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CS 131 - Homework #4 $A \setminus B := \{x \mid x \in A \text{ and } x \notin B\}$

Problem 1 Suppose $A \cap C \subseteq B$, and $a \in C$. Prove that $a \notin A \setminus B$.

a) Given: $A \cap C \subseteq B$, and $a \in C$
goal: $a \notin A/B$

b) $a \notin A/B$ conditional $(\rightarrow)$
$\neg(a \in A \setminus B)) \equiv \neg(x \in A \wedge x \notin B)$
$\equiv x \notin A \vee x \in B$ $(a \rightarrow b) \equiv \neg a \vee b$
$\equiv x \in A \rightarrow x \in B$

c) Direct proof

Given: $A \cap C \subseteq B$, and $a \in C$, $x \in A$.
goal: $x \in B$.

D) Suppose that $A \cap C \subseteq B$ and $a \in C$, and $a \in A$, so we can write $a \in C \wedge a \in A = a \in (A \wedge C)$. Since $(A \cap C)$ is an subset of B. Therefore a is an element of B.

Problem 2.
Let a be an integer, Prove by contradiction that if a^2 is divisible by 3, than a is divisible by 3.
α
r

a is an integer.
α: $a^2 = 3l$, for some integer l
r: $a = 3K$ for some integer K.
$\alpha \rightarrow r$
$\neg r \rightarrow \neg\alpha \rightarrow$ True Given: a be an integer, a is not divisble by 3

Goal: a^2 is not divisle by 3

Contraposition proof.

Suppose a is an Integer, and a is not divisible by 3. So we can write in two cases, case 1 $a = 3k+1$ and case 2 $a = 3k+2$ (for some integer k) By square rooting a, we get for case 1, $a^2 = (3k+1)^2 = 9k^2+6k+1 = 3(3k^2+2k)+1$ Thus, we have expressed $a^2 = 9k^2+6k+1 = 3(3k^2+2k)+1$ which is $3T+1$, a^2 is not divisible by 3. For case 2. $a^2 = (3k+2)^2 = 9k^2+12k+4 = 3(3k^2+4k+1)+1$, Thus we have expressed that $a^2 = 3(3k^2+4k+1)+1 = 3(T)+1$ for some integer T, a^2 is not divisible by 1. Therefore a^2 is not divisible by 3. QED.

Problem 3.

contradiction $\sqrt{3}$ is irrational

↳ $\sqrt{3}$ is rational

a) given: integer m, and a natural number n such that $\sqrt{3} = \frac{m}{n}$

Goal: m is divisible by 3.

For the sake of contradition, we assum that $\sqrt{3}$ is rational we suppose m is an integer and n is a natural number, such that $\sqrt{3} = \frac{m}{n}$, so we can write $\sqrt{3}n = m$, $m^2 = 3n^2$, Since m is an integer and n is a natural number m^2 is divisible by 3. Thus, m^2 is divisible by 3, so m is divisible by 3, There fore. m is divisible by 3. QED

problem 3

b) prove that n is divisible by 3.

we prove that m is divisible by 3 and the sake of the contradition we assume that $\sqrt{3}$ is rational. we suppose m is an integer divisible by 3 and n is a natural number, such that $\sqrt{3} = \frac{m}{n}$

So we can write $m = 3k$ (for some integer k), $m^2 = 3n^2$.

By plugging 3k to m. $m^2 = 9k^2 = 3n^2$ $n^2 = 3k^2$. Since n^2 is divisble by $3k^2$, Thus n^2 is divisible by 3. n is divisible by 3 (prove from problem 2) Therefore. n is divisible by 3. QED

c) Reach a contradiction.

For the sake of contradiction, we assume that $\sqrt{3}$ is rational. There exists an integer m, and natural number n, which both divisible by 3. Since $\sqrt{3}$ is rational we can write $\sqrt{3} = \frac{m}{n}$ and m and n have a common divisor > 1, $\frac{m}{n}$ has to be simplified. However, m and n are both divisible by 3, so there are a common divisor bigger than 1. we have a proof that contradict is false. There fore $\sqrt{3}$ is irrational. QED.

Problem 4.

A and B are equal if and only if $A \subseteq B$ and $B \subseteq A$

$A \subseteq B$ if and only if for all x, $x \in A \rightarrow x \in B$

$A^c = \{x : x \notin A\}$

- $A = B \leftrightarrow (A \subseteq B \land B \subseteq A)$
- $A \subseteq B \leftrightarrow \forall x, x \in A \rightarrow x \in B$

a) prove a "distributive law" for sets $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

First, we will show that $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$. we do this by showing that if x is in $A \cup (B \cap C)$, then it must also be in $(A \cup B) \cap (A \cup C)$. Now suppose that $x \in A \cup (B \cap C)$. By the definition of union, $(x \in A) \lor (x \in (B \cap C))$. Using the definition of intersection $(x \in A) \lor ((x \in B) \land (x \in C))$.

By applying distributive law for proposition, we see that $((x \in A) \lor (x \in B)) \land ((x \in A) \lor (x \in C))$. Using the definition of union, we have $(x \in A \cup B)$ or $(x \in A \cup C)$. Consequently, by the definition of Intersection, we see that $x \in ((A \cup B) \cap (A \cup C))$. now we have shown that $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$

Next, we will show that $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$. we do this by showing that if x is in $(A \cup B) \cap (A \cup C)$, then it must also be in $A \cup (B \cap C)$. Now suppose that $x \in (A \cup B) \cap (A \cup C)$. By the definition of Intersection, we know that $x \in (A \cup B)$ and $x \in (A \cup C)$. Using the definition of Union, we see that ($x \in A$ or $x \in B$) and ($x \in A$ or $x \in C$). Consequently, $((x \in A) \lor (x \in B)) \land ((x \in A) \lor (x \in C))$.

By the distributive law for proposition, we conclude that $(x \in A) \lor (x \in B \land x \in C)$. By the definition of Intersection it follows that $(x \in A) \lor (x \in (B \cap C))$. By the definition of union we now conclude that $x \in (A \cup (B \cap C))$ This shows that $(A \cup B) \cap (A \cup C) \subseteq (A \cup (B \cap C))$.

Because we have shown that each set is a subset of the other the two sets are equal, and the identity is proved

$A \cup (B \cap C) = \{x \mid (x \in A) \vee (x \in (B \cap C))\}$ by definition of Onion

$= \{x \mid (x \in A) \vee (x \in B \wedge x \in C)\}$ by definition of Intersection

$= \{x \mid (x \in A \vee x \in B) \wedge (x \in A \vee x \in C)\}$ by distributive law for logical equivalences

$= \{x \mid (x \in A \cup B) \wedge (x \in A \cup C)\}$ by definition of Union

$= \{x \mid x \in (A \cup B) \cap (A \cup C)\}$ by definition of Intersection

$= (A \cup B) \cap (A \cup C)$ by meaning of set builder notation.

b) prove a "De Morgan's law" for sets: $(A \cup B)^c = A^c \cap B^c$

First, we will show that $(A \cup B)^c \subseteq A^c \cap B^c$. we do this by showing that if x is in $(A \cup B)^c$, then it must be in $A^c \cap B^c$. Now suppose that $x \in (A \cup B)^c$. By the definition of complement, $x \notin A \cup B$. Using the definition of Onion, we see that the proposition $\neg((x \in A) \vee (x \in B))$ is true.

By applying the De Morgan's law for proposition, we have $\neg(x \in A)$ and $\neg(x \in B)$. Using the definition of negation of propositions, we have $x \notin A$ and $x \notin B$. Using the definition of the complement of a set, we have that this implies that $x \in \bar{A}$ and $x \in \bar{B}$. consequently, by the definition of Intersection, we see that $x \in \bar{A} \cap \bar{B}$. we have now ~~that~~ shown that $(A \cup B)^c \subseteq A^c \cap B^c$.

Next, we will show that $A^c \cap B^c \subseteq (A \cup B)^c$. We do this by showing that if x is in $A^c \cap B^c$, then it must be in $(A \cup B)^c$. Now suppose that $x \in A^c \cap B^c$. By the defintion of Intersection, we know that $x \in A^c$ and $x \in B^c$. Using the definition of complement, we see that $x \notin A$ and $x \notin B$. consequently, the proposition $\neg(x \in A) \wedge \neg(x \in B)$ is true.

By De Morgan's Law for proposition, we conclude that $\neg((x \in A) \vee (x \in B))$ is true. By the definition of Onion, it follows that $\neg(x \in (A \cup B))$ we now ~~see~~ use the definition of complement to conclude that $x \in (A \cup B)^c$. This shown that $A^c \cap B^c \subseteq (A \cup B)^c$

Because we have shown that each set is a subset of the other, the two sets are equal, and the identity is proved.

$(A \cup B)^c = \{x \mid x \notin (A \cup B)\}$ by definition of complement
$= \{x \mid \neg(x \in (A \cup B))\}$ by definition of does not belong symbol
$= \{x \mid \neg(x \in A \vee x \in B)\}$ by definition of Union
$= \{x \mid \neg(x \in A) \wedge \neg(x \in B)\}$ by De Morgan's Law for logical equivalences
$= \{x \mid x \notin A \wedge x \notin B\}$ by definition of negation
$= \{x \mid x \in A^c \wedge x \in B^c\}$ by definition of does not belong symbol.
$= \{x \mid x \in (A^c \cap B^c)\}$ by definition of Intersection
$= A^c \cap B^c$ by meaning of set builder notation.

Problem 5. $a > 0$, b two integer
Exists a unique remainder after the division of b by a. $(b \div a)$
prove 1. There is at most one remainder
prove 2. There exists at least one remainder. → Unique one remainder.
$r \geq 0$ or, $r < a$ Integer q, $b = qa + r$.

a) To prove the unique r, First we need to prove that there is at most one remainder.

Given: a, b integers $a > 0$, $0 \leq r < a$
① Existence $b = aq + r$
② Uniqueness
$\boxed{\text{we will use contradiction}}$ For the sake of contradiction, we assume

b) $S = \{$ Integer $s \geq 0$: $\exists$ integer q such that $b = aq + s \}$.

1. S is not empty

Given → 2. part 1 ⇒ there is a minimal element in S

3. $\exists r \in S$ such that $0 \leq r < a$

1. In order to prove that S is not empty, we use direct prove.

Goal : find $s \geq 0$ such that $\exists q$ such that) $b = aq + s$

In other word, find q such that $b - aq = 0$

case 1 $b \geq 0$, $q = 0$

case 2 $b < 0$ $q \leq \frac{b}{a}$

Case 1

If $b \geq 0$, $aq < b$ or $aq < 0$

$aq < 0 \leq b$ $\quad$ $q = 0$

Case 2 If $b < 0$, If $b < 0$ and $aq < 0$

$aq < b < 0$

$q < b/a$

$\exists r \in S$ such that $0 \leq r \leq a$

For some integer, T, If $r = aT$ The it is mutiple of a

$b = aq + aT$

$b = a(q + T)$

r can not be mutiple of T

$b = aq + (a - \{x : 0 \leq r < a\})$

Thats for every number, there is at least 1 remainder

QED