CS 131 – Fall 2019, Assignment 10

Problems must be submitted by Friday December 6, 2019 5:00pm, on Gradescope.

Problem 1. For any integers a, b and any positive integer n define $a \equiv b \pmod{n}$ if and only if $n|(a-b)$.

Last week we used the fact that if $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, then $a + c \equiv b + d \pmod{n}$ and $ac \equiv bd \pmod{n}$. Now you have a chance to prove it.

a) (12 points) Prove that if $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, then $a + c \equiv b + d \pmod{n}$.

b) (12 points) Prove that if $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, then $ac \equiv bd \pmod{n}$.

We also learned that if a is a non-zero integer, n is a positive integer, and a is coprime with n, then there is an integer x, called the inverse of a modulo n, such that $ax \equiv 1 \pmod{n}$. But is x unique? It turns out it is not because if x is a solution, then $x + n$ is also a solution, but it is unique modulo n.

c) (6 points) Prove that if a is a non-zero integer, n is a positive integer, and a is coprime with n, then the solution to $ax \equiv 1 \pmod{n}$ is unique modulo n. That is, if x and y are two solutions, then $x \equiv y \pmod{n}$.

Problem 2.

a) (3 points) A particular brand of shirt comes in 12 colors, has a male version and a female version, and comes in three sizes for each sex. How many different types of this shirt are made?

b) (3 points) How many different three-letter initials with none of the letters repeated can people have? Each letter is lowercase latin letter.

c) (3 points) How many bit strings are there of length six or less, not counting the empty string?

d) (3 points) How many strings are there of four lowercase letters that have the letter x in them?

e) (3 points) How many different functions are there from a set with 10 elements to a set with 6 elements?

f) (3 points) How many different surjective functions are there from a set with 10 elements to a set with 2 elements?

g) (3 points) How many different surjective functions are there from a set with 10 elements to a set with 3 elements?

h) (3 points) You have n socks of different color. How many ways are there to wear a pair of socks if choosing green for the left sock and red for the right sock, and choosing red for the left sock and green for the right sock are considered different choices.

i) (3 points) You have n socks of different color. How many ways are there to wear a pair of socks if choosing green for the left sock and red for the right sock, and choosing red for the left sock and green for the right sock are considered the same choices.

j) (3 points) There is a class of 240 students and you want to pick 4 graders out of them for a future class. How many ways are there to do it? Calculate your answer.

k) (3 points) How many ways are there to choose 5 different integers from the set $\{1, 2, ..., 9\}$ such that the smallest integer with the largest integer sum up to 10?

l) (3 points) How many ways are there to seat four of a group of ten people around a circular table where two seatings are considered the same when everyone has the same immediate left and immediate right neighbor?

m) (Extra credit for 5%, or 10 points) How many bit strings of length 10 contain either five consecutive 0s or five consecutive 1s?

Problem 3.

a) (3 points) A bowl contains 10 red balls and 10 blue balls. A woman selects balls at random without looking at them. How many balls must she select to be sure of having at least three balls of the same color? How many balls must she select to be sure of having at least three blue balls?

b) (6 points) How many numbers must be selected from the set $\{1, 3, 5, 7, 9, 11, 13, 15\}$ to guarantee that at least one pair of these numbers add up to 16? Prove your answer.

c) (9 points) Suppose you have a drawer with cards on which a number 1 through 18 is written. You can pick cards from the drawer with your eyes closed. What is the minimum number of cards you have to draw to guarantee that two of the drawn cards sum up to 9 and two of the other drawn cards sum up to 27.

Problem 4. Give a *combinatorial* proof for each of the following identities:

a) (6 points) $1 + 2 + 3 + ... + n = \binom{n+1}{2}$.

b) (6 points) $k\binom{n}{k} = n\binom{n-1}{k-1}$.

c) (6 points) $\binom{2n}{2} = 2\binom{n}{2} + n^2$.

d) (6 points) $\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$.

Problem 5.

a) (6 points) In a 2D grid, how many distinct paths are there from $(0, 0)$ to (x, y) that follow the gridlines and move only up and to the right. Provide your reasoning.

b) (Extra credit for 5%, or 10 points) Determine the number of ordered pairs (A, B), where $A \subseteq B \subseteq \{1, 2, ..., n\}$.

Problem 6.

a) (3 points) How many different strings can be made from the letters in RASPBERRY, using all the letters?

b) (3 points) How many different strings can be made from the letters in TERRIER, using all the letters?

Problem 7. Consider a graph G with $3n$ vertices names $1, ..., 3n$. When limited to vertices $1, ..., 2n$, G is a complete graph. When limited to vertices $n+1, ..., 3n$, G is a complete graph. Finally, there is an edge between vertex 1 and vertex $3n$. There are no other edges.

a) (3 points) How many edges does G have?

b) (6 points) A set of three vertices is called a triangle if there is an edge between each pair of the vertices.

How many triangles does G have?

Problem 8.

a) (9 points) Let $G = (V, E)$ be graph (where V is the set of vertices and E is the set of edges). Define a relation $\sim$ on V as follows: $u \sim v$ if and only if there is a path from u to v in G.

Prove that the $\sim$ is an equivalence relation.

b) (6 points) Prove that a tree with non-zero number of vertices always contains a leaf (a vertex with degree 0 or 1).

c) (9 points) We know that a tree with $n \geq 1$ vertices has exactly $n-1$ edges. In this problem prove by induction that a tree with $n \geq 1$ vertices has at least $n-1$ edges.

d) (9 points) Prove by induction that a connected graph with $n \geq 1$ vertices and $n-1$ edges must be a tree.

e) (3 points) The length of a path between vertices u and v is the sum of the weights of its edges. A path between vertices u and v is called a shortest path if and only if it has the minimum length among all paths from u to v.

Is a shortest path between two vertices in a weighted graph unique if the weights of edges are distinct? Give a proof.

Problem 9. Suppose that a new company has five employees: Zamora, Agraharam, Smith, Chou, and Macintyre. Each employee will assume one of five responsiblities: planning, publicity, sales, marketing, and industry relations. Each employee is capable of doing one or more of these jobs: Zamora could do planning, sales, marketing, or industry relations; Agraharam could do planning; Smith could do publicity, sales, or industry relations; Chou could do planning, sales, or industry relations; and Macintyre could do planning, publicity, sales, or industry relations.

Model the capabilities of these employees using a bipartite graph.

a) (3 points) Find an assignment of responsibilites such that each employee is assigned one distinct responsibility.

b) (3 points) Now suppose Smith refuses to do publicity and Zamore loses interest in marketing. Can we still assign each employee to distinct responsibilities?

Problem 10. (29 points) Use Hall's theorem to prove the following. Suppose G is a bipartite graph with bipartition (V_1, V_2). Show that the maximum number of vertices of V_1 that are endpoints of a matching of G equals

$$|V_1| - \max_{A \subseteq V_1} (|A| - |N(A)|).$$