CS 191 Homework Set - Two Level Systems and The NMR Quantum Computer

Here are a few things you might need for this assignment. The Pauli matrices are

$$\sigma_x \equiv X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y \equiv Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z \equiv Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The Bloch sphere representation is defined in terms of the polar angle, θ, and the azimuthal angle, ϕ, as

$$|\hat{n}\rangle = \cos(\theta/2)\,|0\rangle + \sin(\theta/2)e^{i\phi}\,|1\rangle\,.$$

In the presence of a time-invariant Hamiltonian, the state of a system at time, t, may be given in terms of the initial state as

$$|\psi(t)\rangle = e^{-iHt/\hbar}\,|\psi(0)\rangle$$

1. **Stern Gerlach Devices**
   You are given a sequence of Stern-Gerlach devices labeled by their orientation. Assuming the intensity of the input beam is 1, give the intensity of the output beam for each of the following configurations. We shall use the convention that a solid line indicates that the emerging beam corresponds to the $+1$ eigenstate of the Stern-Gerlach device, while a dashed line corresponds to the -1 eigenstate.

   (a) $\xrightarrow{|\hat{z}\rangle} \boxed{\hat{z}} \longrightarrow$

   (b) $\xrightarrow{|\hat{z}\rangle} \boxed{\hat{z}} \dashrightarrow$

   (c) $\xrightarrow{|\hat{x}\rangle} \boxed{\hat{z}} \longrightarrow$

   (d) $\xrightarrow{|\hat{x}\rangle} \boxed{\hat{z}} \dashrightarrow$

   (e) $\xrightarrow{|\hat{n}\rangle} \boxed{\hat{m}} \dashrightarrow$

   (f) $\xrightarrow{|\hat{n}\rangle} \boxed{\hat{m}} \longrightarrow \boxed{\hat{n}} \longrightarrow$

   (g) $\xrightarrow{|\hat{x}\rangle} \boxed{\hat{z}} \longrightarrow \boxed{\hat{z}} \longrightarrow$

   (h) $\xrightarrow{|\hat{x}\rangle} \boxed{\hat{z}} \dashrightarrow \boxed{\hat{x}} \longrightarrow$

2. **Bloch Sphere**

   (a) Given the following states, find their corresponding unit vectors on the Bloch sphere. For instance, if given the state $|\psi\rangle = |0\rangle$, you would respond, $\hat{n} = (0, 0, 1)$.

      i. $|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$
      ii. $|\psi\rangle = \frac{i}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle$
      iii. $|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{i}{\sqrt{2}}|1\rangle$
      iv. $|\psi\rangle = e^{i\pi/5}|1\rangle$
      v. $|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{i}{\sqrt{2}}|1\rangle$

   (b) Given the following unit vectors on the Bloch sphere, find the corresponding state vectors. For instance, if given the unit vector $\hat{n} = (0, 0, 1)$, you would respond, $|\psi\rangle = |0\rangle$.

      i. $\hat{n} = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$
      ii. $\hat{n} = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$
      iii. $\hat{n} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

iv. $\hat{n} = \left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$

v. $\hat{n} = (0, 1, 0)$

3. **Single Qubit Gates**

Suppose that a particle is placed in a constant magnetic field, $\vec{B} = B_0\hat{b}$, so that the Hamiltonian for the spin degree of freedom is,

$$H = -\vec{\mu} \cdot \vec{B} = -\mu_0 B_0 \vec{S} \cdot \hat{b}$$

Here we have used $\vec{S} = (\sigma_x, \sigma_y, \sigma_z)$ as the vector of Pauli matrices and $\vec{\mu} = \mu_0\vec{S}$ as the magnetic moment operator.

(a) Suppose the magnetic field is such that $\vec{B} = B_0\hat{z}$. What is the Hamiltonian?

(b) If the initial state of the particle is $|\psi(0)\rangle = |\hat{x}+\rangle$, what is the state's Bloch vector as function of time?

(c) Under the action of this Hamiltonian, how long will it take until the state of the system is $|\psi\rangle = |-\hat{x}\rangle$?

(d) Suppose we now add to this Hamiltonian a weak oscillating term,

$$H \to H(t) = -\mu_0 B_0 \vec{S} \cdot \hat{b} + a\cos(\omega t)\sigma_x$$

What frequency should we choose for ω so that the oscillating term is on resonance?

(e) How long will it take to do a π-pulse about the $\hat{x}$ axis? Note that your answer will be independent of the initial state.