CS 131 - Fall 2019, Assignment 7

problem 1.

$f(n) = af(n/b) + cn^d$

$$f(n) \begin{cases} O(n^d) & \text{if } a < b^d \\ O(n^d \log n) & \text{if } a = b^d \\ O(n^{\log_b a}) & \text{if } a > b^d \end{cases}$$

a) $f(n) = f(n/4) + 2n^{1/2}$

$a = 1,\ b = 4\ \ c = 2\ \ d = 1/2$

$1 < 4^{1/2}$ $\quad 4^{1/2}\ \ 2$

$\therefore O(n^{1/2}) = O(\sqrt{n})$

b) $f(n) = 3f(n/4) + 2n^{1/2}$

$a = 3\ \ b = 4\ \ c = 2\ \ d = 1/2$

$3 > 4^{1/2}$

$\therefore O(n^{\log_4 3})$

problem 2.

a) Root (150, 3)

$\rightarrow$ search (150, 3, 1, 150)

$\rightarrow$ $m = \lceil \frac{151}{2} \rceil = \lceil 75.5 \rceil = 76$

$\rightarrow$ search (150, 3, 1, 75)

$\rightarrow$ $m = \lceil \frac{76}{2} \rceil = 38$

$\rightarrow$ search (150, 3, 1, 37)

$\rightarrow$ $m = \lceil \frac{38}{2} \rceil = 19$

$\rightarrow$ search (150, 3, 1, 18)

$\rightarrow$ $m = \lceil \frac{19}{2} \rceil = 10$

search $(150, 3, 1, 9)$
→ $m = \lceil \frac{10}{2} \rceil = 5$
→ $5^3 \le 150$
→ search $(150, 3, 5, 9)$
→ $m = \lceil \frac{14}{2} \rceil = 7$
→ search $(150, 3, 5, 6)$
→ $m = \lceil \frac{11}{2} \rceil = 6$
→ search $(150, 3, 5, 5)$
→ Return 5

All calls to search, $\boxed{8}$

b) Use master theorem.

$f(n) = af(n/b) + cn^d$

Since it always have 1 step which "if $l = r$ then return l" the constant $c = 1$,

$b = 2$ since if $l \ne r$, there would be only two choices either "if $m^b \le a$ then return search (a, b, m, r)" or "return search $(a, b, l, m-1)$." It split to two part. $b = 2$.

$a = 1$ since, both search has one sub proble.
so $f(n) = 1 \cdot f(n/2) + 1$

$1 = 2^0$
$\therefore$ $O(2^0 \log n) = O(\log n)$

problem 3

a) $(303)_{10}$ to binary

$2\,|\,303$ $= (100101111)_2$

$2\,|\,151 + \boxed{1}$

$2\,|\,75 + \boxed{1}$

$2\,|\,37 + \boxed{1}$

$2\,|\,18 + \boxed{1}$

$2\,|\,9 + \boxed{0}$

$2\,|\,4 + \boxed{1}$

$2\,|\,2 + 0$

$1 + 0$

(b) $(303)_{10}$ to base 3

$3\,|\,303$ $= (102020)_3$

$3\,|\,101 + 0$

$3\,|\,33 + \boxed{2}$

$3\,|\,11$ 0

$3\,|\,3$ 2

1 0

c) $(1001011)_2$ to decimal

1 0 0 1 0 1 1

6 5 4 3 2 1 0

$1\times2^6 + 1\times2^3 + 1\times2^1 + 1\times2^0$

$= 64 + 8 + 2 + 1$

$= 75$

D) $(12012)_3$ to decimal

$$\begin{array}{ccccc} 1 & 2 & 0 & 1 & 2 \\ 4 & 3 & 2 & 1 & 0 \end{array}$$

$= 1 \times 3^4 + 2 \times 3^3 + 1 \times 3^1 + 2 \times 3^0$

$= 81 + 54 + 3 + 2$ 5 59

$= 140$

E) last digit of two multiplier

Befoe we look at the multiple of two number,

123456789012345678 is end with 8, so this number is multiple of 2.

And also the addition of all digit is 1+2+3+4+5+6+7+8+9+1+2+3+4+5+6+7+8 = 81, which is multiple of 3, so this number is multiple of 2 and 3, multip of 6.

when a = multiple of 6, which is $6k$ for some integer k,

when a is multiplied with number b:

ab is also multiple of 6

$a = 6k$

$ab = 6kb$

Since the result of the two number multiplication is multiple of 6

the last digit is 0

$\boxed{\therefore 0}$

Problem 4

Fibonacci number

$F_0 = 0,\ F_1 = 1,\ F_2 = 1,\ F_3 = 2,\ F_4 = 3,\ F_5 = 5,\ F_6 = 8,\ F_7 = 13,\ F_8 = 21$

$F_9 = 34,\ F_{10} = 55,\ F_{11} = 89,\ F_{12} = 144$

a) $1 = F_0 + F_2$

$= 0 + 1$

$= 1$

199

$7 = F_5 + F_3$

$= 5 + 2$

$= 7$

$14 = F_7 + F_2$

$= 13 + 1$

$= 14$

$88 = F_{10} + F_8 + F_6 + F_3$

$= 55 + 21 + 8 + 2$

$= 88$

$199 = F_{12} + F_{10}$

$= 144 + 55$

$= 199$

b) Use Induction that any positive integer can be represented as a sum of Fibonacci numbe's f_2, f_3, f_4

Basis case

let's say predicate $P(n)$ = represented as a sum of fibonacci number

$P(1) = f_1$ ✓

Inductive Step
Now we assume $p(n)$ is true, we prove $p(n+1)$ is true.

we know $p(n)$ can write down as. $n \times f_1$ since it can possibly with repititions or consecutive Fibonacci numbers,

$p(n) = n \times f_1$.

For $p(n+1)$ we can write this as $n+1 \times f_1$
$p(n+1) \overset{IH}{=} p(n) + f_1 = n \cdot f_1 + f_1 = (n+1) f_1$

so it is true QED.

c) $p(n) :=$ n is a positive integer that can be represented as a sum of Fibonacci numbers with no consecutive Fibonacci numbers and no repetitions.

For Basis case:
$p(1) = f_2 = 1$ ✓
$p(2) = f_3 = 2$ ✓
$p(3) = f_4 = 3$ ✓
$p(4) = f_4 + f_2 = 3 + 1 = 4$ ✓

Inductive Step
Assume $p(k)$ $p(1), p(2), p(3), p(4)$ is true

Case 1: If $k+1 = f_n$ then $p(k+1)$ is true
Case 2: If $f_n < k+1 < f_{n+1}$, $x = (k+1) - f_n$

$f_{n-1} + f_n = f_{n+1} \Rightarrow f_{n-1} = f_{n+1} - f_n \rightarrow x < f_{n-1}$
as $x+1 < f_{n+1}$

proving $p(x)$ is true for $x \in \mathbb{Z}^+$
there fore $k+1 = r+fn$ proving $p(k+1)$

proved $P(k+1)$ for both case QED

problem 5.

a) Euclid $(108, 60)$

$\to$ $108 \geq 60$, $108 \div 60 = 1 \cdot 60 + 48$

Euclid $(48, 60)$

$48 \leq 60$, $60 \div 48 = 1 \cdot 48 + 12$

Euclid $(48, 12)$

$48 \geq 12$ $48 \div 12 = 4 \cdot 12 + 0$

Euclid $(0, 12)$

$\to$ $\underline{12}$

b) prove that $gcd(a,b) = gcd(a, b-a)$ for non-negative integers a, and b.

$c = gcd(a,b)$

$c = gcd(a, b-a)$

a, $b-a$ are both multiple of C

$a = C \cdot k$ for some integer k, $b = C \cdot T$ for some integer T

$b - a = CT - Ck = C \cdot (T-k)$ So, a and $b-a$ are both multiple of C.

Now prove that C is the biggest common divisor.

D is common divisor of a and $b-a$,

prove that $D \leq C$.

$a = D \cdot k^*$, $b - a = T^* \cdot D$ $b = D \cdot k^* + T^* \cdot D = D(k^* + T^*)$

so Becuse $D|a$ and $D|b-a$ and $D|b$

D must be $\leq gcd(a,b)$

c) Let $p(n)$ be $\forall a,b\ (\neg(a=0 \wedge b=0)) \wedge a \leq n \wedge b \leq n$
$\rightarrow (\text{Euclid}(a,b) = \gcd(a,b))$

Basis
p(0) is true

p(0): since the premise will be false, becuse $a \neq 0 \wedge b \neq 0$ will be fuse, then the whole predicate conclusion will be true since false $\rightarrow \square$ is always true

p(0) is true

Inductive step
use strong induction

Assume $p(0), p(1), p(2) \ldots p(k)$ is true, prove $p(k+1)$ is true.

Case 1: $a = k+1$, $0 \leq b \leq k$
from this case 1,
there are two cases.
a) $a = k+1$, $b = 0$
$\gcd(k+1, 0) = k+1$
$\text{Euclid}(k+1, 0) = k+1$
$\therefore P(k+1)$ is true for case a)

b) $a = k+1$, $0 < b \leq k$
$\gcd(k+1, b) = \gcd(k+1-b, b)$
Since $((k+1)-b) \leq k$ and $b \leq k$, by strong Induction $\gcd((k+1)-b, b) = \text{Euclid}((k+1)-b, b)$

$\therefore P(k+1)$ is true for case II
$\therefore P(k+1)$ is true for case 1.

Case II $0 \le a < k,\ b = k+1$

Same as Case I It also has two different sub cases.

case a) $a = 0,\ b = k+1$

$gcd(0, k+1) = k+1$

$Euclid(0, k+1) = k+1$

$\therefore\ P(k+1)$ is true

case b. $0 < a \le k,\ b = k+1$

$gcd(a, k+1) = gcd(a, (k+1)-a)$

Since $((k+1)-a) \le k$ and $a \le k$ by strong Induction $gcd(a, (k+1)-a)$

$= Euclid(a, (k+1)-a)$

$\therefore\ P(k+1)$ is true

$\therefore\ P(k+1)$ is true for both cases a, b

case II proved

Case III $a = k+1,\ b = k+1$

$gcd(k+1, k+1) = k+1$

$Euclid(k+1, k+1) = Euclid((k+1)\%(k+1), k+1)$

$= Euclid(0, k+1) = k+1$

$Euclid(k+1, k+1) = gcd(k+1)$

$\therefore\ P(k+1)$ is true for case III

Therefore, $P(k+1)$ is true for all cases proving the correctness of Euclid.