Interpolation functions that satisfy the B.C.'s implicitly

```
u[x_] := c1 x (1 - x) + c2 x^2 (1 - x) + c3 x^3 (1 - x);
```

$$II = \int_0^1 \left(u'[x]^2 - u[x]^2 + 2 x^2 u[x]\right) dx$$

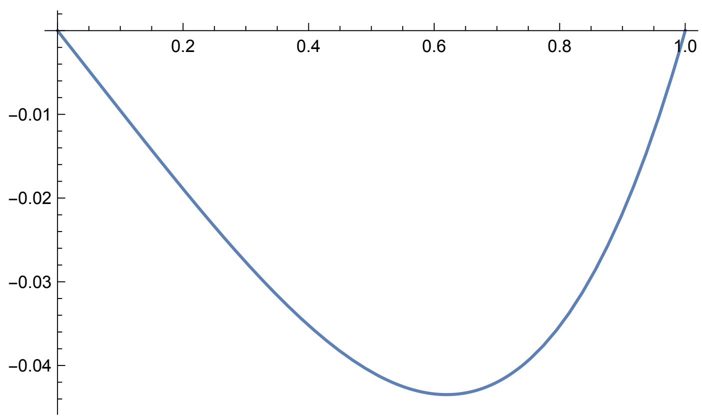
$$\frac{c1}{10} + \frac{3\,c1^2}{10} + \frac{c2}{15} + \frac{3\,c1\,c2}{10} + \frac{13\,c2^2}{105} + \frac{c3}{21} + \frac{19\,c1\,c3}{105} + \frac{79\,c2\,c3}{420} + \frac{103\,c3^2}{1260}$$

```
eqn1 = D[II, c1];
eqn2 = D[II, c2];
eqn3 = D[II, c3];
```

```
sol = First@Solve[{eqn1 == 0, eqn2 == 0, eqn3 == 0}, {c1, c2, c3}]
```

$$\left\{c1 \to -\frac{2335}{24518},\ c2 \to -\frac{1232}{12259},\ c3 \to -\frac{21}{299}\right\}$$

```
plot1 = Plot[u[x] /. sol, {x, 0, 1}]
```



Monomial that satisfies the B.C.'s

In[1]:=
```
u[x_] := c3 (x^2 - x);
```

In[2]:= $$II = \int_0^1 \left(u'[x]^2 - u[x]^2 + 2 x^2 u[x]\right) dx$$

Out[2]= $$-\frac{c3}{10} + \frac{3\,c3^2}{10}$$

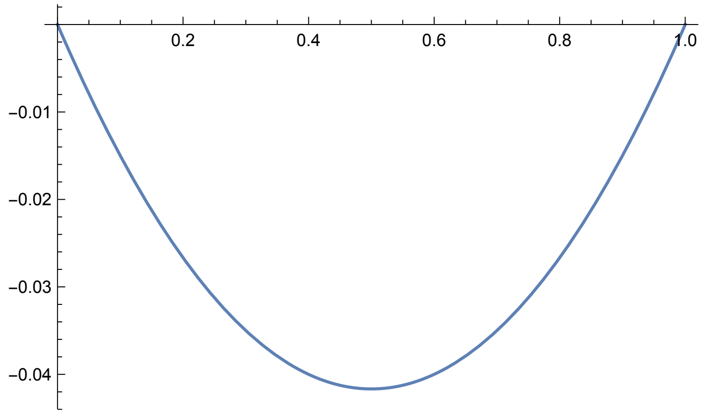
In[3]:=
```
eqn3 = D[II, c3];
```

In[4]:=
```
sol = First@Solve[{eqn3 == 0}, {c3}]
```

Out[4]= $$\left\{c3 \to \frac{1}{6}\right\}$$

In[5]:= `plot2 = Plot[u[x] /. sol, {x, 0, 1}]`

Out[5]=



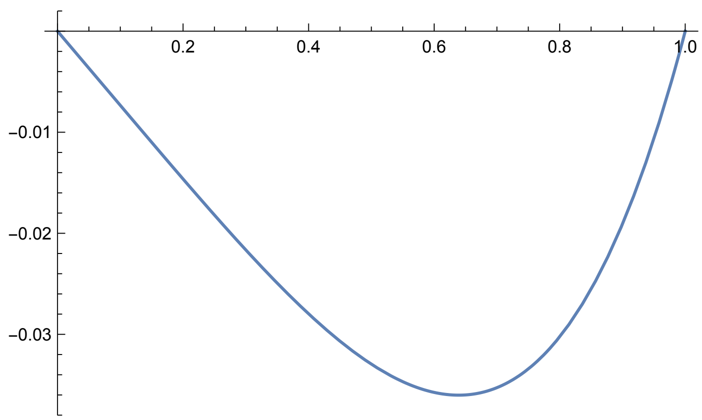
Exact solution

In[8]:= `asol = DSolve[{v''[x] - v[x] + x^2 == 0, v[0] == 0, v[1] == 0}, v[x], x] // First`

Out[8]= $\left\{v[x] \to -\frac{1}{-1 + e^2} e^{-x} \left(-3 e + 2 e^2 + 2 e^x - 2 e^{2 x} - 2 e^{2+x} + 3 e^{1+2 x} + e^x x^2 - e^{2+x} x^2\right)\right\}$

In[10]:= `plot3 = Plot[-v[x] /. asol, {x, 0, 1}]`

Out[10]=



1D elastic bar problem

In[3]:= `A[x_] = A0 (1 - x / 2 / L);`

In[35]:= `uh[x_] = Sum[c[i] * x^i, {i, 0, 3}]`

Out[35]= $c[0] + x\, c[1] + x^2\, c[2] + x^3\, c[3]$

In[36]:= `sol1 = Solve[uh[0] == 0, c[0]] // First`

Out[36]= $\{c[0] \to 0\}$

In[37]:= `uh[x_] = uh[x] /. sol1`

Out[37]= $x\,c[1] + x^2\,c[2] + x^3\,c[3]$

In[38]:= $\mathtt{II} = \int_0^1 \left(\frac{\mathrm{Ey}\,A[x]}{2}\,\mathrm{uh}'[x]^2\right) dx - P\,\mathrm{uh}[x]$

Out[38]= $-P\left(x\,c[1] + x^2\,c[2] + x^3\,c[3]\right) + \frac{1}{120\,L}$
$$\mathrm{A0}\,\mathrm{Ey}\left(15\,(-1+4\,L)\,c[1]^2 + 10\,(-3+8\,L)\,c[2]^2 + 36\,(-2+5\,L)\,c[2]\,c[3] + 9\,(-5+12\,L)\,c[3]^2 + 5\,c[1]\,(8\,(-1+3\,L)\,c[2] + 3\,(-3+8\,L)\,c[3])\right)$$

In[39]:= `sol2 = First@Solve[{D[II, c[1]] == 0, D[II, c[2]] == 0, D[II, c[3]] == 0}, {c[1], c[2], c[3]}]`

Out[39]= $\{c[1] \to \left(8\,L\,P\left(18\,x - 120\,L\,x + 180\,L^2\,x - 30\,x^2 + 225\,L\,x^2 - 360\,L^2\,x^2 + 15\,x^3 - 120\,L\,x^3 + 200\,L^2\,x^3\right)\right) / \left(\mathrm{A0}\,\mathrm{Ey}\left(-1 + 24\,L - 120\,L^2 + 160\,L^3\right)\right),$
$c[2] \to -\left(\left(10\,L\,P\left(24\,x - 180\,L\,x + 288\,L^2\,x - 45\,x^2 + 432\,L\,x^2 - 768\,L^2\,x^2 + 24\,x^3 - 256\,L\,x^3 + 480\,L^2\,x^3\right)\right) / \left(\mathrm{A0}\,\mathrm{Ey}\left(-1 + 24\,L - 120\,L^2 + 160\,L^3\right)\right)\right),$
$c[3] \to \left(40\,L\,P\left(9\,x - 72\,L\,x + 120\,L^2\,x - 18\,x^2 + 192\,L\,x^2 - 360\,L^2\,x^2 + 10\,x^3 - 120\,L\,x^3 + 240\,L^2\,x^3\right)\right) / \left(3\,\mathrm{A0}\,\mathrm{Ey}\left(-1 + 24\,L - 120\,L^2 + 160\,L^3\right)\right)\}$

In[40]:= `uh[x] /. sol2 //. {x → L, L → 1, P → 1, A0 → 1, Ey → 1} // N`

Out[40]= 1.38624

In[5]:= `DSolve[{D[Ey A[x] u'[x], x] == 0, Ey A[x] u'[L] == P, u[0] == 0}, u[x], x] //. {x → L, L → 1, P → 1, A0 → 1, Ey → 1} // N`

Out[5]= {{u[1.] → 1.38629}}

Quadratic shape functions

In[6]:= `X = {1, x, x^2}`

Out[6]= $\{1, x, x^2\}$

In[8]:= `A = {X /. x → 0, X /. x → L / 2, X /. x → L}; MatrixForm[A]`

Out[8]//MatrixForm=
$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & \frac{L}{2} & \frac{L^2}{4} \\ 1 & L & L^2 \end{pmatrix}$$

In[9]:= `NN = X.Inverse[A]`

Out[9]= $\left\{1 - \frac{3\,x}{L} + \frac{2\,x^2}{L^2}, \frac{4\,x}{L} - \frac{4\,x^2}{L^2}, -\frac{x}{L} + \frac{2\,x^2}{L^2}\right\}$

In[10]:=
```
Total[NN]
```

Out[10]= 1