2.14, 2.21

Assignment 3 U27565203 Jue Hong Lee

Problem 1 (2.14)

Compute the values e_1, e_2, e_3 in the proof of Theorem 2.6 in the case where $k=3$, $n_1=3$, $n_2=5$, and $n_3=7$. Also, find an integer a such that $a \equiv 1 \pmod 3$, $a \equiv -1 \pmod 5$, and $a \equiv 5 \pmod 7$.

i) $a \equiv 1 \pmod 3 \rightarrow e_1$ Lets say Answers of i) ~ iii) A_1, A_2, A_3
ii) $a \equiv -1 \pmod 5 \rightarrow e_2$ so $a = A_1 + A_2 + A_3 \pmod{3 \times 5 \times 7}$
iii) $a \equiv 5 \pmod 7 \rightarrow e_3$ In order to arrive A_1 is not divided by 3: $3 \nmid A_1$
Similarly $5 \nmid A_2$, and $7 \nmid A_3$

so. $x \equiv 5 \times 7 \times e_1 + 3 \times 7 \times e_2 + 3 \times 5 \times e_3 \pmod{3 \times 5 \times 7}$

$$\Rightarrow x \equiv \begin{cases} x \equiv 35e_1 + 21e_2 + 15e_3 \pmod 3 \equiv 35e_1 + 0 + 0 \pmod 3 \equiv A_1 \\ x \equiv 35e_1 + 21e_2 + 15e_3 \pmod 5 \equiv 0 + 21e_2 + 0 \pmod 5 \equiv A_2 \\ x \equiv 35e_1 + 21e_2 + 15e_3 \pmod 7 \equiv 0 + 0 + 15e_3 \pmod 7 \equiv A_2 \end{cases}$$

use modular inverse

$1 \pmod 3 \equiv 35 \times (35^{-1} \times 1) \pmod 3$, $35^{-1} \pmod 3 \Rightarrow 2^{-1} \pmod 3 \Rightarrow 2$
$-1 \pmod 5 \equiv 21 \times (21^{-1} \times -1) \pmod 5$, $21^{-1} \pmod 5 \Rightarrow 1^{-1} \pmod 5 \Rightarrow 1$
$5 \pmod 7 \equiv 15 \times (15^{-1} \times 5) \pmod 7$, $15^{-1} \pmod 7 \Rightarrow 1^{-1} \pmod 7 \Rightarrow 1$

$e_1 \equiv 35 \times (2) = 70$
$e_2 = 21 \times (1) = 21$
$e_3 = 15 \times (1) = 15$

145
21

$a = 70 - 21 + 75 = \boxed{124}$

$124 \pmod{105} \equiv 19 \bmod (105)$

$a = 124$ $e_1 = 70$, $e_2 = -21$ $e_3 = 75$

$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}\ \frac{1}{6}\ \frac{1}{?}$ $\quad lcm(a,b) \times gcd(a,b) = ab$

$\frac{6}{5}$

Problem 2 (2.21)
Let p be an odd prime. Show that the numerator of $\sum_{i=1}^{p-1} 1/i$ is divisible by p.

$\sum_{i=1}^{p-1}\frac{1}{i} = \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots\frac{1}{p-1}$ we can swith position by coupling first and last, second and second last, etc.

$$\sum_{i=1}^{p-1}\frac{1}{i} = \underbrace{\frac{1}{1}+\frac{1}{p-1}}_{\text{(first, last)}}+\underbrace{\frac{1}{2}+\frac{1}{p-2}}_{\text{(second, second last)}}+\underbrace{\frac{1}{3}+\frac{1}{p-3}}_{\text{(third, third last)}}+\cdots\frac{1}{\lfloor\frac{p}{2}\rfloor}+\frac{1}{\lfloor\frac{p}{2}\rfloor+1}$$ (since p is the odd prime, p-1 is even.

So all of sequence has the pair

$$\sum_{i=1}^{p-1}\frac{1}{i} = \left(\frac{1}{1}+\frac{1}{p-1}\right)+\left(\frac{1}{2}+\frac{1}{p-2}\right)+\left(\frac{1}{3}+\frac{1}{p-3}\right)+\cdots\frac{1}{\lfloor\frac{p}{2}\rfloor}+\frac{1}{\lfloor\frac{p}{2}\rfloor+1}$$ we can add each pair

$$\sum_{i=1}^{p-1}\frac{1}{i} = \left(\frac{p-1+1}{p-1}\right)+\left(\frac{p-2+2}{2p-4}\right)+\left(\frac{p-3+3}{3p-9}\right)+\cdots+\frac{\lfloor\frac{p}{2}\rfloor+1+\lfloor\frac{p}{2}\rfloor}{(\lfloor\frac{p}{2}\rfloor)\times(\lfloor\frac{p}{2}\rfloor+1)}$$

we can simplify this to $\sum_{i=1}^{p-1}\frac{1}{i} = \left(\frac{p}{p-1}\right)+\left(\frac{p}{2p-4}\right)+\left(\frac{p}{3p-9}\right)+\cdots\frac{\lfloor\frac{p}{2}\rfloor+1+\lfloor\frac{p}{2}\rfloor}{(\lfloor\frac{p}{2}\rfloor)\times(\lfloor\frac{p}{2}\rfloor+1)}$

So we can examine all of the numerators are ps and $\lfloor\frac{p}{2}\rfloor+1+\lfloor\frac{p}{2}\rfloor$

For $\lfloor\frac{p}{2}\rfloor+1+\lfloor\frac{p}{2}\rfloor = 2\times\lfloor\frac{p}{2}\rfloor+1 = p-1+1 = p$

$2\times\lfloor\frac{p}{2}\rfloor = p-1$ Since it is a floor.

So the numerators are all p. In order to make deraminator equal each Added pairs has to be multipled by lcm of all denominator. so all numerators will be multiple of p. Adding multiple of ps it will be also multiple of p.

Show this in eqution. Let's say lcm of $p-1,\ 2p-4,\ 3p-9,\ \ldots\ (\lfloor\frac{p}{2}\rfloor\times(\lfloor\frac{p}{2}\rfloor+1))$ is n. In oredr to make all denominator equal to n there is some value has to be multiple to all pairs.

$$\frac{p \times \frac{n}{p-1}}{p-1 \times \frac{n}{p-1}} + \frac{p \times \frac{n}{2p-4}}{2p-4 \times \frac{n}{2p-4}} + \frac{p \times \frac{n}{3p-9}}{3p-9 \times \frac{n}{3p-9}} + \cdots \frac{p \times \frac{p}{\left(\lfloor \frac{p}{2} \rfloor\right)\left(\lfloor \frac{p}{2} \rfloor+1\right)}}{\left(\lfloor \frac{p}{2} \rfloor\right)\left(\lfloor \frac{p}{2} \rfloor+1\right) \times \frac{n}{\left(\lfloor \frac{p}{2} \rfloor\right)\left(\lfloor \frac{p}{2} \rfloor+1\right)}}$$

the numerators are all multiple of p so the sum up will be divided by p.