Kevin Mueller

1)

$u_t(x,t) = u_{xx}(x,t) + k\,u(x,t) \quad 0<x<\pi$

a) $k=-1$, $u_x(0,t)=0$, $u_x(\pi,t)=0$, $u(x,0)=1$

$u_t(x,t) = u_{xx}(x,t) - u(x,t)$ $\quad C_n(t) = \dfrac{\int_0^\pi u(x,t)\varphi_n(x)}{\int_0^\pi \varphi_n(x)\varphi_n(x)}$ $\quad \varphi_n''(x) = -\mu_n\varphi_n(x)$

$u(x,t) = \sum C_n(t)\varphi_n(x)$

$$\int_0^\pi u_t(x,t)\varphi_n(x)\,dx = \int_0^\pi u_{xx}(x,t)\varphi_n(x)\,dx + k\int_0^\pi u(x,t)\varphi_n(x)\,dx$$

$$\frac{\partial}{\partial t}\left[\int_0^\pi u(x,t)\varphi_n(x)\,dx\right] = \int_0^\pi u_{xx}(x,t)\varphi_n(x)\,dx + k\int_0^\pi u(x,t)\varphi_n(x)\,dx$$

integration by parts & substitution

$$\frac{\partial}{\partial t}\left[C_n(t)\int_0^\pi \varphi_n^2(x)\right] = \left[u_x(x,t)\varphi_n(x)\Big|_{x=0}^{x=\pi} - \int_0^\pi u_x(x,t)\varphi_n'(x)\,dx\right] + k\int_0^\pi u(x,t)\varphi_n(x)$$

$$\frac{\partial}{\partial t}\left[C_n(t)\int_0^\pi \varphi_n^2(x)\right] = \cancel{u_x(x,t)\varphi_n(x)\Big|_{x=0}^{x=\pi}} - u(x,t)\varphi_n'(x)\Big|_{x=0}^{x=\pi} + \int_0^\pi u(x,t)\varphi_n''(x)\,dx + k\int_0^\pi u(x,t)\varphi_n(x)\,dx$$

$$C_n'(t)\int_0^\pi \varphi_n^2(x)\,dx = -u(x,t)\varphi_n'(x)\Big|_{x=0}^{x=\pi} + \int_0^\pi u(x,t)\varphi_n''(x)\,dx + \underset{=-1}{k}\int_0^\pi u(x,t)\varphi_n(x)\,dx$$

$$C_n'(t)\int_0^\pi \varphi_n^2(x)\,dx = -u(x,t)\varphi_n'(x)\Big|_{x=0}^{x=\pi} - \int_0^\pi u(x,t)\,\mu_n\varphi_n(x) - \int_0^\pi u(x,t)\varphi_n(x)\,dx$$

$$C_n'(t)\int_0^\pi \varphi_n^2(x)\,dx = -u(x,t)\varphi_n'(x)\Big|_{x=0}^{x=\pi} - \mu_n\int_0^\pi C_n(t)\varphi_n^2(x)\,dx - \int_0^\pi C_n(t)\varphi_n^2(x)\,dx$$

$$C_n'(t)\int_0^\pi \varphi_n^2(x)\,dx = \underbrace{-u(x,t)\varphi_n'(x)\Big|_0^\pi}_{0 \text{ b/c } \varphi_n'(\pi)=0,\ \varphi_n(0)=0} - (\mu_n+1)\,C_n(t)\cancel{\int_0^\pi \varphi_n^2(x)\,dx}$$

$$C_n'(t) + (\mu_n+1)\,C_n(t) = 0$$

$$C_n'(t) + (\mu_n + 1)C_n(t) = 0 \quad C_n(t) = \frac{\int_0^\pi u(x,t)\,\varphi_n(x)\,dx}{\int_0^\pi \varphi_n^2(x)\,dx}$$

Solve EF ode: $\varphi_n''(x) + \mu_n\varphi_n(x) = 0$, $\varphi_n'(\pi) = \varphi_n'(0) = 0$

$\mu_n > 0$ $\varphi_n(x) = C_1\cos(\sqrt{\mu_n}x) + C_2\sin(\sqrt{\mu_n}x)$

$$\varphi_n'(x) = -C_1\sqrt{\mu_n}\sin(\sqrt{\mu_n}x) + \cancel{C_2\sqrt{\mu_n}\cos(\sqrt{\mu_n}x)}_{x=0}$$

$$\varphi_n'(\pi) = -C_1\sqrt{\mu_n}\sin(\sqrt{\mu_n}\pi) = 0 \Rightarrow \sqrt{\mu_n} = n \Rightarrow \mu_n = n^2$$

$$\boxed{\varphi_n(x) = \cos(nx)} \quad n = 1, 2, 3, 4 \ldots$$

$\mu_n = 0$ $\varphi_n(x) = \cancel{C_1}x + C_2 \Rightarrow$ constant

$$\varphi_n'(x) = C_1 x \Rightarrow \varphi_n(\pi) = \cancel{C_1}\pi = 0$$

$$\boxed{\varphi_n(x) = 1}$$

$\mu_n < 0$, nothing as no periodic solutions

Solve Coefficient ODE:

$$C_n'(t) + (\mu_n + 1)C_n(t) = 0 \Rightarrow \boxed{C_n(t) = a_n e^{-(\mu_n+1)t}}$$

apply I.C's:

$n=0$: $\mu_n = 0$: $$C_n(0) = \frac{\int_0^\pi (1)(1)}{\int_0^\pi (1)^2} = 1, \quad C_n(0) = a_0 \Rightarrow \boxed{a_0 = 1}$$

$\mu_n < 0$ $$C_n(0) = \frac{\int_0^\pi (1)\cos(nx)}{\int_0^\pi \cos^2(nx)} = 0 \Rightarrow a_n = 0$$

$$u(x,t) = e^{-(0+1)t} = \boxed{u(x,t) = e^{-t}}$$

without I.C's

$$\boxed{u(x,t) = a_0 e^{-t} + \sum_{n=1}^{\infty} a_n e^{-(n^2+1)t}\cos(nx)}$$

b) $k=1 \quad u(0,t)=0 \quad u(\pi,t)=0 \quad u(x,0)=\sin(x)+\sin(4x)$

$$C_n'(t)\int_0^\pi \varphi_n^2(x)\,dx = u_x(x,t)\varphi_n(x)\Big|_0^\pi - \cancel{u(x,t)\varphi_n'(x)\Big|_0^\pi} + \int_0^\pi u(x,t)\,\varphi_n''(x)\,dx$$
$$+\int_0^\pi u(x,t)\,\varphi_n(x)\,dx$$

$$C_n'(t)\int_0^\pi \varphi_n^2(x)\,dx = u_x(x,t)\,\varphi_n(x)\Big|_0^\pi - \mu_n\int_0^\pi u(x,t)\,\varphi_n(x) + \int_0^\pi u(x,t)\,\varphi_n(x)\,dx$$

$$C_n'(t)\int_0^\pi \varphi_n^2(x)\,dx = \underbrace{u_x(x,t)\varphi_n(x)\Big|_0^\pi}_{\varphi_n(0)=0,\ \varphi_n(\pi)=0} - (\mu_n-1)\,C_n(t)\int_0^\pi \varphi_n^2(x)\,dx$$

$$\boxed{C_n'(t)+(\mu_n-1)\,C_n(t)=0 \qquad C_n(t)=\frac{\int_0^\pi u(x,t)\,\varphi_n(x)}{\int_0^\pi \varphi_n^2(x)}}$$

Solve EF ode:

$\mu_n>0 \quad \varphi_n(x)=C_1\cos(\sqrt{\mu_n}x)+C_2\sin(\sqrt{\mu_n}x)$

$\varphi_n(0)=\cancel{C_1}\cos(0)+C_2\sin(0)=0$

$\varphi_n(\pi)=C_2\sin(\sqrt{\mu_n}\pi)=0 \qquad \sqrt{\mu_n}=n \Rightarrow \mu_n=n^2$

$\mu_n=0 \quad \varphi_n(0)=\cancel{0}+1=0$

$$\boxed{\varphi_n(x)=\sin(nx) \quad n=1,2,3}$$

$C_n'(t)+(\mu_n-1)\,C_n(t)=0 \Rightarrow C_n(t)=a_n e^{-(\mu_n-1)t}$

$$C_n(0)=a_n=C_n(0)=\frac{\int_0^\pi[\sin(x)+\sin(4x)]\sin(nx)}{\int_0^\pi \sin^2(nx)=\frac{\pi}{2}}$$

$$=\frac{2}{\pi}\int_0^\pi \sin(x)\sin(nx)+\frac{2}{\pi}\int_0^\pi \sin(4x)\sin(nx)$$

$$=\frac{2}{\pi}\,\frac{\pi}{2},\ 0 \quad + \quad \frac{2}{\pi}\,\frac{\pi}{2},\ 0$$

$n=1 \quad n\neq 1 \qquad n=4,\ n\neq 4$

$a_1=1 \qquad a_4=1$

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} a_n e^{-(n^2-1)t} \sin(nx)}, \; u_n = n^2$$

for $u(x,0) = \sin(x) + \sin(4x)$:

$$= \cancel{e^{-(1^2-1)t}} \sin(x) + e^{-(15)t} \sin(4x$$

$$\boxed{u(x,t) = \sin(x) + e^{-15t} \sin(4x)}$$

$u_t(x,t) = u_{xx}(x,t)$

c) $k=0$ $u(0,t) = u(\pi,t)$, $u_x(\pi,t) = u_x(0,t)$

$u(x,0) = \sin(4x)\cos(2x)$, periodic BC's

$\psi_n''(x) + u_n \psi_n(x) = 0$

$u_n > 0$ $c_1 \cos(\sqrt{u_n}x) + c_2 \sin(\sqrt{u_n}x)$

$\psi_n'(x) = -c_1\sqrt{u_n}\sin(\sqrt{u_n}x) + c_2\sqrt{u_n}\cos(\sqrt{u_n}x)$

$\psi(0) = \psi(\pi) \Rightarrow \cancel{c_1}(1) = \cancel{c_1}\cos(\sqrt{u_n}\pi)$ $\sqrt{u_n} = 2n$

$\psi'(0) = \psi'(\pi) \Rightarrow c_2\sqrt{u_n} = \ldots$

Undetermined coefficients. $\boxed{\begin{array}{l}\psi_n^1 = \cos(2nx) \\ \psi_n^2 = \sin(2nx)\end{array}}$

$u_n = 0$;

$\psi_n(x) = c_1 + c_2 x$

$\psi(0) = \psi(\pi) \Rightarrow c_1 + c_2(0) = c_1 + \overset{0}{c_2}\pi$

$\psi_n'(x) = c_1 \Rightarrow \psi_n'(0) = \psi_n'(\pi) \Rightarrow \boxed{\psi_n = 1}$

$c_n'(t)\int_0^\pi \psi_n^2(x) = u_x(x,t)\psi_n(x)\Big|_0^\pi - u(x,t)\psi_n'(x)\Big|_0^\pi - u_n(t)\int_0^\pi u(x,t)\psi_n''(x)$

from I.C

$c_n'(t) + c(t)u_n \Rightarrow \boxed{c_n(t) = a_n e^{-(u_n)t}}$ $u_n = 4n^2$

$$c_n(t) = \frac{\int_0^\pi u(x,t)\psi_n(x)}{\int_0^\pi \psi_n^2(x)}$$

$u(x,t) = \sum C_n(t)\, \psi_n(x)$

$= a_0 + a_n \cos(2nx)\, e^{-4n^2 t} + b_n \sin(2nx)\, e^{-4n^2 t}$

find a_0, a_n, b_n for $u(x,0) = \sin(4x)\cos(2x)$

a_0: $u_n = 0,\ t=0 \Rightarrow \psi_n(x) = 1$

$$C_n(0) = \frac{\int_0^\pi \sin(4x)\cos(2x)}{\quad} = 0 \quad !!$$

$$C_n(0) = a_0 e^{-(n^2)t} \Rightarrow \boxed{a_0 = 0}$$

$$a_n:\ \frac{\int_0^\pi \sin(4x)\cos(2x)\cos(2n\pi)}{\int_0^\pi \cos^2(2nx) = \frac{\pi}{2}} = \frac{\frac{1}{2}\int_0^\pi [\sin(2x)+\sin(6x)]\cos(2nx)}{\pi/2}$$

since orthogonal: $C_n(0) = 0 = a_n$

$b_n \cos x$

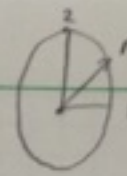
$$b_n:\ \frac{\frac{1}{2}\int_0^\pi [\sin(2x)+\sin(6x)]\sin(2nx)}{\pi/2} = \frac{1}{\pi}\Big[\underbrace{\int_0^\pi \sin^2(2x)}_{\frac{\pi}{2}}{}_{n=1} + \int_0^\pi \sin^2(6x)_{n=3}\Big]$$

$C_n(0) = b_n = \frac{1}{2},\ n = 1,3 \quad b_n = 0\ n \neq 1,3$

$$\boxed{u(x,t) = \frac{1}{2} e^{-4t}\sin(2x) + \frac{1}{2} e^{-36t}\sin(6x)}$$

2.a

$a=1,\ b=2 \quad \nabla u\cdot\hat{n}=1$



$r=\sqrt{a^2\cos^2(\theta)+b^2\sin^2\theta}$
$=\sqrt{\cos^2(\theta)+4\sin^2(\theta)}$

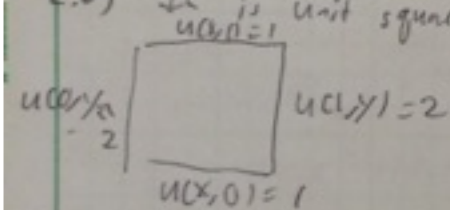
$$\int_{\partial\Omega}\nabla u\cdot\hat{n}\,ds \overset{?}{=} 0$$

$$\int_0^{2\pi}(\nabla u\cdot\hat{n})\,\underbrace{ds}_{r\,d\theta}=\int_0^{2\pi}\cos(\theta)\sqrt{\cos^2(\theta)+4\sin^2(\theta)}\,d\theta=0$$

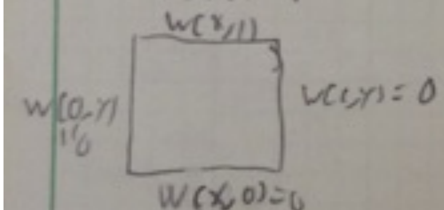
$\underset{\text{Cartesian}}{\nabla u\cdot\hat{n}=1}$ $\quad\hat{n}=\overset{\text{in Polar}}{\begin{bmatrix}\cos\theta\ \hat{i}\\ \sin\theta\ \hat{j}\end{bmatrix}} \Rightarrow \nabla u\cdot\hat{n}=\cos\theta$

$\boxed{\text{Solution Exists}}$

2.b) Ω is unit square $\quad g(x)=0 \quad u(x,0)=1 \quad u(x,1)=1$
$u(0,y)=2 \quad u(1,y)=2$



Suppose $U_1 \neq U_2$ solve $\Delta u=0$
$\Delta w=\Delta u_1-\Delta u_2=0$
$\Rightarrow u_1-u_2=0 \Rightarrow u_1\equiv u_2$



$\boxed{\text{The solution is unique, as expected due to all Dirichlet boundary conditions.}}$

2c) $u_r(R,\theta)=\cos(\theta)$

Solvability criterion:

$$\int_{\partial\Omega}\nabla u\cdot\hat{n}\,ds=0,\quad \int_0^{2\pi}\cos(\theta)\,R\,d\theta=\int_0^{2\pi}\cos(\theta)=-\sin\Big|_0^{2\pi}=-[\sin(2\pi)-\sin(0)]$$
$$=0$$

$\nabla u\cdot\hat{n}=$
$\underset{\text{in }r}{\nabla u}=\frac{1}{r}u_r\hat{r}\Big|_R$

$\boxed{\text{Solution exists}}$

2c continue

d) $\Delta u(\vec{x}) = g(x)$
$\quad\quad\quad\quad \overset{\shortparallel}{0}$

$\Delta u(r,\theta) = 0$

$\Delta u(r,\theta) = \frac{1}{r}(r u_r(r,\theta))' + \frac{1}{r^2} u_{\theta\theta}(r,\theta) = 0$

$\psi_n''(\theta) + \lambda_n \psi_n(\theta) = 0$

$\psi_n'(0) = \psi_n'(2\pi)$

$\psi_n(0) = \psi_n(2\pi)$ ↘ BC's

$$\boxed{\psi_0 = 1 \quad , \quad \psi_n = \cos(n\theta) \quad \psi_n = \sin(n\theta)}$$

$\lim_{r\to 0} \frac{1}{r} C_n' r C_n$

$C_n = \alpha_n r^n + \beta_n r^{-n}$ (0) $\quad C_n'(R) = \frac{\int_0^{2\pi} \cos(\theta)(1)}{\int_0^{2\pi}(1)} = 0 \Rightarrow A_0 = 0$

$C_n' = n\alpha_n r^{n-1} + n\beta_n r^{-n-1}$

$C_n' = \frac{n\alpha_n}{R^{n-1}} = \ldots \quad C_n(R) = \int_0^{2\pi} \cos(\theta)\sin(n\theta) = 0$

$\boxed{\alpha_n = 0}$

$C_n(R) \quad \frac{\int_0^{2\pi}\cos(\theta)\cos(n\theta)}{\int_0^{2\pi}\cos^2(\theta)} = 1 \Rightarrow A_n = 1$

$C_n'(R) = \frac{n\alpha_n}{R^{n-1}} = \frac{\int_0^{2\pi}\sin^2(\theta)\,d\theta}{\pi/2}$

$n \neq 1 \quad \alpha_n = 0$

$\boxed{\alpha_1 = 1}$

$$\boxed{u_r(R,\theta) = \alpha_0 + \cos(\theta) r}$$

Solution exists as shown, and is not unique

d) Ω circle of Radius R, $g(x)=1$ $U_r(R,\theta)=\cos(\theta)$

Consider the poisson equation $\nabla^2 U(\hat{x}) = g(x) = 1$.

Since we have Neuman boundary conditions,

we can claim $U(\hat{x}) = \hat{n}\cdot\nabla U(\hat{x})$ on the boundary.

Suppose there are two solutions u_1 & u_2 to above poisson problem. Then, $V = u_1 - u_2 \Rightarrow \Delta V = \Delta u_1 - \Delta u_2 = \overset{g(x)}{(1)} - \overset{g(x)}{(1)} = 0$ throughout $\partial\Omega$ (inside of boundary), and

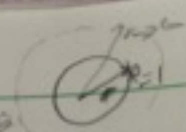
$$V = [\hat{n}\cdot\nabla u(\hat{x})] - [\hat{n}\cdot\nabla(u(\hat{x}))]$$

on the boundary.

Furthermore: $\hat{n}\cdot\overset{\text{in polar}}{\nabla u} = \frac{1}{r}u_r\Big|_R = \left[\frac{1}{R}\cos\theta - \frac{1}{R}\cos(\theta)\right] = 0$

We can conclude that $u_1 = u_2$ and is therefore a $\boxed{\text{unique solution}}$

3.a



$\Delta u = 0$, Ω is outside unit circle $u(1,\theta) = \sin\theta$

$\Delta u(r,\theta) = \frac{1}{r}(r u_r(u,\theta))' + \frac{1}{r^2} u_{\theta\theta}(r,\theta) = 0$

Periodic BC's

$u(r,0) = u(r,2\pi)$

$u_\theta(r,\theta) = u_\theta(r,2\pi)$

$u(1,\theta) = \sin\theta$

$u(r,\theta) = \sum C_n(r)\varphi_n(\theta)$, $\varphi_n''(\theta) + \mu_n \varphi_n(\theta) = 0$

$C_n(r) = \dfrac{\int u(r,\theta)\varphi_n(\theta)\, d\theta}{\int \varphi_n(\theta)\varphi_n(\theta)}$

$\lim_{r\to 0} r u < \infty$, $\lim_{r\to\infty} r u_r < \infty$

$$\int_0^{2\pi} \frac{1}{r}(r u_r)_r \varphi_n(\theta) + \int_0^{2\pi} \frac{1}{r^2} u_{\theta\theta}\varphi_n(\theta)\, d\theta = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)\int_0^{2\pi} u(r,\theta)\varphi_n(\theta)\, d\theta + \frac{1}{r^2}\int_0^{2\pi} u_{\theta\theta}\varphi_n(\theta)\, d\theta = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)\int_0^{2\pi} C_n(r)\int_0^{2\pi}\varphi_n^2(\theta)\, d\theta + \frac{1}{r^2}\Big[\, u_\theta\varphi_n(\theta)\Big|_{\theta=0}^{\theta=2\pi} - u\varphi_n'(\theta)\Big|_{\theta=0}^{\theta=2\pi}$$

$$+ \int_0^{2\pi} u(r,\theta)\varphi_n''(\theta)\, d\theta\,\Big]$$

$$\frac{1}{r}(rC_n'(r))'\int_0^{2\pi}\varphi_n^2(\theta)\, d\theta + \frac{1}{r^2}\left[\begin{matrix} u(r,0)\varphi_n(2\pi) \\ -u_\theta(r,0)\varphi_n(0)\end{matrix} - \begin{matrix} u(r,0)\varphi_n'(2\pi) \\ -u(r,0)\varphi_n'(0)\end{matrix} - \mu_n\int_0^{2\pi} u\varphi_n\, d\theta\right]$$

$\varphi_n(0) = \varphi_n(2\pi)$ $\quad$ $\varphi_n'(2\pi) = \varphi_n'(0)$

$$\frac{1}{r}(rC_n'(r))'\int_0^{2\pi}\varphi_n^2(\theta)\, d\theta - \mu_n\int_0^{2\pi} C_n(r)\,\varphi_n^2\, d\theta$$

$$\boxed{\frac{1}{r}(rC_n'(r))' - \frac{1}{r^2}\mu_n C_n(r) = 0 \quad C_n'(1) = \frac{\int_0^{2\pi}\sin(\theta)\varphi_n(\theta)}{\varphi_n^2(\theta)}}$$

$\Psi_n''(\theta) + \mu_n \Psi_n(\theta) = 0$, Periodic BC: $\Psi_n(0) = \Psi_n(2\pi)$, $\Psi_n'(0) = \Psi_n'(2\pi)$

$\mu_n > 0: \begin{cases} \cos(\sqrt{\mu_n}\,\theta) \\ \sin(\sqrt{\mu_n}\,\theta) \end{cases}$ $\sqrt{\mu_n} = n \Rightarrow \mu_n = n^2$

$\mu_n = 0: \begin{cases} 1 \\ \theta \end{cases} \xrightarrow{\text{BC}} 1$ $\quad \mu_n < 0: \times$

Coefficient ODE

$\mu_n = 0: \quad \frac{1}{r}\left(r C_n'(r)\right)' = 0 \overset{\text{integrate once}}{\Rightarrow} r C_n'(r) = C_1 \Rightarrow C_n'(r) = C_1\left(\frac{1}{r}\right)$

$\overset{\text{integrate}}{\Rightarrow} C_n(r) = \cancel{C_1 \ln(r)} + C_2 \Rightarrow C_n(r) = C_2 \Rightarrow a_0$

because $\ln(r) \to \infty$ blows up; constant

$\mu_n > 0 \quad r^2\left[\frac{1}{r}\left(r C_n'(r)\right)' - \frac{1}{r^2}\mu_n C_n(r)\right] = 0$

$r\left(r C_n'(r)\right)' - \mu_n C_n(r) = 0$

$r\left[1 C_n'(r) + r C_n''(r)\right] - \mu_n C_n(r) = 0$

$r^2 C_n''(r) + r C_n'(r) - \mu_n C_n(r) = 0 \quad C_n = r^m$

$\cancel{r^2} r^{m-2}(m-1)m + \cancel{r r^{m-1}} m - \mu_n \cancel{r^m} = 0$

$(m-1)m + m - \mu_n = 0$, $m = \pm\sqrt{\mu_n}$

$C_n = \underbrace{\cancel{\alpha_n r^n}}_{\text{blows up}}^{0} + B_n r^{-n} \Rightarrow C_n = B_n r^{-n}$

Solution without IC's

$$\boxed{u(r,\theta) = \sum \Psi_n(\theta) C_n(r) = a_0 + \sum_{n=1}^{\infty} a_n r^{-n} \cos(n\theta) + \sum_{n=1}^{\infty} b_n r^{-n} \sin(n\theta)}$$

applying I.C $\quad u(1,\theta) = \sin(\theta)$

$$C_n'(1) = \frac{\int_0^{2\pi} \sin(\theta)\,\Theta(1)}{\int_0^{2\pi}(1)} = -\cos\Big|_0^{2\pi} = 0 = 0$$

$$C_n'(1) = \frac{\int_0^{2\pi} \sin(\theta)\cos(n\theta)}{\int_0^{2\pi}\cos^2(n\theta)} = \frac{2}{\pi}\int_0^{2\pi}\sin(\theta)\cos(n\theta) = 0$$

$$C_n'(1) = \frac{2}{\pi}\int_0^{2\pi}\sin(\theta)\sin(n\theta) = \frac{2}{\pi}\int_0^{2\pi}\sin^2(\theta) = \frac{2}{\pi}\frac{\pi}{2} = 1$$

$n=1$

$n\neq 1 = 0$

$C_n(r) = \alpha_1 r^{-n}$

$C_n'(r) = \alpha_1(-n)r^{-n-1} \Rightarrow C_n'(1) - \alpha_1(1)^{-2} = 1 \Rightarrow \alpha_1 = -(1)^2 = -1$

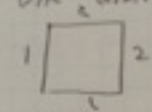
$C_1(1) = \alpha_1 1^{-1} \quad C_1'(1) = -\alpha_1 1^{-2} = -\alpha_1 = 1 \Rightarrow \boxed{\alpha_1 = -1} = b_1$

after I.C

$u(r,\theta) = \boxed{-r^{-n}\sin(\theta)}$

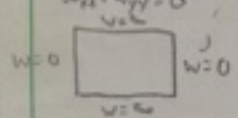
3.6) $\Delta u = 0$ in Ω (rectangular region with width 2 and infinite height), $u(0,y)=1$, $u(2,y)=2$

Nonhomogeneous in y)



$w(x,y) = u(x,y) - \varphi_0(x)$ $\qquad \varphi_0''(x) + 0 \cdot \varphi_0(x) = 0$

$w_{xx} + w_{yy} = 0$, $u_{xx} + u_{yy} = 0$ $\qquad \varphi_0''(x) = 0 \Rightarrow \varphi_0(x) = a_0 x + b_0$



left boundary: $w(0,y) = u(0,y) - \varphi_0(0)$

$= 1 - b_0 = 0$

$\boxed{b_0 = 1}$

right boundary: $w(2,y) = u(2,y) - \varphi_0(2)$

$\boxed{\varphi_0(x) = \frac{1}{2}x + 1}$ $\quad = 2 - (2a_0 + 1) = 0$

$1 - 2a_0 = 0$

$\boxed{a_0 = \frac{1}{2}}$

$w(x,0) = \infty - \varphi_0(x) = \infty$)

$w(x,\infty) = \infty - \varphi_0(x) = \infty$

$w(x,y) = \sum c(y)\varphi_n(x)$ $\qquad c_n(y) = \dfrac{\int_0^2 w(x,y)\varphi_n(x)\,dx}{\int_0^2 \varphi_n^2(x)\,dx}$

$\varphi_n'' + \mu_n \varphi_n = 0$

$\int_0^2 w_{xx}(x,y)\varphi_n(x)\,dx + \int_0^2 w_{yy}\varphi_n(x)\,dx = 0$

switch sides, integrate by parts:

$\frac{\partial^2}{\partial y^2}\left[\int_0^2 w(x,y)\varphi_n(x)\,dx\right] + w_x\varphi_n(x)\Big|_{x=0}^{x=2} - \int_0^2 w_x\varphi_n'(x)\,dx = 0$

$\frac{\partial^2}{\partial y^2}\left[\int_0^2 w(x,y)\varphi_n(x)\,dx\right] + w_x\varphi_n(x)\Big|_{x=0}^{x=2} - \cancel{w\varphi_n'(x)\Big|_{x=0}^{x=2}} + \int_0^2 w\varphi_n''(x)\,dx$

$\varphi_n(0)=0$, $\varphi_n(2)=0$

$\frac{\partial^2}{\partial y^2}\left[c_n(y)\int_0^2 \varphi_n^2(x)\,dx\right]$ $\qquad -\mu_n\int_0^2 \varphi_n(x)w\,dx = -\mu_n c_n(y)\int_0^2 \varphi_n^2(x)\,dx$

$\Rightarrow$ $\boxed{\begin{array}{l} c_n''(y) - \mu_n c_n(y) = 0 \\ c_n(y) = \dfrac{\int w(x,y)\varphi_n(x)}{\int \varphi_n^2(x)\,dx} \end{array}}$

E.F ODE

$\psi_n''(x) + \mu_n \psi_n(x) = 0 \qquad \psi_n(0) = 0 \quad \psi_n(2) = 0$

$\mu_n > 0 \qquad \psi_n(x) = C_1 \cos(\sqrt{\mu_n}\, x) + C_2 \sin(\sqrt{\mu_n}\, x)$

$\psi_n(0) = \cancel{C_1 \cos(\sqrt{\mu_n}\, x)}$

$\psi_n(2) = C_2 \sin(\sqrt{\mu_n}\, 2) = 0 \qquad \sqrt{\mu_n} = \frac{n\pi}{2} \Rightarrow \mu_n = \left(\frac{n\pi}{2}\right)^2$

$\mu_n < 0$ No Eigen Values : $a \sinh(2\sqrt{\mu_n}) \neq 0$

$\mu_n = 0 \qquad \psi_n(x) = ax + b$

$\psi_n(0) = 0 + b = 0$

$\psi_n(2) \; a(2) = 0$

no E.V

$$\boxed{\psi_n(x) = \sin\left(\frac{n\pi x}{2}\right)} \quad n = 1, 2, 3 \ldots$$

$C_n''(y) - \mu_n C_n(y)$ $\quad \mu_n > 0$ better representation than hyperbolics. due to assumption

$C_n(y) = a_n e^{-\sqrt{\mu_n} y} + b_n e^{\sqrt{\mu_n} y}$

$\boxed{\lim_{y \to \infty} W(x,y) \overset{\text{assume}}{=} 0}$

$$\boxed{w(x,y) = \sum_{n=1}^{\infty} a_n e^{-\left(\frac{n\pi}{2}\right)^2 y} \sin\left(\frac{n\pi x}{2}\right)}$$

Since $u(x,y) = \psi_0(x) + w(x,y)$

$$= \boxed{\left(\frac{1}{2}x + 1\right) + \sum_{n=1}^{\infty} a_n e^{-\left(\frac{n\pi}{2}\right) y} \sin\left(\frac{n\pi x}{2}\right)}$$