Proofs

- What is a proof?

- A proof is a valid argument that establishes the truth of a mathematical statement, using the hypotheses of the theorem, if any, axioms assumed tobe true, and previously proven theorems.

- Using these ingredients and rules of inference, the proof establishes thetruth of the statement being proved.

Direct proofs – Final version of a proof versus reasoning work

Question: How do we prove the following theorem?

> **Theorem:** Suppose a, b are real numbers. If $0 < a < b$ then $a^2 < b^2$.

- What is given to us as hypothesis?

- What is the conclusion?

Direct proofs – Final version of a proof versus reasoning work

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Theorem: Suppose a, b are real numbers. If $0 < a < b$ then $a^2 < b^2$.

- What is given to us as hypothesis?

  Givens: a, b are real numbers

- What is the conclusion we want to prove?

  Goal: $P \rightarrow Q$ where P is $0 < a < b$ and Q is $a^2 < b^2$.

Direct proofs – Final version of a proof versus reasoning work

- **Direct proof** technique for $P \to Q$: Add P to the set of hypotheses. Then prove Q.
- **Let's apply it!** What is given to us as hypotheses now?
- **Givens:** a, b are real numbers, $0 < a < b$
- **Goal:** Show that $a^2 < b^2$

Direct proofs – Final version of a proof versus reasoning work

Let's write the formal proof now.

- **Proof:** Suppose $0 < a < b$. Multiplying the inequality $a < b$ by the positive number a we can conclude $a^2 < ab$, and similarly multiplying by b we get $ab < b^2$. Therefore,

$$a^2 < ab < b^2,$$

so, $a^2 < b^2$ as required. **QED**[1]

[1] "quod erat demonstrandum", literally meaning "what was to be shown".

Direct proofs

Prove the following theorems.

1. If n is an odd integer, then n^2 is odd.
2. Suppose m, n are natural numbers. If m, n are both perfect squares, then nm is also a perfect square.

Solutions on blackboard

Proof by contraposition

Question: How do we prove the following theorem?

Theorem: Suppose a, b, c are real numbers, and $a > b$. Prove that if $ac \leq bc$ then $c \leq 0$.

- What is given to us as hypothesis?
- What is the conclusion?

Proof by contraposition

Question: How do we prove the following theorem?

Theorem: Suppose a, b, c are real numbers, and $a > b$. Prove that if $ac \leq bc$ then $c \leq 0$.

- What is given to us as hypothesis?

  Givens: a, b, c are real numbers, $a > b$

- What is the conclusion?

  Goal: $P \rightarrow Q$ where P is $ac \leq bc$ and Q is $c \leq 0$.

Proof by contraposition

- **Proof by contraposition** technique for $P \to Q$: Add $\neg Q$ to the set of hypotheses. Then prove $\neg P$.
- What is given to us as hypothesis?

  Givens: a, b, c are real numbers, $a > b$, $c > 0$

  Goal: $ac > bc$

So, the proof structure using contraposition would look like this:

Suppose $c > 0$

[Proof that $ac > bc$ goes here]

Therefore, if $ac \leq bc$ then $c \leq 0$.

Proof by contraposition

This is how the final/formal proof by contrapositive would look like on the paper:

Theorem: Suppose a, b, c are real numbers, and $a > b$. Prove that if $ac \leq bc$ then $c \leq 0$.

Proof: We will prove by contrapositive. Suppose $c > 0$. Then we can multiply both sides of the given inequality $a > b$ by c and conclude that $ac > bc$. Therefore, if $ac \leq bc$, then $c \leq 0$.

Important remark!

Even if we have used logic in the scratch work, we have not used them in the final form. While logic is essential to figure out a proof strategy, in the final write-up of the proof, mathematicians avoid using the notation and rules of logic.

Direct vs contraposition

- **When do we use** a **direct** proof, and when a proof by **contraposition**?

- Rule of thumb: Evaluate first if a direct proof looks promising. If it does not seem to go anywhere, try alternative strategies, including proof by contraposition.

- **Example:** Prove that if n is an integer, and $3n + 2$ is odd, then n is odd. (Blackboard)

Proof by counterexample

SHORTEST KNOWN PAPER PUBLISHED

March 22, 2017 · *by* randomprojection · *in Think..., Uncategorized* · *Leave a comment*

COUNTEREXAMPLE TO EULER'S CONJECTURE ON SUMS OF LIKE POWERS

BY L. J. LANDER AND T. R. PARKIN

Communicated by J. D. Swift, June 27, 1966

A direct search on the CDC 6600 yielded

$$27^5 + 84^5 + 110^5 + 133^5 = 144^5$$

as the smallest instance in which four fifth powers sum to a fifth power. This is a counterexample to a conjecture by Euler [1] that at least n nth powers are required to sum to an nth power, $n>2$.

REFERENCE

1. L. E. Dickson, *History of the theory of numbers*, Vol. 2, Chelsea, New York, 1952, p. 648.

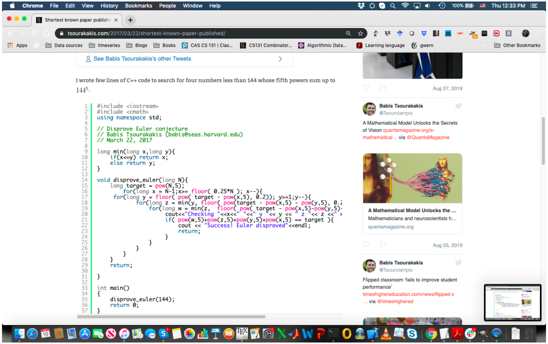
I recently saw a post from OpenCulture, that I explored, and tweeted about: the shortest known paper published in a serious math journal.

Babis Tsourakakis
@Tsourolampis

The Shortest-Known Paper Published in a Serious Math Journal: Two Succinct Sentences goo.gl/x4seZE via @openculture

Remark

Nowadays computers give us a lot of power. A small piece of C++ code would be able to find the counterexample from that paper.



https://tsourakakis.com/2017/03/22/shortest-known-paper-published/

Proof by contradiction

Exercise: Prove that $\sqrt{2}$ is irrational.

Ideas?

What does it mean to be rational to begin with?

Proof by contradiction [Scratch work]

- *Irrational* means not rational, so our goal is a negative statement. This fact already suggests that a proof by contradiction might be the right choice.

- What would it mean for $\sqrt{2}$ to be rational? $\frac{p}{q} = \sqrt{2}$, where $p, q \neq 0$ are integers.

- In general a fra
- *without loss of generality*, we may assume that p, q are both positive (since $\sqrt{2} > 0$), and that the fraction is in lowest terms (i.e., p, q have no common factors)

- What do we infer by squaring?

Proof by contradiction [Scratch work]

- What do we infer by squaring? That both p, q are even!
- By squaring we obtain that $p^2 = 2q^2$.
- This means that p^2 is even, and therefore $p = 2a$ for some integer a, i.e., p is even.
- By substituting $p = 2a$ we obtain that q^2 and hence q is also even since $2q^2 = 4a^2 \rightarrow q^2 = 2a^2$. Therefore $q = 2b$ for some integer b.

Proof by contradiction [Scratch work]

- So we have shown that p, q have to both be even.
- What does this mean?
  - That they share 2 as a common factor
- Therefore, our assumption that 2 is rational ($\neg p$) leads to the contradiction that
  1. 2 does not divide p, q (lower terms)
  2. p, q are even, so 2 divides both of them
- Thus, $\sqrt{2}$ is rational

Proof by contradiction – $\sqrt{2}$ is irrational

Read carefully the way the proof is also written in Rosen, p. 90, 91

- **Remark:** Writing nice proofs requires practice
- Additional reading: Mathematical writing (sections 1,2,3) `http://jmlr.csail.mit.edu/reviewing-papers/knuth_mathematical_writing.pdf`