Jacobs University Bremen

Sensor fusion 2

Automation

Course CO23-320203

Recap: complementary filter

- In situations where two or more sensors measure related quantities but show complementary characteristics in terms of noise (bias, etc)
- Dissimilar filters can be chosen in such way that $F_1(s)+F_2(s) = 1$, for example a low pass and a high pass filter pair
- A perfect estimation of the searched quantity can be obtained while the noises get efficiently filtered out.
- Such characteristics can be exploited also when the sensors measure different quantities related through kinematics
- For example: acceleration due to gravity measured by an accelerometer burdened by high frequency noise and rate of rotation measured by a gyroscope burdened by a (near)constant bias.

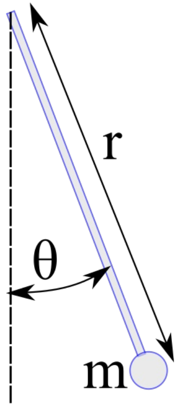
Equilibrium points

- A simple (autonomous, time invariant) system represented in space-state as

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

- can have an equilibrium point x* such that if once x is equal to x*, the state remains x* for all future trajectory
- From this we can imply that $\mathbf{f}(\mathbf{x}^*) = \mathbf{0}$
- Simple example: what is are equilibrium points of $\dot{x} = -x^3 + \sin^4 x$?
- Example #2: consider a rigid pendulum with the following dynamics of its tilt angle θ:

$$mr^2\ddot{\theta} + b\dot{\theta} + mgr\sin(\theta) = 0$$



- What are its equilibrium points?

Stability of equilibrium points/states

- The equilibrium state $x=0$ is said to be stable if, for any $R>0$, there exist some $r>0$ such that if $||x(0)|| < r$, then $||x(t)|| < R$ for all $t \geq 0$. Otherwise, the equilibrium point is unstable.
- (of course this definition does not only apply to the $x=0$ point! Think of a substitution $x \leftarrow x - x_0$ if you want to analyse the system for $x=x_0$)
- [Reference: J-J. E. Slotine, W. Li, “Applied nonlinear control”, Prentice Hall 1991, Chapter 3]

Asymptotic stability

- The equilibrium state $x=0$ is said to be asymptotically stable if it is stable and in addition, there exists some $r>0$ such that $\|x(0)\| < r$ implies that $x(t) \rightarrow 0$ as $t \rightarrow \infty$
- In other words: if when the system is in a state a certain distance away from the equilibrium point (on a certain orbit near this point), its trajectory converge to the equilibrium state in time, then it is asymptotically stable.
- The pendulum example revisited
  - What can we say about the stability of its equilibrium points?

Stability and equilibrium in systems

- We have seen so far:
  - Equilibrium points x* of a simple system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ (condition: $\mathbf{f}(\mathbf{x}^*) = \mathbf{0}$)
  - Stability and asymptotic stability of an equilibrium point
- Similar criteria apply to non-autonomous (= time-dependent) systems
- For a system of a form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$$

- Equilibrium points x* are defined by $\mathbf{f}(\mathbf{x}^*, t) = \mathbf{0}, \forall t \geq t_0$

Lyapunov stability analysis

- Lyapunov's global stability theorem: a powerfull theory which applies equally to LTI and non-linear systems
- For a system of a form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$$

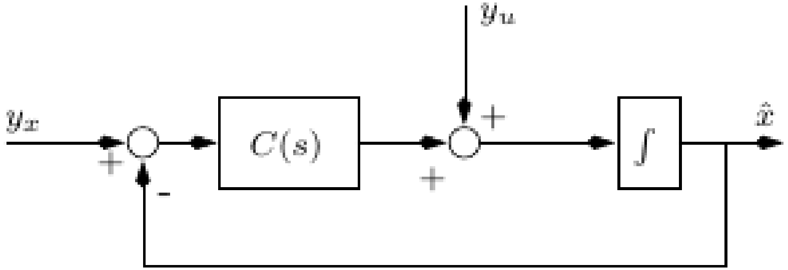
- Assume that there exist a scalar function $\mathcal{L}$ of the system's state x with continuous first order derivatives such that
- $\mathcal{L}(x)$ is positive definite
- $\dot{\mathcal{L}}(x)$ is negative definite
- $\mathcal{L}(x) \to 0$ as $||x|| \to \infty$ then, the equilibrium at the origin is globally asymptotically stable.
- If the derivative of $\mathcal{L}(x)$ is only negative semi-definite, then the system's origin is stable (not asymptotically)
- (of course, it can be used to show stability of any point, not just 0 – think of a variable substitution)

Lyapunov stability analysis

- Barbalat's lemma: a way to deal with situations where $\dot{\mathcal{L}}(x)$ is not negative definite in case of time-varying systems (=dependent on t)
- For a system of a form $$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$$
- If a scalar function $V(x,t)$ can be found which satisfies the following conditions:
  - $V(x,t)$ is lower-bounded
  - $\dot{V}(\mathbf{x}, t)$ is negative semi-definite, i.e. $\dot{V}(\mathbf{x}, t) \leq 0$
  - $\dot{V}(\mathbf{x}, t)$ is uniformly continuous in time (= $\ddot{V}(\mathbf{x}, t)$ is bounded)
- Then, $\dot{V}(\mathbf{x}, t) \rightarrow 0$ as $t \rightarrow \infty$

Lyapunov stability analysis

- Let us apply it to the complementary filter to confirm its convergence



(done on the blackboard)

Filtering continued: adding a bias estimator

- We can deal with the (near) constant offset in the gyro reading with an explicit method
- The all-pass filter $C(s)$ can be augmented with an integral component:

$$C(s) = K_P \rightarrow C(s) = K_P + K_I/s$$

(remember your PID controller lectures: the action of the integral term and coefficient K_I)

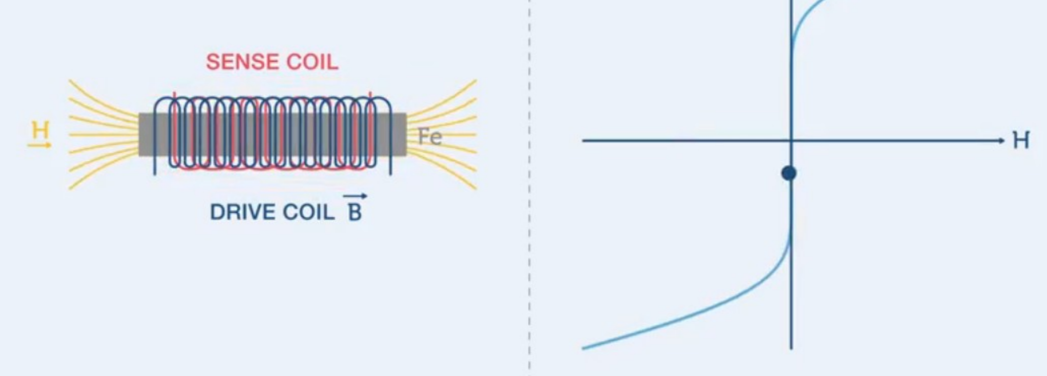
- Stability of the new system?

  (shown on the blackboard)

Filtering continued: adding Compas / Magnetometers

- Another common sensor which helps estimate orientation
- Technology: a 3D magnetometer can be designed and a orthogonal arrangement of Hall effect sensors
- Fluxgate magnetometer is a popular choice as well

  (see: https://www.youtube.com/watch?v=314GUagR7o4)



Filtering continued

- Let us introduce rotation matrices for expressing orientation
  - More robust notation (no gimbal lock)

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \qquad R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \qquad R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{R} = \mathbf{R}_{z,\phi}\mathbf{R}_{y,\theta}\mathbf{R}_{x,\psi}$$

- Watch out for the convention!
- Rotate any vector in one reference frame to express it in another
- Properties
  - Orthogonal matrix => $R^{\mathrm{T}} = R^{-1}$
  - *det(R)* = +/- 1

Filtering continued

- Dynamics of the rotation expressed by a rotation matrix (rough sketch):

$$R(t+dt) = R(t)\begin{bmatrix} 1 & -\omega_3 dt & \omega_2 dt \\ \omega_3 dt & 1 & -\omega_1 dt \\ -\omega_2 dt & \omega_1 dt & 1 \end{bmatrix} = R(t)\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} dt\right)$$

- The result:

$$^{A}_{B}\dot{\mathbf{R}} = {}^{A}_{B}\mathbf{R}\,[\,^{B}\boldsymbol{\omega}_{\mathbf{B/A}}\times]$$

with $^{A}_{B}\mathbf{R}$ - rotation of ref. frame $\mathcal{B}$ in frame $\mathcal{A}$, $^{B}\boldsymbol{\omega}_{\mathbf{B/A}}$ - rotational velocity vector of frame $\mathcal{B}$ in frame $\mathcal{A}$, expressed in $\mathcal{B}$ (ex. as read by a gyro)

- Reminder – the skew-symmetric operator x :

$$^{B}\boldsymbol{\omega}_{\mathbf{B/A}}\times \;=\; \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

Vectors a, b: $a_\times\, b = a \times b$

Filtering input

- Vector of gravity measured by the accelerometer (when the proper motions are low):

$$ {}^{B}\mathbf{a} = -{}^{A}_{B}\mathbf{R}^{\mathsf{T}}(g\ {}^{A}\hat{\mathbf{z}}) \qquad {}^{B}\hat{\mathbf{a}} = \frac{{}^{B}\mathbf{a}}{\|{}^{B}\mathbf{a}\|} = -{}^{A}_{B}\mathbf{R}^{\mathsf{T}}\ {}^{A}\hat{\mathbf{z}} = -{}^{A}_{B}\mathbf{R}^{\mathsf{T}}\begin{bmatrix}0\\0\\1\end{bmatrix} $$

  with $\hat{\mathbf{a}}$ and $\hat{\mathbf{z}}$ the unit vector pointing along the direction of gravity in body and world reference system respectively.

- Given the best orientation estimate $\hat{\mathbf{R}}$, we can calculate the estimated equivalent of $\hat{\mathbf{a}}$:

$$ \hat{\mathbf{d}} = -{}^{B}_{A}\hat{\mathbf{R}}^{\mathsf{T}}\begin{bmatrix}0\\0\\1\end{bmatrix} $$

- The error between the two is:

$$ \mathbf{e}_a \triangleq {}^{B}\hat{\mathbf{a}} \times {}^{B}\hat{\mathbf{d}} $$

Filtering input

- The gyroscope measures ${}^{B}\omega_{\mathbf{B/A}}$ augmented by an unknown quasi-constant bias b_0:

$$\boldsymbol{\Omega} \triangleq {}^{B}\omega_{\mathbf{B/A}} + \mathbf{b}_0$$

- Let us assume that we can estimate the bias as $\hat{\mathbf{b}}$.

Filtering input

- The direction of the local “north” of the Earth magnetic field (when the local disturbances are low):

$$ {}^{B}\mathbf{c} = {}^{A}_{B}\mathbf{R}^{\mathsf{T}} (m \; {}^{A}\hat{\mathbf{x}}) \qquad {}^{B}\hat{\mathbf{c}} = \frac{{}^{B}\mathbf{c}}{\| {}^{B}\mathbf{c} \|} = {}^{A}_{B}\mathbf{R}^{\mathsf{T}} \; {}^{A}\hat{\mathbf{x}} = {}^{A}_{B}\mathbf{R}^{\mathsf{T}} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} $$

- With the best orientation estimate, we can calculate its equivalent:

$$ {}^{B}\hat{\mathbf{m}} = {}^{B}_{A}\hat{\mathbf{R}}^{\mathsf{T}} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} $$

- The error between the two is:

$$ \mathbf{e}_{m} = {}^{B}\hat{\mathbf{c}} \times {}^{B}\hat{\mathbf{m}} $$

Combined estimation error

- The total combined error vector of the accelerometer and the magnetometer can be written as:

$$\mathbf{e}_{tot} = k_a \mathbf{e}_a + k_m \mathbf{e}_m$$

Where k_a and k_m are positive constants which are proportional to the respective reliabilities of the two sensors.

Candidate for a sensor fusion scheme

- Let us define the filter through its state-space representation:

$$\dot{\hat{\mathbf{R}}} = \hat{\mathbf{R}}\left[(\boldsymbol{\Omega} - \hat{\mathbf{b}} + K_p \mathbf{e}_{tot})\times\right], \qquad \hat{\mathbf{R}}(0) = \hat{\mathbf{R}}_{init}$$

$$\dot{\hat{\mathbf{b}}} = -K_I\, \mathbf{e}_{tot}, \qquad \hat{\mathbf{b}}(0) = \hat{\mathbf{b}}_{init}$$

[Source: “Non-linear complementary filters on the special orthogonal group”, Mahony et al, IEEE Transactions on Automatic Control, Vol. 53, No. 5, June 2008]

- Observer-based filtering scheme, with innovation term based on the e_{tot} error variable
- The same filter is used in open-source libraries for hobbyist-built and low cost intertial measurement units
- (example: MinIMU-9 + Arduino AHRS)

State feedback / state observer

$$\dot{\hat{\mathbf{R}}} = \hat{\mathbf{R}}\left[(\mathbf{\Omega} - \hat{\mathbf{b}} + K_p \mathbf{e}_{tot})\times\right], \qquad \hat{\mathbf{R}}(0) = \hat{\mathbf{R}}_{init}$$

$$\dot{\hat{\mathbf{b}}} = -K_I\, \mathbf{e}_{tot}, \qquad \hat{\mathbf{b}}(0) = \hat{\mathbf{b}}_{init}$$

- Our system is expressed in state space (as oposed to Laplace space)
  - The output is the orientation estimation, the input is the sensor data
- We have chosen to consider the bias estimation to be a part of the state so that we can include it into the feedback term
- Bias cannot be measured directly → bias estimation is a form of state observer
  - State observer provides estimate of part of the system’s state using system’s input and output

Discretisation

- Typical system in space state:

$$\dot{x} = f(x)$$

- Let's recall the fundamental definition of derivative:

$$\dot{x} := \lim_{\Delta t \to 0} \frac{x_{t+\Delta t} - x_t}{\Delta t}$$

- If we take a simple linear system $\dot{x} = Ax$ as an example, solving for the update term, we obtain the forward difference scheme sometimes also called the Euler method of numerical integration:

$$\frac{x_{t+\Delta t}}{\Delta t} \approx \frac{x_t}{\Delta t} + Ax_t$$

$$x_{t+\Delta t} \approx x_t + \Delta t A x_t$$

Fusion scheme – the big picture

- Reading sensor values and normalise gravity/magnetic field vectors
  - $\Omega[k] = [\omega 1, \omega 2, \omega 3]^T$ from gyroscope
  - $\hat{a}[k]$ from accelerometer
  - $\hat{c}[k]$ from magnetometer
- From the estimate $R[k]$ compute $\hat{\mathbf{d}}[k], \mathbf{e}_a[k], \hat{\mathbf{m}}[k], \mathbf{e}_m[k], \mathbf{e}_{tot}[k]$
- Apply discretized filter dynamics

$$\hat{\mathbf{b}}[k+1] = \hat{\mathbf{b}}[k] - \Delta T\, K_I \mathbf{e}_{tot}[k]$$
$$\hat{\mathbf{R}}[k+1] = \hat{\mathbf{R}}[k] + \Delta T\, \hat{\mathbf{R}}[k] \left[\left(\Omega[k] - \hat{\mathbf{b}}[k] + K_P \mathbf{e}_{tot}[k] \right) \times \right]$$

- Renormalise $R[k+1]$
- Repeat

Renormalisation of R

- Due to the errors of numerical integration, $\hat{\mathbf{R}}$ will eventually loose the properties of a rotation matrix. It must be corrected by renormalisation
- As an orthogonal matrix, $\hat{\mathbf{R}}$ should have each of its constituting column vectors mutually orthogonal. This can be enforced:

$$\mathbf{X} = \begin{bmatrix} r_{xx} \\ r_{xy} \\ r_{xz} \end{bmatrix} \qquad \mathbf{Y} = \begin{bmatrix} r_{yx} \\ r_{yy} \\ r_{yz} \end{bmatrix} \qquad error = \mathbf{X} \cdot \mathbf{Y} = \mathbf{X}^{\mathbf{T}}\mathbf{Y} = \begin{bmatrix} r_{xx} & r_{xy} & r_{xz} \end{bmatrix} \begin{bmatrix} r_{yx} \\ r_{yy} \\ r_{yz} \end{bmatrix}$$

$$\mathbf{X}_{orthogonal} = \mathbf{X} - \frac{error}{2}\mathbf{Y} \qquad \mathbf{Y}_{orthogonal} = \mathbf{Y} - \frac{error}{2}\mathbf{X}$$

$$\mathbf{Z}_{orthogonal} = \mathbf{X}_{orthogonal} \times \mathbf{Y}_{orthogonal}$$

Renormalisation of R

- Last stage of renormalisation of $\hat{\mathbf{R}}$ is to ensure that the component vectors have unit length:

$$\mathbf{X}_{normalized} = \frac{1}{2}\left(3 - \mathbf{X}_{orthogonal} \cdot \mathbf{X}_{orthogonal}\right)\mathbf{X}_{orthogonal}$$

$$\mathbf{Y}_{normalized} = \frac{1}{2}\left(3 - \mathbf{Y}_{orthogonal} \cdot \mathbf{Y}_{orthogonal}\right)\mathbf{Y}_{orthogonal}$$

$$\mathbf{Z}_{normalized} = \frac{1}{2}\left(3 - \mathbf{Z}_{orthogonal} \cdot \mathbf{Z}_{orthogonal}\right)\mathbf{Z}_{orthogonal}$$

Fusion scheme – why does it work?

- The convergence of the filter can be proven by examining the error terms $\tilde{R} = \hat{R}^T R$ and $\tilde{b} = b - \hat{b}$.
- Lyapunov theory can be then applied on the system by finding the right $V(\mathbf{x}, t)$ and $\dot{V}(\mathbf{x}, t)$ based on these error terms and showing their properties.
- The (not so straightforward) proof is available in “Non-linear complementary filters on the special orthogonal group”, Mahony et al, IEEE Transactions on Automatic Control, Vol. 53, No. 5, June 2008