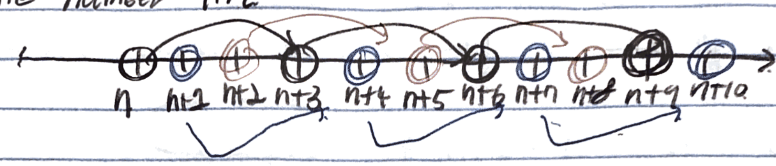
131- Homework #6.

Problem 1. $n \in \mathbb{N}^+$

$p(n) \rightarrow p(n+3)$.

(a) since p(n) is true then p(n+3) is true.
from tle number lire



There are always, 2 numbers are empty so we need to prowe n+1, and n+2 is also true, so tlere are 3 different base cases And they have to be. consecutive number, so it is 2. p(0), p(1), and p(2).

1. no
2. Yes
3. $p(2) \rightarrow p(5) \rightarrow p(8) \rightarrow p(11) \rightarrow \ldots$
   $p(4) \rightarrow p(7) \rightarrow p(10) \rightarrow$
   $p(5) \rightarrow p(8) \rightarrow \ldots$
   NO
4. Yes
   $p(3) \rightarrow p(6) \rightarrow p(9) \rightarrow p(12)$
   $p(5) \rightarrow p(8) \rightarrow p(11) \rightarrow p(14)$
   $p(7) \rightarrow p(10) \rightarrow p(13) \rightarrow \ldots$
   It allo cover $n \geq 5$,

(b)

1. $p(n)$ holds for all $n \geq 5$ $\boxed{NO}$
Since only $p(5)$ holds. so only $5+3n$ for some intege $n \geq 0$ holds

2. $p(3n)$ holds for all $n \geq 5$ $\boxed{NO}$
Sinle $p(5)$ is true (hold), It only holds $p(3n+5)$.

3. $p(n)$ holds for $n = 8, 11, 14, \ldots$ $\boxed{Yes}$
$8 = 3+5$ $11 = 3 \cdot 2 + 5$, $14 = 3 \cdot 3 + 5$ It is $3n+5$
So It is true

4. $p(n)$ does not hold for any $n < 5$. $\boxed{Yes}$
$p(n)$ only hold for 5 from the question

5. $p(3n+5)$ holds for all $n \geq 0$ $\boxed{Yes}$
Since It hold $p(5)$ and if $p(n)$ is true $p(n+3)$ is true, $3n+5$ is true for all $n \geq 0$,

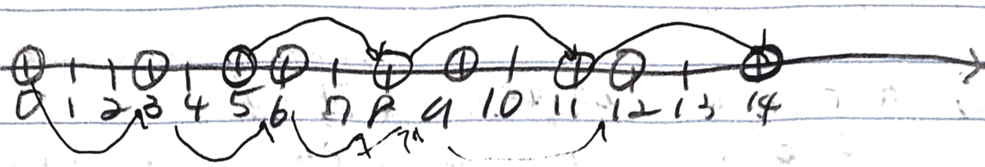
6. $P(3n-1)$ holds for all $n > 2$. $\boxed{Yes}$
When $n = 2$, $6-1 = 5$
$n = 3$, $9-1 = 8$
$n = 4$, $12-1 = 11$
$n = 5$, $15-1 = 14$

$3n - 1 = 3n' + 5$ $n' \geq 0$
$3n = 3n' + 6$
$3n = 3(n'+2)$
$n = n'+2$,
$n' = n-2$ $\therefore n \geq 2$

So It is true for $3n-1$ $n > 2$

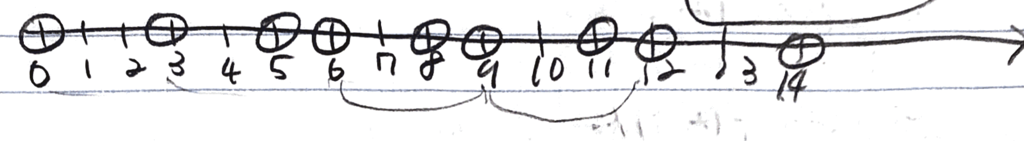
7. $p(0) \to (\forall n \geq 0 . p(3n+2))$. NO

If $p(0)$ is true, then $p(3)$, $p(6)$, $p(9)$, $p(12)$ ... is true.
so $p(5)$ holds at the same time.



when $n=0$ $p(2)$ is not true. so it's false.

8. $p(0) \to (\forall n \geq 0 . P(3n))$. Yes



since $p(0)$ is True all the multiple of 3 is true

problem 2.

a). $F(n,m)$ output: n if $m=0$ $T(n,$
or
$F(n, m-1)+1$

1. $F(10,6)$
$F(10,5)+1$
$F(10,4)+2$
$F(10,3)+3$
$F(10,2)+4$
$F(10,1)+5$
$F(10,0)+6$
$= 10+6$
$= 16$.

2. T: How many steps in Recursion function T

if $T(n,0) = 1$
if $T(n,m), m \neq 0, = T(n,m-1)+1$

so in this case, $T(10,6)$
$T(n,m) = m+1$

Runntime is $\boxed{7}$

prove $T(n,m) = m+1$
basis $T(n,0) = 1$ ✓
induction:
we assume $T(n,m) = m+1$ is true.
prove that $T(n,m+1) = m+2$
$T(n,m+1) = T(n,m)+1 = m+1+1 = m+2$ ✓

3. What does $F(n,m)$ compute?

$F(n,m) = n+m$

b) Let $F(n,m)$ output: 1 if $m=0$

or

$F(n, m-1) * n$

1. Evaluate $F(2,7)$

$= F(2,6) \times 2$

$= F(2,5) \times 2^2$

$= F(2,4) \times 2^3$

$= F(2,3) \times 2^4$

$= F(2,2) \times 2^5$

$= F(2,1) \times 2^6$

$= F(2,0) \times 2^7$

$= 1 \times 2^7$

$= 2^7$

2. T = How many Steps of Recursion

Function T

if $T(n,0) = 1$

if $T(n,m) = T(n,m-1)+1$

so $T(n,m) = m+1$

$p(n) := T(n,m) = m+1$

Basis step $T(n,0) = 0+1 = 1$ ✓

Inductive step

Assume $T(n,m)$ is True. I need to prove $T(n, m+1)$ is true

$T(n, m+1) = T(n,m)+1 = m+1+1 = m+2$ ✓

So the runtime of this Function F(2, 7) is $\underline{8}$

3. what does F(n, m) compute?

$F(n, m) = n^m$

C). function F(n, m) outputs: 1 if $m = 0$

or

if $m \neq 0$ and m is even

$F(n, m) = F(n, m/2)^2$

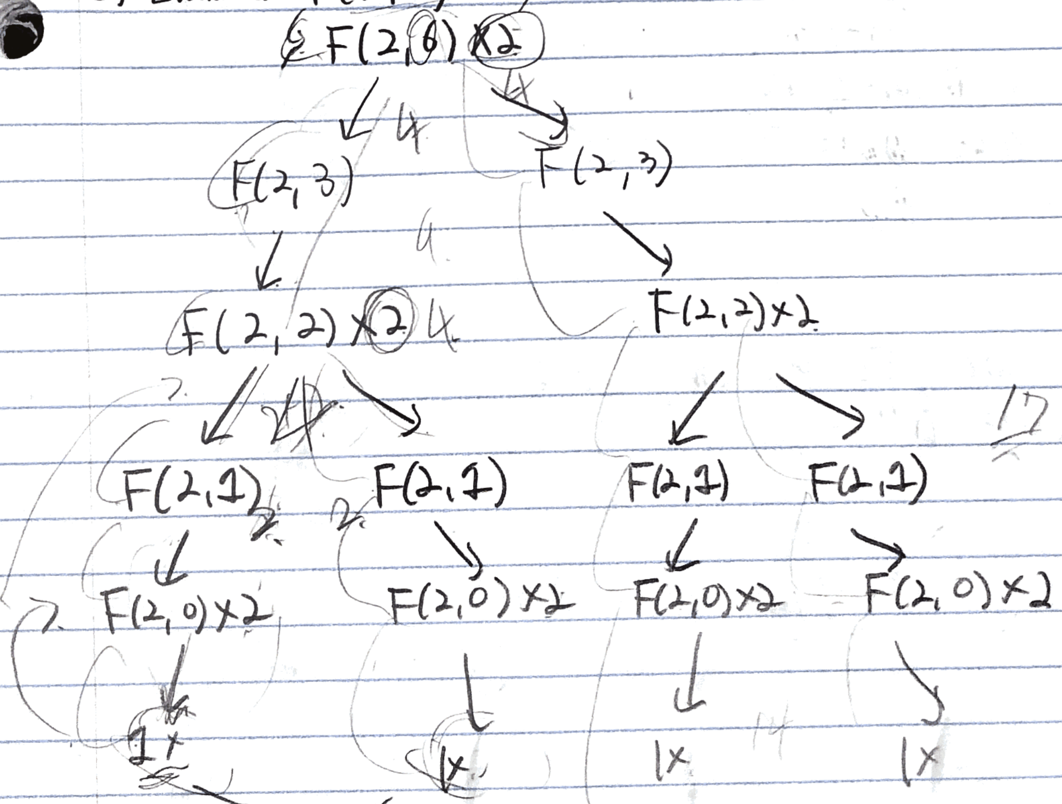
or

if $m \neq 0$ and m is odd,

$F(n, m) = F(n, m-1) * n$

$2^7 = 128$

1) Evaluate F(2, 7)



$= ((1 \times 2)^2 \times 2)^2 \times 2$

$= (8)^2 = \underline{64} \times 2 = 128$

2) White a recursion of the running time and solve it

$$T(n,m) = \begin{cases} T(n, m-1)+1 & m \text{ is odd} \quad \lceil \log_2 m-1 \rceil + 3,\ k\%2=1 \\ T(n,\frac{m}{2}) \times 2 & m \text{ is even} \quad \lceil \log_2 m \rceil + 2,\ k\%2=0 \\ 1 & m \text{ is } 0 \\ 2 & m=1 \end{cases}$$

H is true for all $m \in Z^+$ and any integer n
lets prove $T(n, m+1)$

Case 1: If $m+1$ is odd
$T(n, m+1) = T(n,m)+1$
Since $m+1$ is odd m is even
so $T(n, m+1) \overset{IH}{=} (\lceil \log_2 m \rceil + 2) + 1$
$\lceil \log_2 m \rceil + 3 = \lceil \log_2 m \rceil + 3$ ✓

Case 2: If $m+1$ is even
$T(n, m+1) = T(n, \frac{m}{2}) + 1$

3. What does $F(n,m)$ compute ?
$F(n,m) = n^m$

problem 3
Country X, one way road.

P(n): A country X with n cities, there exists a city that can be reached by all cities either directly or via a route that goes through at most one other city in country X which has n cities

Base case

P(2) $\bullet A \longrightarrow \bullet B$

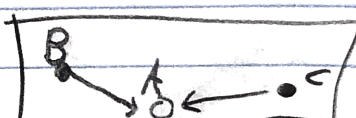
B can be reach from A directly ✓

P(2): Suppose you have two cities A and B in country X such that you have a one way road from A to B. from the picture above. In this case, there exists city B which acts as a hub for all the other cities (A) in that it can be reached from A directly

<u>Inductive step</u>

$P(n) \to P(n+1)$

we assume that there exists a city out of N cities in country X that is reachable either directly or via a route that goes through at most one another city. There are three possible cases for where the n+1 city could go.

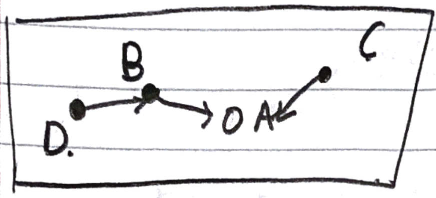


Case 1

The added city belongs to the set of cities that are directly linked to the center hub. Suppose we have arbitrary cities B and C such that B has a road leading to A. B is the set of cities that are directly linked to A, By Adding a city C to this set, we are also directly linking it to B thus B is still our center hub in this case

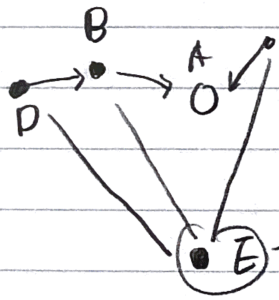
Case 2)

The added City belongs to the set of cities that we 1 stop (2 hops) away. In this case, the center city or hub would still be A because all the New City from case 1 is connected to A as a hub. Using the previous example, by adding another city with a one way road to B would now make it two hops away from the center A. Thus the center remains the same as in case 1



Case 3)

The added city is neither directly linked to the hub city nor is it one stop away from the hub city. In this case, the added city would now have a one way road from all other cities, making it one hop away and would now be 2 hops away from the previous center A. This newly added City will become the new hub.



You don't have to connect City A since It is 1 hop away if you connect all other city except A.

- New city

Problem 4.

For which value of n can be a group of people be divided into teams, where each team consists of exactly 4 or 7 people? Use Induction.

To fulfill that n is divided to group of 4 or 7 people, which means n need to be represented in the form of $n = 4 \cdot x + 7 \cdot y$ where x and y are number of 4 people and 7 people groups respectively.

We start from trying to construct Inductive Step.

For p(n) to be true, we need to have a representation of n+1 $= 4x' + 7y'$ which previous cases could help us to make an inference? we could subtract 4 or 7 from this amount. Thus, if we find representation $n-4 = 4 \cdot x'' + 7 \cdot y''$ or $n-7 = 4 \cdot x''' + 7 \cdot y'''$ we can add to them 4 or 7 respectively and get the representation for n.

Thus we have Induction step $(p(n-4) \vee p(n-7)) \to p(n)$. Thus, if we have p(n) being true for 4 consecutive integers, the statement will be true for any integer greater or equal those 4 numbers.

However, because we have group of two different number, if we have representation for first two numbers, it will be true for the second pair of numbers. Now we need to start search for 2 integers starting from 0, It is true for 18, 19, 20, 21 so it works for all number of $\underline{\underline{n \geq 18}}$.