Lectures 3 and 4 (9/10 and 9/12) Outline

- A quick remark
- Logic puzzles and satisfiability (from Lecture 2)
- Quantification logic (Rosen 1.4, 1.5)
  - Quantifiers
  - Equivalences involving quantifiers
- Rules of inference (Rosen 1.6)

Quick remark

- Practice implications!
- Practice logical equivalencies!
- Produce table truths for them (good exercise).

Implication and its contrapositive:

1. Statement: if xy is even, then x is even or y is even.

2. Statement: If x is odd and y is odd, then xy is odd.

Quick remark

- Let P is "xy is even", Q is "x is even", and R is "y is even".
- Logical form of statement 1: $P \rightarrow (Q \wedge R)$
- Contrapositive:

$$\neg(Q \wedge R) \rightarrow \neg P \equiv \neg Q \wedge \neg R \rightarrow \neg P.$$

- This is precisely statement 2!

Logical puzzle and satisfiability

Puzzle: As a reward for saving his daughter from pirates, the King has given you the opportunity to win a treasure hidden inside one of three trunks. The two trunks that do not hold the treasure are empty. To win, you must select the correct trunk.

- Trunks 1 and 2 are each inscribed with the message "This trunk is empty"
- Trunk 3 is inscribed with the message "The treasure is in Trunk 2."

The Queen, who never lies, tells you that only one of these inscriptions is true, while the other two are wrong. Which trunk should you select to win?

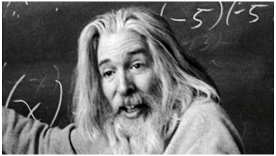
Logical puzzle and satisfiability

- p_i=treasure is in trank i, $i = 1, 2, 3$
- The inscriptions of the three trunks are respectively $\neg p_1, \neg p_2, p_2$
- According to the Queen

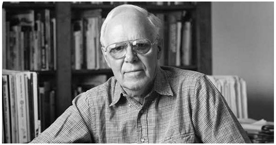
$$\underbrace{(\neg p_1 \wedge \neg(\neg p_2) \wedge \neg p_2)}_{\text{inscription 1 is the only true one}} \vee \underbrace{(\neg(\neg p_1) \wedge \neg p_2 \wedge \neg p_2)}_{\text{inscription 2 is the only true one}} \vee$$
$$\underbrace{(\neg(\neg p_1) \wedge \neg(\neg p_2) \wedge p_2)}_{\text{inscription 3 is the only true one}}.$$

  Notice that only one of the terms of the disjunction can be true.
- By using equivalence laws, this is logically equivalent to p_1
- Hence, the treasure is in trunk 1, and the inscription of trunk 2 is the only true one.

For those interested in puzzles..



Raymond Smullyan



Martin Gardner

- Many great books on puzzles for those who have fun with puzzles
- Raymond Smullyan and Martin Gardner are two important figures with many cool books

Quantificational/First order logic/Predicate logic

- Propositional logic has its limitations. It deals with simple declarative propositions.

- First order logic (aka predicate or quantificational logic) covers predicates and quantification.

  1. What is a predicate?

  2. What is quantification?

Predicate

- $P(x) = \underbrace{x}_{\text{variable}} \underbrace{\text{is greater than 3}}_{\text{predicate}}$
- $P(x)$ is a propositional function of x.
- Depending on x, the truth value of $P(x)$ may change. For example:
  - $P(4)$ is true, since $4 > 3$
  - but $P(2)$ is false
- *Predicate* is a property that the subject of a statement may have.

Predicate

- A predicate may be a function of two or more variables.

- E.g., $Q(x, y) =$ "$x = y + 3$"
  - $Q(2, 2)$ is false since $2 \neq 2 + 3$
  - $Q(3, 0)$ is true since $3 = 0 + 3$.

- $P(x_1, \ldots, x_n)$ is a propositional function of n variables.

Universal quantifier $\forall$

- Goal: Say that $P(x)$ is true for every value of x in the universe of discourse
- "For all $x, P(x)$"
- We write $\forall x P(x)$.
- $\forall$ is called the universal quantifier since it states that $P(x)$ is universally true.

Universal quantifier $\forall$

Let's see some examples. Let's translate them in English and decide if they are true or false.

1. $\forall x(x^2 \geq 0)$ where the universe of discourse is the set of all real numbers

2. Let $M(x)$ be "x is a man", and $B(x)$ "x has blue eyes". Let the universe of discourse be the set of all humans, i.e., domain of x.

$$\forall x(M(x) \to B(x)).$$

Universal quantifier $\forall$

Let's see some examples. Let's translate them in English and decide if they are true or false.

1. $\forall x(x^2 \geq 0)$ where the universe of discourse is the set of all real numbers
   This means that for every real number x, x^2 is non-negative. This is true.
2. Let $M(x)$ be "x is a man", and $B(x)$ "x has blue eyes".

$$\forall x(M(x) \rightarrow B(x)).$$

   For every human x, if x is a man, then x has blue eyes. But $M(Prof.\ Tsourakakis)$ =true, $B(Prof.\ Tsourakakis)$=false. This immediately proves that not all men have blue eyes. (Counterexample)

Remark

- A statement $\forall x P(x)$ is false when there exists an x in the domain for which $P(x)$ is false
- The domain is **very** important. Actually without being precise about the domain, asking the truth value of a universal statement does not make sense
- **Example**: $\forall x(x^2 \geq x)$
    - if the domain are reals, then it is false.
    - If the domain is the set of integers, then it is true.
- If the domain is finite (e.g., $\{val_1, \ldots, val_k\}$ for finite k) then $\forall x P(x)$ is equivalent to $P(val_1) \wedge \ldots \wedge P(val_k)$.

Existential quantifier $\exists$

- To write $\exists x P(x)$ means that there is at least one value of x in the universe for which $P(x)$ is true.

- $\exists$ is the existential quantifier

- Examples
  - $\exists (M(x) \wedge B(x))$: there exists a blue-eyed man
  - $\exists x(x^2 - 2x + 5 = 0)$ with domain the real numbers

- If the domain is finite (e.g., $\{val_1, \ldots, val_k\}$ for finite k) then $\exists x P(x)$ is equivalent to $P(val_1) \vee \ldots \vee P(val_k)$.

Important remarks

- **Precedence** $\forall, \exists$: The quantifiers $\forall, \exists$ have higher precedence than all logical operators from propositional calculus
- E.g., $\forall x P(x) \vee Q(x)$ means $(\forall x P(x)) \vee Q(x)$.
  Contrast this to $\forall x (P(x) \vee Q(x))$

- **Definition:** When a quantifier is used on the variable x, we say that this occurrence of the variable is **bound**. An occurrence of a variable that is not bound by a quantifier or set equal to a particular value is said to be **free**.
- **Examples**: Identify the free, and bound variables.
  - $\exists x(x + y = 1)$
  - Let $L(x, y)$ be "x likes y", and the domain be the set of people. The statement is $\forall x L(x, y)$.

Logical equivalence involving quantifiers

Statements involving predicates and quantifiers are *logically equivalent* if and only if they have the same truth value no matter which predicates are substituted into these statements and which domain of discourse is used for the variables in these propositional functions. We use the notation $S \equiv T$ to indicate that two statements S and T involving predicates and quantifiers are logically equivalent.

You need to be very careful about distributing quantifiers over conjunctions and disjunctions:

1. True or false: $\forall x(P(x) \wedge Q(x)) \equiv \forall x P(x) \wedge \forall x Q(x)$?
2. True or false $\forall x(P(x) \vee Q(x)) \equiv \forall x P(x) \vee \forall x Q(x)$?
3. True or false: $\exists x(P(x) \wedge Q(x)) \equiv \exists x P(x) \wedge \exists x Q(x)$?
4. True or false $\exists x(P(x) \vee Q(x)) \equiv \exists x P(x) \vee \exists x Q(x)$?

Logical equivalence involving quantifiers

1. $\forall x(P(x) \wedge Q(x)) \equiv \forall x P(x) \wedge \forall x Q(x)$ is **true**

   To show that these statements are logically equivalent, we must show that they always take the same truth value, no matter what the predicates are, and no matter which domain of discourse is used.

   We prove that if $\forall x(P(x) \wedge Q(x))$ is true, then $\forall x P(x) \wedge \forall x Q(x)$ is true, and **vice versa**.

   Details on blackboard (see Rosen p.49)

Logical equivalence involving quantifiers

2. $\forall x(P(x) \vee Q(x)) \equiv \forall xP(x) \vee \forall xQ(x)$ is **false**

Let's prove it by counterexample that illustrates that it is false.

- Let domain be the set of integers
- Let $P(x) = x$ is even
- Let $Q(x) = x$ is odd
- $\forall x(P(x) \vee Q(x))$ is true (every integer is either odd or even)
- $\forall xP(x) \vee \forall xQ(x)$ is false. (since $\forall xP(x)$ is false as not all integers are even, and $\forall xQ(x)$ is also false since not all integers are odd, hence disjunction is false)

Logical equivalence involving quantifiers

3 : $\exists x(P(x) \land Q(x)) \equiv \exists x P(x) \land \exists x Q(x)$? is **false**

Let's prove it by counterexample that illustrates that it is false.

- Let domain be the set of integers
- Let $P(x) = x$ is even
- Let $Q(x) = x$ is odd
- $\exists x(P(x) \land Q(x)$ is false (no integer can be both and even)
- $\exists x P(x) \land \exists x Q(x)$ is true. (2 is even, hence $\exists x P(x)$ is true, and 3 is odd, hence $\exists x Q(s)$ is also true. Therefore the conjunction is true.)

Logical equivalence involving quantifiers

4 : $\exists x(P(x) \vee Q(x)) \equiv \exists x P(x) \vee \exists x Q(x)$ is **true**

Homework/lab problem.

Logical equivalence involving quantifiers – Negation of quantified expressions

Let's first see some sentences in English and their negation. Let's negate the following statements:

- Everybody is perfect (universal quantifier)

- Every student in CS131 has taken a course in calculus (universal quantifier)

$$\neg \forall P(x) \equiv \exists x \neg P(x)$$

Logical equivalence involving quantifiers – Negation of quantified expressions

Let's again see some sentences in English and their negation. Let's negate the following statement:

- There does not exist a student in CS131 who meets the calculus prerequisites (existential quantifier)

$$\neg \exists P(x) \equiv \forall x \neg P(x)$$

DeMorgan's laws for quantifiers

TABLE 2 De Morgan's Laws for Quantifiers.

Negation	*Equivalent Statement*	*When Is Negation True?*	*When False?*
$\neg \exists x P(x)$	$\forall x \neg P(x)$	For every x, $P(x)$ is false.	There is an x for which $P(x)$ is true.
$\neg \forall x P(x)$	$\exists x \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.

DeMorgan's laws for quantifiers – Examples

What are the negations of the statements ?

1. There is an honest politician
2. $\forall x(x^2 > x)$
3. $\exists x(x^2 = 2)$

DeMorgan's laws for quantifiers – Examples

What are the negations of the statements? We apply DeMorgan's laws everywhere!

1. Let $H(x)$ denote x is a mortal human being. The statement is $\exists x H(x)$. Its negation is:
   $\neg \exists x H(x) \equiv \forall x \neg H(x)$. In English, this simply means that every human being is immortal
2. The negation of $\forall x(x^2 > x)$ is as follows:
   $\neg \forall x(x^2 > x) \equiv \exists x \neg (x^2 > x) \equiv \exists x(x^2 \leq x)$.
3. The negation of $\exists x(x^2 = 2)$ is:
   $\neg \exists x(x^2 = 2) \equiv \forall x \neg (x^2 = 2) \equiv \forall x(x^2 \neq 2)$

Practice

Analyze the logical forms for the following statements

1. Someone did not do the homework.
2. Nobody is perfect
3. Everything in that store is either overpriced or poorly made
4. Everybody in the dorm has a roomate he does not like.
5. All married couples have fights
6. John likes (*exactly*) one person.

Ideas?

Practice (solutions)

1. Someone did not do the homework.

   Let $H(x)$ stand for "x did the homework". Then we can rewrite the statement as $\exists x \neg H(x)$. Equivalently, $\neg \forall x H(x)$.

2. Nobody is perfect

   Let $P(x)$ stand for "x is perfect". The statement is $\neg \exists x P(x)$ or equivalently $\forall x \neg P(x)$.

Practice (solutions)

3. Everything in that store is either overpriced or poorly made

   We can rephrase this as "If something is in that store, then it is either overpriced or poorly made".

   Let $S(x)$ stand for "x is in that store"

   $O(x)$ stand for "x is overpriced"

   $P(x)$ stand for "x is poorly made".

$$\forall x(S(x) \to (O(x) \vee P(x)).$$

Practice (solutions)

4. Everybody in the dorm has a roomate he does not like.

   We wish to say $\forall x$ (if x lives in the dorm, then x has a roomate he does not like)

   Let $D(x)$ stand for x lives in the dorm

   Let $L(x, y)$ stand for x likes y

   Let $R(x, y)$ stand for x, y are roomates

$$\forall x(D(x) \rightarrow \exists y(R(x, y) \wedge \neg L(x, y))).$$

Practice (solutions)

5. All married couples have fights.

- We need to define predicates that take two arguments x, y.
- Let $M(x, y)$ mean "x and y are married to each other".
- Let $F(x, y)$ mean "x and y fight with each other"
- Then we can write

$$\forall x \forall y (M(x, y) \rightarrow F(x, y)).$$

Practice (solutions)

6. Remark: John likes one person is not the same as John likes exactly one person

   Let $L(x, y)$ mean x likes y

7. John likes one person means $\exists x L(John, x)$.

8. John likes exactly one person means that person is unique and this is expressed as

$$\exists x(L(j, x) \wedge \neg \exists y(L(John, y) \wedge y \neq x)).$$

9. We introduce a new quantifier to denote uniqueness of x

$$\exists ! x P(x)$$

Nesting quantifiers

As we have seen already, we can use more than one quantifiers in one statement, or we can even *nest* them. Nesting order of the quantifiers **matters**!

Exercise: Decide if the following statements are true or false. We assume the domain is the set of natural numbers.

1. $\forall x \exists y (x < y)$
2. $\exists y \forall x (x < y)$
3. $\exists x \forall y (x < y)$
4. $\forall y \exists x (x < y)$
5. $\exists x \exists y (x < y)$
6. $\forall x \forall y (x < y)$

Nesting quantifiers

1. $\forall x \exists y (x < y)$
   True. Set $y = x + 1$
2. $\exists y \forall x (x < y)$
   False. No matter what y we pick, we can always find an x that is larger.
3. $\exists x \forall y (x < y)$
   False. (why not $x = 0$?)
4. $\forall y \exists x (x < y)$
   False (for $y = 0$ no such x exists)
5. $\exists x \exists y (x < y)$
   True.
6. $\forall x \forall y (x < y)$
   False. There is not even one value for x for which $\forall y (x < y)$

Negating nesting quantifiers

Exercise: Let's express the following negation so that no negation precedes a quantifier.

1. $\forall x \exists y (xy = 1)$

   Rule: We apply successively the rules for negating statements involving a single quantifier.

2. Therefore,

$$\neg \forall x \exists y (xy = 1) \equiv \exists x \neg \exists y (xy = 1) \equiv \exists x \forall y \neg (xy = 1) \equiv \exists x \forall y (xy \neq 1).$$

Rules of inference

Some important tautologies give rise to some important rules of inference.

TABLE 1 Rules of Inference.

Rule of Inference	*Tautology*	*Name*
p $p \rightarrow q$ $\therefore q$	$(p \land (p \rightarrow q)) \rightarrow q$	Modus ponens
$\neg q$ $p \rightarrow q$ $\therefore \neg p$	$(\neg q \land (p \rightarrow q)) \rightarrow \neg p$	Modus tollens
$p \rightarrow q$ $q \rightarrow r$ $\therefore p \rightarrow r$	$((p \rightarrow q) \land (q \rightarrow r)) \rightarrow (p \rightarrow r)$	Hypothetical syllogism
$p \lor q$ $\neg p$ $\therefore q$	$((p \lor q) \land \neg p) \rightarrow q$	Disjunctive syllogism
p $\therefore p \lor q$	$p \rightarrow (p \lor q)$	Addition
$p \land q$ $\therefore p$	$(p \land q) \rightarrow p$	Simplification
p q $\therefore p \land q$	$((p) \land (q)) \rightarrow (p \land q)$	Conjunction
$p \lor q$ $\neg p \lor r$ $\therefore q \lor r$	$((p \lor q) \land (\neg p \lor r)) \rightarrow (q \lor r)$	Resolution

Rules of inference – Remark

Resolution.

- The resolution rule of inference relies on the tautology

$$((p \lor q) \land (\neg p \lor r)) \to (q \lor r).$$

- Plays an important role in programming languages based on logic, e.g., PROLOG
- **Example**: Let's apply the resolution rule on the hypotheses Jasmine is skiing or it is not snowing and It is snowing or Bart is playing hockey

Rules of inference – Remark

Example:

- Let p be the proposition "it is snowing"
- Let q be the propositon "Jasmine is skiing"
- Let r be the proposition "Bart is playing hockey".
- Our hypothesis is $((p \vee q) \wedge (\neg p \vee r))$
- Therefore, by the resolution rule we conclude $(q \vee r)$ which means

  Jasmine is skiing or Bart is playing hockey