Model

Let us build a model of elastic 2D grid.

$U(x, y, t) \to \mathbb{R}^2$,

$U(x, y, t) = (u(x, y, t),\ v(x, y, t))\,;$

U is vector function which represents displacement field, consisting at each spacetime point of horizontal and vertical displacement.

Wave equation

The well-known wave equation for our displacement function:

$$U_{tt} = c^2\left(U_{xx} + U_{yy}\right);$$

where c is the wave propagation speed which parametrizes medium properties.

Discretization and integration

We will discretize out model over even space grid with step h and even time intervals with step d. Thus we get values $U^n_{i,j} = \left(u^n_{i,j}, v^n_{i,j}\right)$, indexed by space indexes i, j and timestep index n.

Finite difference approximation for spatial laplacians, using the well-known five-point stencil:

$$\Delta U^n_{i,j} \approx \frac{1}{h^2}\left(U^n_{i+1,j} + U^n_{i-1,j} + U^n_{i,j+1} + U^n_{i,j-1} - 4 \cdot U^n_{i,j}\right).$$

Backward finite difference approximation for second time derivative:

$$(U_{tt})^n_{i,j} \approx \frac{1}{d^2}\left(2U^n_{i,j} - 5U^{n-1}_{i,j} + 4U^{n-2}_{i,j} - U^{n-3}_{i,j}\right).$$

Explicit scheme:

$$(U_{tt})^{n+1}_{i,j} = c^2 \cdot \Delta U^n_{i,j}.$$

Substituting the approximations into it we get explicit solution:

$$\frac{2U^{n+1}_{i,j} - 5U^n_{i,j} + 4U^{n-1}_{i,j} - U^{n-2}_{i,j}}{d^2} = c^2 \cdot \Delta U^n_{i,j}$$

$$U^{n+1}_{i,j} = \frac{1}{2}\left(c^2 d^2 \cdot \Delta U^n_{i,j} + 5U^n_{i,j} - 4U^{n-1}_{i,j} + U^{n-2}_{i,j}\right),$$

$$U^{n+1}_{i,j} = \frac{1}{2}\left(c^2 \cdot \frac{d^2}{h^2}\left(U^n_{i+1,j} + U^n_{i-1,j} + U^n_{i,j+1} + U^n_{i,j-1} - 4 \cdot U^n_{i,j}\right) + 5U^n_{i,j} - 4U^{n-1}_{i,j} + U^{n-2}_{i,j}\right).$$

Stability considerations

Explicit solution blows up. It's unstable as long as time and grid steps are not small enough.

Implicit solution

$$(U_{tt})_{i,j}^{n+1} = c^2 \cdot \Delta U_{i,j}^{n+1};$$

$$\frac{2U_{i,j}^{n+1} - 5U_{i,j}^{n} + 4U_{i,j}^{n-1} - U_{i,j}^{n-2}}{d^2} = c^2 \cdot \Delta U_{i,j}^{n+1};$$

$$2U_{i,j}^{n+1} - 5U_{i,j}^{n} + 4U_{i,j}^{n-1} - U_{i,j}^{n-2} = c^2 \cdot \frac{d^2}{h^2}\left(U_{i+1,j}^{n+1} + U_{i-1,j}^{n+1} + U_{i,j+1}^{n+1} + U_{i,j-1}^{n+1} - 4 \cdot U_{i,j}^{n+1}\right);$$

abbreviate $r = c^2 \cdot \frac{d^2}{h^2}$;

$$(2+4r)\, U_{i,j}^{n+1} - r\left(U_{i+1,j}^{n+1} + U_{i-1,j}^{n+1} + U_{i,j+1}^{n+1} + U_{i,j-1}^{n+1}\right) = 5U_{i,j}^{n} - 4U_{i,j}^{n-1} + U_{i,j}^{n-2}.$$