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MATHEMATICS

MATHEMATICAL ANALYSIS OF STEADY BOUNDARY LAYER FLOW AND HEAT TRANSFER OF A VISCOUS INCOMPRESSIBLE FLUID AND HOMOTOPY ANALYSIS METHOD

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Abstract*:*

In this research paper, the heat transfer of a viscous incompressible fluid due to a stretching porous sheet in the presence of heat source under the influence of viscous dissipation on laminar flow is studied. The analytical expressions for dimensionless stream function and dimensionless temperature are derived using the Homotopy analysis method. And also recovery temperature are solved analytical and graphically for the various effective parameters. Our analytical and graphical results are compared with the previous works and a satisfactory agreement is noted.

Keywords: Heat transfer; Viscous dissipation; Porous medium, Homotopy analysis method.

1. Introduction:

Flow of fluid and heat transfer on a moving surfaces are interesting due to many of its applications in manufacturing processes such as glass-fibre production, metal extrusion,cooling of metallic paths rapidly, paper production, drawing plastic films and aerodynamic extraction of plastic sheet.One of the earliest studies on the boundary layer flow of a viscous fluid due to a motion of a plate was initiated by Sakiadis [1]. The solution of Sakiadis problem was exprementally verified by Tsou et al [2]. The extended of this problem in the case of suction or blowing of fluid was carried by Erickson et al [3] and Gupta et al [4]. Ali [5] who investigated solutions for a thermal boundary layer over a power-law stretching surface with suction or injection. The hydromagnetic flow and heat transfer have been studied by many researchers such as Abo-Eldahab [6] and Chakarabati et al [7]. The aerodynamic boundary layer was first defined by Ludwig Prandtl. The fluid flow through non-porous medium are studied by these author and internal heat generation are neglected. The analysis of flow through porous medium have several engineering applications such as the flow through packed beds, environmental pollution, blood theology. Varjravelu et al [8] studied the heat transfer in the boundary layer of a viscous fluid over a stretching surface with viscous dissipation. Convective heat and mass transfer in the fluid passes through porous medium over a stretching sheet with variable viscosity was studied by Abel et al.[9]. T. A. Abdelhafez[10] studied the skin Friction and heat transfer on a continuous flat surface moving in a parallel free stream. Kaviany[11] found the heat transfer on a laminar flow through a porous medium in isothermal parallel plates.L. Swain et al[12] has studied the heat and mass transfer effect in a boundary layer MHD flow of an electrically conducting viscous fluid subject to transverse magnetic field on an exponentially stretching sheet through porous medium. Swati Mukhopadhyay[13]studied the laminar boundary layer flow of a viscous incompressible fluid and heat transfer towards a stretching cylinder embedded in a porous medium. Jaber [14] examined the effects of viscous dissipation and joule heating on magneto hydrodynamics flow of a fluid with variable properties past a stretching vertical plate. The effect of Hall current on MHD couette flow between thick arbitrarily conducting Plates in a rotating system was studied by Mandal, et al [15] .H. Kumar[16] found the heat transfer due to stretching porous sheet in presence of heat source. Jaber [17] concluded the effect of viscous dissipation on flow over a stretching porous sheet in presence of heat source. Ananthaswamy et al[18] found the analytical expressions of mass transfer effects on unsteady flow past an accelerated vertical porous plate with suction.

In the present study, heat transfer and viscous dissipation of fluid due to stretching porous sheet in the presence of heat source is discussed. Dimensionless velocity and dimensionless temperature are solved analytically and graphically and also compared

2.Mathematical formulation of the problem

Consider the fluid flow over the plate containing the porous sheet caused the viscous dissipation in the presence of heat source. The plate is placed horizontally coincide with the plane $y=0$ and it is subjected to a power law heat flux. The flow is along the x -axis and its speed is propotional to its distance from the slit see fig. 1. The plate is continuously stretching by equal powers with opposite direction.

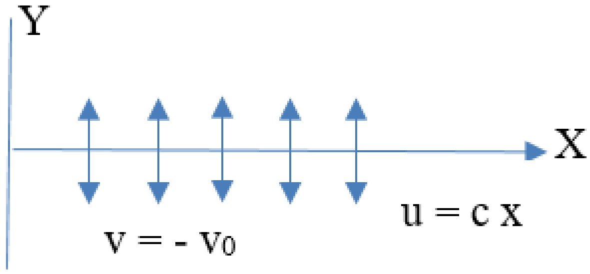


Fig.1: Physical coordinate system

Under Boussinesq's approximation approximation, the governing boundary layer equations are

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial x}=0 \tag{1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=v\frac{\partial^2 u}{\partial y^2}-\frac{v}{K}u \tag{2}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{k}{\rho C_p}\frac{\partial^2 T}{\partial y^2}+\frac{Q}{\rho C_p}(T_\infty-T)-\frac{v}{C_p}\left(\frac{\partial u}{\partial y}\right)^2 \tag{3}$$

The boundary conditions for the problem are as follows

$$\text{At } y=0: u=cx,\ v=-v_0,\ \frac{\partial T}{\partial y}=Bx^2 \tag{4}$$

$$\text{At } y\to\infty: u=v=0,\ T=T_\infty \tag{5}$$

Where c is the stretching rate and it is positive.

The suggested solution for the above equations satisfying the boundary conditions is:

$$u=cxf'(\eta),\quad v=-\sqrt{vc}\,f(\eta),\quad \eta=\sqrt{cv^{-1}}\,y,\quad T=T_\infty+Bx^2\sqrt{vc^{-1}}\,\theta(\eta)$$

The eqns.(1) and (2) can be converted into the following non-linear ordinary differential equations with dimensionless parameters

$$f'''+ff''-f'^2-\lambda f'=0 \tag{6}$$

$$\theta''+Pr\,f\theta'-2Pr\,f'\theta-\beta\theta+Pr\,Ecf''^2=0 \tag{7}$$

The corresponding boundary conditions are as follows:

$$\text{At the surface of the plate } f'(0)=1,\quad f(0)=1,\quad \theta'(0)=1 \tag{8}$$

$$\text{As } \eta\to\infty:\quad f'=f=\theta=0, \tag{9}$$

Where $Pr=\frac{\rho v C_p}{k}$, $Ec=\frac{c^{5/2}}{BC_p\sqrt{v}}$, $\lambda=\frac{v}{ck}$,denotes the Prandtl number, Eckert number, and porosity parameters respectively.

The local shear stress τ_w are defined as

$$q_w=-k\left(\frac{\partial T}{\partial y}\right)_{y=0} \tag{10}$$

Then the Nusselt number is defined as

$$Nu=\frac{q_w x}{k}=-Bx^3\theta'(0) \tag{11}$$

The shear stress on the wall is given by

$$\tau_w=\mu\left(\frac{\partial u}{\partial y}\right)_{y=0}=\frac{\mu x c^{3/2}}{\sqrt{v}}f''(0) \tag{12}$$

The numerical values of $-f''(0)$ and $\theta(0)$ for various parameter such as Prandtl number, Eckert number are tabulated in table1.

3.Approximate analytical solution for the non-linear differential equations using Homotopy analysis method

In recent days, Homotopy analysis method (HAM) is used for solving non-linear differential equations which was proposed by Liao[19]. HAM is based on the concept of Homotopy(in topology) by Hilton[20]which is applied in numerical techniques as in [21-24]It is successfully applied to numerous problems in science and engineering such as heat transfer [25],viscous flows[26],non-linear water waves[27]. HAM contains the auxiliary parameter h ,which is used to adjust the solution region.

The analytical expression for the non-linear differential equations (6) and (7) are solved using Homotopy analysis method. Dimensionless stream function $f(\eta)$ and dimensionless temperature $\theta(\eta)$ (see Appendix B)are obtained as follows:

$$f(\eta)=2-\frac{exp(-\sqrt{\lambda}\eta)}{\sqrt{\lambda}}-\frac{1}{\lambda}+\frac{exp(-\sqrt{\lambda}\eta)}{\lambda}+\frac{\eta\, exp(-\sqrt{\lambda}\eta)}{\sqrt{\lambda}} \tag{13}$$

$$\begin{aligned}\theta(\eta)=&\frac{exp(-A\eta)}{A}+C_1\, exp(-\sqrt{\beta}\eta)-\frac{Pr\, Ec\lambda\, exp(-2\sqrt{\lambda}\eta)}{4\lambda-\beta}\\&-\frac{2\,Pr\, exp(-A\eta)}{A(A^2-\beta)^2}\Big((1-\eta)(A^2-\beta)-2A\Big)\\&+\frac{Pr\, exp(-\sqrt{\lambda}-A)\eta}{A\sqrt{\lambda}((\sqrt{\lambda}+A)^2-\beta)^2}\begin{pmatrix}(1-\eta)((\sqrt{\lambda}+A)^2-\beta)-2(\sqrt{\lambda}+A)\\+2\sqrt{\lambda}(\eta((\sqrt{\lambda}+A)^2-\beta)+2(\sqrt{\lambda}+A))\end{pmatrix}\end{aligned} \tag{14}$$

Where

$$\begin{aligned}C_1=&\frac{Pr}{A\sqrt{\lambda}\Big((\sqrt{\lambda}+A)^2-\beta\Big)^2}\Big[(\sqrt{\lambda}-1-A)((\sqrt{\lambda}+A)^2-\beta)-2(\sqrt{\lambda}+A)^2(2\sqrt{\lambda}-1)\Big]\\&+\frac{2\,Pr}{(A^2-\beta)^2}\Big[(1/A+1)(A^2-\beta)-2A\Big]+\frac{2\,Pr\, Ec\lambda\sqrt{\lambda}}{4\lambda-\beta}\end{aligned} \tag{15}$$

$$A=Pr\, Ec\lambda\beta \tag{16}$$

The recovery temperature at the stretching plate is given by

$$\begin{aligned}\theta(0)=&\frac{1}{A}+C_1-\frac{Pr\, Ec\lambda}{4\lambda-\beta}-\frac{2\,Pr}{A(A^2-\beta)^2}\Big((A^2-\beta)-2A\Big)\\&+\frac{Pr}{A\sqrt{\lambda}((\sqrt{\lambda}+A)^2-\beta)^2}\Big(((\sqrt{\lambda}+A)^2-\beta)-2(\sqrt{\lambda}+A)+2\sqrt{\lambda}(2(\sqrt{\lambda}+A)\Big)\end{aligned} \tag{17}$$

4.Result and discussion

The system of two non-linear ordinary differential equations (5) and (6) corresponding to the boundary conditions(7) and (8) are solved using Homotopy Analysis method. Figures 2 – 3 represents the dimensionless velocity and dimensionless temperature profiles are shown for the various parameters. From Fig.2, it is clear that when the permeability parameter increases the corresponding dimensionless velocity profile decreases. Fig. 3(a) shows that increasing of permeability parameter causes increasing in temperature profile for some fixed values of other parameter. Fig. 3(b) shows that when Eckert number increases the corresponding dimensionless temperature also increases for some fixed values of other parameters. Fig. 3(c) it is easily seen that increasing of heat source parameter leads to decreases of dimensionless temperature profile for some other parameter have fixed values. Fig.3(d) shows that when Prandtl number increases the corresponding dimensionless temperature profile also increases. Figure 4 represent the dimensionless skin friction versus permeability parameter. From fig. 4 it is depict that when permeability parameter increases it also increases the skin friction coefficient.

Figure 5 represent the recovery temperature versus Prandtl number for various parameters. Fig. 5(a) it is noted that recovery temperature increases when permeability parameter increases. Fig. 5(b) shows that effect of Eckert number, increasing of Eckert number leads to increase of recovery temperature. Fig. 5(c) it is clear that when heat source increases the corresponding recovery temperature decreases.

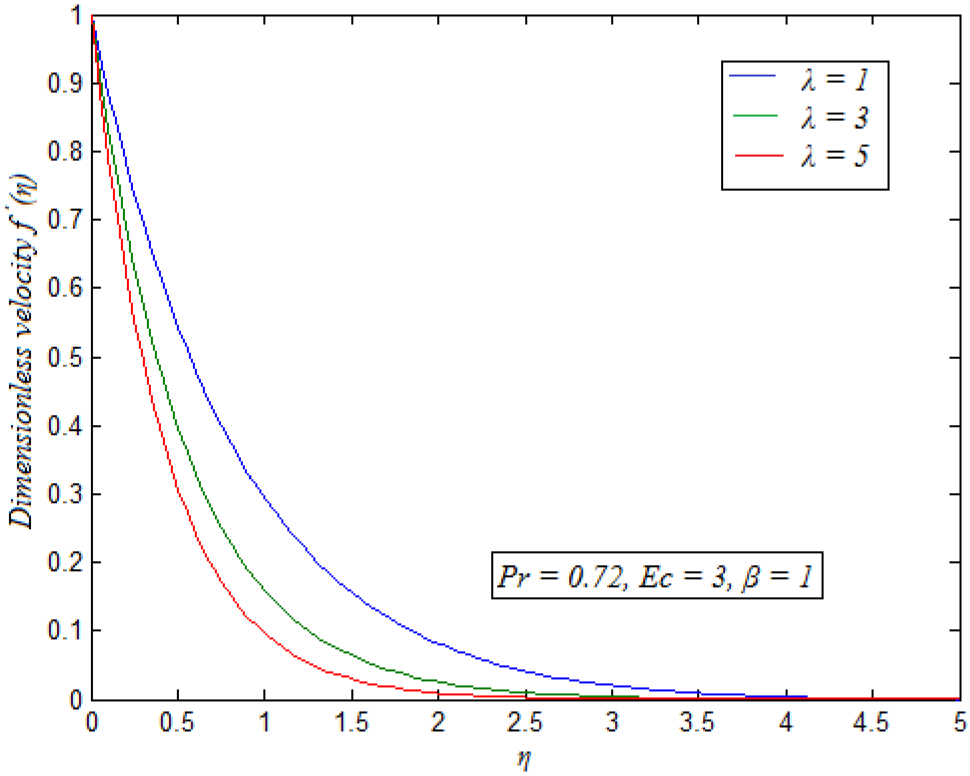
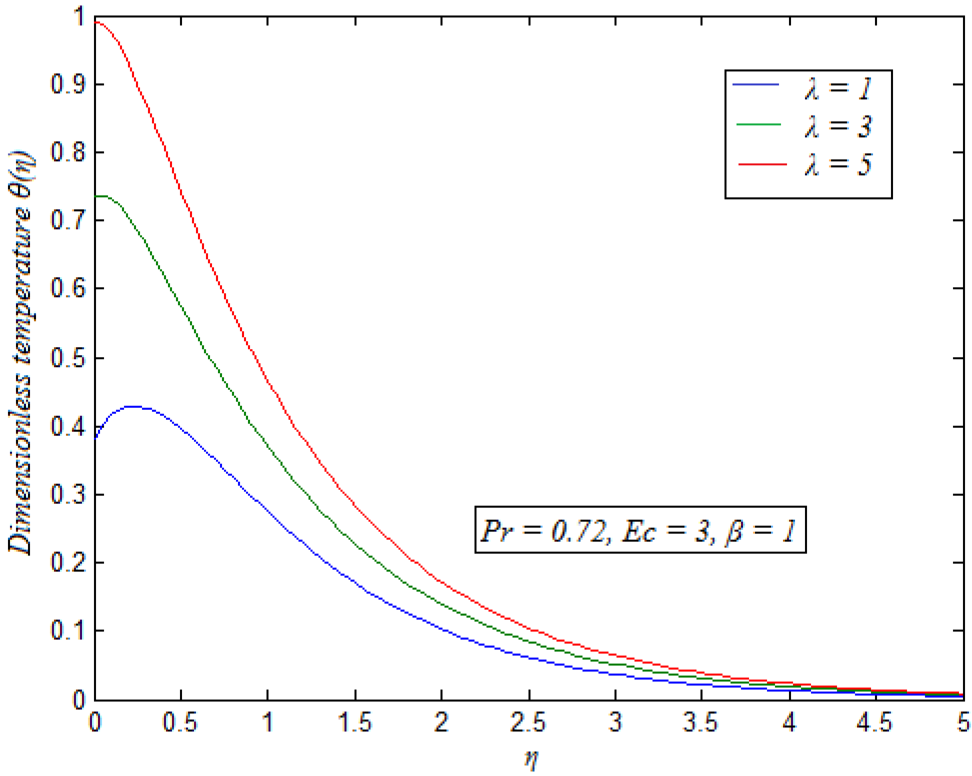
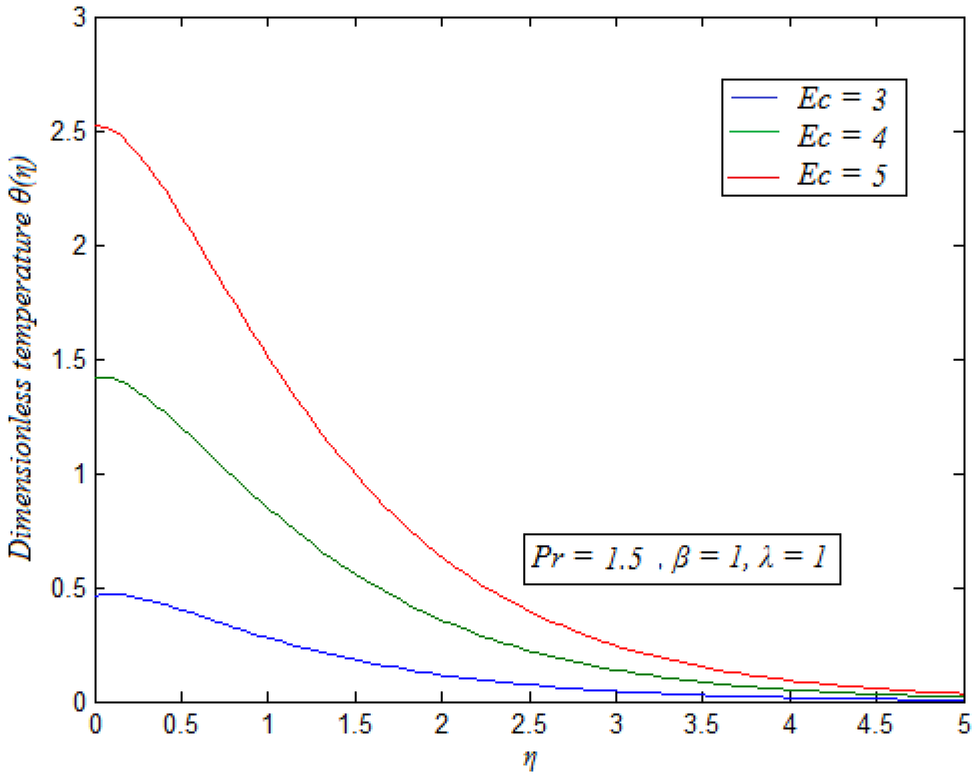


Fig.2: Dimensionless velocity $f'(\eta)$ versus η. The curves are plotted for various values of the permeability parameter λ with some fixed values of $\Pr = 0.72, \beta = 1, Ec = 3.$

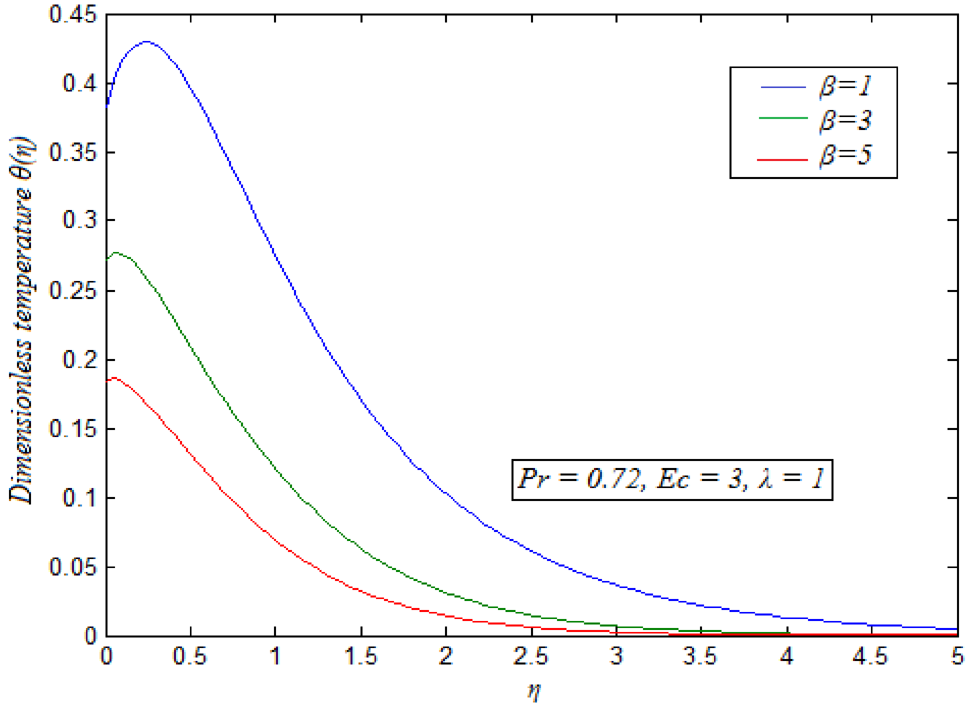
(a)



(b)


Ec = 3
Ec = 4
Ec = 5
Pr = 1.5 , β = 1, λ = 1
Dimensionless temperature θ(η)
η
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5

(c)


β=1
β=3
β=5
Pr = 0.72, Ec = 3, λ = 1
Dimensionless temperature θ(η)
η
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5

(d)

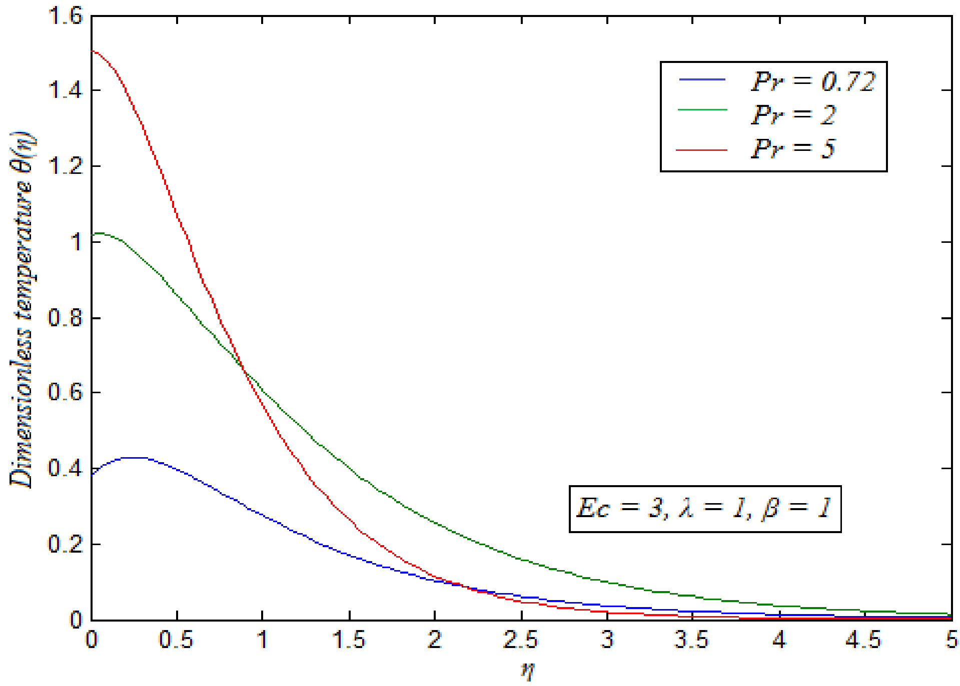


Fig.3: Dimensionless axial temperature $\theta(\eta)$ against η. The curves are plotted using the eqn. (11) for various values of the dimensionless parameter $\lambda, \beta, \mathrm{Pr}, Ec$ and in some fixed values of the other dimensionless parameters, when $(a)\,\mathrm{Pr}=0.72, \beta=1, Ec=3;\ (b)\ \mathrm{Pr}=1.5, \beta=1, \lambda=1;$ $(c)\,\mathrm{Pr}=0.72, Ec=3, \lambda=1;\ (d)\,\beta=1, Ec=3, \lambda=1$

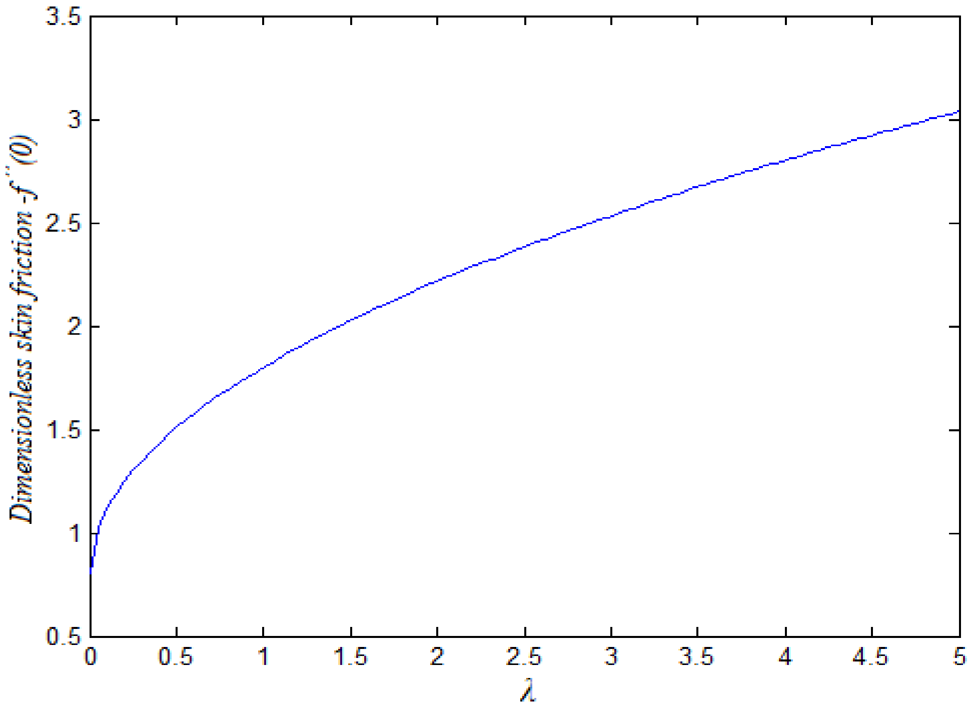
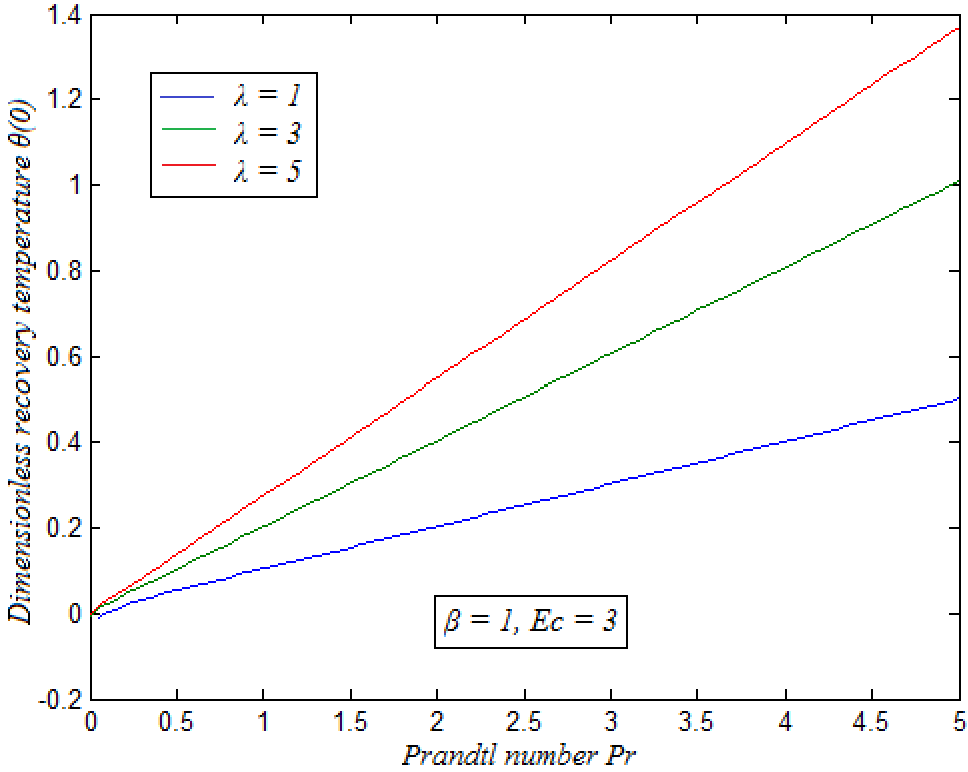
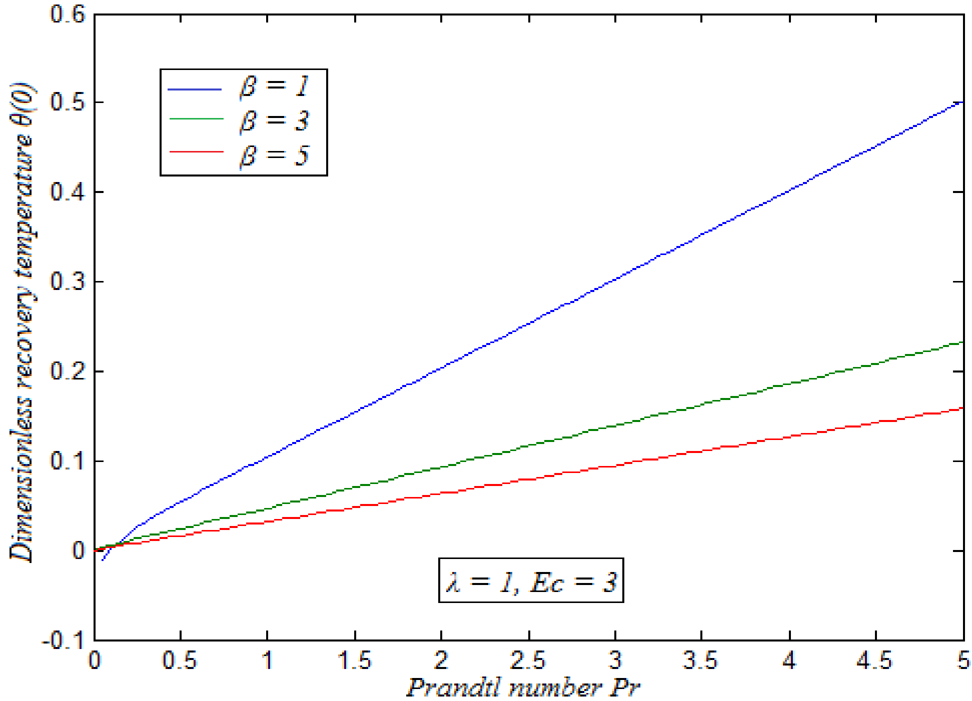


Fig. 4: Variation of dimensionless skin friction $-f''(0)$ against permeability parameter λ.

(a)


λ = 1
λ = 3
λ = 5
β = 1, Ec = 3
Dimensionless recovery temperature θ(0)
Prandtl number Pr
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5

(b)


β = 1
β = 3
β = 5
λ = 1, Ec = 3
Dimensionless recovery temperature θ(0)
Prandtl number Pr
0.6
0.5
0.4
0.3
0.2
0.1
0
-0.1
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5

(c)

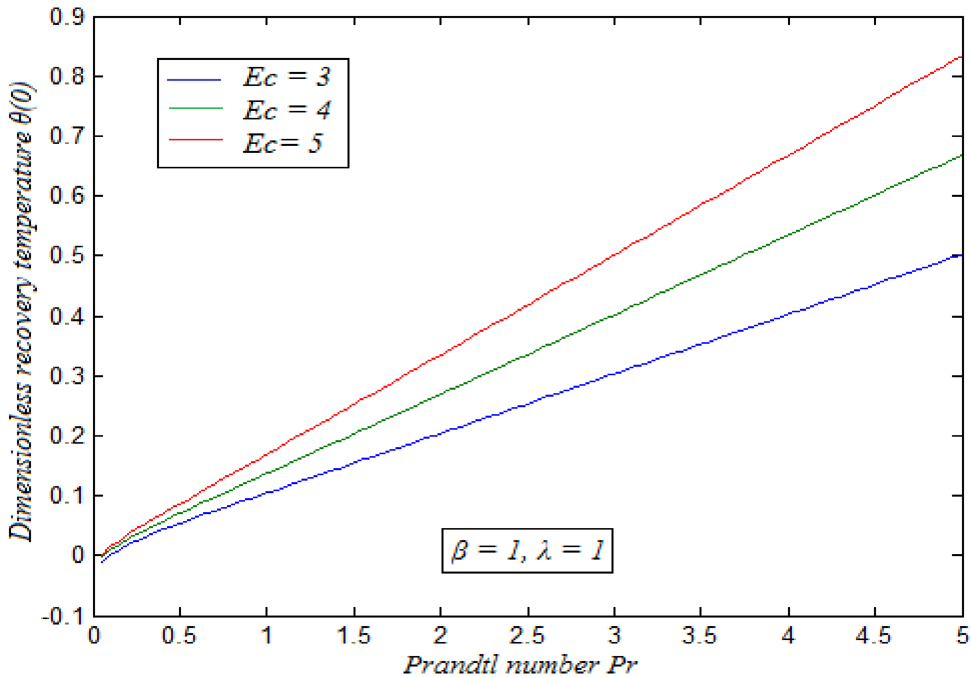


Fig.5: Dimensionless recovery temperature $\theta(0)$ against Prandtlnumber $\Pr$. The curves are plotted using the eqn. (12) for various values of the dimensionless parameter λ, β, Ec and in some fixed values of the other dimensionless parameters, when

$(a)\beta = 1, Ec = 3;\ (b)\ Ec = 3, \lambda = 1;$

$(c)\beta = 1, \lambda = 1$

Table 1 Variation of the dimensionless skin friction and recovery temperature

$\Pr$	Ec	λ	β	$-f''(0)$		$\theta(0)$	
				Previous work[14]	Current work	Previous work[14]	Current work
0.72	3	1	1	2.00007	2	0.395499	0.39548
2	3	1	1	2.00007	2	1.08867	1.08859
5	3	1	1	2.00007	2	1.50712	1.50749
0.72	3	3	1	2.56155	2.56155	0.735769	0.735714
0.72	3	5	3	3	3	1.01252	1.01245
0.72	3	1	5	2.00007	2	0.219914	0.219909
0.72	3	1	5	2.00007	2	0.145518	0.14564

5.Conclusion

In this paper the effect of heat transfer and viscous dissipation of fluid due to stretching porous sheet in the present of heat source is studied. Analytical expression for dimensionless velocity, dimensionless temperature, and recovery temperature are derived and compared. The effects of the different parameters involved in the equations are observed through graphs. Further it is observed that inclusion of porosity increases the temperature, recovery temperature and dimensionless skin friction and decreasing the velocity. Our physical quantities for various parameters are compared. This method can be easily extended to solve the other non-linear initial and boundary value problems in physical and biological sciences.

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Appendix: A

Basic concept of the Homotopy analysis method [18–30]

Consider the following differential equation:

$$N[u(t)] = 0 \qquad \text{(A.1)}$$

Where N is a nonlinear operator, t denotes an independent variable, $u(t)$ is an unknown function. For simplicity, weignore all boundary or initial conditions, which can be treated in the similar way. By means of generalizing the conventionalHomotopy method, Liao [33] constructed the so-called zero-order deformation equation as:

$$(1-p)L[\varphi(t;p) - u_o(t)] = phH(t)N[\varphi(t;p)] \qquad \text{(A.2)}$$

where $p \in [0,1]$ is the embedding parameter, $h \neq 0$ is a nonzero auxiliary parameter, $H(t) \neq 0$ is an auxiliary function, L an auxiliary linear operator, $u_o(t)$ is an initial guess of $u(t), \phi(t:p)$ is an unknown function. It is important to note thatone has great freedom to choose auxiliary unknowns in HAM. Obviously, when $p = 0$ and $p = 1,$ it holds:

$$\varphi(t;0) = u_o(t)\ text and\ \varphi(t;1) = u(t) \qquad \text{(A.3)}$$

respectively. Thus, as p increases from 0 to 1, the solution $\varphi(t;p)$ varies from the initial guess $u_o(t)$ to the solution $u(t)$.

Expanding $\varphi(t;p)$ in Taylor series with respect to p, we have:

$$\varphi(t;p) = u_o(t) + \sum_{m=1}^{+\infty} u_m(t)p^m \qquad \text{(A.4)}$$

$$u_m(t) = \frac{1}{m!} \left. \frac{\partial^m \varphi(t;p)}{\partial p^m} \right|_{p=0} \qquad \text{(A.5)}$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter h, and the auxiliary function are so properly chosen,the series (A.4) converges at $p = 1$ then we have:

$$u(t) = u_o(t) + \sum_{m=1}^{+\infty} u_m(t) \qquad \text{(A.6)}$$

Differentiating (A.2) for m times with respect to the embedding parameter p, and then setting $p = 0$ and finally dividing them by m!, we will have the so-called mth -order deformation equation as:

$$L[u_m - \chi_m u_{m-1}] = hH(t)\Re_m(\vec{u}_{m-1}) \qquad \text{(A.7)}$$

Where

$$\Re_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N[\varphi(t;p)]}{\partial p^{m-1}} \qquad \text{(A.8)}$$

And

$$\chi_m = \begin{cases} 0, m \leq 1, \\ 1, m > 1. \end{cases} \qquad \text{(A.9)}$$

Applying L^{-1} on both side of equation (A7), we get

$$u_m(t) = \chi_m u_{m-1}(t) + hL^{-1}[H(t)\Re_m(\overrightarrow{u_{m-1}})] \quad \text{(A.10)}$$

In this way, it is easily to obtain u_m for $m \geq 1$, at M^{th} order, we have

$$u(t) = \sum_{m=0}^{M} u_m(t) \quad \text{(A.11)}$$

When $M \to +\infty$, we get an accurate approximation of the original equation (A.1). For the convergence of the above method we refer the reader to Liao [34]. If equation (A.1) admits unique solution, then this method will produce the unique solution.

Appendix: B

Solution of the equations (6) - (9) using Homotopy analysis method

The analytical expressions involves in the equation (10) and (11) for $f(\eta)$ and $\theta(\eta)$ are derived using HAM Method. The equations (6) - (9) are as follows,

$$f''' + ff'' - f'^2 - \lambda f' = 0 \quad \text{(B.1)}$$

$$\theta'' + Pr\ f\theta' - 2Pr\ f'\theta - \beta\theta + Pr\ Ecf''^2 = 0 \quad \text{(B.2)}$$

The corresponding boundary conditions are

$$f'(0) = 1, \quad f(0) = 1, \quad \theta'(0) = 1 \quad \text{(B.3)}$$

As $\eta \to \infty : \quad f' = f = \theta = 0$ (B.4)

We construct the Homotopy for the equations (B.1) and (B.2) as follows:

$$(1-p)(f''' - \lambda f') = hp(f''' + ff'' - f'^2 - \lambda f') \quad \text{(B.5)}$$

$$(1-p)(\theta'' - \beta\theta) = hp(\theta'' + Pr\ f\theta' - 2Pr\ f'\theta - \beta\theta + Pr\ Ecf''^2) \quad \text{(B.6)}$$

The approximate solution for the above two eqns. are as follows:

$$f = f_0 + pf_1 + p^2 f_2 + ... \quad \text{(B.7)}$$

$$\theta = \theta_0 + p\theta_1 + p^2\theta_2 + ... \quad \text{(B.8)}$$

By substituting the eqms/(B.7) and (B.8) into the eqns. (B.5) and (B.6) respectively, we get

$$\begin{aligned}(1-p)&\left[(f_0''' + pf_1''' + p^2 f_2''' + ...) - \lambda(f_0 + pf_1 + p^2 f_2 + ...)\right] \\ &= hp\begin{bmatrix}(f_0''' + pf_1''' + p^2 f_2''' + ...) + (f_0 + pf_1 + p^2 f_2 + ...)(f_0'' + pf_1'' + p^2 f_2'' + ...) \\ -(f_0' + pf_1' + p^2 f_2' + ...)^2 - \lambda(f_0 + pf_1 + p^2 f_2 + ...)\end{bmatrix}\end{aligned} \quad \text{(B.9)}$$

$$\begin{aligned}(1-p)&\left[(\theta_0'' + p\theta_1'' + p^2\theta_2'' + ...) - \beta(\theta_0 + p\theta_1 + p^2\theta_2 + ...)\right] \\ &= hp\begin{bmatrix}(\theta_0'' + p\theta_1'' + p^2\theta_2'' + ...) \\ + Pr(f_0 + pf_1 + p^2 f_2 + ...)(\theta_0' + p\theta_1' + p^2\theta_2' + ...) \\ - 2Pr(f_0' + pf_1' + p^2 f_2' + ...)(\theta_0 + p\theta_1 + p^2\theta_2 + ...) \\ - \beta(\theta_0 + p\theta_1 + p^2\theta_2 + ...) \\ + Pr\ Ec(f_0'' + pf_1'' + p^2 f_2'' + ...)^2\end{bmatrix}\end{aligned} \quad \text{(B.10)}$$

Equating the coefficients of p^0 and p^1 for the equations (B.9) and (B.10) we get

$$p^0 : f_0''' - \lambda f_0' = 0 \quad \text{(B.11)}$$

$$p^0 : \theta_0'' - \beta\theta_0 = 0 \quad \text{(B.12)}$$

$$p^1 : f_1''' - \lambda f_1' - hf_0 f_0'' + hf_0'^2 = 0 \quad \text{(B.13)}$$

$$p^1 : \theta_1'' - \beta\theta_1 - hPr\ f_0\theta_0' + 2hPr\ f_0'\theta_0 - hPr\ Ecf_0''^2 = 0 \quad \text{(B.14)}$$

Consider the initial solution as,

$$f_0 = \frac{-\exp(-\sqrt{\lambda}\eta)}{\sqrt{\lambda}} + 2 \quad \text{(B.15)}$$

$$\theta_0 = \frac{\eta \exp(-Pr\, Ec\beta\lambda\eta)}{Pr\, Ec\beta\lambda} \quad \text{(B.16)}$$

Solving the eqns.(B.13) and (B.14) using the boundary conditions (B.3) and (B.4), we get

$$f_1 = -\frac{1}{\lambda} + \frac{1}{\lambda}\exp(-\sqrt{\lambda}\eta) + \frac{\eta \exp(-\sqrt{\lambda}\eta)}{\sqrt{\lambda}} \quad \text{(B.17)}$$

$$\begin{aligned}\theta_1 = {} & C_1 \exp(-\sqrt{\beta}\eta) - \frac{Pr\, Ec\lambda \exp(-2\sqrt{\lambda}\eta)}{4\lambda - \beta} \\ & - \frac{2\, Pr \exp(-A\eta)}{A(A^2-\beta)^2}\Big((1-\eta)(A^2-\beta) - 2A\Big) \\ & + \frac{Pr \exp(-\sqrt{\lambda} - A)\eta}{A\sqrt{\lambda}((\sqrt{\lambda}+A)^2 - \beta)^2}\begin{pmatrix}(1-\eta)((\sqrt{\lambda}+A)^2 - \beta) - 2(\sqrt{\lambda}+A) \\ + 2\sqrt{\lambda}(\eta((\sqrt{\lambda}+A)^2 - \beta) + 2(\sqrt{\lambda}+A))\end{pmatrix}\end{aligned} \quad \text{(B.18)}$$

where C_1 and A are defined in the text eqns. (15) and (16) respectively.

By Homotopy analysis method we have

$$f = \lim_{p\to 1} f(\eta) = f_0 + f_1 \quad \text{(B.19)}$$

$$\theta = \lim_{p\to 1} \theta(\eta) = \theta_0 + \theta_1 \quad \text{(B.20)}$$

Using the eqns. (B.15) and (B.17) into an eqn.(B.19) we get the solution obtained in the eqn.(13). Similarly we get eqn. (14) by using the eqns. (B.16) and (B.18) into an eqn.(B.20).

Appendix C:

Nomenclature

Symbols	Meaning
u	Velocity component in $x-$direction
v	Velocity component in $y-$direction
Ec	Eckert number
$\Pr$	Prandtl number
C_p	Specific heat at constant pressure
β	Heat source parameter
λ	Permeability parameter
k	Thermal conductivity
Nu	Nusselt number
τ	Shear stress
x	Stream wise coordinate
y	Direction normal to the plate
ρ	Density
η	Pseudo similar variable
μ	Dynamical viscosity
θ	Dimensional temperature
T	Temperature
c	Stretching rate
B	Constant
q_w	The local heat flux
f'	Dimensionless velocity