

Ex

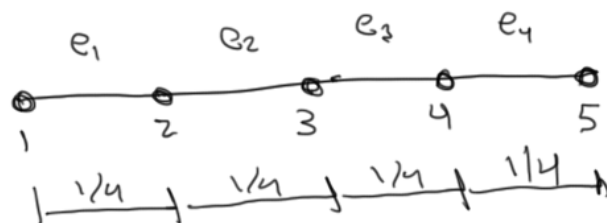
$$a = 1, \quad c = -1, \quad f = -x^2$$

$$-\frac{d^2 u}{dx^2} - u + x^2 = 0 \quad 0 < x < 1$$

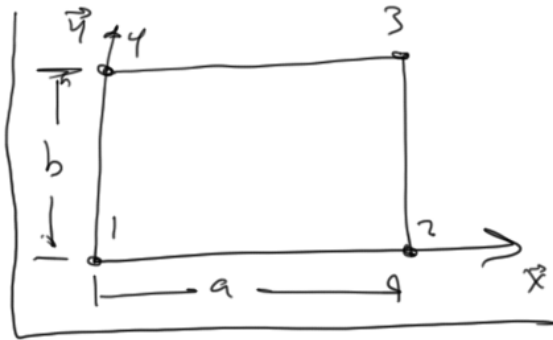
$$u(0) = 0 \quad u(1) = 0$$

$$K_{ij} = \int_{x_a}^{x_b} \left( \frac{dN_i}{dx} \frac{dN_j}{dx} - N_i N_j \right) dx$$

$$f_i = \int_{x_a}^{x_b} (-x^2) N_i dx$$



## Linear Rectangler Element (Quad4)



$$u_n(x, y) = C_1 + C_2 x + C_3 y + C_4 xy$$

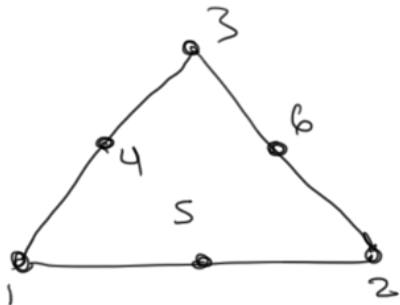
$$N_1 = \left(1 - \frac{\bar{x}}{a}\right) \left(1 - \frac{\bar{y}}{b}\right)$$

$$N_2 = \frac{\bar{x}}{a} \left(1 - \frac{\bar{y}}{b}\right)$$

$$N_3 = \frac{\bar{x}}{a} \frac{\bar{y}}{b}$$

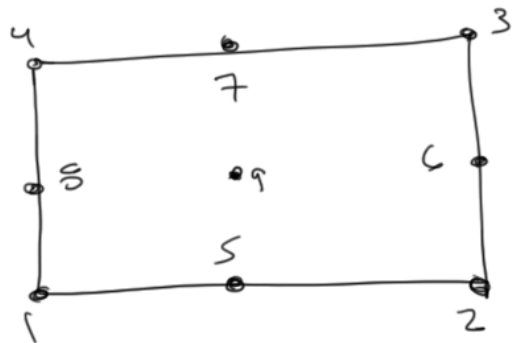
$$N_4 = \left(1 - \frac{\bar{x}}{b}\right) \frac{\bar{y}}{b}$$

## Quadratic Triangle



$$u^h(x, y) = C_1 + C_2 x + C_3 y + C_4 xy + C_5 x^2 + C_6 y^2$$

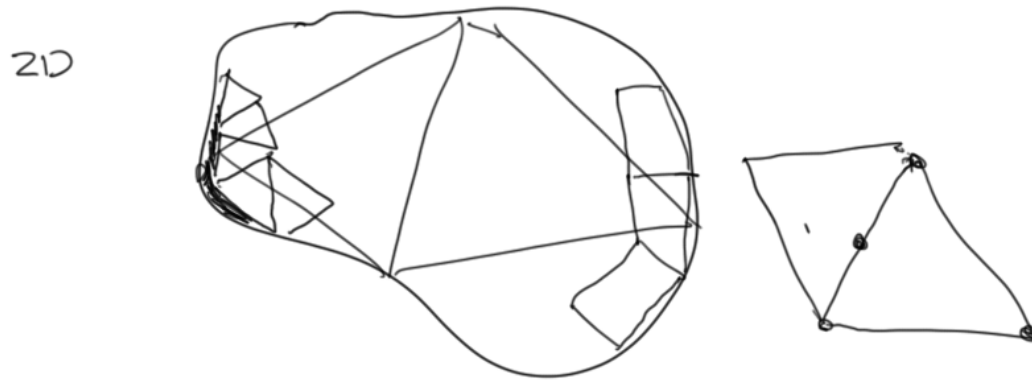
(Quad 8)



$$u_n(x, y) = C_1 + C_2 x + C_3 y + C_4 xy + C_5 x^2 + C_6 y^2 + C_7 xy^2 + C_8 x^2 y + C_9 x^2 y^2$$


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# Problems in 2D with scalar field variables, $u$



## General rules

1. Element should tie governing eqn.
2. #, shape, type  $\rightarrow$  accurate
3. The mesh density should cover areas of high gradients

Element type

Triangles

Quad

# elements

Degree of accuracy - interpolant

- no magic



4. Grade away gradually,

$$\underbrace{-\frac{\partial}{\partial x} \left( a_{11} \frac{\partial u}{\partial x} + a_{12} \frac{\partial u}{\partial y} \right) - \frac{\partial}{\partial y} \left( a_{21} \frac{\partial u}{\partial x} + a_{22} \frac{\partial u}{\partial y} \right)}_{\nabla \cdot (A \nabla u)} + a_{00} u - f = 0$$

$u = u(x, y)$        $a_{ij}$  are the data with B.C.'S

### Models

Heat transfer in 2D

Irrrotational flow of an ideal fluid

Groundwater flow through permeable geology

### Weak Form

$$0 = \int_{\Omega} \delta u \left[ -\frac{\partial}{\partial x} (F_1) - \frac{\partial}{\partial y} (F_2) + a_{00} u - f \right] dx dy$$

where

$$F_1 = a_{11} \frac{\partial u}{\partial x} + a_{12} \frac{\partial u}{\partial y}, \quad F_2 = a_{21} \frac{\partial u}{\partial x} + a_{22} \frac{\partial u}{\partial y}$$

Integrate - by - parts

$$\frac{\partial}{\partial x}(\delta u F_1) = \frac{\partial u}{\partial x} F_1 + \delta u \frac{\partial F_1}{\partial x} \Rightarrow -\delta u \frac{\partial F_1}{\partial x} = \frac{\partial \delta u}{\partial x} F_1 - \frac{\partial}{\partial x}(\delta u F_1)$$

$$\frac{\partial}{\partial y}(\delta u F_2) = \frac{\partial \delta u}{\partial y} F_2 + \delta u \frac{\partial F_2}{\partial y} \Rightarrow -\delta u \frac{\partial F_2}{\partial y} = \frac{\partial \delta u}{\partial y} F_2 - \frac{\partial}{\partial y}(\delta u F_2)$$

Diverge. Theorem

$$\int_{\Omega} \frac{\partial}{\partial x}(\delta u F_1) dx dy = \oint_{\Gamma} \delta u F_1 n_x dS_{\Gamma}$$

$$\int_{\Omega} \frac{\partial}{\partial y}(\delta u F_2) dx dy = \oint_{\Gamma} \delta u F_2 n_y dS_{\Gamma}$$

where  $n_x$  &  $n_y$  are components of the unit normal

$$\mathbf{n} = n_x \hat{e}_x + n_y \hat{e}_y \quad \text{on } \Gamma_c$$

$$\begin{aligned} 0 = \int_{\Omega} & \left[ \frac{\partial \delta u}{\partial x} \left( a_{11} \frac{\partial u}{\partial x} + a_{12} \frac{\partial u}{\partial y} \right) + \frac{\partial \delta u}{\partial y} \left( a_{21} \frac{\partial u}{\partial x} + a_{22} \frac{\partial u}{\partial y} \right) + a_{00} \delta u u \right. \\ & \left. - \delta u f \right] - \oint_{\Gamma} \underbrace{\left[ n_x \left( a_{11} \frac{\partial u}{\partial x} + a_{12} \frac{\partial u}{\partial y} \right) + n_y \left( a_{21} \frac{\partial u}{\partial x} + a_{22} \frac{\partial u}{\partial y} \right) \right]}_{\equiv q_n} dS \end{aligned}$$

$$0 = \int_{\Omega} \left[ \frac{\partial \delta u}{\partial x} \left( a_{11} \frac{\partial u}{\partial x} + a_{12} \frac{\partial u}{\partial y} \right) + \frac{\partial \delta u}{\partial y} \left( a_{21} \frac{\partial u}{\partial x} + a_{22} \frac{\partial u}{\partial y} \right) + a_{00} \delta u u \right. \\ \left. - \delta u f \right] dx dy - \int_{\Gamma} \delta u g_n dS \quad \text{III}$$

$$B(\delta u, u) = \int_{\Omega} \left[ \frac{\partial \delta u}{\partial x} \left( a_{11} \frac{\partial u}{\partial x} + a_{12} \frac{\partial u}{\partial y} \right) + \frac{\partial \delta u}{\partial y} \left( a_{21} \frac{\partial u}{\partial x} + a_{22} \frac{\partial u}{\partial y} \right) + a_{00} \delta u u \right] dx dy$$

$$l(\delta u) = \int_{\Omega} \delta u f dx dy + \int_{\Gamma} \delta u g_n dS$$

$$\text{If } a_{21} = a_{12}$$

$$I(u) = \frac{1}{2} B(u, u) - l(u)$$

$$I(u) = \frac{1}{2} \int_{\Omega} \left[ a_{11} \left( \frac{\partial u}{\partial x} \right)^2 + 2a_{12} \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} + a_{22} \left( \frac{\partial u}{\partial y} \right)^2 + a_{00} u^2 \right] dx dy \\ + \int_{\Omega} u f dx dy + \int_{\Gamma} u g_n dS$$

$$B(\vec{\delta u}, \vec{v}) = l(\vec{\delta u})$$

Let

$$C = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & a_{00} \end{bmatrix}, \quad D = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ 1 \end{bmatrix}$$

$$B(\vec{\delta u}, \vec{v}) = \int_{\Omega} \begin{bmatrix} \frac{\partial \delta u}{\partial x} \\ \frac{\partial \delta u}{\partial y} \\ \delta u \end{bmatrix}^T \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & a_{00} \end{bmatrix} \begin{bmatrix} \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \\ v \end{bmatrix} dx dy$$

$$l(\delta u) = \int_{\Omega} \{ \delta u \}^T \{ f \} dx dy + \int_{\Gamma} \{ \delta u \}^T \{ g_n \} dS$$

$$B(\vec{\delta u}, \vec{v}) = \int_{\Omega} (D \delta u)^T C D v dx dy$$

$$l(\vec{\delta u}) = \int_{\Omega} \vec{\delta u}^T \vec{f} dx dy + \int_{\Gamma} \vec{\delta u}^T \vec{g}_n dS \quad \left. \vphantom{\int_{\Omega}} \right\} \underline{\text{Weak Form}}$$



FE model in 2D

$$u(x, y) \approx u_h(x, y) = \sum_{j=1}^n N_j u_j$$

$$0 = \int_{\Omega} \left[ \frac{\partial \delta u}{\partial x} \left( a_{11} \frac{\partial N_j}{\partial x} u_j + a_{12} \frac{\partial N_j}{\partial y} u_j \right) + \frac{\partial \delta u}{\partial y} \left( a_{21} \frac{\partial N_j}{\partial x} u_j + a_{22} \frac{\partial N_j}{\partial y} u_j \right) + a_{00} \delta u N_j u_j - \delta u f \right] dx dy - \int_{\Gamma} \delta u q_n dS$$

or

$$0 = \int_{\Omega} (D \delta u)^T C D (N^T u) dx dy - \int_{\Omega} \delta \vec{u}^T \vec{f} dx dy - \int_{\Gamma} \delta \vec{u} \vec{q}_n dS$$

Let  $\delta u_i = N_i$   $K_{ij}$

$$0 = \sum_{i=1}^n \left\{ \int_{\Omega} \left[ \frac{\partial N_i}{\partial x} \left( a_{11} \frac{\partial N_j}{\partial x} + a_{12} \frac{\partial N_j}{\partial y} \right) + \frac{\partial N_i}{\partial y} \left( a_{21} \frac{\partial N_j}{\partial x} + a_{22} \frac{\partial N_j}{\partial y} \right) + a_{00} N_i N_j \right] dx dy \right\} u_j - \int_{\Omega} f N_i dx dy - \int_{\Gamma} N_i q_n dS \Rightarrow K_{ij} u_j = f_i + Q_i$$

$$K \bar{u} = \vec{F} = \vec{F} + \vec{Q}$$

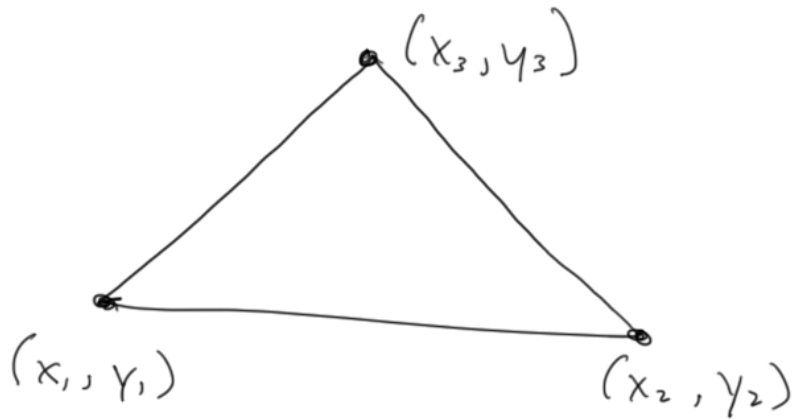
where

$$K = \int_{\Omega} B^T C B \, dx dy, \quad \vec{F} = \int_{\Omega} N^T \vec{f} \, dy dx, \quad Q = \oint_{\Gamma} N^T \hat{g}_n \, dS$$

$$B = D N^T = \begin{bmatrix} N_{1,x} & N_{2,x} & \dots & N_{n,x} \\ N_{1,y} & N_{2,y} & \dots & N_{n,y} \\ N_1 & N_2 & \dots & N_n \end{bmatrix}$$


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Constant Strain (CST) 3-nodes



$$u_n = C_1 + C_2 x + C_3 y$$

$$X = \begin{bmatrix} 1 & x & y \end{bmatrix}$$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}}_A \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \end{Bmatrix}$$

$$N = X A^{-1}$$

