Optimization
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Course CO24-300491
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Homework 1
(related to lectures 1-3)

Homework problems are supposed to help you digest the content of the lecture. It is important that you manage to solve them on your own. Before you write your solutions, you may of course ask questions, and discuss things. In order to prepare for the exam, already now, make an effort to explicitly write down your solutions – clearly and easy to read. Apply definitions properly, and give explanations for what you are doing. That will help you to understand them later when you prepare for the final exam.

Task 1 Expand a Function *(lecture 2)*

Consider the function $f(x) = (x_1 + 3x_2)^4$.

a) Find an approximation close to the point $(x_1^* = 1, x_2^* = 2)$ in terms of $\Delta x_1 = x_1 - x_1^*$ and $\Delta x_2 = x_2 - x_2^*$. Write terms up to and including 2nd order, that is, expressions like $(\Delta x_1)^2$, $(\Delta x_2)^2$, $(\Delta x_1) \cdot (\Delta x_2)$.

b) Now, find the Hessian matrix at $(x_1^* = 1, x_2^* = 2)$, that is, write all the second partial derivatives of f at that point in a 2x2 matrix form:

$$H = \left.\begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{pmatrix}\right|_{x_1^*=1, x_2^*=2}$$

, and compare the second order part of your expression from part a) with $\frac{1}{2}(\Delta x_1 \quad \Delta x_2) H \begin{pmatrix} \Delta x_1 \\ \Delta x_2 \end{pmatrix}$.

c) Similarly, compare the first order terms with $(\nabla(f))^T \cdot \begin{pmatrix} \Delta x_1 \\ \Delta x_2 \end{pmatrix}$

where $\nabla(f) = \left.\begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{pmatrix}\right|_{x_1^*=1, x_2^*=2}$

Task 2 Simple Two-Dimensional Problem: 2nd Order Criteria *(lecture 3)*

Consider the function $f(x, y) = x^4 + y^4 - 2x^2 - 2y^2$ for $(x, y) \in \mathbb{R}^2$.

a) (15) Use the gradient of $f(x, y)$ in order to find all its critical (also called stationary) points.

b) (15) Use the respective Hessian matrices $H = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$ at those critical points to identify local maxima / minima / saddle points. In order to do so, use the following criteria: H positive definite ⇒ min, H negative definite ⇒ max, H indefinite ⇒ saddle.

Use Sylvester's criterion to check for positive definiteness. Also, a matrix M is negative definite iff its negative (–M) is positive definite. It is indefinite if there are vectors x,y such that $x^T Mx < 0$, and $y^T My > 0$.

Task 3 Positive Definite Matrices *(lecture 3)*

A matrix M is positive definite iff for all vectors $x \neq 0$, $x^T Mx > 0$.
Use Sylvester's criterion to check for positive definiteness, that is, show that all the leading principal minors are strictly positive. In order to demonstrate that a matrix is NOT positive definite, you can simply write a vector $x \neq 0$ with $x^T Mx \leq 0$ – if you happen to "see" one.

$$M_1 = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 3 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 1 & -1 & 5 \\ -1 & 20 & 3 \\ 5 & 3 & -3 \end{pmatrix}$$