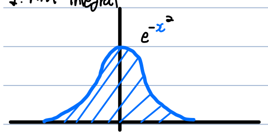
How is area under Bell curve a $\sqrt{\pi}$

Eugene wigner.

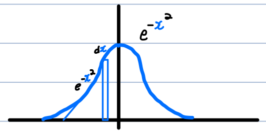
Step 1 why area under e^{-x^2} is $\sqrt{\pi}$

1. First integral



$\int_{-\infty}^{\infty} e^{-x^2} dx$

2.



3.

integral

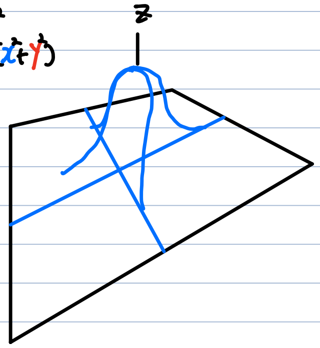
$$\int e^{-x^2} dx = \boxed{\text{impossible}}$$

differentiate

4 Get the volume (increase the dimension)

$f_1(x) = e^{-x^2}$

$f_2(x,y) = e^{-(x^2+y^2)}$

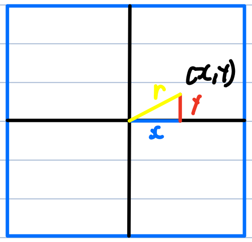


5. get the Adjacent (피타고라스 정리)

$f_1(x) = e^{-x^2}$

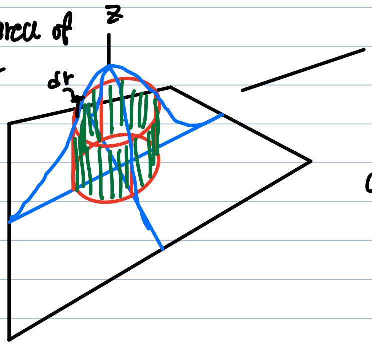
$f_2(x,y) = e^{-(x^2+y^2)} = e^{-r^2}$

$x^2+y^2 = r^2$



Rotate!

6. get the area of one cylinder



$$\text{area} = \underbrace{2\pi r}_{\text{원의둘레}} \cdot e^{-r^2} \quad \times \underbrace{(dr)}_{\text{두께}}$$

7) Add up all the cylinder

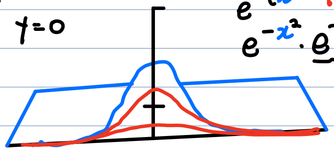
$$\int_0^\infty 2\pi r \cdot e^{-r^2} dr$$

$$= \pi \int_0^\infty \underbrace{2r \cdot e^{-r^2}}_{\substack{-e^{-r^2} \\ \text{integral}}} dr = \pi \left[\underbrace{-e^{-\infty^2}}_{0} - \underbrace{(-e^{-0^2})}_{-1} \right] = \pi$$

∴ 3차원 원형 bell curve의 부피는 π

8.

$y=0$



$e^{-(x^2+y^2)}$

$e^{-x^2} \cdot \underbrace{e^{-y^2}}_{\text{just constant 가 된다.}} = 1 \times e^{-x^2}$

change y ↗ 모양은 쭐쭉한 퍼짐정도가 달라진다.
since C is just an constant

$$\int_{-\infty}^{\infty} C \cdot e^{-y^2} dy \quad = C \int_{-\infty}^{\infty} e^{-y^2} dy = C^2$$

$$\boxed{C = \int_{-\infty}^{\infty} e^{-x^2} dx}$$

∴ 넓이 $= C$ 부피 $= C^2$ 부피 $= \pi$

$$\int_{-\infty}^{\infty} e^{-x^2} dx = C \qquad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-y^2} e^{-x^2} dx dy = C^2 \qquad \int_0^{\infty} 2\pi r \cdot e^{-r} dr = \pi$$

∴ $C = \sqrt{\pi}$ $\qquad C^2 = \pi \quad \longleftarrow$

step2 why e^{-x^2} is importanc

성질 1

확률(밀도)은 오직 원점으로 부터의
거리에만 종속이다.

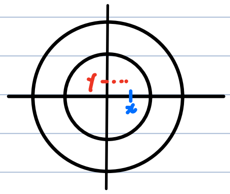
성질 2

x와 y 좌표는 서로 독립이다.

$f_2(x, y) = \underline{g(x)g(y)}$

서로비례 분포가 같으므로

$= f(r)$



$n = p/q$

$h(p/q) = h(p)/h(q)$

$h(\sqrt{p/q} =$ $\qquad h(p) = p^n$

$\sqrt{\frac{q}{p}} \quad \sqrt{q}/\sqrt{p} \qquad h(q) = q^n \quad = \left(\frac{p}{q}\right)^n$

Helper function $h(x) = f(\sqrt{x})$, $h(x^2) = f(x)$

$h(x^2+y^2) = h(x^2)h(y^2)$

$\Downarrow$

$h(x_1+x_2+\cdots+x_n) = h(x_1)h(x_2)\cdots h(x_n)$

$\Downarrow$ $h(x) = b^x$ $\left(h(p/q) = b^{p/q} \right)$

$h(x) = e^{\alpha}$ $\quad$ $f(x) = e^{cx^2}$

C is always negative

area of circle

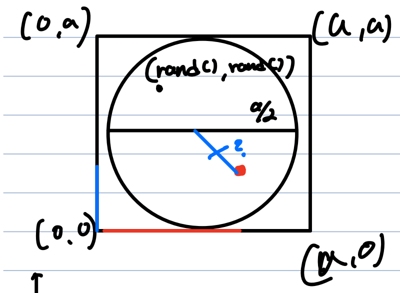
$$\frac{\left(\frac{a^2}{4}\pi\right)}{a^2} = \frac{\pi}{4}$$

$\left(\frac{a}{2}\right)^2$

square

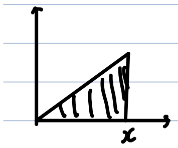
Lecture 11 30

$\pi = 3.141592$



area of circle πr^2

$$\frac{\pi\left(\frac{a}{2}\right)^2}{a^2}$$



Indicator Random variable

Random variable indicate a fact

if a Random point in the square is inside or outside of a circle

compute te distance fom te center point of circle and if It is smaller than radius then inside

$\sqrt{(x-0.5)^2+(y-0.5)^2}$

given (x,y) if $\sqrt{(x-\frac{1}{2})^2+(y-\frac{1}{2})^2} \leq \frac{1}{2}$ ten p is inside of circle

---

Generate 4 points $P_1, \dots, P_n = (x_n, y_n)$

$$X_i = \begin{cases} 1 \text{ if } P_i \text{ is inside} \\ 0 \text{ if } P_i \text{ is outside} \\ (o/w) \end{cases}$$

with this dataframe.

$X_i \sim Ber(P)$ $p = \frac{\pi}{4}$ $\quad$ iid independent, and identically distributed.

$X_1, \dots, X_n$ iid

$E(X_i) = 1 \cdot \frac{\pi}{4} + 0\left(1-\frac{\pi}{4}\right) = \frac{\pi}{4}$

→ # of point inside C_1

$S_n = X_1 + \cdots + X_n \sim \mathrm{Bin}(n, p)$

$Pr(S_n = k) = \binom{n}{k}\left(\frac{\pi}{4}\right)^k\left(1-\frac{\pi}{4}\right)^{n-k}$

Expectation of $S_n = np = n\frac{\pi}{4}$

$E(S_n) = E\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} E(x) = np$

$E(S_n) = n\frac{\pi}{4} \Rightarrow \pi = \frac{4E(S_n)}{n}$

→ code

4 x mean (df[1, inside]) = $4 \times \frac{E(S_n)}{\pi}$

Linearilty of Expectation

$E\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} E(x)$

$var(x_1 + x_2) = var(x_1) + (x_2)$

$Var(x_1) = E\left((\underbrace{x - E(x)}_{z})^2\right) = E(x^2 + \mu^2 - 2\mu x) = E(x^2) + E(\mu^2) - E(2\mu x)$

$= E(x^2) + \mu^2 - 2\mu \underbrace{E(x)}_{\mu} = E(x^2) + \mu^2 - 2\mu^2 = E(x^2) - \mu^2$

$var(2x) = E((2x)^2) - (E(2x))^2$

$= E(4x^2) - (2E(x))^2$

$= 4E(x^2) - 4E(x)^2$

$= 4\underbrace{(E(x^2) - E(x)^2)}_{variance}$

$X \sim Ber(p)$

$E(x) = p$

$Z = x - p = \begin{cases} 1-p & p = p \\ -p & p = 1-p \end{cases}$ $E(Z) = 0$ $((1-p)p + (-p)(1-p) = 0$

$E(Z^2) = (1-p)^2 p + (-p)^2(1-p) = p(1-p)$

$= p \cancel{-2p^2} + \cancel{p^3} + p^2 - \cancel{p^3} = p - p^2 = p(1-p)$

$-2p + p^2$

Simple exercise

$x = \begin{cases} 1 & \frac{1}{2} \\ 2 & \frac{1}{4} \\ 10 & \frac{1}{4} \end{cases}$ $E[x] = 1 \times \frac{1}{2} + 2 \times \frac{1}{4} + 10 \times \frac{1}{4} \approx 3.5$

$E[x^2] = 1 \times \frac{1}{2} + 2^2 \times \frac{1}{4} + 10^2 \times \frac{1}{4}$

$$\boxed{Var\ x = E(x^2) - (E(x))^2}$$

$Z = x - E[x] = \begin{cases} -2.5 & \frac{1}{2} \\ -1.5 & \frac{1}{4} \\ +6.5 & \end{cases}$ P

$\frac{}{4}$

$var(x) = E(Z^2) = (-2.5)^2\times\frac{1}{2} + (-1.5)^2\times\frac{1}{4} + (6.5)^2\times\frac{1}{4}$

$= \frac{25}{4}\times\frac{1}{2} + \frac{9}{4}\times\frac{1}{4} + \frac{169}{4}\times\frac{1}{4}$

moment of Z: central moment

Exercise 1

7
182
147
25

$\frac{7}{2}$ $\frac{49}{4}$ 2

147

a. $E[x] = 3.5$

b. $var(x) = E(x^2) - (E[x])^2$

$= 1\times\frac{1}{6} + 4\times\frac{1}{6} + 9\times\frac{1}{6} + 16\times\frac{1}{6} + 25\times\frac{1}{6} + 36\times\frac{1}{6}$

$= \frac{91}{6}$ 10 20 61 12

$= \frac{91}{6} - (3.5)^2$

1 − 3.5

2 − 3.5

3 − 3.5

4 − 3.5

5 − 3.5

6 − 3.5

$\frac{1}{6}$

$(-2.5)^2\times\frac{1}{6} + (-1.5)^2\times\frac{1}{6} + (-0.5)^2\times\frac{1}{6} \times (0.5)^2\times\frac{1}{6} + (1.5)^2\frac{1}{6} + (2.5)^2\frac{1}{2}$

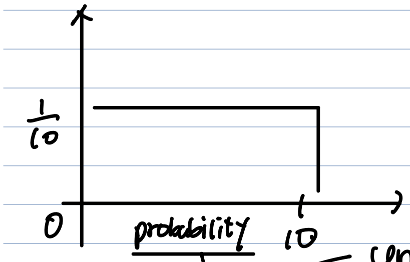
$= \frac{}{3}$

$\frac{3}{2}$ $\frac{9}{4}$ $\frac{1}{2}$ $\frac{1}{4}$

$\frac{5}{2}$ $\frac{25}{4} +$ $\frac{35}{4}\times\frac{1}{3} = \boxed{\frac{35}{12}}$

Exercise 2

Exercise 2



$\int_0^{10} x\frac{1}{10}\,dx$ uniform = constant

Not a function

$= \frac{1}{10}\int_0^{10} x\,dx$

$= \frac{1}{2}x^2$ $\frac{1}{2}(10)^2 - \frac{1}{2}(0)^2$

$\boxed{5}$ 50 100 $= 5$

$E[x^2] = \int_0^{10} x^2\frac{1}{10}\,dx$

$$= \frac{1}{10}\int_0^{10} \frac{1}{3}x^3 \cancel{dx} = \frac{1}{10}\left(\frac{1}{3}(10^3) - \frac{1}{3}(0)'\right.$$

$$\frac{100\cancel{0}}{3} \times \frac{1}{1\cancel{0}}$$

$$\frac{100}{3} - 25$$

$(x-5)^2 \frac{1}{10}$ $\frac{75}{3}$ $\boxed{\frac{25}{3}}$ — variance.

$\frac{1}{10}$ $\frac{x^2-10x+25}{10}$

Conditional probability

X $E(x) \to +\infty$

$x=0$ $p \to 1$

$$X = \begin{cases} n^2 & \frac{1}{n} \\ 0 & 1-\frac{1}{n} \end{cases}$$

$E[x] = n^2 \times \frac{1}{n} = \underline{n}$

$\underset{\sim}{n} \to \underline{\infty}$ then $1-\frac{1}{n} \approx 1$

$\frac{1}{n} \approx 0$

$E(x^2) = (n^2)^2 \times \frac{1}{n} = n^3$

variance $= n^3 - n^2$

$P(E_1 \cap E_2) = P(E_2|E_1) \times P(E_1)$

$P(E_1 \cap E_2|E_3) = P(E_1|E_3)\,P(E_2|E_3)$

$E_1' = E_1|E_3$

$E_2' = E_2|E_3$

$P[E_1 \cap E_2|E_3] = P(E_1|E_2, E_3] \cdot P[E_2|E_3)$

lecture 21(

$Pr(A|B) = Pr(A)$

$A \cap B_i$

Chain Rule

mutual independent

$$\prod_{i=1}^{n} Pr(A_i)$$

pairwise independent

$$Pr(A_i \cap A_j) = Pr(A_i) \times Pr(A_j)$$

mutually independent all events are independent

pairwise independence

xor

X	Y	Z
0	0	0
0	1	1
1	0	1
1	1	0

$Pr(X=x, Z=x) = P(X=x)\, Pr(Z=x)$

$Pr(Z=0) = P(X=0, Y=0) + P(X=1, Y=1) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

Try ae independence

However if you give condition, te probability change

ex) $Pr(X=0, Y=0, Z=1) = 0$ It shoulde not be $\frac{1}{8}$ so it is not mutual independence

Bayes Theorem

discret RVs. $Pr(A|B) = Pr(A|B) = \frac{Pr(B|A) Pr(A)}{Pr(B)}$, if $Pr(B) > 0$

Continuous RVs. $f_{X|Y=y}(x) = \frac{f_{Y|X=x}(y) f_X(x)}{f_Y(y)}$

Kahneman - Tversky Taxi Accident

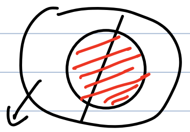
$Pr(G) = \frac{85}{100}$ $Pr(B) = \frac{15}{100}$

$Pr(r) = \frac{80}{100}$ $Pr(w) = \frac{20}{100}$

$$Pr(A=B / w=w) = \frac{Pr(A=B \cap W=b)}{Pr(W)} = \frac{Pr(W=b|A=B) Pr(B)}{Pr(w)}$$

$$\frac{Pr(W=b|B)\, Pr(B)}{Pr(W=b)}$$

$$Pr(W=b|B) Pr(B) + Pr(W=b|G) Pr(G)$$
$$= Pr(W=b)$$



Bayes' rule

inference likehood prior

$$Pr(H|D) = \frac{\underline{Pr(D|H)}\,\underline{Pr(H)}}{Pr(D)}, \text{ and } Pr(D) > 0$$

Hypothesis Data

final Resut

posterior $\propto$ likelihood $\times$ prior

$$PE\{\underbrace{0.1}_{H_1}, \underbrace{0.5}_{H_2}, \underbrace{0.9}_{H_3}\}$$

H : coin toss $|pr(H) = \frac{1}{3}|$

$(55H, 45T)$

$Pr(H_1|D|$ $Pr(H_2|D)$ $Pr(H_3|D)$

$$Pr(D) = Pr(55H, 45T) = Pr(D, H_1) + Pr(D, H_2) + Pr(D, H_3)$$

$$= Pr(D, H_1) = Pr(D|H_1)\,Pr(H_1)$$

$$= \binom{100}{55}(0.1)^{55}(0.9)^{45} \cdot \frac{1}{3}$$

$$Pr(A_1) + Pr(A_2) + Pr(A_3) - Pr(A_1 \cap A_2) - Pr(A_2 \cap A_3) - Pr(A_3 \cap A_1) + Pr(A_1 \cap A_2 \cap A_3)$$

Union bound

$$Pr(F_1 \cup F_2) \le Pr(F_1) + Pr(F_2)$$

$$= P_1 + P_2 - P_1 \cap P_2 \quad = P_1 + P_2$$

Hats off

$$X_i = \begin{cases} 1 & \text{if the nth man gets his own hat} \\ 0 & \text{o/w} \end{cases}$$

Not mutually independent.

$$S = x_1 + x_2 + \cdots + X_n$$

$$E(S) = E\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} E(X_i)$$

$$= \sum_{i=1}^{n} \frac{1}{n} = \frac{n}{n} = 1$$