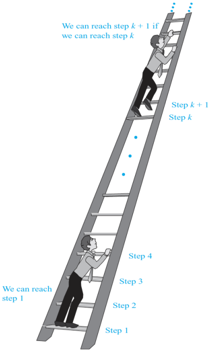
Lecture 10 (10/3)
Outline

- Mathematical induction [Rosen 5.1]

- **Remark** Everything up to this point (including 5.1) will be tested in midterm 1.

Stairway to heaven



1. We can reach the first rung of the ladder.
2. If we can reach a particular rung of the ladder, then we can reach the next rung.

   Can we reach every rung of this infinite ladder?

Principle of mathematical induction

- **Goal:** The typical goal is to prove statements of the form "$P(n)$ is true for all positive integers n".

1. **Basis step:** We verify that $P(1)$ is true.
   Analogy: first rung of the ladder
2. **Inductive step:** We show that the conditional statement $P(k) \to P(k+1)$ is true for all positive integers k.
   Analogy: If we can reach a particular rung of the ladder, then we can reach the next rung.

As a rule of inference we can write:

$$P(1) \wedge \forall k\big(P(k) \to P(k+1)\big) \to \forall n P(n).$$

Practice problems

Use mathematical induction to prove the following statements.

1. $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.
2. Conjecture a formula for the sum of the first n positive odd integers. Then prove your conjecture using mathematical induction
3. $2^n < n!$ for every integer $n \geq 4$
4. $\sum_{j=0}^{n} ar^j = a\frac{r^{n+1}-1}{r-1}, r \neq 1$
5. $\overline{\cap_{j=1}^{n} A_j} = \cup_{j=1}^{n} \overline{A_j}$ for $n \geq 2$ (generalization of De Morgan's law)

Creative uses of mathematical induction

- **Theorem:** Let n be a positive integer. Show that every $2^n \times 2^n$ checkerboard with one square removed can be tiled using right triominoes, where these pieces cover three squares at a time.

Basis

- Let $P(n)$ be the proposition that every $2^n \times 2^n$ checkerboard with one square removed can be tiled using right triominoes.
- We will use induction.
- Basis $n = 1$: $P(1)$ is true. There are four cases (i.e., defined by where is the missing square), and for each one we can tile the board with one right triomino.

Inductive step

- The inductive hypothesis is the assumption that $P(k)$ is true for the positive integer k
- It must be shown that under the assumption of the inductive hypothesis, that $P(k+1)$ must also be true
- **Question:** What is $P(k+1)$? (in English)

Ideas?

Inductive step

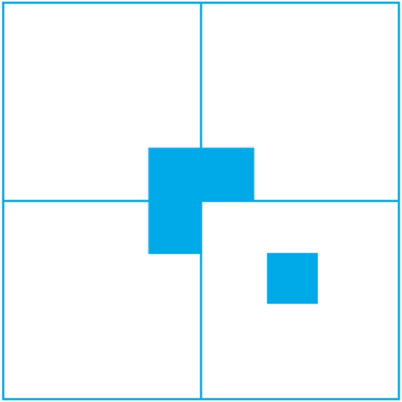


FIGURE 7 **Tiling the $2^{k+1} \times 2^{k+1}$ Checkerboard with One Square Removed.**