)
Dimensionless normal distance y

(d)



Bi = 1
Bi = 2
Bi = 3
Bi = 5
Br = α = 1 , a = 1 ,b = 0
Dimensionless temperature θ(y)
Dimensionless normal distance y

(e)



Br = 0.1
Br = 0.5
Br = 1.0
Br = 1.5
Br = α = 1 , a = 0 , b = 1
Dimensionless temperature θ(y)
Dimensionless normal distance y

(f)

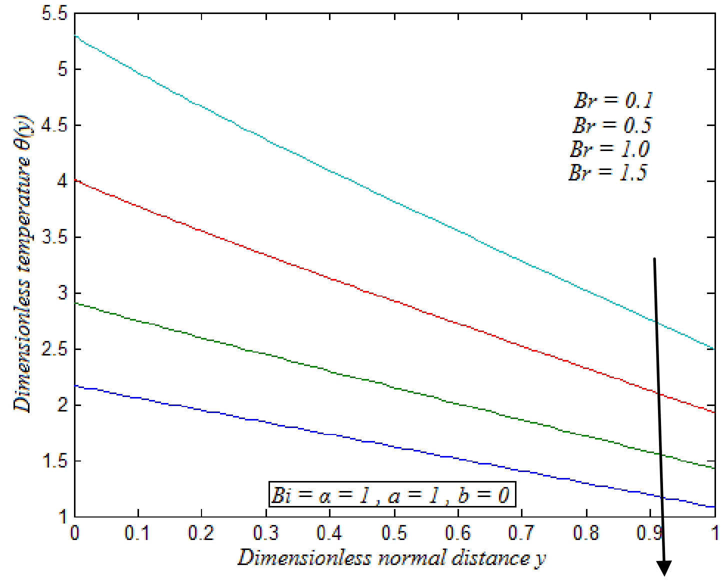
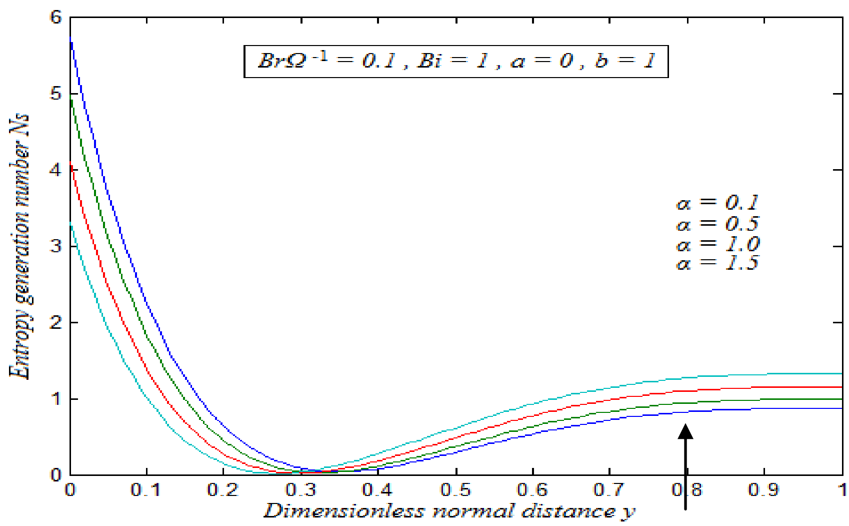


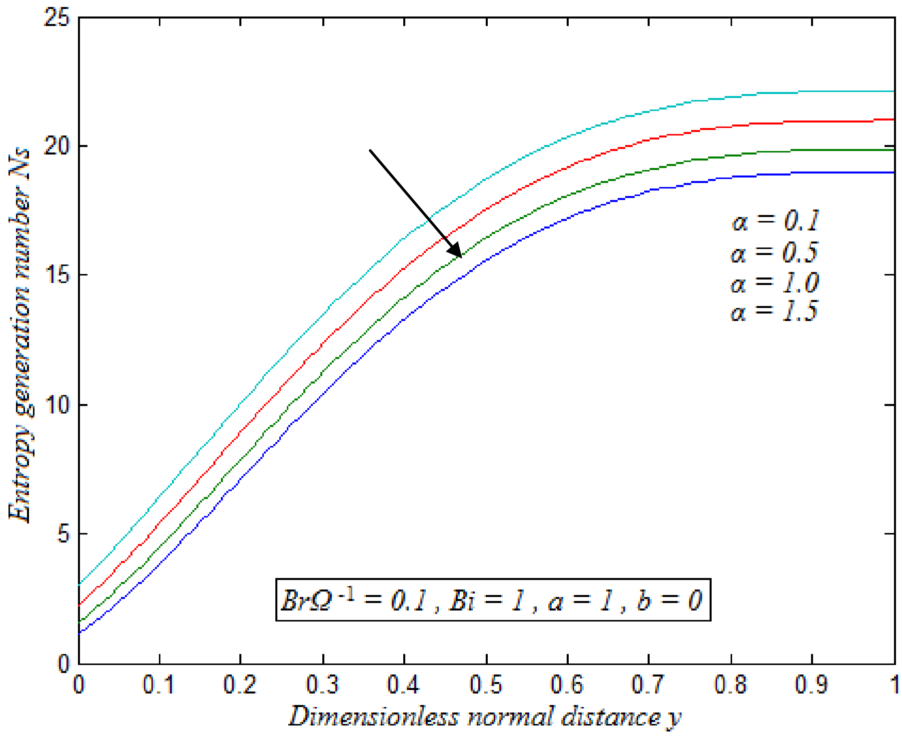
Fig. 3: Dimensionless temperature $\theta(y)$ versus dimensionless normal distance y . The curves are plotted using the eqn.(13)for various values of the dimensionless parameters α, Bi, Br, and in some fixed values of the other dimensionless parameters, when

$(a) Br = Bi = 1, a = 0, b = 1,$ (b) $Br = Bi = 1, a = 1, b = 0,$ (c) $Br = \alpha = 1, a = 0, b = 1;$
$(d) Br = \alpha = 1, a = 1, b = 0;$ (e) $Bi = \alpha = 1, a = 0, b = 1;$ $(f) Bi = \alpha = 1, a = 1, b = 0.$

(a)

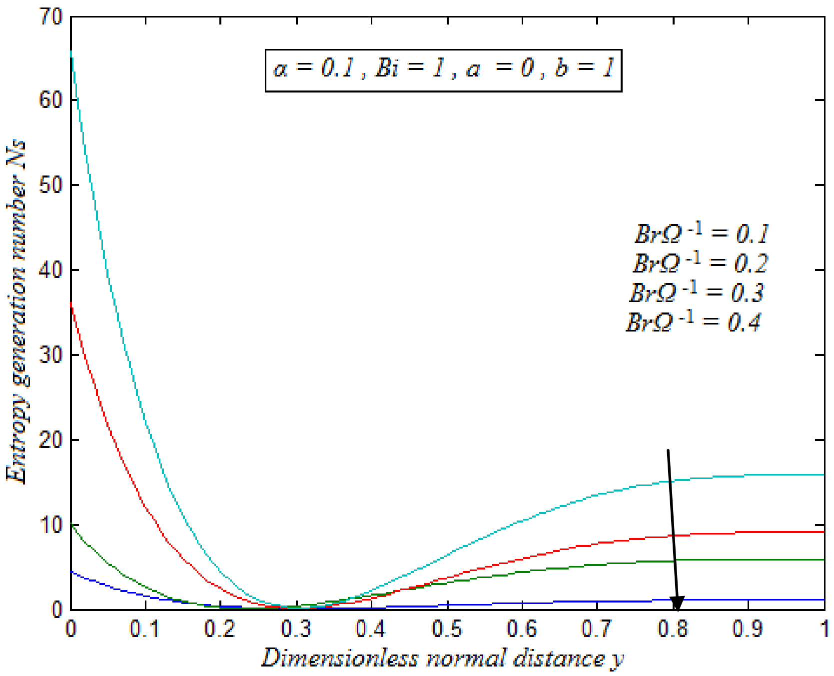


(b)



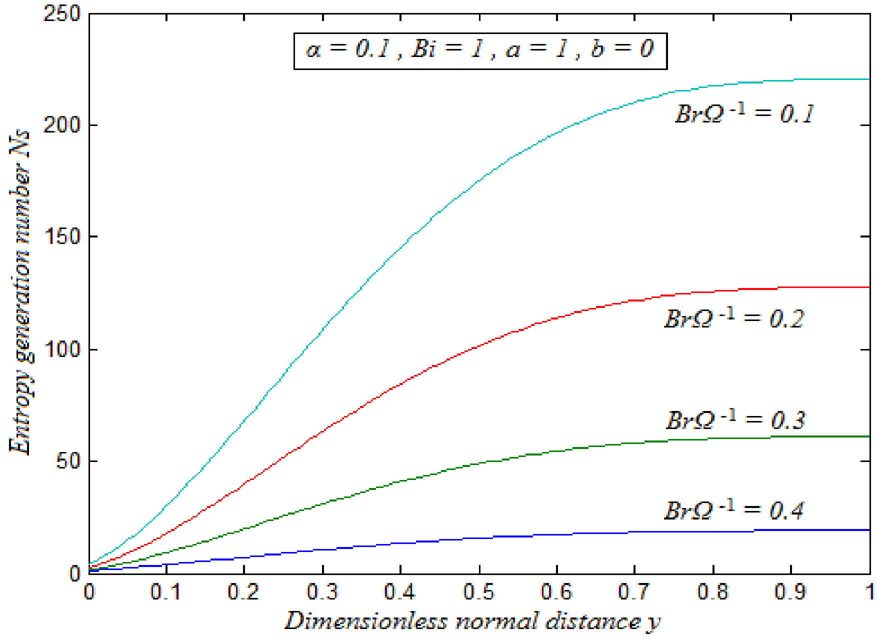
Entropy generation number Ns
Dimensionless normal distance y
α = 0.1
α = 0.5
α = 1.0
α = 1.5
BrΩ -1 = 0.1 , Bi = 1 , a = 1 , b = 0

(c)



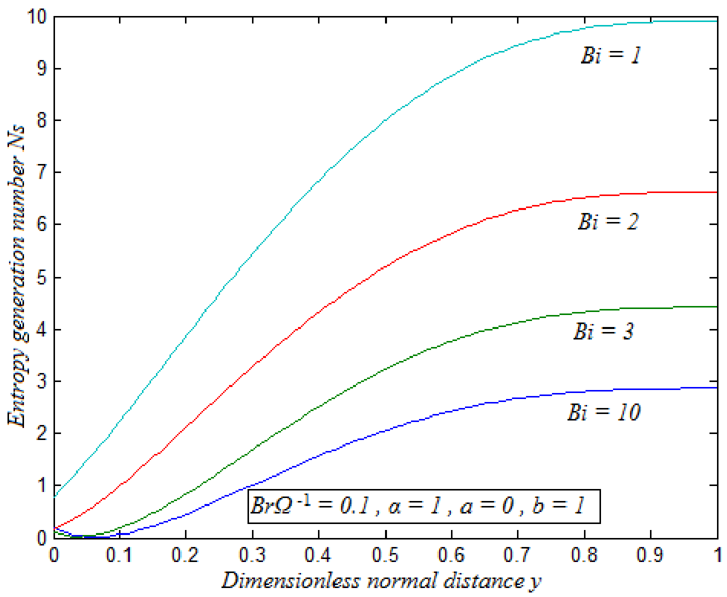
Entropy generation number Ns
Dimensionless normal distance y
α = 0.1 , Bi = 1 , a = 0 , b = 1
BrΩ -1 = 0.1
BrΩ -1 = 0.2
BrΩ -1 = 0.3
BrΩ -1 = 0.4

(d)



$\alpha = 0.1$, $Bi = 1$, $a = 1$, $b = 0$
$Br\Omega^{-1} = 0.1$
$Br\Omega^{-1} = 0.2$
$Br\Omega^{-1} = 0.3$
$Br\Omega^{-1} = 0.4$
Entropy generation number Ns
Dimensionless normal distance y

(e)



$Bi = 1$
$Bi = 2$
$Bi = 3$
$Bi = 10$
$Br\Omega^{-1} = 0.1$, $\alpha = 1$, $a = 0$, $b = 1$
Entropy generation number Ns
Dimensionless normal distance y

(f)

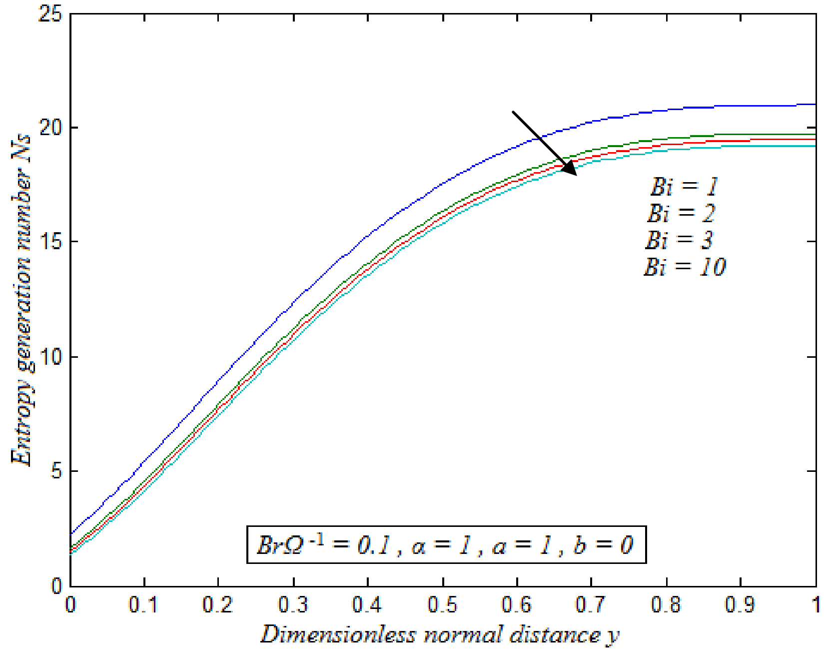
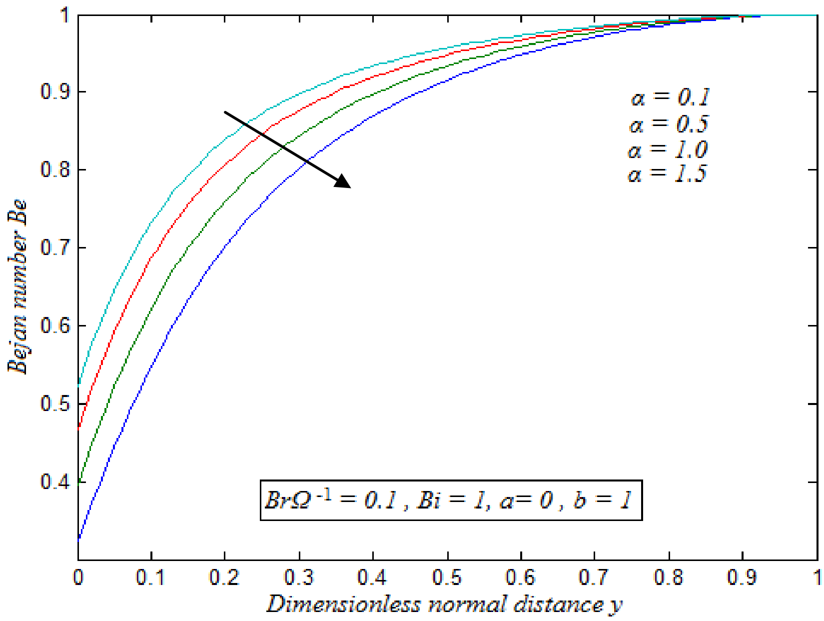


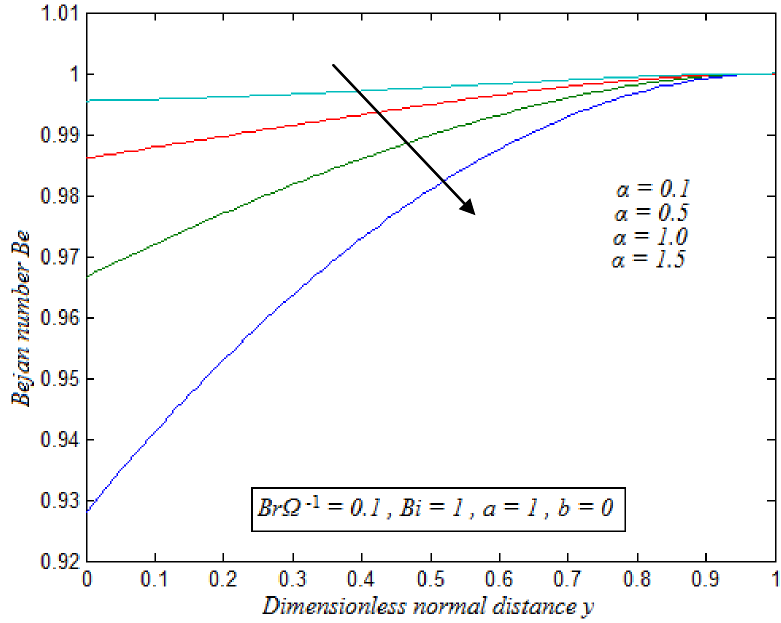
Fig.4 : Entropy generation number Ns, versus dimensionless normal distance y. The curves are plotted using the eqn.(25) for various values of the other dimensionless parameters, with

$(a) Br\,\Omega^{-1} = 0.1, Bi = 1, a = 0, b = 1$ (b) $Br\,\Omega^{-1} = 0.1, Bi = 1, a = 1, b = 0;$ $(c) \alpha = 0.1, Bi = 1, a = 0, b = 1$ $(d) \alpha = 0.1, Bi = 1, a = 1, b = 0;$ $(e) Br\,\Omega^{-1} = 0.1, \alpha = 1, a = 0, b = 1;$ $(f) Br\,\Omega^{-1} = 0.1, \alpha = 1, a = 1, b = 0.$

(a)

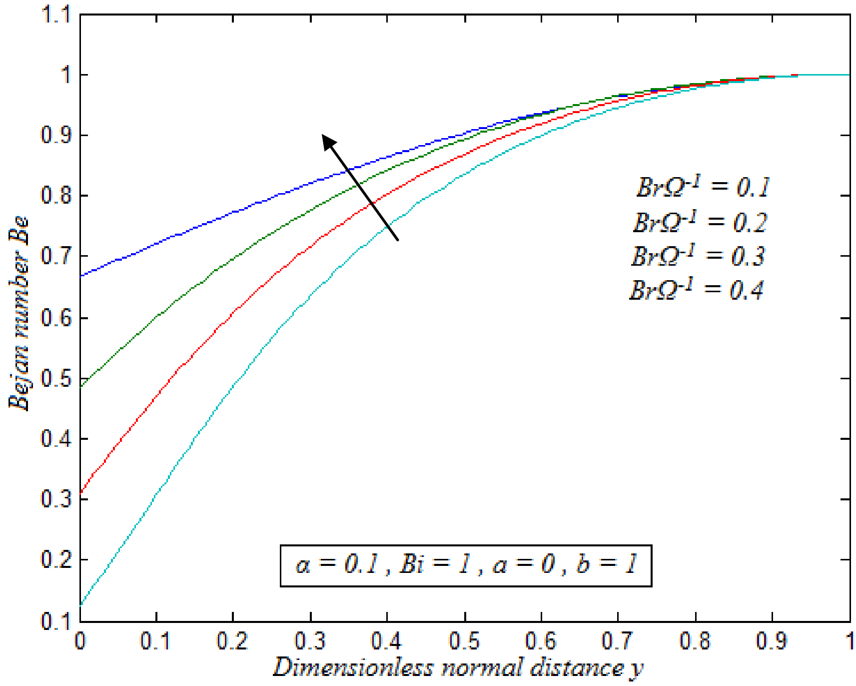


(b)



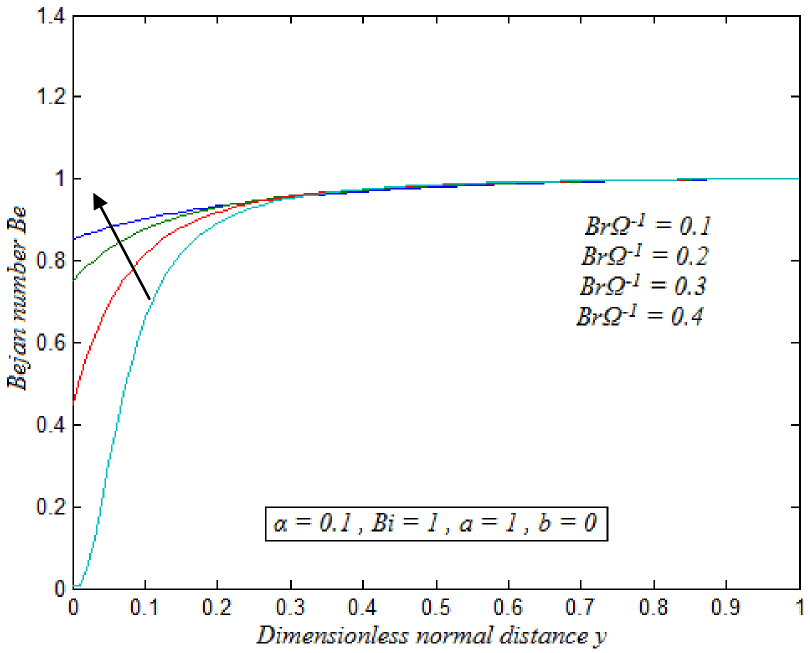
1.01
1
0.99
0.98
0.97
0.96
0.95
0.94
0.93
0.92
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Bejan number Be
Dimensionless normal distance y
α = 0.1
α = 0.5
α = 1.0
α = 1.5
BrΩ -1 = 0.1 , Bi = 1 , a = 1 , b = 0

(c)

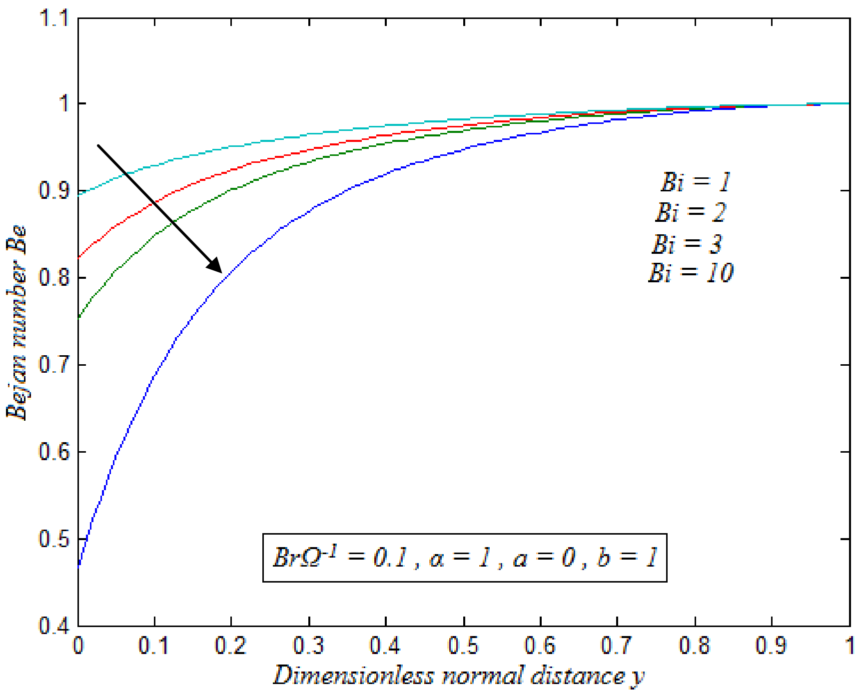


1.1
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Bejan number Be
Dimensionless normal distance y
BrΩ-1 = 0.1
BrΩ-1 = 0.2
BrΩ-1 = 0.3
BrΩ-1 = 0.4
α = 0.1 , Bi = 1 , a = 0 , b = 1

(d)


Bejan number Be
Dimensionless normal distance y
$Br\Omega^{-1} = 0.1$
$Br\Omega^{-1} = 0.2$
$Br\Omega^{-1} = 0.3$
$Br\Omega^{-1} = 0.4$
$\alpha = 0.1$, $Bi = 1$, $a = 1$, $b = 0$

(e)


Bejan number Be
Dimensionless normal distance y
$Bi = 1$
$Bi = 2$
$Bi = 3$
$Bi = 10$
$Br\Omega^{-1} = 0.1$, $\alpha = 1$, $a = 0$, $b = 1$

(f)

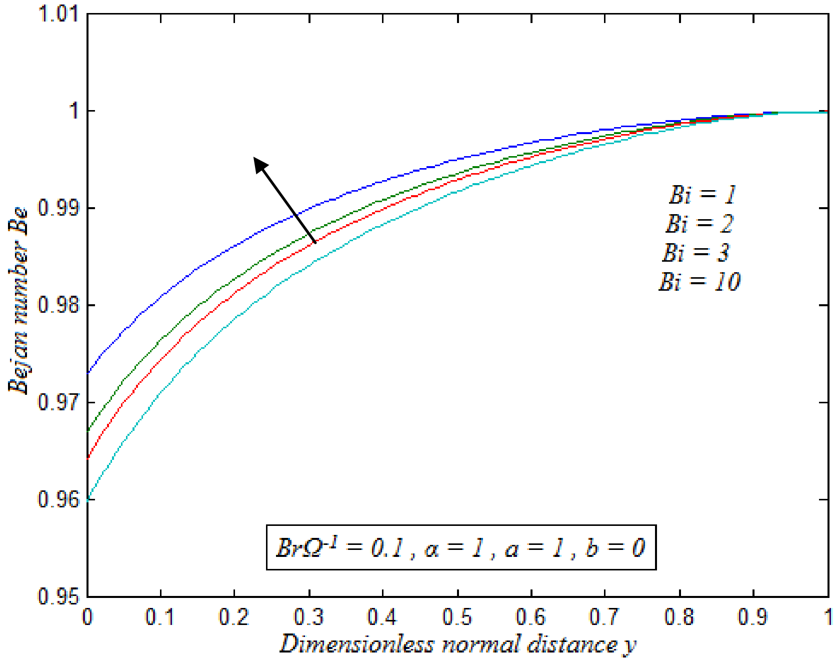


Fig. 5: Dimensionless normal distance y versus Bejan number Be.The curves are plotted using the eqn.(27) for various values of the other dimensionless parameters, with

$(a)Br\,\Omega^{-1}=0.1, Bi=1, a=0, b=1$ $(b)Br\,\Omega^{-1}=0.1, Bi=1, a=1, b=0$ $(c)\alpha=0.1, Bi=1, a=0, b=1$
$(d)\alpha=0.1, Bi=1, a=1, b=0$ $(e)Br\,\Omega^{-1}=0.1, \alpha=1, a=0, b=1$
$(f)Br\,\Omega^{-1}=0.1, \alpha=1, a=1, b=0.$

7. Conclusion

In this paper the modified Homotopy analysis method is employed to get the analytical solution for the gravity driven variable viscosity liquid film is derived mathematically and graphically. We depict the analytical solution for the velocity and temperature profiles are obtained and it is used to find the shear stress and Bejan number. Our results revealed that both the velocity and temperature decreased with increasing convective cooling (Bi) and increased with increasing values of α , Br . The flow system over an isothermal-heated inclined plate surface is more thermally stable than the case of the isoflux-heated plate surface. We also derived the mathematical and graphical representations of the entropy generation number and the Bejan number.

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Appendix: A

Basic Concept of the Homotopy analysis method

Consider the following differential equation:

$$N[u(t)] = 0 \tag{A.1}$$

Where N is a nonlinear operator, t denotes an independent variable, $u(t)$ is an unknown function. For simplicity, weignore all boundary or initial conditions, which can be treated in the similar way. By means of generalizing the conventional Homotopy method, Liao constructed the so-called zero-order deformation equation as:

$$(1-p)L[\varphi(t;p)-u_o(t)] = phH(t)N[\varphi(t;p)] \tag{A.2}$$

where $p \in [0,1]$ is the embedding parameter, $h \neq 0$ is a nonzero auxiliary parameter, $H(t) \neq 0$ is an auxiliary function, L an auxiliary linear operator, $u_o(t)$ is an initial guess of $u(t), \phi(t:p)$ is an unknown function. It is important to note thatone has great freedom to choose auxiliary unknowns in HAM. Obviously, when $p=0$ and $p=1,$ it holds:

$$\varphi(t;0) = u_o(t) \ text and \ \varphi(t;1) = u(t) \tag{A.3}$$

respectively. Thus, as p increases from 0 to 1, the solution $\varphi(t;p)$ varies from the initial guess $u_o(t)$ to the solution $u(t)$.

Expanding $\varphi(t;p)$ in Taylor series with respect to p, we have:

$$\varphi(t;p) = u_o(t) + \sum_{m=1}^{+\infty} u_m(t)p^m \tag{A.4}$$

$$u_m(t) = \frac{1}{m!}\frac{\partial^m \varphi(t;p)}{\partial p^m}\bigg|_{p=0} \tag{A.5}$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter h, and the auxiliary function are so properly chosen, the series eqn.(A.4) converges at $p=1$ then we have:

$$u(t) = u_o(t) + \sum_{m=1}^{+\infty} u_m(t) \tag{A.6}$$

Differentiating the enq.(A.2) for m times with respect to the embedding parameter p, and then setting $p=0$ and finally dividing them by m!, we will have the so-called mth-order deformation equation as:

$$L[u_m - \chi_m u_{m-1}] = hH(t)\Re_m(\vec{u}_{m-1}) \tag{A.7}$$

Where

$$\Re_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1} N[\varphi(t;p)]}{\partial p^{m-1}} \tag{A.8}$$

And

$$\chi_m = \begin{cases} 0, m \leq 1, \\ 1, m > 1. \end{cases} \tag{A.9}$$

Applying L^{-1} on both side of equation (A7), we get

$$u_m(t) = \chi_m u_{m-1}(t) + hL^{-1}[H(t)\Re_m(\vec{u}_{m-1})] \tag{A.10}$$

In this way, it is easily to obtain u_m for $m \geq 1$, at M^{th} order, we have

$$u(t) = \sum_{m=0}^{M} u_m(t) \tag{A.11}$$

When $M \to +\infty,$ we get an accurate approximation of the original eqn. (A.1). For the convergence of the above method we refer the reader to Liao [19]. If equation (A.1) admits unique solution, then this method will produce the unique solution.

Appendix: B

Solution of the non-linear differential eqns.(9)-(11) using the Homotopy analysis method

This appendix contains the derivation of the analytical expressions eqns.(13) and (12) for $u(y)$ and $\theta(y)$ using the Homotopy analysis method. The eqns.(9)-(11) are as follows:

$$\frac{d^2\theta}{dy^2} + Br(1-y)^2(1+\alpha\theta) = 0 \tag{B.1}$$

$$\frac{du}{dy} = (1-y)(1+\alpha\theta) \tag{B.2}$$

With the following boundary conditions

$$\frac{d\theta}{dy}(1) = -Bi\theta(1)\ ,\ a\frac{d\theta}{dy}(0) - b\theta(0) = -1\ ,\ u(0) = 0 \tag{B.3}$$

We construct the Homotopy for the eqn. (B.1) is as follows:

$$(1-p)\frac{d^2\theta}{dy^2} + Br(1-y)^2 = hp\left(\frac{d^2\theta}{dy^2} + Br(1-y)^2(1+\alpha\theta) = 0\right) \tag{B.4}$$

The approximate solution for the eqn.(B.4) is given by

$$\theta = \theta_o + p\theta_1 + p^2\theta_2 + \ldots\ldots \tag{B.5}$$

Substituting the eqn.(B.5) into an eqn. (B.4) we get

$$(1-p)\frac{d^2(\theta_o + p\theta_1 + p^2\theta_2 + \ldots\ldots)}{dy^2} + Br(1-y)^2$$

$$= hp\left(\begin{array}{l}\dfrac{d^2(\theta_o + p\theta_1 + p^2\theta_2 + \ldots\ldots)}{dy^2} \\ + Br(1-y)^2(1+\alpha(\theta_o + p\theta_1 + p^2\theta_2 + \ldots\ldots))\end{array}\right) \tag{B.6}$$

Now equating the coefficients of θ_o and θ_1 we get the following equations:

$$p^o = \frac{d^2\theta_o}{dy^2} + Br(1-y)^2 \tag{B.7}$$

$$p^1 = \frac{d^2\theta_1}{dy^2} + Br(1-y)^2 - \frac{d^2\theta_o}{dy^2} + Br(1-y)^2 - h\left(\begin{array}{l}\dfrac{d^2(\theta_o)}{dy^2} \\ + Br(1-y)^2(1+\alpha(\theta_o))\end{array}\right) \tag{B.8}$$

The initial approximations are as follows:

$$\frac{d\theta_0}{dy}(1) = -Bi\theta_0(1)\ ,\ a\frac{d\theta_0}{dy}(0) - b\theta_0(0) = -1\ ,\ u_0(0) = 0 \tag{B.9}$$

$$\frac{d\theta_i}{dy}(1) = -Bi\theta_i(1)\ ,\ a\frac{d\theta_i}{dy}(0) - b\theta_i(0) = 0\ ,\ u_i(0) = 0\ \ ,i = 1,2,3,\ldots\ldots \tag{B.10}$$

Solving the eqns. (B.7) and (B.8) using the eqns. (B.9) and (B.10), we obtain the following solutions

$$\theta_o = -Br\left\{\frac{y^2}{2} - \frac{y^3}{3} + \frac{y^4}{12}\right\} + Ay + B \tag{B.11}$$

$$\theta_1 = -\frac{Br\alpha C_2 y^2}{2} + E_1\frac{y^3}{6} + E_2\frac{y^4}{12} + E_3\frac{y^5}{20} + Br^2\alpha\left[\frac{3y^6}{72} - \frac{y^7}{84} + \frac{y^8}{672}\right] + C_3 y + C_4 \tag{B.12}$$

We construct the Homotopy for the eqn.(B.2) is as follows:

$$(1-p)\frac{du}{dy} - (1-y) = hp\left(\frac{du}{dy} - (1-y)(1+\alpha\theta)\right) \tag{B.13}$$

The approximate solution of the eqn.(B.13) is given by,

$$u = u_o + pu_1 + p^2u_2 + \ldots\ldots \tag{B.14}$$

Substituting the eqn.(B.14) into an eqn.(B.13) we get

$$(1-p)\frac{d\left(u_o+pu_1+p^2u_2+......\right)}{dy}-(1-y)=hp\left(\frac{d\left(u_o+pu_1+p^2u_2+......\right)}{dy}-(1-y)(1+\alpha\theta)\right) \quad (B.15)$$

Now equating the coefficients of p^0 and p^1 we get the following equations:

$$p^0=\frac{du_0}{dy}-(1-y) \quad (B.16)$$

$$p^1=\frac{du_1}{dy}-(1-y)-\frac{du_0}{dy}-(1-y)-h\left(\frac{d(u_0)}{dy}-(1-y)(1+\alpha\theta_0)\right) \quad (B.17)$$

Solving the eqns.(B.16) and (B.17) using the eqns.(B.9) and (B.10), we obtain following solutions:

$$u_0=y-\frac{y^2}{2} \quad (B.18)$$

$$u_1=\alpha\left(\frac{C_1y^2}{2}+C_2y-\frac{C_1y^3}{3}-\frac{C_2y^2}{2}\right) \quad (B.19)$$

Where

$$C_1=\frac{\frac{bBr}{3}+\frac{bBiBr}{4}-Bi}{\left(b(1+Bi)+aBi\right)} \quad (B.20)$$

$$C_2=\frac{1}{Bi}\left[\frac{Br}{3}+\frac{BiBr}{4}-(1+Bi)C_1\right] \quad (B.21)$$

$$C_2=\frac{1}{Bi}\left[\frac{Br}{3}+\frac{BiBr}{4}-(1+Bi)C_1\right] \quad (B.22)$$

$$C_3=\frac{\left(\begin{array}{l}Br\alpha C_2-\frac{15Br^2\alpha}{84}+\frac{C_2BiBr\alpha}{2}+\frac{21BiBr^2\alpha}{672}-\frac{E_1}{2}-\frac{E_2}{3}-\frac{E_3}{4}\\-\frac{BiE_1}{6}-\frac{BiE_2}{12}-\frac{BiE_3}{20}\end{array}\right)}{b(1+Bi)+a\,Bi} \quad (B.23)$$

$$C_4=\frac{\left(\left(\begin{array}{l}Br\alpha C_2-\frac{15Br^2\alpha}{84}+\frac{C_2BiBr\alpha}{2}+\frac{21BiBr^2\alpha}{672}-\frac{E_1}{2}-\frac{E_2}{3}\\-\frac{E_3}{4}-\frac{BiE_1}{6}-\frac{BiE_2}{12}-\frac{BiE_3}{20}\end{array}\right)-C_3(1+Bi)\right)}{Bi} \quad (B.24)$$

$$E_1=-\alpha BrC_1+2Br\alpha C_2 \quad (B.25)$$

$$E_2=\frac{Br^2\alpha}{2}+2\alpha BrC_1-\alpha BrC_2 \quad (B.26)$$

$$E_3=\frac{-4Br^2\alpha}{3}-\alpha BrC_1 \quad (B.27)$$

According to Homotopy analysis method we have

$$\theta=\lim_{p\to1}\theta(y)=\theta_0+\theta_1 \quad (B.28)$$

$$u = \lim_{p \to 1} u(y) = u_0 + u_1 \qquad \text{(B.29)}$$

Using the eqns. (B.11) and (B.12) in to an eqn. (B.28) and using the eqns. (B.18) and (B.19) into an eqn. (B.29), we get the solution in the text eqns. (13) and (12) respectively.

Appendix C:
Nomenclature

Symbol	Meaning
T	Fluid temperature
T_0	Plate reference temperature
$\bar{y}$	Normal distance
k	Thermal conductivity
h	Heat transfer coefficient
$\bar{u}$	Axial velocity component
T_a	Ambient temperature
g	Gravitational acceleration
Br	Brinkmann number
Bi	Biot number
Be	Bejan number
m	Variable viscosity parameter
u	Dimensionless axial velocity
a	Isoflux heating parameters
b	Isothermal heating parameter
Ns	Entropy generation number
y	Dimensionless normal distance
ϕ	Inclination angle
δ	Liquid film thickness
θ	Dimensionless temperature
μ	Fluid dynamic viscosity
α	Variable viscosity parameter
ρ	Fluid density
μ_0	Viscosity at ambient temperature
Φ	Irreversibility distribution ratio
Ω	Temperature difference parameter
$\bar{\mu}$	Fluid dynamic viscosity