CS 131 – Fall 2019, Assignment 2

Problems must be submitted by Friday September 20, on Gradescope.

Problem 1. (12 Points) Write logical expressions in first-order logic for the following sentence:

a) (4 Points) Every human has a stomach.

Solution. Let $S(x)$ denote x is a human that has a stomach, where x is in the set of humans H. Then, $\forall x S(x)$.

b) (4 Points) Everyone is a friend of someone.

Solution. Let $F(x, y)$ denote x is a friend of y, where x and y are in the set of humans H. Then, $\forall x \exists y F(x, y)$.

c) (4 Points) Nobody likes everybody.

Solution. Let $L(x, y)$ denote x likes y, where x and y are in the set of humans H. Then, $\neg \exists x \forall y L(x, y)$.

Problem 2. (12 Points) Negate the following logical statements

a) (4 Points) $\forall x \exists y P(x, y)$ (Assume that x and y belong to the same domain, and this domain is arbitrary).

Solution.

$$\neg \forall x \exists y P(x, y) \tag{1}$$
$$\equiv \exists x \forall y \neg P(x, y) \tag{2}$$

b) (8 Points) $\exists x\Big(F(x) \rightarrow \forall y\big(\neg P(y) \wedge \forall z Q(z)\big)\Big)$. (Simplify this expression until you have no negation operator).

Solution.

$$\neg \exists x\Big(F(x) \rightarrow \forall y\big(\neg P(y) \wedge \forall z Q(z)\big)\Big) \tag{3}$$
$$\equiv \neg \exists x\Big((\neg F(x)) \vee \forall y\big(\neg P(y) \wedge \forall z Q(z)\big)\Big) \tag{4}$$
$$\equiv \forall x F(x) \wedge \neg \forall y\big(\neg P(y) \wedge \forall z Q(z)\big) \tag{5}$$
$$\equiv \forall x F(x) \wedge \exists y\big(P(y) \vee \exists z \neg Q(z)\big) \tag{6}$$
$$\equiv \forall x \exists y \exists z F(x) \wedge \big(Q(z) \rightarrow P(y)\big) \tag{7}$$

Problem 3. (13 Points) State whether the following sentences are true or false. Justify your answer or try to give a counter example when necessary.

a) (6 Points) Let x be a real number. Then $\forall x\ 0 \leq \frac{1}{x}$ is always true.

Solution. False. Let $x = -1$ then $0 \leq -1$ which is wrong. The statement is true if the domain is the positive real numbers.

b) (7 Points) $\forall x\ x^2 + 4x + 5 > 0$ where x is in the set of real numbers

Solution. True. You can plot the function, or you can simplify the polynomial as $x^2 + 4x + 5 = (x + 2)^2 + 1 > 0$ and it's strictly greater than zero because the first term is positive (because it's the square of a real number), and the second term is also positive.

Problem 4. (15 Points) Prove by using the laws of deduction that the following logical statement is unsatisfiable:

$$\varphi := \neg(A \to C) \wedge \Big(\big((A \wedge B) \to C\big) \wedge (A \to B) \Big)$$

Solution.

$$\begin{aligned}
\varphi &:= \neg(A \to C) \wedge \Big(\big((A \wedge B) \to C\big) \wedge (A \to B) \Big) && (8)\\
&\equiv \neg(\neg A \vee C) \wedge \Big(\big(\neg A \vee \neg B \vee C\big) \wedge (\neg A \vee B) \Big) \quad \text{Conditional identity} && (9)\\
&\equiv (A \wedge \neg C) \wedge \Big((\neg A \wedge \neg A) \vee (\neg A \wedge B) \vee (\neg B \wedge \neg A) \vee (\neg B \wedge B) \vee (C \wedge \neg A) \vee (C \wedge B) \Big) \quad \text{DeMorgan's law} && (10)\\
&\equiv (A \wedge \neg C) \wedge \Big((\neg A) \vee (\neg A \wedge B) \vee (\neg B \wedge \neg A) \vee (\neg B \wedge B) \vee (C \wedge \neg A) \vee (C \wedge B) \Big) \quad \text{Distributive laws} && (11)\\
&\equiv (A \wedge \neg C) \wedge \Big((\neg A) \vee (\neg A \wedge B) \vee (\neg B \wedge \neg A) \vee (F) \vee (C \wedge \neg A) \vee (C \wedge B) \Big) \quad \text{Complement laws} && (12)\\
&\equiv (A \wedge \neg C) \wedge \Big((\neg A) \vee (\neg A \wedge B) \vee (\neg B \wedge \neg A) \vee (C \wedge \neg A) \vee (C \wedge B) \Big) \quad \text{identity laws} && (13)\\
&\equiv (A \wedge \neg C) \wedge \Big((\neg A) \vee \big(\neg A \wedge (B \vee \neg B)\big) \vee (C \wedge \neg A) \vee (C \wedge B) \Big) \quad \text{Associative laws} && (14)\\
&\equiv (A \wedge \neg C) \wedge \Big((\neg A) \vee \big(\neg A \wedge (T)\big) \vee (C \wedge \neg A) \vee (C \wedge B) \Big) \quad \text{Complement laws} && (15)\\
&\equiv (A \wedge \neg C) \wedge \Big((\neg A) \vee \neg A \vee (C \wedge \neg A) \vee (C \wedge B) \Big) \quad \text{identity laws} && (16)\\
&\equiv (A \wedge \neg C) \wedge \Big(\neg A \vee (C \wedge \neg A) \vee (C \wedge B) \Big) \quad \text{Idempotent laws} && (17)\\
&\equiv \Big((A \wedge \neg C) \wedge \neg A \Big) \vee \Big((A \wedge \neg C) \wedge (C \wedge \neg A) \Big) \vee \Big((A \wedge \neg C) \wedge (C \wedge B) \Big) \quad \text{Distributive laws} && (18)\\
&\equiv \Big(A \wedge (\neg C \wedge \neg A) \Big) \vee \Big(A \wedge (\neg C \wedge C) \wedge \neg A \Big) \vee \Big(A \wedge (\neg C \wedge C) \wedge B \Big) \quad \text{Associative laws} && (19)\\
&\equiv \Big(A \wedge (\neg A \wedge \neg C) \Big) \vee \Big(A \wedge (\neg C \wedge C) \wedge \neg A \Big) \vee \Big(A \wedge (\neg C \wedge C) \wedge B \Big) \quad \text{Commutative laws} && (20)\\
&\equiv \Big((A \wedge \neg A) \wedge \neg C \Big) \vee \Big(A \wedge (\neg C \wedge C) \wedge \neg A \Big) \vee \Big(A \wedge (\neg C \wedge C) \wedge B \Big) \quad \text{Associative laws} && (21)\\
&\equiv \Big((F) \wedge \neg C \Big) \vee \Big(A \wedge (F) \wedge \neg A \Big) \vee \Big(A \wedge (F) \wedge B \Big) \quad \text{Complement laws} && (22)\\
&\equiv \Big(F \Big) \vee \Big(F \Big) \vee \Big(F \Big) \quad \text{Complement laws} && (23)\\
&\equiv F && (24)
\end{aligned}$$

Problem 5.

a) (18 Points) Write a first-order logical expression for the following sentences by using the predicate $Cup(x)$ to mean x is a cup.

i) (9 Points) There are at most two cups.

Solution. You can think about this problem as if you are pulling cups from a bag that has at most two cups with replacement. If you make three pulls from the bag, then you must have pulled the same cup more than once.

$$\forall x \forall y \forall z \Big((Cup(x) \wedge Cup(y) \wedge Cup(z)) \rightarrow ((x = y) \vee (y = z) \vee (x = z)) \Big) \tag{25}$$

ii) (9 Points) There are exactly two cups.

Solution.

$$\exists x \exists y \Big((x \neq y) \wedge Cup(x) \wedge Cup(y) \wedge \forall z \big(Cup(z) \rightarrow ((z = x) \vee (z = y)) \big) \Big) \tag{26}$$

b) (30 Points) Let the predicate $P(x, y)$ mean x is a parent of y. You are allowed to use the quantifiers, the predicates P, $=$, and $\neq$, logical operators and variables, and the predicates from the previous subproblems. For the purposes of the problem below, the notion of "sibling" includes half-siblings. For definitions of "first cousin" or "first cousin once removed," please take a look here `https://en.wikipedia.org/wiki/Cousin#Non-blood_relations`.

i) (5 Points) Define a predicate $G(a, b)$ which is true if and only if a is a grandparent of b.

Solution. $G(a, b) \stackrel{\text{def}}{=} \exists c (P(a, c) \wedge P(c, b))$

ii) (5 Points) Define a predicate $S(a, b)$ which is true if and only if a is a sibling of b (recall that a person is not his/her own sibling).

Solution. $S(a, b) \stackrel{\text{def}}{=} (a \neq b) \wedge \exists c (P(c, a) \wedge P(c, b))$

iii) (5 Points) Define a predicate $A(a, b)$ which is true if and only if a is an aunt or uncle of b.

Solution. $A(a, b) \stackrel{\text{def}}{=} \exists c (P(c, b) \wedge S(a, c))$

iv) (5 Points) Define a predicate $C(a, b)$ which is true if and only if a is a first cousin of b (think about the warning in part (b) and how it generalizes to this case).

Solution. $\exists c \exists d (P(c, a) \wedge P(d, b) \wedge S(c, d))$

v) (5 Points) Define a predicate $R(a, b)$ which is true if and only if a is a first cousin once removed of b.

Solution. $R(a, b) \stackrel{\text{def}}{=} \exists c ((P(c, a) \wedge C(c, b)) \vee (P(c, b) \wedge C(c, a)))$

vi) (5 Points) Let a be a man and b be a woman. Massachusetts law prohibits marriage between them if they are related in certain ways. Define a predicate $I(a, b)$ which is true if and only if the marriage between a and b is prohibited according to `https://malegislature.gov/Laws/GeneralLaws/PartII/TitleIII/Chapter207/Section1` and/or `https://malegislature.gov/Laws/GeneralLaws/PartII/TitleIII/Chapter207/Section2`. For the purposes of this problem, ignore prohibitions that involve the word "husband," "wife," or the prefix "step" (because we have no language to express them). To get full credit, avoid making your statement too repetitive – strive for simplicity.

Solution. $I(a,b) \stackrel{\text{def}}{=} P(a,b) \vee P(b,a) \vee G(a,b) \vee G(b,a) \vee S(a,b) \vee A(a,b) \vee A(b,a)$