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OBSERVATION ON THE HYPERBOLA $y^2 = 220x^2 + 9$

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Abstract

The binary quadratic equation represented by the positive pellian $y^2 = 220x^2 + 9$ is analyzed for its distinct integer solutions. A few interesting relations among the solutions are also given. Further, employing the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbolas, parabolas and special Pythagorean triangle.

Keywords: Binary quadratic, hyperbola, parabola, integral solutions, pell equation, Special numbers.
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NOTATIONS:

$$P_n^m = \frac{[n(n+1)(m-2)n + (5-m)]}{6}$$

$$pr_n = n(n+1)$$

$$cp_{6,n} = n^3$$

$$t_{m,n} = n\frac{(n+1)}{2}$$

INTRODUCTION:

The binary quadratic equation of the form $y^2 = Dx^2 + 1$, where D is non- square positive integer, has been studied by various mathematicians for its non- trivial integral solutions when D takes different integral values [1-4]. In [5] infinitely many Pythagorean triangle in each of which hypotenuse is four times the product of the generators added with unity are obtained by employing the non-integral solutions of binary quadratic equation of $y^2 = 3x^2 + 1$. In [6], a special Pythagorean triangle is obtained by employing the integral solutions of $y^2 = 10x^2 + 1$. In [7], different patterns of infinitely many Pythagorean triangle are obtained by employing the non-trivial solutions of $y^2 = 12x^2 + 1$. In this context one may also refer [8-16]. These results have motivated us to search for the integral solutions of yet another binary quadratic equation $y^2 = 220x^2 + 9$ representing a hyperbola. A few interesting properties among the solutions are presented. Employing the integral solutions of the equation under the consideration, a special Pythagorean triangle is obtained.

METHODS OF ANALYSIS:

Consider the binary quadratic equation

$$y^2 = 220x^2 + 9 \qquad (1)$$

with the least positive integer solutions x_0 =18, y_0 =267

To obtain the other solutions of equation (1) ,Consider the pellian equation

$$y^2 = 220x^2 + 1$$

whose general solution,

$$x_n = \frac{g_n}{4\sqrt{55}},\ y_n = \frac{f_n}{2}$$

in which

$$f_n = \left(89 + 12\sqrt{55}\right)^{n+1} + \left(89 - 12\sqrt{55}\right)^{n+1}$$

$$g_n = \left(89 + 12\sqrt{55}\right)^{n+1} - \left(89 - 12\sqrt{55}\right)^{n+1}$$

where n= -1, 0, 1, 2……

Applying Brahmagupta lemma between the solution s of (x_0, y_0) and (x_n, y_n) the general solutions of equation (1) are found to be

$$x_{n+1} = 9f_n + \frac{267g_n}{4\sqrt{55}} \quad (2)$$

$$y_{n+1} = \frac{267f_n}{2} + 18\sqrt{55}g_n \quad (3)$$

Thus, (2) and (3) represent the non-zero distinct integer solutions of (1) which represent a hyperbola.

The recurrence relations satisfied by the values of x and y are respectively

$$178x_{n+2} - x_{n+1} - x_{n+3} = 0$$

$$178y_{n+2} - y_{n+1} - y_{n+3} = 0$$

A few numerical examples are presented in the table below

n	x_{n+1}	y_{n+1}
-1	18	267
0	3204	47523
1	570294	8458827
2	101509128	1505623683
3	18068054490	267992556747
4	3216012190092	47701169477283

A few interesting properties are given below:

- The values of x are even while the values of y are odd
- Each of the following is a Nasty number

$$801\left[267x_{2n+4} - 8458827x_{2n+2} + 9612\right]$$

$$6\left\{2\left[267y_{2n+2} - 3960x_{2n+2}\right] + 18\right\}$$

$$6\left\{2\left[47523y_{2n+3} - 704880x_{2n+3}\right] + 18\right\}$$

- $4(267y_{n+1} - 3960x_{n+1}) * \left[47523y_{n+2} - 704880x_{n+2}\right]$ is a Square number
- Each of the following is a cubical integer

$$267x_{3n+4} - 47523x_{3n+3} + 18\left[267y_{n+1} - 3960x_{n+1}\right]$$

$$3\left[95046y_{3n+4}-140976x_{3n+4}+267x_{n+2}-47523x_{n+1}\right]$$

$$\frac{1}{178}\left[267x_{3n+5}-8458827x_{3n+3}+142578x_{n+2}-25377282x_{n+1}\right]$$

- $267x_{n+2}=1602y_{n+1}+23763x_{n+1}$
- $267y_{n+1}-3960x_{n+1}=47523y_{n+2}-704880x_{n+2}$
  - $x_{n+1}\equiv 0(\bmod 9)$
  - $y_{n+1}\equiv 0(\bmod 3)$
  - $\left[x_{2n+1}\equiv 0(\bmod 4)\right]$

REMARKABLE OBSERVATIONS:

1. Employing the solutions (x, y) of (1), each of the following expressions among the special polygonal and pyramidal numbers is a congruent to zero under modulo 9

$$\left(\frac{2p^5{}_{y-1}}{t_{4,y-1}}\right)^2-220\left(\frac{3p^3{}_x}{t_{3,x+1}}\right)^2;\left(\frac{6p^3{}_{y-2}}{pr_{y-2}}\right)-220\left(\frac{6p^4{}_{x-1}}{t_{3,2(x-1)}}\right)^2$$

2. Employing the linear combinations among the integer solution of (1), one may obtain integer solutions for different geometrical representations. For simplicity and clear understanding, a few illustrations are presented in the table 1 below:

Table 1:

3. Using the integer solutions of (1), one may obtain relations among special polygonal numbers. A few relations are exhibited in the following table 2.

X_1	X_2	Y_1	Relation between X_1, X_2 and Y_1	Geometrical representation
$534y_{n+1}-7920x_{n+1}$	-	$95046y_{n+1}-7920x_{n+2}$	$Y_1=89X_1$	Straight line
$178y_{n+1}-2640x_{n+1}$	$534y_{2n+2}-7920x_{2n+2}+18$	$58740x_{n+1}-3960y_{n+1}$	$Y_1^2=495X_2-17820$ **&** $X_2=X_1^2$	Parabola
$178y_{n+1}-2640x_{n+1}$	$534y_{2n+2}-7920x_{2n+2}+18$	$58740x_{n+1}-3960y_{n+1}$	$Y_1^2=495X_1^2-17820$	Hyperbola

Table 2:

m	n	Relations among special numbers
$\frac{y_{3N-3}-2}{2}$	$\frac{y_{3N-3}+2}{5}, N=1,2,3,..$	$2t_{3,m}=275t_{12,n}+222$
$\frac{y_{s+1}-1}{2}$	x_{s+1}	$t_{3,m}=\frac{55np_n^5}{pr_n}+1$
$\frac{y_{s+1}-1}{2}$	x_{s+1}	$2(t_{3,m}-1)\times n=55cp_{6,n}$

$\frac{y_{s+1}-1}{2}$	$x_{s+1}-1$	$2(t_{3,m}-1)t_{3,n+1}=55(n+1)p_{n+1}^{5}$

4. Consider $r = x_{n+1} + y_{n+1}$, $s = x_{n+1}$. Observe that $r > s > 0$. Treat *r,s* as the generators of the Pythagorean triangle T(α, β, γ),where $\alpha = 2rs, \beta = r^2 - s^2, \gamma = r^2 + s^2$.

   Let A, P represent the area and perimeter of T(α, β, γ)

   Then the following interesting relations are observed.

$$\alpha - 110\beta + 109\gamma = -9$$

$$111\alpha - \gamma + 9 = 440\frac{A}{P}$$

CONCLUSION:

In this paper, we have presented infinitely many integer solutions for the hyperbola represented by the positive Pell equation $y^2 = 220x^2 + 9$. As the binary quadratic Diophantine equations are rich in variety, one may search for the other choices of negative Pell equations and determine their integer solutions along with suitable properties.

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