Exercise 5.16

use Stirling's approximation (previous exercise) to show that

$$\binom{2m}{m} = \Theta(4^m/\sqrt{m})$$

In the book, Stirling approximation from 5.16 is that $n! \sim (n/e)^n\sqrt{2\pi n}$

Big Theta's definition is that f is bounded both above and below by g asymptotically
$f(n) = \Theta(g(n))$ so $\lim_{n\to\infty} \frac{f(n)}{g(n)} =$ constant, which constant is bigger than 0

So From the question, $\lim_{n\to\infty} \binom{2m}{m} / (4^m/\sqrt{m}) \Rightarrow \lim_{n\to\infty} \frac{\binom{2m}{m} \times \sqrt{m}}{4^m}$

Since $\binom{x}{y} = \frac{x!}{y! \times (x-y)!}$,

$\Rightarrow \lim_{n\to\infty} \frac{(2m)! \times \sqrt{m}}{(m!)^2 \times 4^m}$ $\Rightarrow$ Substitute Stirling approximation.

$$\Rightarrow \lim_{n\to\infty} \frac{(2m/e)^{2m}\sqrt{4\pi m} \times \sqrt{m}}{((m/e)^m\sqrt{2\pi m})^2 \times 4^m} \Rightarrow \lim_{n\to\infty} \frac{2^{2m} \cdot m^{2m}\sqrt{4\pi m^2} \times e^{2m}}{e^{2m} \cdot m^{2m}(\sqrt{2\pi m})^2 \times 4^m} \Rightarrow \lim_{n\to\infty} \frac{2m\sqrt{\pi}}{2\pi m}$$

$$\Rightarrow \lim_{n\to\infty} \frac{\sqrt{\pi}}{\pi} = \lim_{n\to\infty} \sqrt{\frac{1}{\pi}}.$$

Since $\lim_{n\to\infty} \frac{f(n)}{g(n)}$ has a constant value $\sqrt{\frac{1}{\pi}}$, $\binom{2m}{m} = \Theta(4^m/\sqrt{m})$

Exercise 5.20 Using the prime number theorem, show that $\vartheta(x) \sim x$.

From Chebyshev's theta function we have

$$\vartheta(x) = \sum_{p \le x} \log p$$

Use theorem 5.5 $\vartheta(x) = \sum_{p \le x} \log p \le \log x \sum_{p \le x} 1 = \pi(x) \log x$

also from prime number theorem, $\pi(x) \sim x/\log x$

$$\vartheta(x) = \sum_{p \le x} \log p \le \pi(x) \log x = (x/\log x) \cdot \log x = x$$

so $\vartheta(x) \sim x$.