CS 131 #5

problem 1

a) $\sum_{i=1}^{\infty} i^3$

b) $\sum_{i=1}^{10} 3i-1$ 30

c) $\sum_{i=1}^{5} (-1)^i \times (3i+5)$

D) $\sum_{i=0}^{\infty} (-1)^i \times 3(i+1)$

E) $\sum_{i=0}^{3} (-1)^i \times \frac{1}{(2i+3)}$

f) $\prod_{i=0}^{4} \frac{1}{(4)^i}$

g) $\sum_{i=0}^{n} (n-i)$

h) $\sum_{i=1}^{5} \prod_{j=1}^{4} f(i,j) = \sum_{i=1}^{5} \prod_{j=1}^{4} (j \times i)$

$i=1$ $j=1$ 2

$j=2$ 4

$j=3$

j 4

$i=2$

problem 2

a) $$\sum_{i=1}^{10}\left(\sum_{j=1}^{5} j^2 i\right) = \sum_{i=1}^{10}(i+4i+9i+16i+25i)$$

26

5 14 30

$$= \sum_{i=1}^{10}(55i)$$

$$= 55\sum_{i=1}^{10} i = 55(1+2+3+\dots+10)$$

$$= 3025$$

b) $$\prod_{i=1}^{3}\frac{1}{i^{2i}} = \left(\frac{1}{1^2}\times\frac{1}{2^4}\times\frac{1}{3^6}\right)$$

$$= \frac{1}{1}\times\frac{1}{16}\times\frac{1}{729} = \frac{1}{11664}$$

$$= 8.573\times 10^{-5}$$

c) $$\sum_{i=1}^{3}\left(\prod_{j=1}^{3}\frac{i}{j}\right) = \sum_{i=1}^{3}\left(\frac{i}{1}\times\frac{i}{2}\times\frac{i}{3}\right)$$

$$= \sum_{i=1}^{3}\left(\frac{i^3}{6}\right)$$

$$= \frac{1^3}{6}+\frac{2^3}{6}+\frac{3^3}{6}$$

$$= \frac{1+8+27}{6} = \frac{36}{6} = 6$$

d) $\prod_{i=1}^{3}\prod_{j=1}^{3}\left(\frac{j+i}{2j}\right) = \prod_{i=1}^{3}\left(\frac{1+i}{2}\times\frac{2+i}{4}\times\frac{3+i}{6}\right)$

$2+i+2i+i^2$

$(i^2+3i+2)(3+i)$

$3\cancel{i^2}+i^3+9i+\cancel{3i^2}+6)$

$= i^3+6i^2+11i+6$

$= \prod_{i=1}^{3}\left(\frac{(1+i)(2+i)(3+i)}{48}\right)$

(i^2+3i+2)

$= \prod_{i=1}^{3}\left(\frac{(2+i+2i+i^2)(3+i)}{48}\right)$

$= \prod_{i=1}^{3}\left(\frac{3i^2+i^3+9i+3i^2+6+2i}{48}\right)$

$= \prod_{i=1}^{3}\left(\frac{i^3+6i^2+11i+6}{48}\right)$ 114

7 18 | 32+22 54 | 81+33+6

$= \left(\frac{1+6+11+6}{48}\times\frac{8+24+22+6}{48}\times\frac{27+54+33+6}{48}\right)$

$= \frac{(24)\times(60)\times(120)}{48^3}$

$= \frac{172800}{110592}$

$= \frac{25}{16} \approx 1.5625$

problem 3

a) Use induction to prove that for all $n \geq 0$, $f_0 + f_1 + \cdots + f_n = f_{n+2} - 1$

proof

$P(n) = (f_0 + f_1 + f_2 + \cdots + f_n) = \sum_{i=0}^{n} f_i = (f_{i-1} + f_{i-2}, f_0 = 0, f_1 = 1) = f_{n+2} - 1$

for all integer $n \geq 0$

Basis prove the smallest n that bigger than 0 and 1 (since $f_0 = 0, f_1 = 1$.) which is 2

$$\sum_{i=0}^{2} f_i = 0 + 1 + (f_1 + f_0)$$
$$= 0 + 1 + 1 + 0$$
$$= 2$$
$$= \boxed{f_4 - 1}$$

$f_{i-2} + f_{i-1}$

$f_4 = f_3 + f_2 = (f_2 + f_1) + f_2 = 2 + 1 + 2 = 5$

$f_4 - 1 = 5 - 1 = 4$

$$\sum_{i=0}^{2} f_{i-1} + f_{i-2} = 4 = f_4 - 1$$

Induction step Assume $p(n)$ is true for some integer $n \geq 0$,

i.e, $\sum_{i=0}^{n} f_{i-1} + f_{i-2} = f_{n+2} - 1$

we will prove $p(n+1)$ is true

$P(n+1) = f_{n+3} - 1 \quad P(n) + f_{n+1}$

$$\sum_{i=0}^{n+1} (f_{i-1}+f_{i-2}) = \sum_{i=0}^{n} f_{i-1}+f_{i-2} + (f_{n+1-1}+f_{n+1-2})$$

$$\overset{IH}{=} f_{n+2} - 1 + f_{n+1}$$

$$= \underbrace{f_{n+2}+f_{n+1}} - 1$$

$$= f_{n+3} - 1$$

$$= f_{n+3} - 1$$

b) Use Induction to prove that for every $n \geq 2$, $f_n \geq (1.5)^{n-2}$

proof

$f_n = f_{n-1} + f_{n-2} \geq (1.5)^{n-2}$ for every $n \geq 2$

Basis 1 step.

prove f_2 is true

$$f_2 \geq (1.5)^{n-2} = f_{2-1}+f_{2-2} \geq (1.5)^{2-2}$$

$$= f_1 + f_0 \geq (1.5)^0$$

$$= 1+0 \geq$$

$$= 1 \geq 1$$

$\therefore f_n \geq (1.5)^{n-2}$ is true

☆ Basis 2

prove f_3 is true.

$f_3 = f_2 + f_1$, $f_2 + f_1 \geq (1.5)^{3-2}$
$= 2 + 1 \geq (1.5)^1$
$= 3 \geq (1.5)$

$\therefore f_n \geq (1.5)^{n-2}$ is true

Induction step

$P(k) \land P(k-1) \rightarrow P(k+1)$

we assume $P(k) \land P(k-1)$ is true then $P(k+1)$ is true

Since $P(k) = f_k \geq (1.5)^{k-2}$ and $P(k-1) = f_{k-1} \geq (1.5)^{k-3}$

$P(k) + P(k-1) = P(k+1)$ from fibonacci

so $f_k + f_{(k-1)} \geq (1.5)^{k-2} + (1.5)^{k-3}$
$f_{(k+1)} \geq (1.5)^{k-3}(1 + 1.5)$
$\geq (1.5)^{k-3}(2.5)$

Since $P(k+1) = f_{k+1} \geq (1.5)^{k-1}$,
(2.5) is bigger than $(1.5)^2$

$f_{k+1} \geq (1.5)^{k-3}(2.5) \geq (1.5)^{k-1}$
$(1.5)^{k-3} \times (1.5)^2$
(2.25)

the prove is done QED

c). Use Induction to prove that for every $n \geq 0$, $f_n \leq 2^{n-1}$

prove, for every $n \geq 0$, $f_n = f_{n-1} + f_{n-2} \leq 2^{n-1}$

Basis case 1

prove $P(0)$ is true

$n = 0$

$f_0 \leq 2^{0-1}$ $\qquad 2^{-1} = \frac{1}{2}$

$0 \leq \frac{1}{2}$ is true ✓

Basis case 2

prove $P(1)$ is true

$n = 1$

$f_1 \leq 2^{1-1}$

$1 \leq 2^0 = 1$

$1 \leq 1$ is true ✓

Induction step

we assume if $P(k) \wedge P(k-1)$ is true then $P(k+1)$ is true

$f_{k+1} \leq 2^k$

Since $P(k) = f_k \leq 2^{k-1}$ and $P(k-1) = f_{k-1} \leq 2^{k-2}$ and $f_{k+1} = f_k + f_{k-1}$ (Fibonacci)

is true

$\underbrace{f_k + f_{k-1}}_{\text{fibonacci}} \leq 2^{k-1} + 2^{k-2}$

$f_{k+1} \leq 2^{k-2}(1+2)$

$f_{k+1} \leq 2^{k-2}(3)$

we know f_{k+1} is smaller or equal to $2^{k-2} \cdot 3$ which is smaller than 2^k. Therefore for every $n \geq 0$, $f_n \leq 2^{n-1}$ is true QED.

problem 4.

a). Use Induction to prove $\forall n \in \mathbb{Z}^+ \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

For every positive integer n, $n \geq 1$, $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

Basis step

we prove for the smallest n, 1, $P(n)$ is true

$$P(1):= \sum_{i=1}^{1} i^2 = 1 = \frac{(1)(2)(3)}{6} = \frac{\cancel{6}}{\cancel{6}} = 1$$

The base case is true ✓

Induction Step

we assume $p(n)$ is true, then prove $p(n+1)$ is true

$$P(n) = \sum_{i=1}^{n} i^2 = 1^2+2^2+3^2 \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$P(n+1):= \sum_{i=1}^{n+1} i^2 = \underbrace{1^2+2^2+3^2 \cdots + n^2}_{\frac{n(n+1)(2n+1)}{6}} + (n+1)^2$$

n^2+2n+1

$$\frac{n(n+1)(2n+1)}{6} + (n+1)^2 \overset{IH}{=} \frac{2n^3+n^2+2n^2+n+6n^2+12n+6}{6}$$

$(n^2+n)(2n+1$

$$= \frac{2n^3+9n^2+13n+6}{6}$$

$(n^2+3n+2)(2n+3)$

$$\frac{2n^3+3n^2+6n^2+9n+4n+6}{6} = \frac{(n+1)(n+2)(2n+3)}{6}$$

Therefore, we prove that for every $n \geq 1$, $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

b.) Use Induction to prove $\forall n \in \mathbb{Z}^+ \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$

For every positive Integer n, $n \geq 1$, $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$

Basis step

we prove that for the smallest n, 1 $p(n)$ is true

$$P(1) := \sum_{i=1}^{1} i^3 = 1^3 = 1 = \left(\frac{1(2)}{2}\right)^2 = 1^2 = 1$$

The base case is true ✓

Induction Step

we assume that $P(n)$ is true, then prove $p(n+1)$ is true.

$$p(n) = \sum_{i=1}^{n} i^3 = 1^3+2^3+3^3+\cdots+n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

$$P(n+1) := \sum_{i=1}^{n+1} i^3 = \underbrace{1^3+2^3+3^3+\cdots+n^3}_{\left(\frac{n(n+1)}{2}\right)^2} + (n+1)^3$$

$$\underbrace{\left(\frac{n(n+1)}{2}\right)^2}_{(n^2+n)^2} + \underbrace{(n+1)^3}_{n^3+3x^2+3x+1} = \frac{n^4+2n^3+n^2}{4} + \frac{4n^3+12n^2+12n+4}{4}$$

$$= \frac{n^4+6n^3+13n^2+12n+4}{4}$$

$n^4+6n^3+13n^2+12n+4$
$= (n^2+2n+1)(n^2+4n+4)$
$= (n+1)^2(n+2)^2$

$$= \frac{(n+1)^2(n+2)^2}{4}$$

$$= \left(\frac{(n+1)(n+2)}{2}\right)^2$$

Therefore, we prove that for every $n \geq 1$, $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$ QED problem

5a. Assume that x, Y, and Z are sets, prove
$X \setminus (Y \setminus Z) \subseteq (X \setminus Y) \cup Z$

$X \setminus (Y \setminus Z) \subseteq (X \setminus Y) \cup Z$

If $x \in X \setminus (Y \setminus Z)$, then $x \in ((X \setminus Y) \cup Z)$

$$
\begin{aligned}
& x \in X \setminus (Y \setminus Z) \\
\equiv\ & (x \in X) \land (x \notin (Y \setminus Z)) \\
\equiv\ & (x \in X) \land \neg (x \in (Y \setminus Z)) \\
\equiv\ & (x \in X) \land \neg ((x \in Y) \land (x \notin Z)) \\
\equiv\ & (x \in X) \land (x \notin Y) \lor (x \in Z) \\
\equiv\ & x \in (X \setminus Y) \lor (x \in Z) \\
\equiv\ & x \in (X \setminus Y) \cup Z
\end{aligned}
$$

∴ This is true.

5b. Let A an arbitrary set, prove by using induction that the cardinality of the power set of A is equal to $2^{|A|}$

let n as number of element in Set A
For set A, the Size of power set is 2^n (# element in powerset)

Predicate
$P(n)$ = For arbitrary set A, of Size n, the power set = 2^n

Basis step

For $P(n)$ let $n = 0$, which means $\{\emptyset\}$ (empty set)

so there is only one set of Size $0 = \emptyset$
The power set for this $\{\emptyset\}$; the cardinality for $\{\emptyset\}$ $= 2^0 = 1$

So for the base step $P(0)$ is true.

QED

Induction step

we assume $p(n)$ is true, then we prove $p(n+1)$ is true.

$n=1$,	$2^1=2$	$n^1=2$
$n=2$	$2^2=4$	$n^2=n^1+n^1$
$n=3$	$2^3=8$	$n^3=n^2+n^2$
$n=4$	$2^4=16$	$n^4=n^3+n^3$
$n=5$	$2^5=32$	

we know for arbitrary set A of size n, the power set has $\underline{2^n}$ subsets.

The element of the set A, $A=\{e_1, e_2, e_3 \cdots e_n\}$.

and for arbitrary set B of size $n+1$, we try to prove the power set has 2^{n+1} subsets.

The elemet of the set B, $B=\{\underbrace{e_1, e_2, e_3 \cdots e_n}_{\text{set } A}, e_{n+1}\}$

so by Inductive hypothesis, $B = A \cup \{e_{n+1}\}$
so every subset of A is subset of B,

so every subset either contains e_{n+1} or not.

when a subset contain e_{n+1}, then it is formed my have one of subset of A and e_{n+1} and if it doesn't contain e_{n+1} then it is the subset of 2^n.

so the number of subset containing e_{n+1} and is not
containing e_{n+1} is both 2^n.

so the # of subset of B is $2^n + 2^n = 2^{n+1}$.