Lecture 5 (9/17) Outline

- (Finish off) rules of inference (Rosen 1.6)

- Introduction to proofs (Rosen up to 1.7.6)
    - How do we write proofs?
    - Direct
    - Contraposition

Rules of inference

Consider the following argument. Is it valid?

- **Logical premise 1:** If $\underbrace{\text{you have a current password}}_{p}$, then $\underbrace{\text{you can log onto the network}}_{q}$.

  $p \rightarrow q$
- **Logical premise 2:** $\underbrace{\text{You have the current password}}_{p}$.

  p
- **Conclusion:** Therefore ($\therefore$), q

Rules of inference: *modus ponens*

- Yes, it is valid. The following tautology

$$(p \wedge (p \to q)) \to q,$$

leads to the this valid argument.

This rule of inference is called MODUS PONENS.

- **Remark:** If $\sqrt{2} > \frac{3}{2}$ then $(\sqrt{2})^2 > (\frac{3}{2})^2$. We know that $\sqrt{2} > \frac{3}{2}$. Therefore, $(\sqrt{2})^2 > (\frac{3}{2})^2$, i.e., $2 > \frac{9}{4} = 2.25$.

- **What is the issue here?**

Rules of inference

- Some important tautologies give rise to some frequently used valid arguments/rules of inference.

- Addition

- "It is below freezing now, therefore it is below freezing or raining now"

  p therefore $p \vee q$.

  The tautology $p \to (p \vee q)$.

Rules of inference

- Some important tautologies give rise to some frequently used valid arguments/rules of inference.

- Simplification

- "It is below freezing and raining, therefore it raining"

  The argument is of the form p therefore $p \vee q$.

  The tautology $p \rightarrow (p \vee q)$.

Rules of inference

TABLE 1 Rules of Inference.

Rule of Inference	*Tautology*	*Name*
p $p \to q$ $\therefore q$	$(p \wedge (p \to q)) \to q$	Modus ponens
$\neg q$ $p \to q$ $\therefore \neg p$	$(\neg q \wedge (p \to q)) \to \neg p$	Modus tollens
$p \to q$ $q \to r$ $\therefore p \to r$	$((p \to q) \wedge (q \to r)) \to (p \to r)$	Hypothetical syllogism
$p \vee q$ $\neg p$ $\therefore q$	$((p \vee q) \wedge \neg p) \to q$	Disjunctive syllogism
p $\therefore p \vee q$	$p \to (p \vee q)$	Addition
$p \wedge q$ $\therefore p$	$(p \wedge q) \to p$	Simplification
p q $\therefore p \wedge q$	$((p) \wedge (q)) \to (p \wedge q)$	Conjunction
$p \vee q$ $\neg p \vee r$ $\therefore q \vee r$	$((p \vee q) \wedge (\neg p \vee r)) \to (q \vee r)$	Resolution

Rules of inference – Resolution

Resolution.

- The resolution rule of inference relies on the tautology

$$((p \lor q) \land (\neg p \lor r)) \to (q \lor r).$$

- Plays an important role in programming languages based on logic, e.g., PROLOG
- **Example**: Let's apply the resolution rule on the hypotheses Jasmine is skiing or it is not snowing and It is snowing or Bart is playing hockey

Rules of inference – Resolution

Example:

- Let p be the proposition "it is snowing"
- Let q be the propositon "Jasmine is skiing"
- Let r be the proposition "Bart is playing hockey".
- Our hypothesis is $((p \vee q) \wedge (\neg p \vee r))$
- Therefore, by the resolution rule we conclude $(q \vee r)$ which means

  Jasmine is skiing or Bart is playing hockey

Rules of inference – Fallacy

- If you do every problem in this book, then you will learn discrete mathematics.
- You learned discrete mathematics.
- Therefore, you did every problem in this book.

Is this a valid argument?

Rules of inference – Fallacy

- If you do every problem in this book, then you will learn discrete mathematics.
- You learned discrete mathematics.
- Therefore, you did every problem in this book.

No!

- $((p \to q) \land \neg p) \to \neg q$ is not a tautology (set $p = F, q = T$)

Rules of inference – Quantified statements

TABLE 2 Rules of Inference for Quantified Statements.

Rule of Inference	*Name*
$\dfrac{\forall x P(x)}{\therefore\ P(c)}$	Universal instantiation
$\dfrac{P(c) \text{ for an arbitrary } c}{\therefore\ \forall x P(x)}$	Universal generalization
$\dfrac{\exists x P(x)}{\therefore\ P(c) \text{ for some element } c}$	Existential instantiation
$\dfrac{P(c) \text{ for some element } c}{\therefore\ \exists x P(x)}$	Existential generalization

Proofs

- What is a proof?

- A proof is a valid argument that establishes the truth of a mathematical statement, using the hypotheses of the theorem, if any, axioms assumed tobe true, and previously proven theorems.

- Using these ingredients and rules of inference, the proof establishes thetruth of the statement being proved.

Direct proofs – Final version of a proof versus reasoning work

Question: How do we prove the following theorem?

> **Theorem:** Suppose a, b are real numbers. If $0 < a < b$ then $a^2 < b^2$.

- What is given to us as hypothesis?
- What is the conclusion?

Direct proofs – Final version of a proof versus reasoning work

Question: How do we prove the following theorem?

Theorem: Suppose a, b are real numbers. If $0 < a < b$ then $a^2 < b^2$.

- What is given to us as hypothesis?
  Givens: a, b are real numbers

- What is the conclusion we want to prove?
  Goal: $P \rightarrow Q$ where P is $0 < a < b$ and Q is $a^2 < b^2$.

Direct proofs – Final version of a proof versus reasoning work

- **Direct proof** technique for $P \to Q$: Add P to the set of hypotheses. Then prove Q.
- **Let's apply it!** What is given to us as hypotheses now?
- **Givens:** a, b are real numbers, $0 < a < b$
- **Goal:** Show that $a^2 < b^2$

Direct proofs – Final version of a proof versus reasoning work

Let's write the formal proof now.

- **Proof:** Suppose $0 < a < b$. Multiplying the inequality $a < b$ by the positive number a we can conclude $a^2 < ab$, and similarly multiplying by b we get $ab < b^2$. Therefore,

$$a^2 < ab < b^2,$$

so, $a^2 < b^2$ as required. **QED**[1]

[1] "quod erat demonstrandum", literally meaning "what was to be shown".

Direct proofs

Prove the following theorems.

1. If n is an odd integer, then n^2 is odd.
2. Suppose m, n are natural numbers. If m, n are both perfect squares, then nm is also a perfect square.

Solutions on blackboard

Proof by contraposition

Question: How do we prove the following theorem?

> **Theorem:** Suppose a, b, c are real numbers, and $a > b$. Prove that if $ac \leq bc$ then $c \leq 0$.

- What is given to us as hypothesis?
- What is the conclusion?

Proof by contraposition

Question: How do we prove the following theorem?

Theorem: Suppose a, b, c are real numbers, and $a > b$. Prove that if $ac \leq bc$ then $c \leq 0$.

- What is given to us as hypothesis?

  Givens: a, b, c are real numbers, $a > b$

- What is the conclusion?

  Goal: $P \rightarrow Q$ where P is $ac \leq bc$ and Q is $c \leq 0$.

Proof by contraposition

- **Proof by contraposition** technique for $P \to Q$: Add $\neg Q$ to the set of hypotheses. Then prove $\neg P$.
- What is given to us as hypothesis?

  Givens: a, b, c are real numbers, $a > b$, $c > 0$

  Goal: $ac > bc$

So, the proof structure using contraposition would look like this:

Suppose $c > 0$

[Proof that $ac > bc$ goes here]

Therefore, if $ac \leq bc$ then $c \leq 0$.

Proof by contraposition

This is how the final/formal proof by contrapositive would look like on the paper:

Theorem: Suppose a, b, c are real numbers, and $a > b$. Prove that if $ac \leq bc$ then $c \leq 0$.

Proof: We will prove by contrapositive. Suppose $c > 0$. Then we can multiply both sides of the given inequality $a > b$ by c and conclude that $ac > bc$. Therefore, if $ac \leq bc$, then $c \leq 0$.

Important remark!

Even if we have used logic in the scratch work, we have not used them in the final form. While logic is essential to figure out a proof strategy, in the final write-up of the proof, mathematicians avoid using the notation and rules of logic.