Jacobs University Bremen

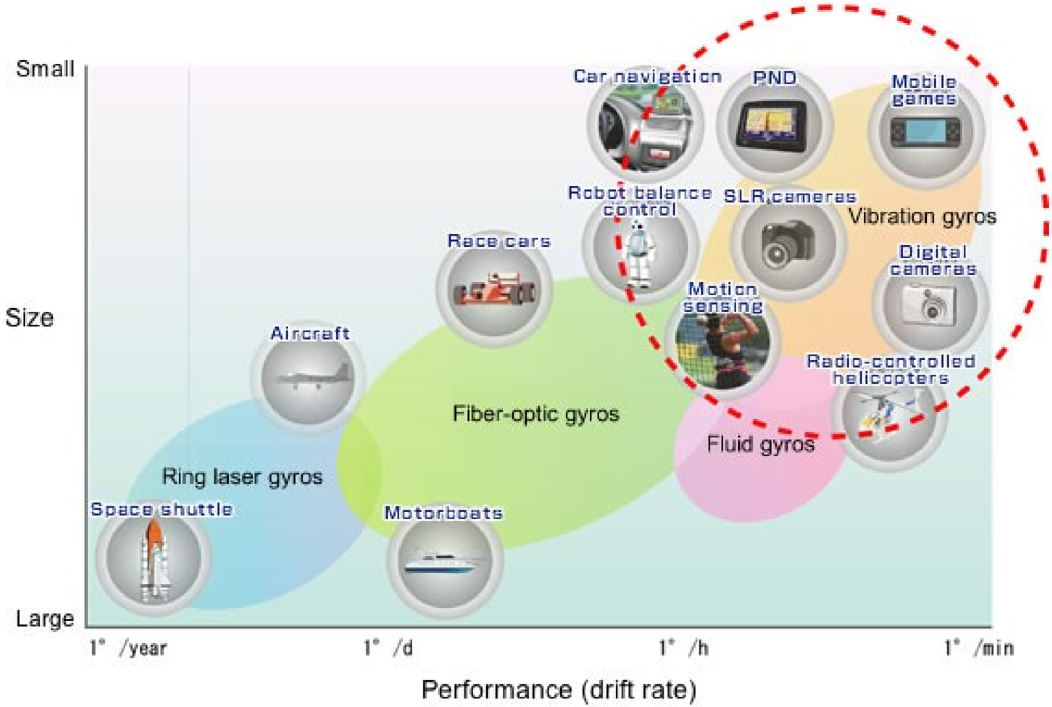
# Sensors & Filtering


## Input to output analysis of a CVG

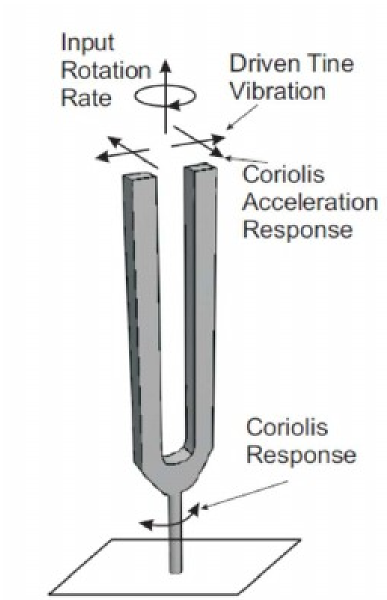

Automation
CO23-320203

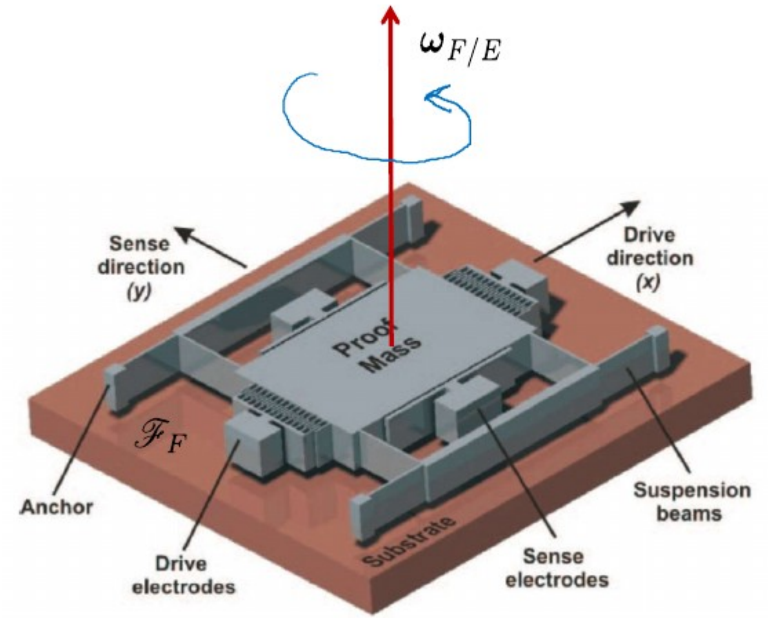
# Back to gyroscopes

- Many different existing gyro technologies strive for their market share



# Coriolis Vibratory Gyroscope

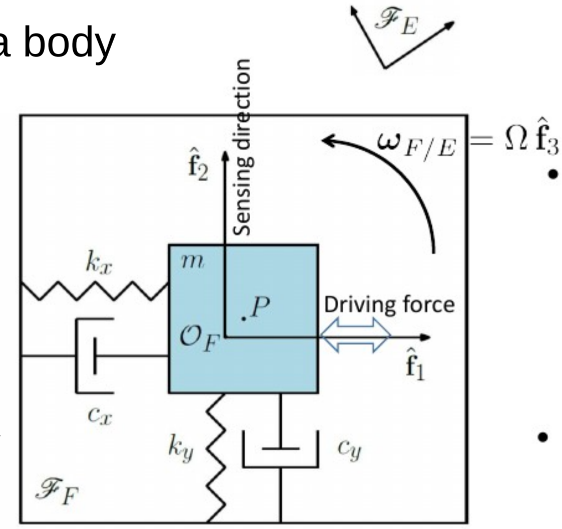


# Coriolis Vibratory Gyroscope idealisation

- Using Coriolis acceleration, which appears on a body in motion in a rotating reference frame, we can estimate the magnitude of rotation!
  - We need to force motion of this mass along one axis (through some drive force F(t))
- The elements of the sensor:
  - Mass $m$, free to move in the x and y directions
  - (Massless) springs with elastic constants $k_x$ and $k_y$
  - Dampers with damping coefficients $c_x$ and $c_y$
- Two reference frames will be necessary in the analysis
  - Inertial $E$
  - Mobile $F$, attached to the base slab of the sensor, with unit basis vectors

$$\hat{f}_1, \hat{f}_2, \hat{f}_3$$

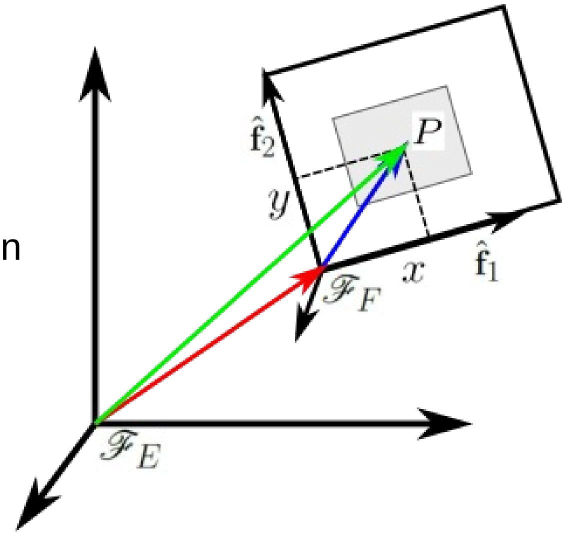


# CVG reference frames setup

- Let us analyse the dynamics of the center of mass *P* in the intertial frame of reference
- Define the vector $r_{O_E P}$ as a sum of two other characteristic vectors:

$$r_{\mathcal{O}_E P} = r_{\mathcal{O}_E \mathcal{O}_F} + r_{\mathcal{O}_F P}$$

- The quantities in ref. frame F can be expressed in such way:




$$\mathbf{r}_{\mathcal{O}_F P} = x\,\hat{\mathbf{f}}_1 + y\,\hat{\mathbf{f}}_2$$

$$\mathbf{v}_{P/F} = \dot{x}\,\hat{\mathbf{f}}_1 + \dot{y}\,\hat{\mathbf{f}}_2$$

$$\mathbf{a}_{P/F} = \ddot{x}\,\hat{\mathbf{f}}_1 + \ddot{y}\,\hat{\mathbf{f}}_2$$

(notation: $a_{P/F}$ – acceleration of point $P$ in frame $F$, etc)

# CVG dynamics

- The rotation around axis $\hat{\mathbf{f}}_3$ which we want to measure can be expressed in the following way:

$$\boldsymbol{\omega}_{F/E} = \Omega\,\hat{\mathbf{f}}_3$$

- The only way to use Newton's laws is to apply them in an inertial reference frame. For mass m:

$$m\,\mathbf{a}_{P/E} = \mathbf{F}_{ext}$$

- Let us derive this acceleration. We must start with velocity

$$\mathbf{v}_{P/E} \triangleq \left(\frac{d\mathbf{r}_{O_E P}}{dt}\right)_E$$

$$\mathbf{a}_{P/E} \triangleq \left(\frac{d\mathbf{v}_{P/E}}{dt}\right)_E$$

# CVG dynamics continued

$$v_{P/E} = \dot{r}_{\mathcal{O}_E\mathcal{O}_F} + \dot{r}_{\mathcal{O}_F P} = v_{F/E} + \frac{d}{dt}\left(|r_{\mathcal{O}_F P}|\hat{r}_{\mathcal{O}_F P}\right)$$

$$= v_{F/E} + \frac{d}{dt}\left(|r_{\mathcal{O}_F P}|\right)\hat{r}_{\mathcal{O}_F P} + |r_{\mathcal{O}_F P}|\frac{d\hat{r}_{\mathcal{O}_F P}}{dt} = v_{F/E} + \underbrace{\hat{f}_1\dot{x} + \hat{f}_2\dot{y}}_{v_{P/F}} + \omega_{F/E}\times r_{\mathcal{O}_F P}$$

$$a_{P/E} = \dot{v}_{P/E} = \dot{v}_{F/E} + \dot{v}_{P/F} + \frac{d}{dt}\left(\omega_{F/E}\times r_{\mathcal{O}_F P}\right)$$

$$= a_{F/E} + \frac{d}{dt}\left(|v_{P/F}|\hat{v}_{P/F}\right) + \frac{d\omega_{F/E}}{dt}\times r_{\mathcal{O}_F P} + \omega_{F/E}\times\frac{dr_{\mathcal{O}_F P}}{dt}$$

$$= a_{F/E} + a_{P/F} + \alpha_{F/E}\times r_{\mathcal{O}_F P} + \omega_{F/E}\times v_{P/F} +$$

$$+ \omega_{F/E}\times\omega_{F/E}\times r_{\mathcal{O}_F P} + \omega_{F/E}\times v_{P/F}$$

with $\alpha_{F/E}$ – the angular acceleration of frame F in frame E

- Collection of terms of the last line yields a term $2\omega_{F/E}\times v_{P/F}$ called the Coriolis acceleration
- (see https://www.youtube.com/watch?v=RBVi-4aftNo and related videos to see details of derivation)

# CVG dynamics continued

- We can assume that the accelerations perceived by our gyroscope in E are not very high:

$$a_{F/E} \approx 0$$

# CVG Coriolis effect

- We have obtained the term which expresses the Coriolis acceleration:

$$2\,\boldsymbol{\omega}_{F/E} \times \mathbf{v}_{P/F}$$

- Substitute the x/y expressions into the terms of the acceleration equation:

$$2\,\boldsymbol{\omega}_{F/E} \times \mathbf{v}_{P/F} = 2\,(\Omega\,\hat{\mathbf{f}}_3) \times (\dot{x}\,\hat{\mathbf{f}}_1 + \dot{y}\,\hat{\mathbf{f}}_2) = 2(\Omega\,\dot{x}\,\hat{\mathbf{f}}_2 - \Omega\,\dot{y}\,\hat{\mathbf{f}}_1)$$

$$\boldsymbol{\omega}_{F/E} \times \boldsymbol{\omega}_{F/E} \times \mathbf{r}_{O_F P} = \Omega^2\,\hat{\mathbf{f}}_3 \times (\hat{\mathbf{f}}_3 \times (x\,\hat{\mathbf{f}}_1 + y\,\hat{\mathbf{f}}_2)) \;= -\Omega^2\,(x\,\hat{\mathbf{f}}_1 + y\,\hat{\mathbf{f}}_2)$$

- Let us look at the external force in more detail now:

$$\mathbf{F}_{ext} = -(k_x x + c_x \dot{x})\hat{\mathbf{f}}_1 - (k_y y + c_y \dot{y})\hat{\mathbf{f}}_2 + F(t)\,\hat{\mathbf{f}}_1$$

# CVG dynamics continued

- After all the substitutions and plugging into Newton's law we obtain:

$$a_{P/E} = \frac{1}{m} F_{ext}$$

- When we collect the terms containing $\hat{\mathbf{f}}_1$ and $\hat{\mathbf{f}}_2$

$$(\ddot{x} - 2\Omega\dot{y} - \dot{\Omega}y - \Omega^2 x)\hat{\mathbf{f}}_1 + (\ddot{y} + 2\Omega\dot{x} + \dot{\Omega}x - \Omega^2 y)\hat{\mathbf{f}}_2$$
$$= -(\frac{k_x}{m}x + \frac{c_x}{m}\dot{x})\hat{\mathbf{f}}_1 - (\frac{k_y}{m}y + \frac{c_y}{m}\dot{y})\hat{\mathbf{f}}_2 + \frac{1}{m}F(t)\,\hat{\mathbf{f}}_1$$

→ forced motion along the $\hat{\mathbf{f}}_1$ axis

$$\ddot{x} - 2\Omega\dot{y} - \dot{\Omega}y - \Omega^2 x + \frac{k_x}{m}x + \frac{c_x}{m}\dot{x} = \frac{1}{m}F(t)$$

→ dynamics of the output axis

$$\ddot{y} + 2\Omega\dot{x} + \dot{\Omega}x - \Omega^2 y + \frac{k_y}{m}y + \frac{c_y}{m}\dot{y} = 0$$

- some cross-terms are present

# CVG dynamics continued

- For the input axis:

$$\ddot{x} - 2\Omega\dot{y} - \dot{\Omega}y - \Omega^2 x + \frac{k_x}{m}x + \frac{c_x}{m}\dot{x} = \frac{1}{m}F(t)$$

- Simplification #2: let us assume that $\Omega$ is slowly varying, thus $\dot{\Omega} \approx 0.$ We obtain:

$$\ddot{x} + (\frac{k_x}{m} - \Omega^2)x + \frac{c_x}{m}\dot{x} - 2\Omega\dot{y} = \frac{1}{m}F(t)$$

- Simplification #3: equal coefficients c and k on both axes:

$$k_x = k_y = k,\ c_x = c_y = c$$

- Let us write the characteristic frequency of this system as $\omega_n$ and the quality factor as $Q$. Observe that

$$\omega_n = \sqrt{\frac{k}{m}} \qquad Q = \frac{\sqrt{km}}{c} \qquad \frac{\omega_n}{Q} = \sqrt{\frac{k}{m}} \times \frac{c}{\sqrt{km}} = \frac{c}{m}$$

# CVG dynamics continued

- We can rewrite the input axis equation as:

$$\ddot{x} + (\omega_n^2 - \Omega^2)x + \frac{\omega_n}{Q}\dot{x} - 2\Omega\dot{y} = \frac{1}{m}F(t)$$

- Simplification #4: suppose that the measured angular rate (~ 1 rad/s) will be greatly inferior to the natural frequency of the system (~ $10^4$ – $10^5$ rad/s)

$$\ddot{x} + \omega_n^2 x + \frac{\omega_n}{Q}\dot{x} - 2\Omega\dot{y} = \frac{1}{m}F(t)$$

- If we “pump” a sinusoidal force at the natural frequency in the x-axis of the form $F(t) = F_D \sin(\omega_n t)$, we can expect a steady oscillation in this direction:

$$x(t) = x_D \sin(\omega_n t + \phi)$$

# CVG - input to output

- What about the output equation?

$$\ddot{y} + 2\Omega\dot{x} + \dot{\Omega}x - \Omega^2 y + \frac{k}{m}y + \frac{c}{m}\dot{y} = 0$$

- Using simplifications #2 and #4 we obtain:

$$\ddot{y} + \frac{\omega_n}{Q}\dot{y} + \omega_n^2 y = -2\Omega\dot{x}$$

- By applying Laplace transform to both sides, we can look at the frequency characteristics of the system:

$$\frac{Y(s)}{X(s)} = \frac{-2\Omega s}{s^2 + \frac{\omega_n}{Q}s + \omega_n^2}$$

- From here, we are close to the result of the input vs the output amplitude:

$$\frac{Y_D}{X_D} = \frac{2\,\Omega\,Q}{\omega_n}$$