Lecture 8, 9 (9/26, 10/1)
Outline

- Sets and set operations [Rosen 2.1, 2.2]
- Sequences and summations [Rosen 2.4]

Sets

- $\{0, 3, 1\}$ is a set
- $\{0, 1, 3\}$ is a set and it is the same as $\{0, 3, 1\}$
- $(0, 1, 3)$ is not a set
- $\{a, b, c, d, ..., z\}$ is a set
- $\{\{a, b\}, \{b, c\}\}$ is a set
- $\{a, b, b, c\}$ is not a set
- $\mathbb{N} = \{0, 1, 2, \ldots\}$ is the set of naturals
- $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ is the set of integers
- $\mathbb{Z}^+ = \{1, 2, \ldots\}$ is the set of positive integers
- $\mathbb{R}$ is the set of reals

Question: Can you define what a set is?

Sets

Definition: A set is an **unordered collection** of **distinct objects**.

- Some remarks.
  1. These objects are called elements or members of the set.
  2. The elements could be sets themselves, or sets containing other sets etc.!
  3. We write $a \in S$ to denote that a is a member of the set S.
  4. We write $a \notin S$ to denote that a is not a member of the set S.
  5. It may be impractical to define a set by listing all its elements.
     - $P = \{2, 3, 5, 7, \ldots\}$
     - Using dots is a common practice but requires the pattern to be clear.
     - A better practice: $P = \{x | \text{x is a prime number}\}$ (set builder)

Exercise: Rewrite the following sets using the set builder notation.

- $E = \{2, 4, 6, 8, 10, \ldots\}$
- $A = \{\text{Brad Pitt, Matt Damon, Meryl Streep}, \ldots\}$

Sets

Exercise: Rewrite the following sets using the set builder notation.

- $E = \{2, 4, 6, 8, 10, \ldots\}$
  $E = \{n | n \text{ is a positive even integer}\}$

- $A = \{\text{Brad Pitt, Matt Damon, Meryl Streep}, \ldots\}$
  $A = \{z | z \text{ is a Holywood actor}\}$

- The set of rationals
  $\mathbb{Q} = \{\frac{p}{q} | p \in \mathbb{Z}, q \in \mathbb{Z}, q \neq 0\}$

Sets

Three definitions and a question.

1. Subset/superset: The set A is a subset of B (and B a superset of A) if and only if every element of A is an element of B, i.e.,

$$\forall(x \in A \to x \in B).$$

   To denote this, we write $A \subseteq B$.

2. We say that A is a proper subset of B (we write $A \subset B$) if

$$\forall x(x \in A \to x \in B) \land \exists x(x \in B \land x \notin A).$$

3. Equal sets: Two sets A, B are equal if and only

$$\forall x(x \in A \leftrightarrow x \in B).$$

   We write $A = B$. Equivalently, this means A is a subset of B and B is a subset of A

Sets

- **Exercise:** Prove that for any subset S, $\emptyset \subseteq S$. (blackboard)
- Continuing with definitions...
- **Size/cardinality of a set:** If there are exactly n distinct elements, we say that the set is finite and the cardinality is n. We write $|S| = n$ to denote the size. When a set is not finite, it is infinite.
- Can two sets be equal if they have different cardinalities? (blackboard)
- **Power set:** Given a set S, the power set $\mathcal{P}(S)$ is the set of all possible subsets of S.

  Example: What is the power set of $\{0, 1, 2\}$? (blackboard)

Truth set

- A truth set is a special type of a set.
- **Definition:** The truth set of a statement $P(x)$ is the set of all values of x that make the statement $P(x)$ true, i.e.,

$$\text{Truth set of } P(x) := \{x | P(x)\}.$$

- **Example 1:** $P(n) := n$ is an even prime number
  The truth set is $\{2\}$, since 2 is the only even prime number
- **Example 2:** Let $Q(x)$ be $x + 1 = 0$
  - If the domain of x is the set of naturals, the truth set is the empty set $\{\}$ denoted as $\emptyset$.
  - If the domain is the set of integers, the truth set is $\{-1\}$.

Operations on sets

- The intersection of two sets A, B is denoted $A \cap B$ and is defined as follows:

$$A \cap B := \{x | x \in A \text{ and } x \in B\}.$$

- The union of A, B is the set of $A \cup B$ and is defined as follows:

$$A \cup B := \{x | x \in A \text{ or } x \in B\}.$$

- The difference of A, B is the set $A \setminus B$ (also denoted as $A - B$) defined as follows:

$$A \setminus B := \{x | x \in A \text{ and } x \notin B\}.$$

- The complement $\bar{A}$ of a set A is defined as $\bar{A} := \text{Domain} \setminus A$. We refer to the domain frequently as *universe* and we denote it as U.

Venn diagrams

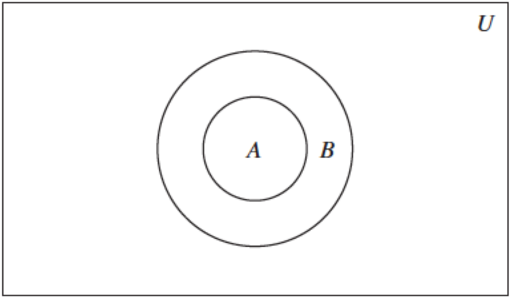
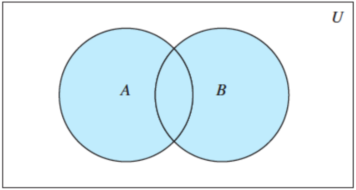


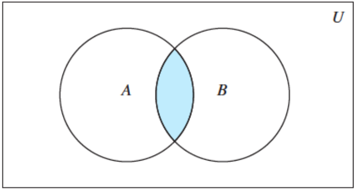
FIGURE 2 **Venn diagram showing that A is a subset of B.**

Venn diagrams



$A \cup B$ is shaded.

FIGURE 1 **Venn diagram of the union of A and B.**



$A \cap B$ is shaded.

FIGURE 2 **Venn diagram of the intersection of A and B.**

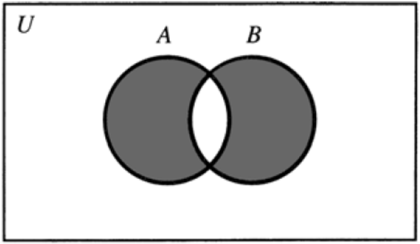
Problems on sets – Exercise

- Suppose $A = \{1, 2, 3, 4, 5\}$, $B = \{2, 4, 6, 8, 10\}$.
  - Visualize the sets using Venn diagrams
  - List the elements of the following sets
    1. $A \cap B$
    2. $A \cup B$
    3. $A \backslash B$
    4. $(A \backslash B) \cup (B \backslash A)$
    5. $(A \backslash B) \cap (B \backslash A)$
  - Prove that $|A \cup B| = |A| + |B| - |A \cap B|$. Generalize.

[Proof on blackboard]

Symmetric difference

- The set $(A \backslash B) \cap (B \backslash A)$ is an important set.
- The corresponding operation is also known as the symmetric difference of A, B and is denoted as $A \triangle B$



$$(A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A)$$

Problems on sets – Exercise

- Let A, B be sets such that $A \cap B = A$. Prove that $A \subseteq B$.

  To prove this, we follow the steps we have seen in class

  1. Read carefully. What is given to you, and what is asked? **Understand the problem**!
  2. Design a proof strategy.
  3. Complete the proof.

- Ideas?

Problems on sets – Exercise

Let's identify what is given, and what we are being asked to prove.

- Givens: $A \cap B = A$
- Goal: $\forall x(x \in A \rightarrow x \in B)$

Therefore, we may design a direct proof, where we consider an arbitrary $x \in A$, and prove $x \in B$.

- Givens: $A \cap B = A$, arbitrary $x \in A$
- Goal: $x \in B$

Problems on sets – Exercise

Therefore a direct proof outline would like this:

- Suppose $A \cap B = A$.
- Choose an arbitrary x
- Prove that if $x \in A$ then $x \in B$
- Since x was arbitrary we can conclude that $A \subseteq B$.

- Now that we have designed the proof, and filled all the details, we write it down nicely.

Proof: Suppose $A \cap B = A$, and $x \in A$. Since $A = A \cap B = A$, $x \in A \cap B$ and therefore $x \in B$ as well. Therefore, $A \subseteq B$. **QED**

Problems on sets – Exercise

- Prove that $\overline{(A \cap B)} = \bar{A} \cup \bar{B}$ (first De Morgan law for sets.)

$$
\begin{aligned}
\overline{A \cap B} &= \{x \mid x \notin A \cap B\} && \text{by definition of complement} \\
&= \{x \mid \neg(x \in (A \cap B))\} && \text{by definition of does not belong symbol} \\
&= \{x \mid \neg(x \in A \wedge x \in B)\} && \text{by definition of intersection} \\
&= \{x \mid \neg(x \in A) \vee \neg(x \in B)\} && \text{by the first De Morgan law for logical equivalences} \\
&= \{x \mid x \notin A \vee x \notin B\} && \text{by definition of does not belong symbol} \\
&= \{x \mid x \in \overline{A} \vee x \in \overline{B}\} && \text{by definition of complement} \\
&= \{x \mid x \in \overline{A} \cup \overline{B}\} && \text{by definition of union} \\
&= \overline{A} \cup \overline{B} && \text{by meaning of set builder notation}
\end{aligned}
$$

Set identities

TABLE 1 Set Identities.

Identity	*Name*
$A \cap U = A$ $A \cup \emptyset = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

try proving a couple for practice, example on blackboard

Sequence

- **Definition:** A sequence is a special type of a function....
  from a subset of the set of integers (usually either the set $\{0, 1, 2, \ldots\}$ or the set $\{1, 2, 3, \ldots\}$)
  to a set S.
  We frequently denote the sequence as $\{a_n\}$.
  We use the notation a_n to denote the image of the integer n.

- a_n is a *term* of the sequence.

- **Question**: what is the n-th term of a sequence?
  - **Answer:** Depends where the index starts from!
    - If the sequence is $a_0, a_1, \ldots$ the n-th term is a_{n-1}
    - If the sequence is $a_1, a_2, \ldots$ the n-th term is a_n

Sequences

- $x_n = \frac{1}{n}, n \in \mathbb{Z}^+$
- $\{y_n\}_{n \geq 0}$ where $y_n = 1 + 2n$

  List the first 3 terms.

  - $y_0 = 1 + 2 * 0 = 1, y_1 = 1 + 2 = 3, y_2 = 1 + 2 * 2 = 5$
- $\{z_n\}_{n \geq 0}$ where $z_n = 10^5 - 31n$

  Arithmetic progression (AP):

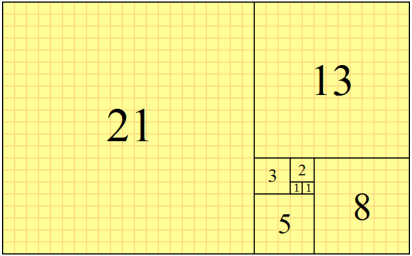
  $x_0 = a, x_1 = a + d, x_2 = a + 2d, \ldots, x_n = a + nd, \ldots$

  a initial term, d difference
- $\{a_n\}_{n \geq 0}$ where $a_n = 3 * 7^n$
  - Geometric progression (GP):

    $x_0 = a, x_1 = ar, x_2 = ar^2, \ldots, x_n = ar^n, \ldots$
  - a initial term, r common ratio

Sequences

- A sequence can be specified via a a recurrence relation.
  We express a_n as a function of one or more previous terms of the sequence:
  - $a_n = a_{n-1} + 3, a_1 = 20$. List the first three terms of the sequence.
    $a_1 = 20, a_2 = a_{2-1} + 3 = a_1 + 3 = 23, a_3 = a_2 + 3 = 26$
    Can we get a **closed formula**? What kind of sequence is $\{a_n\}$?
- Can you define a sequence after looking the next Figure?



Fibonacci sequence

Italian mathematician (12th century)

Definition: The Fibonacci sequence $f_0, f_1, f_2, \ldots$ is defined by the initial conditions $f_0 = 0, f_1 = 1$ and the recurrence

$$f_n = f_{n-1} + f_{n-2}, n = 2, 3, \ldots.$$

List the five first terms of the Fibonacci sequence. Blackboard

https://www.youtube.com/watch?v=DRjFV_DETKQ

Summations and multiplications

- To express the sum of $a_m, a_{m+1}, \ldots, a_n$ we write

$$\sum_{k=m}^{n} a_k,$$

or

$$\sum_{m \leq k \leq n} a_k,$$

or

$$\sum_{k=0}^{n-m} a_{m+k}.$$

- Variable k is index of summation. We could have used any other letter, namely $\sum_{k=m}^{n} a_k = \sum_{i=m}^{n} a_i$.

Summations and multiplications

- Express the following sums and products using the $\Sigma, \prod$ notation:

1. $1 + 2 + 3 + \ldots + 100$
2. $7 + 11 + 15 + 19 + 23$
3. $1 + 3 + 9 + 27 + 81$

   and ... and "outlier" example
4. $100 + 150 + 219 + 220$
5. $1 * 3 * 5 * 7$

Summations and multiplications

- Express the following sums using the Σ notation:
  Rule of thumb: The key is to identify the sequence whose terms we are summing

1. $1 + 2 + 3 + \ldots + 100 = \sum_{i=1}^{100} i$
2. $7 + 11 + 15 + 19 + 23 = \sum_{k=0}^{4}(7 + 4k)$
3. $1 + 3 + 9 + 27 = \sum_{j=0}^{3} 3^j$

   and ... and "outlier" example where there is no clear sequence
4. $100 + 150 + 219 + 220 = \sum_{i \in \{100,150,219,220\}} i$
5. $1 * 3 * 5 * 7 = \prod_{i=1}^{4}(2 * i - 1) = \prod_{i=0}^{3}(2 * i + 1)$

Summations and multiplications

- To express the product of $a_m, a_{m+1}, \ldots, a_n$ we write

$$\prod_{k=m}^{n} a_k,$$

or

$$\prod_{m \leq k \leq n} a_k,$$

or

$$\prod_{k=0}^{n-m} a_{m+k}.$$

Evaluating summations and multiplications

1. $\sum_{i=1}^{100} i$

Carl Friedrich Gauss (1777-1855)

1. 1+100=101
2. 2+99=101

$\vdots$

49. 49+52=101
50. 50+51 =101

Therefore the sum is $50 \times 101 = 5050$.

Evaluating summations and multiplications

1. In general when we sum the n terms of an AP $\{a_n\}$ with initial term a_1 and difference d the sum is

$$\sum_{i=1}^{n} a_i = n\frac{a_1 + a_n}{2} = \frac{n}{2}(2a_1 + (n-1)d).$$

2. Compute the sum $\sum_{i=51}^{100} i$:
   - $\sum_{i=1}^{100} i - \sum_{i=1}^{50} i = 50 * 101 - 25 * 51 = 3775$

     or ... just
   - $\sum_{i=51}^{100} i = \frac{50}{2}(51 + 100) = 25 * 151 = 3775$
3. For summing terms of a geometric series with initial term a and ratio $r \neq 0, 1$ we have (proof [Rosen p. 174, and blackboard])

$$\sum_{j=0}^{n} ar^j = \frac{ar^{n+1} - a}{r - 1}.$$

Higher order sums

- Double, triple and higher order summations appear in many contexts.
- One needs to be careful, i.e., understand what they are summing over.
- Example.

$$\sum_{i=1}^{4}\sum_{j=1}^{3} ij = \sum_{i=1}^{4}(i + 2i + 3i) =$$

$$\sum_{i=1}^{4}(6i) = 6\sum_{i=1}^{4} i = 6(1+2+3+4) = 60.$$

Another double summation

Evaluate $\sum_{i=0}^{2} \sum_{j=0}^{3} (2i + 3j)$.

- The inner sum is:

$$\begin{aligned}
\sum_{j=0}^{3} (2i + 3j) &= \sum_{j=0}^{3} 2i + \sum_{j=0}^{3} 3j \\
&= 2i \sum_{j=0}^{3} 1 + 3 \sum_{j=0}^{3} j \\
&= 2i(4) + 3\left(0 + 1 + 2 + 3\right) \\
&= 8i + 18.
\end{aligned}$$

Another double summation

Evaluate $\sum_{i=0}^{2}\sum_{j=0}^{3}(2i+3j)$.

- The outer sum is therefore:

$$\sum_{i=0}^{2}\left(\sum_{j=0}^{3}(2i+3j)\right) = \sum_{i=0}^{2}(8i+18) = \sum_{i=0}^{2}8i + \sum_{i=0}^{2}18$$

$$= 8\sum_{i=0}^{2}i + 18\sum_{i=0}^{2}1 = 8(0+1+2)+18(3)$$

$$= 24+54 = 78.$$

Lecture 10 (10/3)
Outline

- Mathematical induction [Rosen 5.1]

- **Remark** Everything up to this point (including 5.1) will be tested in midterm 1.