Joint distributions

Lab 3

Practice Problems

P1. Let X and Y be two jointly continuous random variables with join PDF

joint

$$f_{XY}(x,y) = \begin{cases} x + cy^2 & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & otherwise \end{cases}$$

(a). Find the constant c $\frac{3}{2}$

(b). Find $Pr[0 \le x \le 0.5, 0 \le y \le 0.5]$

$$\int_0^1 \int_0^1 \underline{x + cy^2}\, dx\, dy$$

$$\int_0^1 \left[\frac{c}{3} y^3 + xy \right]_0^1 dx$$

$$= \int_0^1 \frac{c}{3} + x \quad dx$$

$$= \left[\frac{c}{3} x + \frac{1}{2} x^2 \right]_0^1 = \frac{c}{3} + \frac{1}{2} - 0 = 1$$

$$= \frac{2c+3}{6} \qquad c = \frac{3}{2}$$

(b) $$\int_0^{0.5} \int_0^{0.5} x + \frac{3}{2} y^2 \, dx\, dy$$

$$\int_0^{0.5} \left[xy + \frac{3}{2} \times \frac{1}{3} y^3 \right]_0^{0.5} dx$$

$$= \int_0^{0.5} 0.5x + \frac{1}{16} \, dx$$

$$\left(\frac{1}{2}\right)^2$$

$$= \left[\frac{1}{4}x^2 + \frac{1}{16}x \right]_0^{0.5} = \quad \frac{1}{4} \times \frac{1}{4} + \frac{1}{16} + \frac{1}{2} = \frac{1}{16} + \frac{1}{32} = \frac{3}{32}$$

P2. Suppose a straight stick is broken in three at two points chosen independently at random along its length. What is the chance that the three sticks so formed can be made into the sides of a triangle?

Assume $x < Y$

$$\begin{cases} (1-Y) + (Y-x) > x \\ 1 + Y + x > Y - x \\ Y - x + x > 1 - Y \end{cases} = \begin{cases} 1 > 2x \\ 1 > 2(Y-x) \\ 2Y > 1 \end{cases}$$

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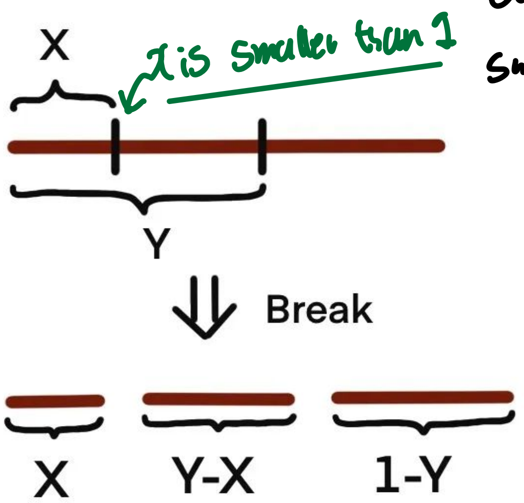
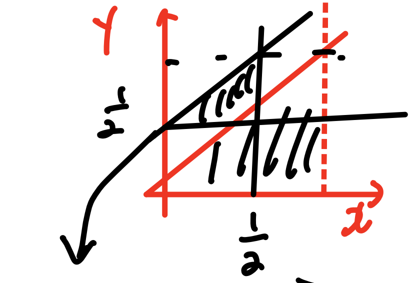
Biggest one has to be smaller than last two



Y

Summation of two has to be bigger than other one.

$x < Y$



$Y > \frac{1}{2}$

$2Y - 2x < 1$

$1 - 2Y < 2x$

$\frac{1}{2} - Y < x$

$Y < x + \frac{1}{2}$ $\rightarrow$ $1 > 2x$

Covariance of X,Y

A measure of how well X and Y vary together.

We are often interested in two or more random variables at the same time.

Consider the following examples:

1. The relationship between height and weight
2. The frequency of exercise and the rate of heart disease
3. Air pollution levels and rate of respiratory illness

Covariance Computation from joint PDF

P3. Let X and Y be continuous random variables with joint pdf

$$f_{X,Y}(x,y) = 3x, \qquad 0 \leq y \leq x \leq 1,$$

and zero otherwise.

And we want to compute the covariance of X,Y. Cov(X,Y)

Definition of Covariance

$$Cov_{f_{X,Y}}[X,Y] \quad = E_{f_{X,Y}}[XY] - E_{f_X}[X]\,E_{f_Y}[Y]$$

Definition of Covariance

$$Cov_{f_{X,Y}}\left[X,Y\right] \quad = E_{f_{X,Y}}\left[XY\right] - E_{f_X}\left[X\right] E_{f_Y}\left[Y\right]$$

Expected Value of XY with joint probability density function (pdf), f of X,Y

Definition of Covariance

$$Cov_{f_{X,Y}}[X,Y] = E_{f_{X,Y}}[XY] - E_{f_X}[X]E_{f_Y}[Y]$$

Expected Value of XY with joint probability density function (pdf), f of X,Y

Expected value of X with pdf, f of X

$$f_{X,Y}(x,y) = 3x, \qquad 0 \le y \le x \le 1,$$

Definition of Covariance

$$Cov_{f_{X,Y}}[X,Y] = E_{f_{X,Y}}[XY] - E_{f_X}[X] E_{f_Y}[Y]$$

Expected Value of XY with joint probability density function (pdf), f of X,Y

Expected value of X with pdf, f of X

Expected value of Y with pdf, f of Y

PDF of x and E[x]

$$f_{X,Y}(x,y) = 3x, \qquad 0 \le y \le x \le 1,$$

and zero otherwise.

$$f_X(x) \;= \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy = \int_0^x 3x dy = 3x^2, \qquad 0 \le x \le 1,$$

$$E_{f_X}[X] \;= \int_{-\infty}^{\infty} x f_X(x)dx = \int_0^1 x \times 3x^2 dx = \left[\frac{3}{4}x^4\right]_0^1 = \frac{3}{4},$$

PDF of Y and E[Y]

$$f_{X,Y}(x,y) = 3x, \qquad 0 \le y \le x \le 1,$$

and zero otherwise.

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx = \int_y^1 3x dx = \left[\frac{3}{2}x^2\right]_y^1 = \frac{3}{2}(1-y^2), \quad 0 \le y \le 1,$$

$$E_{f_Y}[Y] = \int_{-\infty}^{\infty} y f_Y(y) dy = \int_0^1 y \times \frac{3}{2}(1-y^2)dy = \left[\frac{3}{2}\left(\frac{y^2}{2} - \frac{y^4}{4}\right)\right]_0^1 = \frac{3}{2}\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{3}{8},$$

E[XY]

$$f_{X,Y}(x, y) = 3x, \qquad 0 \leq y \leq x \leq 1,$$

and zero otherwise.

$$E_{f_{X,Y}}[XY] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f_{X,Y}(x, y) dy dx = \int_0^1 \int_0^x xy \times 3x dy dx$$

$$= \int_0^1 \left\{ \int_0^x y dy \right\} 3x^2 dx = \int_0^1 \left[\frac{y^2}{2} \right]_0^x 3x dx = \int_0^1 \frac{x^2}{2} \times 3x^2 dx$$

$$= \frac{3}{2} \left[\frac{x^5}{5} \right]_0^1 = \frac{3}{10},$$

Cov[XY] final

$$Cov_{f_{X,Y}}\left[X,Y\right] \quad = E_{f_{X,Y}}\left[XY\right] - E_{f_X}\left[X\right]E_{f_Y}\left[Y\right] = \frac{3}{10} - \frac{3}{4} \times \frac{3}{8} = \frac{3}{160}$$

μ σ

$N \sim (0,1)$

P4. X, Y be independent random variables that have normal distribution ~ (0,1)

U=min(X,Y), V=max(X,Y). Find E[U] and calculate the Cov(U,V)

$E[U \cdot V] - E[U]E[V]$

$E[X]=0$

$E[Y]=0$

$E[V] = E[\max(X,Y)] = E[\frac{1}{2}(X+Y+|X-Y|)$

?

Hints:

$$E[U] = E[\min(X,Y)] = E\left[\frac{1}{2}(X + Y - |X - Y|)\right]$$

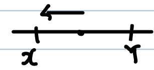
$\frac{1}{2}X + \frac{1}{2}Y$

What is the distribution of X-Y?

$$\Pr_{X \sim N(0,\sigma^2)}[X = x] = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(\frac{-x^2}{2\sigma^2}\right)$$ Normal

$$E_{Z \sim N(0,\sigma^2)}[Z] = \int_{-\infty}^{\infty} z \cdot \Pr[Z = z] dz$$

$E[V] = E[\min(X,Y)] = E[\frac{1}{2}(X+Y - |X-Y|)]$

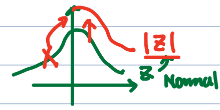


$= \frac{1}{2}(\underbrace{E[X+Y]}_{=0} - E[|X-Y|])$

$X - Y = Z \sim \underline{N(0,2)}$ sum of two variance

$Var[X+Y] = Var[X] + Var[Y] + Cov(X,Y)$

$E[|Z|] = \int_0^\infty \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{z^2}{2\cdot\sigma^2}} \cdot 2z \cdot dz$



$= \int_0^\infty \frac{1}{\sqrt{2\pi 2}} e^{-\frac{z^2}{2\cdot 2}} \; 2z\,dz \rightarrow \frac{dz^2}{z}$

$d(\frac{z^2}{2})$

$w = \frac{z^2}{2}$

$\Rightarrow \boxed{\frac{dz^2}{dz} = 2z} \; ?$

$\frac{dz^2}{2} = z\,dz$

$= \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-\frac{w}{2}} \, dw$

$= \frac{1}{\sqrt{\pi}} [2e^{-\frac{w}{2}}]_0^\infty = \frac{2}{\sqrt{\pi}}$

$\frac{1}{2}(E[X+Y] - E[|X-Y|]) =$

$\frac{1}{2}(0 - \frac{2}{\sqrt{\pi}}) = -\frac{1}{\sqrt{\pi}}$

$\therefore$

$E[V] = \frac{1}{\sqrt{\pi}}$

$Cov(U,V) = E(U - E[U])\cdot(V - E[V])$

$= E[U \cdot V] - E[U] \cdot E[V]$

$= E[XY] - (-\frac{1}{\sqrt{\pi}} \times \frac{1}{\sqrt{\pi}}) = +\frac{1}{\pi}$

0

$E[X] \cdot E[Y]$