CS 131 - Fall 2019, Assignment 9

problem 1.

a) $\varphi(7^n)$

$\varphi(7^n) = 7^n - 7^{n-1}$

$7^{n-1}(7-1)$

$= 6 \cdot 7^{n-1}$

b) $\varphi(5^n 2^{n+3})$

$= \varphi(5^n) \cdot \varphi(2^{n+3})$

$= (5^n - 5^{n-1}) \times (2^{n+3} - 2^{n+2})$

$5^{n-1}(5-1) \times 2^{n+2}(2-1)$

$= 4 \times 2^{n+2} \times 5^{n-1}$

$= 2^{n+4} \times 5^{n-1}$

problem 2

prove the following Statement:

a) For $n \in Z^+$, $3 \mid n^3 + 2n$

prove by induction $p := 3 \mid n^3 + 2n$ for $n \in Z^+$

base case when $n = 1$ $n^3 + 2n = 1 + 2 = 3$

It is divisible by 3 → base case proved.

Inductive step:

we assume that $p(n)$ is true, then we prove that $p(n+1)$ is true.

$p(n) = 3 \mid n^3 + 2n$ is true. For $p(n+1) = (n+1)^3 + 2(n+1)$

$= n^3 + 3n^2 + 3n + 1 + 2n + 2$

$= (n^3 + 2n) + (3n^2 + 3n + 3)$

$3 \mid n^3 + 2n$, $3 \mid 3n^2 + 3n + 3$ so $3 \mid (n+1)^3 + 2(n+1)$

Inductive step proved.

prove that

(b) $n \in \mathbb{Z}^+$, $n^3-n$ is divisible by 6

$n^3-n = n(n^2-1) = n(n+1)(n-1)$, so we figure $n^3-n$ is multiple of three consecutive integer. In three consecutive integers, one of three possible integers are disible by 3 since there are three consecutive numbers, and we know there exists at least one even number, so $n^3-n$ is divisible by 6.

Problem 3 Non-negative integers.

a) $(x^2-y^2) = (x+y)(x-y) = 221$, so divisiors of 221 is $(1,221), (13,17)$, when $x+y=221$, $x-y=1$ and when $x+y=17$ and $x-y=13$ it works. when $x=111$, $y=110$ $x+y=221$, $x-y=1$ and when $x=15$, $y=2$ $x+y=17$ and $x-y=13$ So there are two answers

$(111,110)$ or $(15,2)$

b) $a+b=ab$

$ab-a-b=0$, $a(b-1)-b=0$ $a(b-1)-b+1=+1$

$(a-1)(b-1)=1$. So when $a=2$ and $b=2$

It is the answer of the equation integer

and $(0,0)$ for $(a,b)$ since $a, b$ are non-negative $(2,2)$ and $(0,0)$

c) gcd(a,b) lcm(a,b) = b+9

From the number theory, we know that when gcd of two number multiple lcm of those two numbers, the answer is multiple of two number ( $\gcd(a,b) \cdot \operatorname{lcm}(a,b) = ab$ )

So we know $\gcd(a,b) \cdot \operatorname{lcm}(a,b) = b+9 = ab$.

$ab = b+9 \quad ab-b-9=0 \quad b(a-1)=9$

$9 = (1\times 9), (3\times 3)$ $\therefore$ when $b=1$ $a=10$

when $a=2$ $b=8$, when $b=3$ $a=4$ $\therefore$ the answers are $(4,3)$, $(10,1)$, $(2,8)$

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d) $x^4+2x^3-y^2(1+2x)+x^2(1-y^2)=2299$.

$= \cancel{x^4}+\cancel{2x^3}-y^2-2xy^2+\cancel{x^2}-x^2y^2 = 11\cdot 11\cdot 19$

$= x^2(x^2+2x+1)-y^2(1+2x+x^2) = 11\cdot 11\cdot 19$

$= (x+1)^2(x^2-y^2) = 11^2\cdot 19$

$= (x+1)^2(x+y)(x-y) = 11^2\cdot 19$

Since $(x+1)^2(x^2-y)^2 = 11^2\cdot 19$ $\quad (x+1)=11 \quad x=10$

$(x+y)(x-y) = 19$

$19 = (1\times 19)$ $\quad \therefore x+y=19,\ x-y=9.\quad y=9.$

$(x,y) = (10,9)$

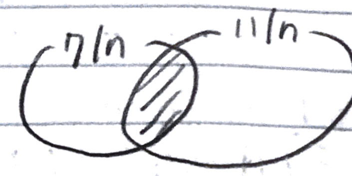
problem 4 - How many positive integers less than 1000

a) $7 \mid n$, $11 \nmid n$, there are $\left\lfloor \frac{999}{7} \right\rfloor = 142$ of n that

divided by 7 but out of that $\operatorname{lcm}(7,11) = 77$.
in 1000, there are $\left\lfloor \frac{999}{77} \right\rfloor = 12$ so there are 12 of multiple

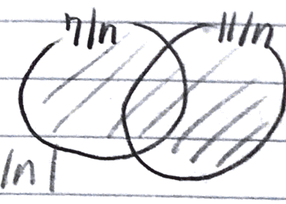
of 11 so $142 - 12 = 130$

$\therefore$ 130




b) $7 \mid n$ and $11 \mid n$.
If there ae divisible by both 11 and 7, then
It should be multiple of $\operatorname{lcm}(7,11)$ which is 77.
$\left\lfloor \frac{999}{77} \right\rfloor = 12$ so then answer is 12

$\therefore 12$




c) divisible by either 7 or 11
$|A \cup B| = |A| + |B| - |A \cap B|$
$\therefore |7 \mid n \cup 11 \mid n| = |7 \mid n| + |11 \mid n| - |7 \mid n \cap 11 \mid n|$
from the letter a $|7 \mid n| = 142$, $|7 \mid n \cap 11 \mid n| = 12$
$|11 \mid n| = \left\lfloor \frac{999}{11} \right\rfloor = 90$ $\therefore 142 + 90 - 12 = 220$

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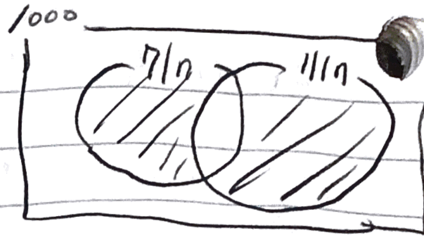
D divisible by exactly one of 7 and 11




when it is divisible by exactly one of 7 and 11

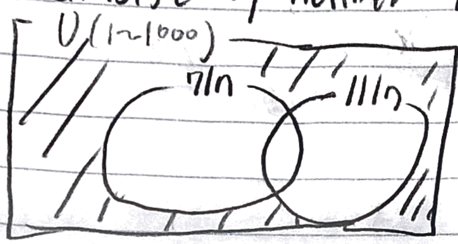
$= |7|n \cup 11|n| - 2|7|n \cap 11|n|$

$= \left\lfloor \frac{999}{7} \right\rfloor + \left\lfloor \frac{999}{11} \right\rfloor - 2\cdot\left\lfloor \frac{999}{77} \right\rfloor$

$= 142 + 90 - 2(12)$

$= 232 - 24 = 208$

E) divisible by neither 7 nor 11.




from the total number 1~999 which is 999, subtract $|7|n \cup 11|n|$ which we got in letter (C) 220 is that

$999 - 220 = 779$

$\therefore 779$

f) have distinct digits?

For 1 digit distinct digit number = 9

for 2 digits distinct digit number = $9 \cdot 9 = 81$

$\boxed{2}\ \boxed{1}$

1~9 1

one of 9 one of 9 - 1 + 1 (0)

for 3 digit distint digit number = $9 \cdot 8 \cdot 8 = 648$

$9 + 81 + 648 = 738$

g) have distinct digits and are even?

For 1 digit distinct digit number = 2, 4, 6, 8 ~ $\boxed{4}$

For 2 digits distinct digit number = $9 \times 1 + 8 \cdot 4 = 41$

For 3 digits distinct digit number = $9 \cdot 8 \cdot 1 + 8 \cdot 8 \cdot 4 = 328$

$4 + 41 + 328 = 373$.

Problem 5.

a) multiplicative inverse of 3 mod 11 $\quad 3 \cdot X = 1 \mod 11$

| | 0 | 1 | 2 | 3 | $\boxed{4}$ | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 0 | 3 | 6 | 9 | 12 | 15 | 18 | 21 | 24 | 27 | 30 |
| | 0 | 3 | 6 | 9 | 1 | 4 | 7 | 10 | 2 | 5 | 8 |

$\boxed{\text{multiplicative inverse: } 4}$

b) multiplicative inverse of 6 mod 11 $\quad 6 \cdot X = 1 \mod 11$

| | 0 | 1 | $\boxed{2}$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 6 | 0 | 6 | 12 | 18 | 24 | 30 | 36 | 42 | 48 | 54 | 60 |
| | 0 | 6 | 1 | 7 | 2 | 8 | 3 | 9 | 4 | 9 | 5 |

multiplicative inverse: 2.

C. multiplicative inverse of 2 mod 12

$2 \cdot x = 1 \bmod 12$

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 0 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 | 22 |
| | $\downarrow$ | $\downarrow$ | $\downarrow$ | $\downarrow$ | $\downarrow$ | $\downarrow$ | $\downarrow$ | $\downarrow$ | $\downarrow$ | $\downarrow$ | $\downarrow$ | $\downarrow$ |
| | 0 | 2 | 4 | 6 | 8 | 10 | 0 | 2 | 4 | 6 | 8 | 10 |

$\boxed{\text{multiplicative Inverse : None}}$

D. multiplicative inverse of 247 mod 154

$\gcd(247, 154) = s \cdot 247 + t \cdot 154$

$247 = (1)154 + 93 \longrightarrow 93 = (1)247 - (1)154$
$154 = (1)93 + 61 \longrightarrow 61 = (1)154 - (1)93$
$93 = (1)61 + 32 \longrightarrow 32 = (1)93 - (1)61$
$61 = (1)32 + 29 \longrightarrow 29 = (1)61 - (1)32$
$32 = (1)29 + 3 \longrightarrow 3 = (1)32 - (1)29$
$29 = (9)3 + 2 \longrightarrow 2 = (1)29 - (9)3$
$3 = (1)2 + 1 \qquad 1 = (1)3 - (1)2$
$2 = (2)1 + 0$

$$
\begin{aligned}
1 &= (1)3 - (1)2 \\
&= (1)3 - (1)\big((1)29 - (9)3\big) \\
&= (1)3 - (1)29 + (9)3 \\
&= (10)3 - (1)29 \\
&= (10)\big((1)32 - (1)29\big) - (1)29 \\
&= (10)32 - (10)29 - (1)29 \\
&= (10)32 - (11)29
\end{aligned}
$$

$= (10)32 - (11)((1)61 - (1)32)$

$= (10)32 - (11)61 + (11)32$

$= (21)32 - (11)61$

$= (21)((1)93 - (1)61) - (11)61$

$= (21)93 - (21)61 - (11)61$

$= (21)93 - (32)61$

$= (21)93 - (32)((1)154 - (1)93)$

$= (21)93 - (32)154 + (32)93$

$= (53)93 - (32)154$

$= (53)((1)247 - (1)154) - (32)154$

$= (53)247 - (53)154 - (32)154$

$= \boxed{(53)}247 - (85)154$

$\boxed{\text{Multiplicative Inverse : } 53}$