CS 131 - Fall 2019, Assignment 20

Problem 1

$a, b \in \mathbb{Z}$, $n \in \mathbb{Z}^+$ define $a \equiv b \pmod{n} \leftrightarrow n \mid (a-b)$

$a \equiv b \pmod{n}$ $\quad$ $a+c \equiv b+d \pmod{n}$

$c \equiv d \pmod{n}$ $\quad$ $ac \equiv bd \pmod{n}$

a) prove that if $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, then $a+c \equiv b+d \pmod{n}$

By $a \equiv b \pmod{n}$, we know $(b-a)$ is multiple of n
so $(b-a) = sn$ for some integers s,
and
By $c \equiv d \pmod{n}$, we know $(d-c)$ is also multiple of n
so $(d-c) = tn$ for some integers,

with the same principle $a+c \equiv b+d \pmod{n} = (b+d-(a+c)) \mid n$
which means it is also multiple of n. $(b+d-a-c) = zn$ for some integer z

$b = sn + a$, $d = tn + c$
$\therefore$ Substitute $sn+a$ to b and $tn+c$ to d.
$sn + a + tn + c - a - c = zn$
$sn + tn = zn$
$(s+t)n = zn$ $\quad$ It is true.

b) prove that if $a \equiv b \pmod n$ and $c \equiv d \pmod n$, then $ac \equiv bd \pmod n$

From problem a we get $b-a = sn$, $d-c = tn$ for some integers s and t.

$ac \equiv bd \pmod n$ if and only if $n \mid bd-ac$ so $(bd-ac) = zn$ for some integer z.

Substitute $b = a+sn$ and $d = tn+c$ to b and d is that

$$(a+sn)(c+tn) - ac = ac + atn + scn + stn^2 - ac$$
$$= (at+sc)n + stn^2 = n(at+sc+stn) \text{ so}$$
$$z = at+sc+stn \text{ and statement is proven } \checkmark$$

c) prove that if a is 1. non-zero integer 2. n is a positive integer and a is coprime with n, then the solution to $ax \equiv 1 \pmod n$ is unique modulo n. That is, if x and y are two solutions.

By Bezout $sa + tn = 1$, s and $t \in \mathbb{Z}$, a and n are coprime

$sa \equiv sa \bmod n$
$tn \equiv 0 \bmod n$
thus $sa+tn \equiv sa \bmod n$ by proof in a)
Because of $sa+tn \equiv sa \bmod n$ and $sa+tn = 1$, $sa \equiv 1 \bmod n$.
There exists a value s such that $sa \equiv 1 \bmod n$.

Prove of uniqueness.
contradiction to Assume that $b \neq s$, such that $b \,\&\, s \in \mathbb{Z}$ and $ab \equiv 1 \bmod n$ and $as \equiv 1 \bmod n$.
$as - ab = 1 \bmod n - 1 \bmod n$
$= a(s-b) \equiv 0 \bmod n$

Thus, $n \mid a(s-b)$. Since a and n are coprime, $n \mid (s-b)$. Assume $s > b$, since $0 \leq s \leq n$ and $0 \leq b \leq n$ the biggest value for $(s-b)$ is $n-1$ and the smallest it is 1.

No integer between 1 and $n-1$ divided n contradiction

problem 2

a) 12 colors, 3 sizes, 2 versions.

$12 \times 3 \times 2 = \boxed{72 \text{ types}}$

b) total 23 Latin letters,

$23 \times 22 \times 21 = \boxed{10,626}$ ways

C) for length 6

1 or 0 2way 2way 2way 2way 2way $= 2^6$

1 or 0 2 2 2 2 $= 2^5$

⋮

$2^6 + 2^5 + 2^4 + 2^3 + 2^2 + 2$

$= 64 + 32 + 16 + 8 + 4 + 2 = 126$ bit strings

D) ①②③④

there are 4 letters string so 26^4 and subtract when it does not have x in it so 25^4

$\boxed{\therefore 26^4 - 25^4}$

E)

1, 2, 3, 4, 5, 6, 7, 8, 9, 10

1, 2, 3, 4, 5, 6

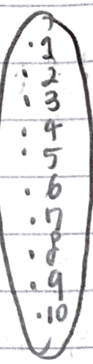
each 10 element has 6 ways of range

$\therefore \boxed{6^{10}}$

f) Surjective functions

$= \forall y \in B, \exists x \in A, f(x) = y$

which means all of the range (2 elements has corresponding domain)



all of 10element has 1st element → 1

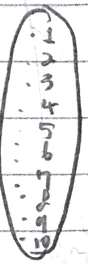
// 2nd element → 1

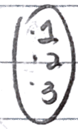
total the are 2^{10} so

$\boxed{2^{10}-2}$

$\therefore \boxed{2^{10}-2}$

g)





• There are total 3^{10} cases.

• Out of that subtract when all of 10 element has one element as their range

$\boxed{3}$

• And there is case when all of the element has only two out of 3 elements as range

$= (2^{10})\,3 \rightarrow (1,2)\ (2,3),\ (1,3)$

$\boxed{\therefore 3^{10} - 3(2^{10}) - 3}$

h) There are n socks of different color

so you have n choices of left and also n choices for right

so first choice n and the second choice n-1

$\boxed{\therefore n(n-1)}$

$C(n,r) = \frac{n!}{r! \times (n-r)!}$

$P(n,r) = \frac{n!}{(n-r)!}$

i) Since the order matter for this question

$\boxed{\frac{n(n-1)}{2}}$

j) $\frac{(240) \times (239) \times (238) \times (237) \times 236}{4 \times 3 \times 2 \times 1}$ 0 0 0 0

$= 134810340$

k) 1, 9 → choose 3 in between 2, 3, 4, 5, 6, 7, 8 $\frac{7!}{3!\,4!} = \frac{7 \times 6 \times 5}{3 \times 2 \times 1} = 35$

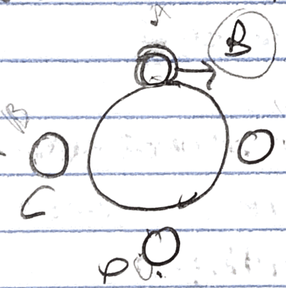
2, 8 → choose 3 in between 3, 4, 5, 6, 7 $\frac{5!}{3!\,2!} = \frac{5 \times 4}{2} = 10$

3, 7 → choose 3 in between 4, 5, 6 → 1.

4, 6 → No way

$35 + 10 + 1 = 46$

l)



10 people

$\frac{10 \times 9 \times 8 \times 7}{4} = 1260$

m) 00000 in 10 bits or 11111

00000XXXXX

X00000XXXX

XX00000XXX

XXX00000XX

XXXX00000X

XXXXX00000 $\boxed{6} + 5 = \boxed{11}$

Problem 3.

a) 10 red balls R B r B r
10 Blue balls. 4/8 5/9

It needs at least $\boxed{5}$ select to have at least three balls
so I need to prove 4 select is inefficeint and 5 select is efficient.
when there are 4 selects, with 2 blue balls and 2 red balls is not meet the statement ✓

With the pigeon hole principle, ball is pigeon and color is hole. In the pigeon hole principle when there is n holes and $n \cdot k+1$ pigeons. There must be a hole with $k+1$ pigeons
In this if we have 5 selects then we have 5 pigeons and 2 holes (2 colors) so we have 2 holes and $2 \cdot 2+1$ pigeons, ($k=2$)
so there must be a hole with $k+1 = 3$ pigeons which is 3 same color ball

$\boxed{\text{5 selects}}$

Since there are 10 red balls and 10 Blue balls,
we are sure to know if we have 13 selects. (First 10 are red balls), last 3 are blue balls. then we have 3 blue balls

$\boxed{\therefore \text{13 selects}}$

b) {1, 3, 5, 7, 9, 11, 13, 15} -5

(1, 15), (3, 13) (5, 11), (7, 9) so there are only 4 pairs that adds up to 16.

Since there is 4 possibles pairs to make sum of 16, we can use pijeon hole principle. 4 possible pairs will be 4 different holes and the minimum selection of 5 number will be pijeon. From the pijeon hole principle when there is n holes, then there will be n·k+1 pijeons. There is 4 holes so n=4 and k=1, so there must exist one hole with k+1 = 2 numbers which makes up 16
∴ the miniumm choice is 5

$\boxed{\therefore \text{ 5 numbers selected}}$

c) sum up to 9 → (1, 8), (2, 7), (3, 6), (4, 5)
sum up to 27 → (9, 18), (10, 17), (11, 16), (12, 15), (13, 14)
There are 4 pairs to sum up to 9, and 5 pairs to sum up to 27.

But 1~8 does not affect the pairs which sum up to 27. Same as the question b, the pijeon hole principle which has n holes and n·k+1 pijeon, there will be at least one hole which has k+1 pijeon.

In this problem there are 4 holes to sum up 9, and 5 holes to sum up to 27. so at least one hole from 4 pairs to sum up 9 has to have 2 pijeons and one hole from 5 pairs to sum up 27 has to have 2 pijeons. The minimum pijeons you need to sure that one pairs from each sum up 9 and 27 is that when u have 15 pijeons (select). so if I breake to two parts which 5 and 10, 5 pijeons will make sure at least one hole from 4 pairs hole of sum up 9 has two pijeons, so It will have one pair and 10 pijeons will meet pair in 5 pairs of sum up 27. so the condition is met

$\therefore$ 15 selection

problem 4.

a) Image a set of $n+1$ element, the choose 2 of them from the set. Start from 1, 1 can pair with 2, 3, 4... n, which is n pairs. Then 2, It can pair with 3, 4, 5... n, which is $n-1$ pairs. If we keep do this and the total possibility, which is sum of possible pair is $n+(n-1)+(n-2)+(n-3) \cdots +1$
$= (1+2+3+4+\ldots+n)$ so it is the same.
prove is done.

b) $k\binom{n}{k} = n\binom{n-1}{k-1}$

In this question looking at LHS (left hand side), we choose the group of k elements from the group of n element. From this group $\binom{n}{k}$ we choose a certain one element which represent $\binom{k}{1}$. With the product rule, It is $\binom{k}{1}\binom{n}{k}$

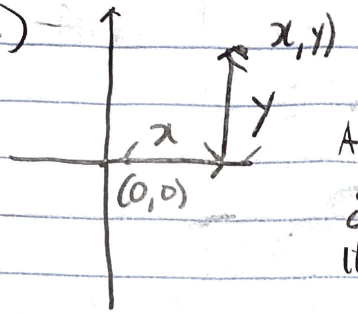
Looking at Right hand side $n\binom{n-1}{k-1}$, we also choose one element But since we want a set of k term and we already choose one we choose an $k-1$ element of set out of the $n-1$ element set. so which is $\binom{n-1}{k-1}$ By the product rule this is $\binom{n}{1} \cdot \binom{n-1}{k-1}$

The Right and left hand side represent how to pick a certain element in a group. in two different ways.

$\binom{k}{2}\binom{n}{2}$

c) $\binom{2n}{2} = 2\binom{n}{2} + n^2$

problem 5.

a) 

from $(0,0)$ to (x,y) we have to move x times to the right and y times to up. And it does not matter when x sets get place to reach (x,y) so it is $\binom{x+y}{x}$ and it is the sure way as

$\binom{x+y}{y}$

$$\therefore \binom{x+y}{x} = \binom{x+y}{y}$$

b) Start from $B = \{1,2,\cdots n\}$

Subset A of B is 2^n.

when $B = \{1,2,3\ldots n-1\}$

Subset A of B is 2^{n-1}

$\vdots$

$$2^n+2^{n-1}+2^{n-2}\cdots+2^0 = (2^n+1)\frac{n}{2}$$

problem 6

a) RASPBERRY is 9 letters but 3 of R,
1 2 3 4 5 6 7 8 9
(Rs are indistinguishable)

, so it is $\boxed{\frac{9!}{3!}}$

b) In Terrier, it is 7 letters, but 3 rs, 2 es, so
(es and rs are indistinguishable)

It is $\boxed{\frac{7!}{3! \times 2!}}$

problem 7.

perfect graph is when all of the vertices are connected with every vertices.

a) when there is 3n vertices, number of all of the edges

are $2\binom{2n}{2} - \binom{n}{2} + 1$

b) $_{2n}C_3 + {}_{2n}C_3 - {}_{n}C_3 + {}_{n}C_3$

$= 2 \cdot {}_{2n}C_3 = \frac{4n(2n-1)}{3}$

problem 8

a) To prove equivalence relation, we need to prove all 3 factors relations, which is reflexive, Symmetric, and Transitive

Reflexive (a=a) : to go from a point U~U or V~V, there is no need of edges to connect a point to itself, we proved reflexive relation

Symmetric (if a=b then b=a) : to go from U to V from the graph A set of intermediary points are needed. In the case that the backward (V to U), we also can use the same set of points to go backwards.
we have proved the symmetric relation

Transitive (if a=b and b=c then a=c) : suppose we have two paths u to v and V to l exists, Since we have u and V, V and l connected, u and l is connected
we have proved the transitive relation

By proving all 3 relations, we equivalence relation.

In the sake of contradiction

b) Suppose there is a graph exists that has n vertices and n-1 edges with no leaves (tree with no leaves). Then The least number of the sum of all degrees of vertices are at least 2n. (since each vertices have at least 2 edges). However, this sum counted the same edges twice, and since the graph is undirected graph, divide 2n by 2, which the sum of the number of all degrees are n. Then there are at least n edges, which is a contradiction.

C) prove by induction.

Base case:

When there is a tree with 1 vertices, there must be 0 edges (1-1=0).

Base case proved ✓

Induction Step:

There is any tree with n+1 vertices with $n \geq 1$ and we know There is a new Tree graph N by reasoning no simple Circuits. so From the graph O with n vertices which has leaf graph T has to connect new vertices to one of the leaf from graph O to make no Simple Circuits. By the Inductive hypothesis, O has the most n-1 edges, since we add one edges to make graph N, N has at n edges d)

Base case:

when $n=1$, A graph with 1 vertice, has 0 edge is a tree.

Base case is proved.

Induction case:

Assume $p(n)$ has a domain of all n, prove that $p(n+1)$ is True.

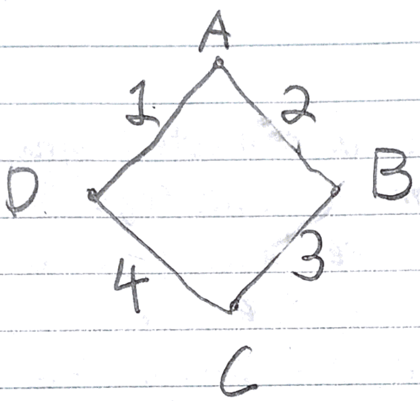
Suppose that we have a connected graph T with $n+1$ vertices and n edges. If we remove one edge from the graph T, it would no longer a tree because it is not connected. However we have created 2 separated graphs, which are subsets of graph T. We can call one graph as graph A and the other one graph B. Because each graph has less than $n+1$ vertices and n edges, we can use inductive hypothesis that graph A, B are connected and they are trees. By adding them back together we have a one graph which is disconnected but a tree. To connect it we have to add one edge previously removed. Then we have connected tree graph.

e) A weighted graph is a graph in which each branch is given a numerical weight.

Suppose that the shortest path between two vertices in a weighted graph is unique if the weights of the edges are distinct.

For an example, in a graph of 4 vertices, 2 sides coordinates do not sum up to the other 2 sides.

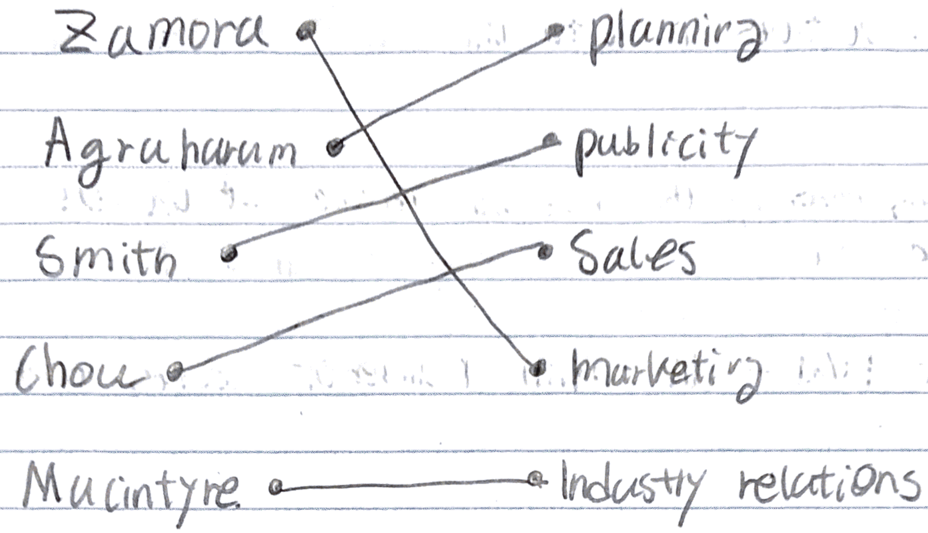
ex)



There are two different ways to get to A to C which go through B or D. These two paths has the same weights on the way to A to C. Through B: 2+3=5, Through D= 1+4=5. Therefore, the shortest path between 2 vertices in a distinct weights of edges is <u>not</u> unique.

problem 9.

A)



B) No it is not possible, becuse there is no one working for marketing

problem 10

Suppose G is a graph (V,E), a bipartite graph with bipartition (V_1, V_2)

Let A be a subset of V_1, $A \subseteq V_1$
number of and points in V_2 from A is N(A). So in a matching |X points in A can have the most $|N(A)|$ edges.
And at most $|X_1| - |A|$ edges can have end points in V_1 those are not in A. so at most $|N(A)| + |V_1| - |A|$ edges will have and points in V_1.

Then $\deg(A) = |A| - N(A)$ the most $|V_1|$ edges will have end points in $V1$, The max number of edges will be when A is bigger subset of V_1.

Next Add deg(A) number vertices in V_2 join all these new vertices in V_2 to all vertices in V_1 result in the new graph T.

It has matching that has all vertices of V_1 as end points. There will be at most deg(A) number of edges in the matching which don't belong to G. so there will be G least $|V_1| - (|A| - N(A))$ number of edges that in G