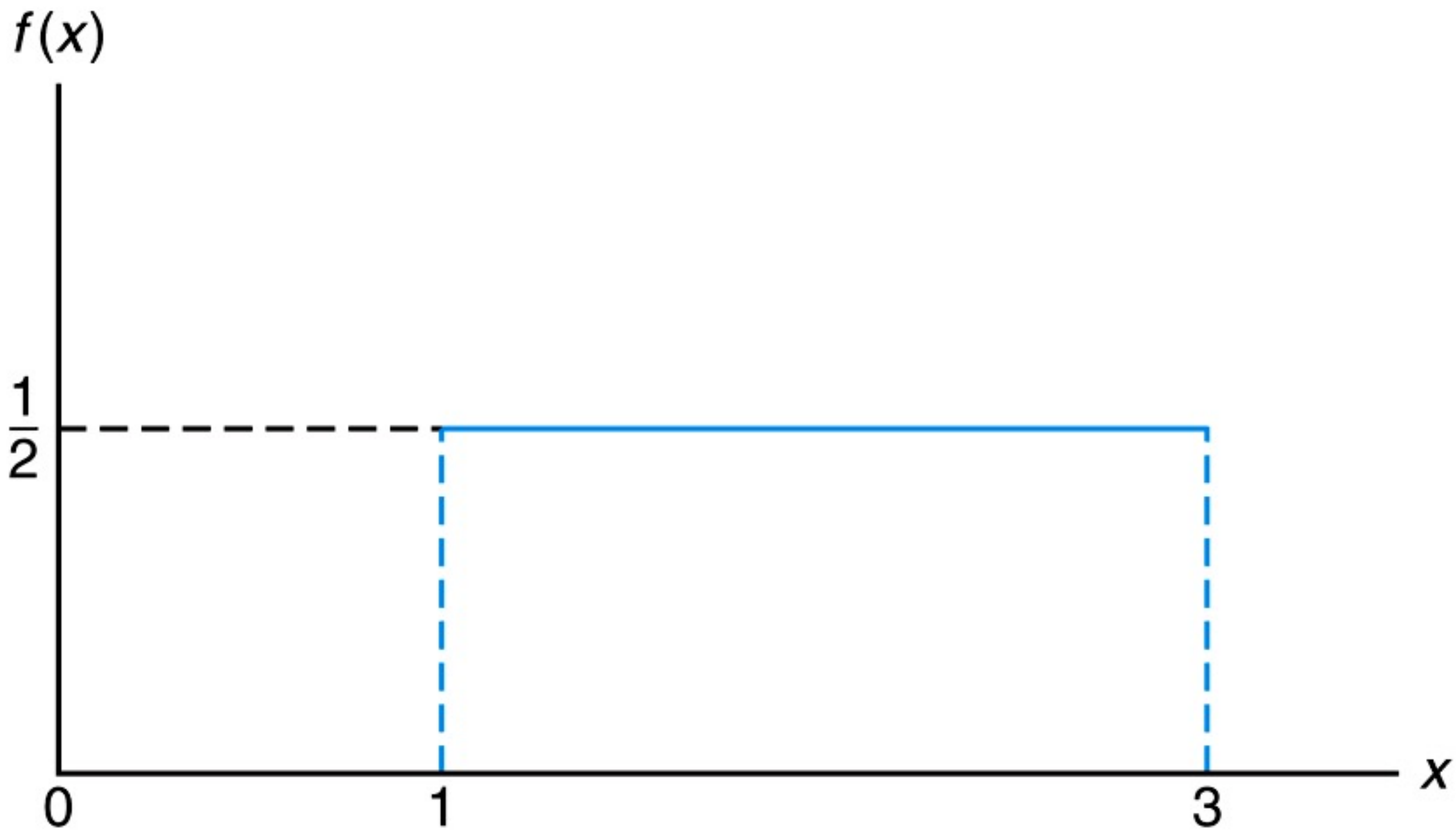
Chapter 6

Some Continuous Probability Distributions

Section 6.1

Continuous
Uniform
Distribution

Figure 6.1 The density function for a random variable on the interval $[1,3]$



Uniform distribution (continuous)

If X has uniform distribution on $[a, b]$, then

$$f(x) = \begin{cases} \frac{1}{b-a}, & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

Uniform distribution (discrete)

- If X has uniform distribution on finite set A, then

$$P(X = a) = \frac{1}{|A|}$$

Theorem 6.1

The mean and variance of the uniform distribution are

$$\mu = \frac{A + B}{2} \text{ and } \sigma^2 = \frac{(B - A)^2}{12}.$$

Theorem 6.1

Example 6.1: Suppose that a large conference room at a certain company can be reserved for no more than 4 hours. Both long and short conferences occur quite often. In fact, it can be assumed that the length X of a conference has a uniform distribution on the interval $[0, 4]$.

(a) What is the probability density function?

(b) What is the probability that any given conference lasts at least 3 hours?

Section 6.2

Normal
Distribution

Normal (Gaussian) distribution

Normal Distribution The density of the normal random variable X, with mean μ and variance σ^2, is

$$n(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}, \quad -\infty < x < \infty,$$

Figure 6.2 The normal curve

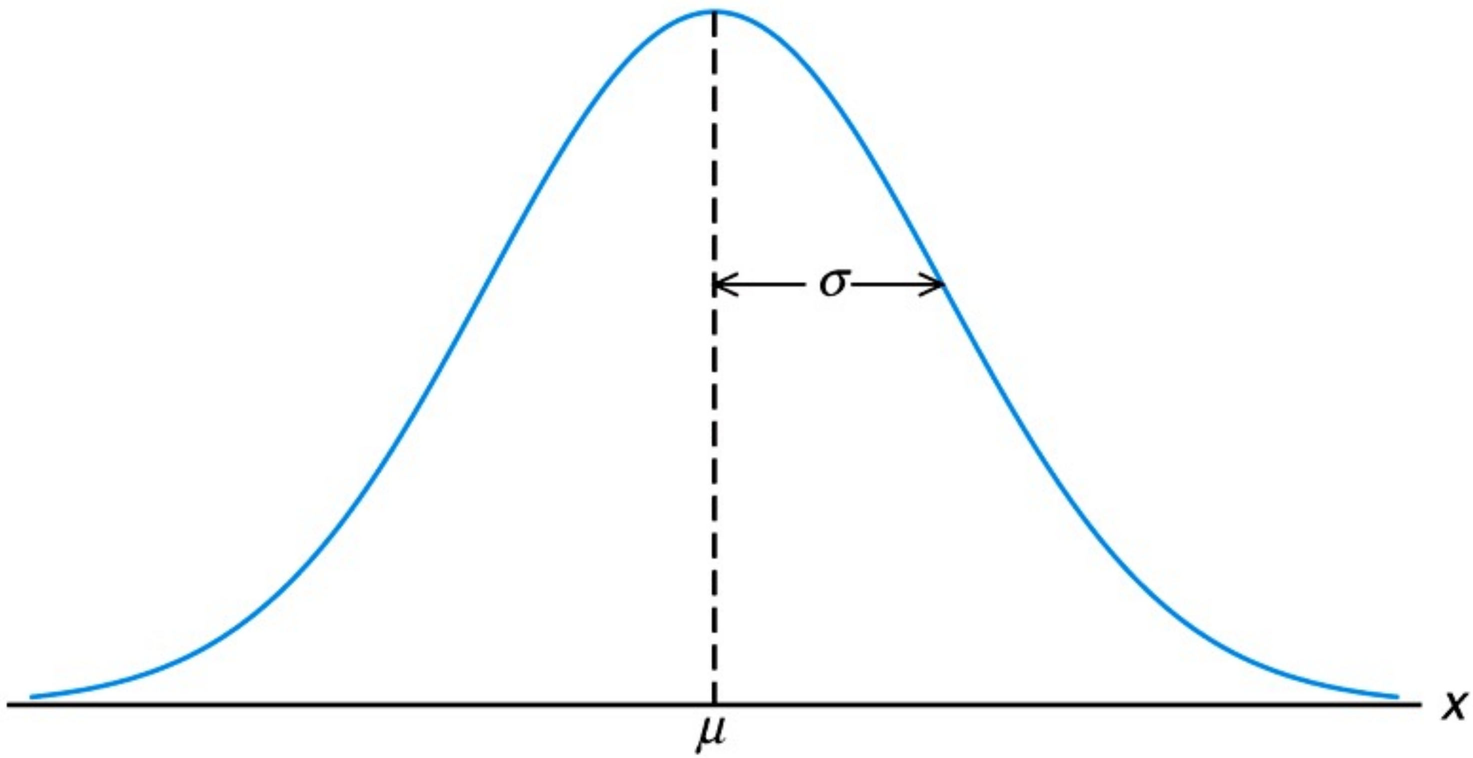


Figure 6.3 Normal curves with $\mu_1 < \mu_2$ and $\sigma_1 = \sigma_2$

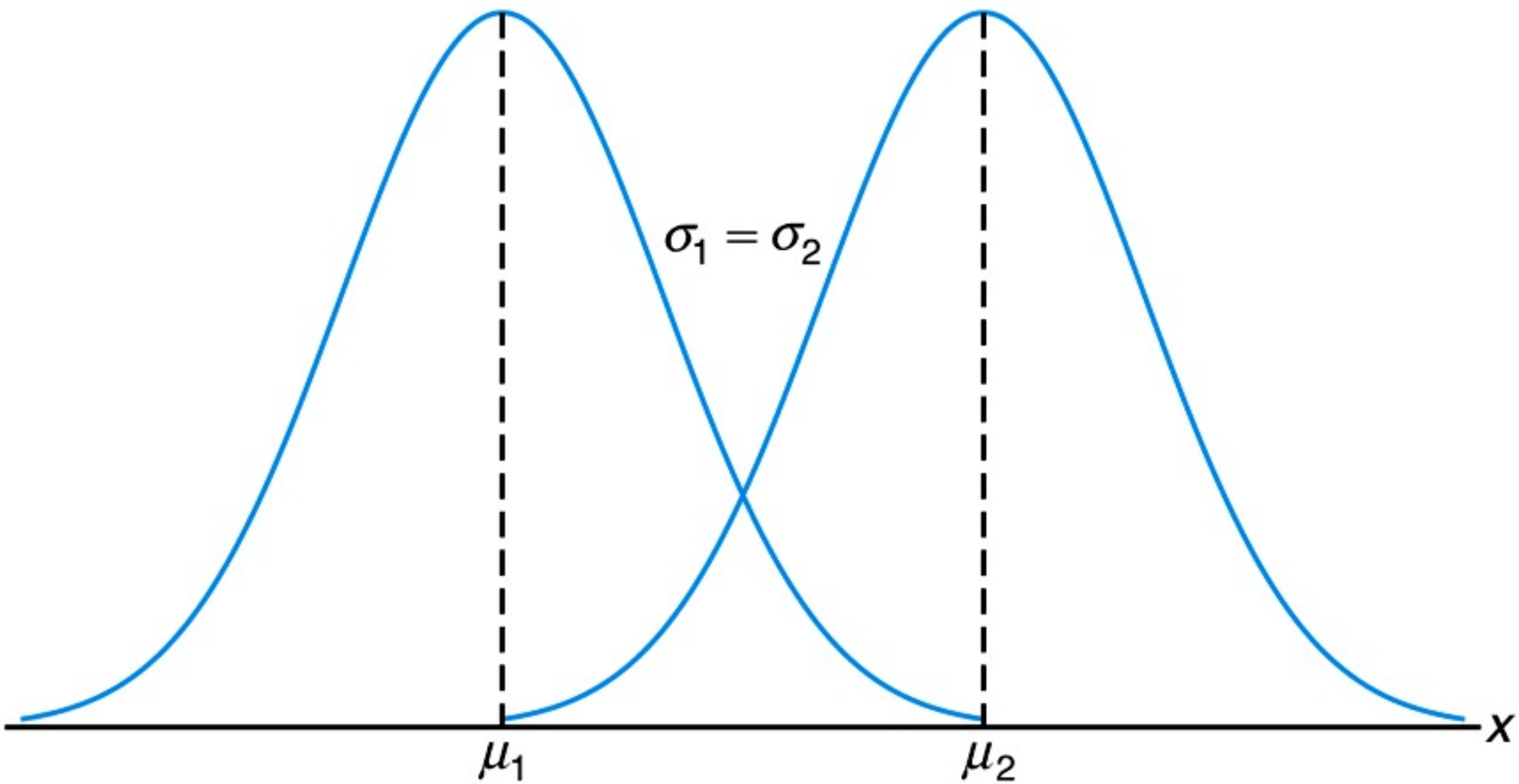


Figure 6.4 Normal curves with $\mu_1 = \mu_2$ and $\sigma_1 < \sigma_2$

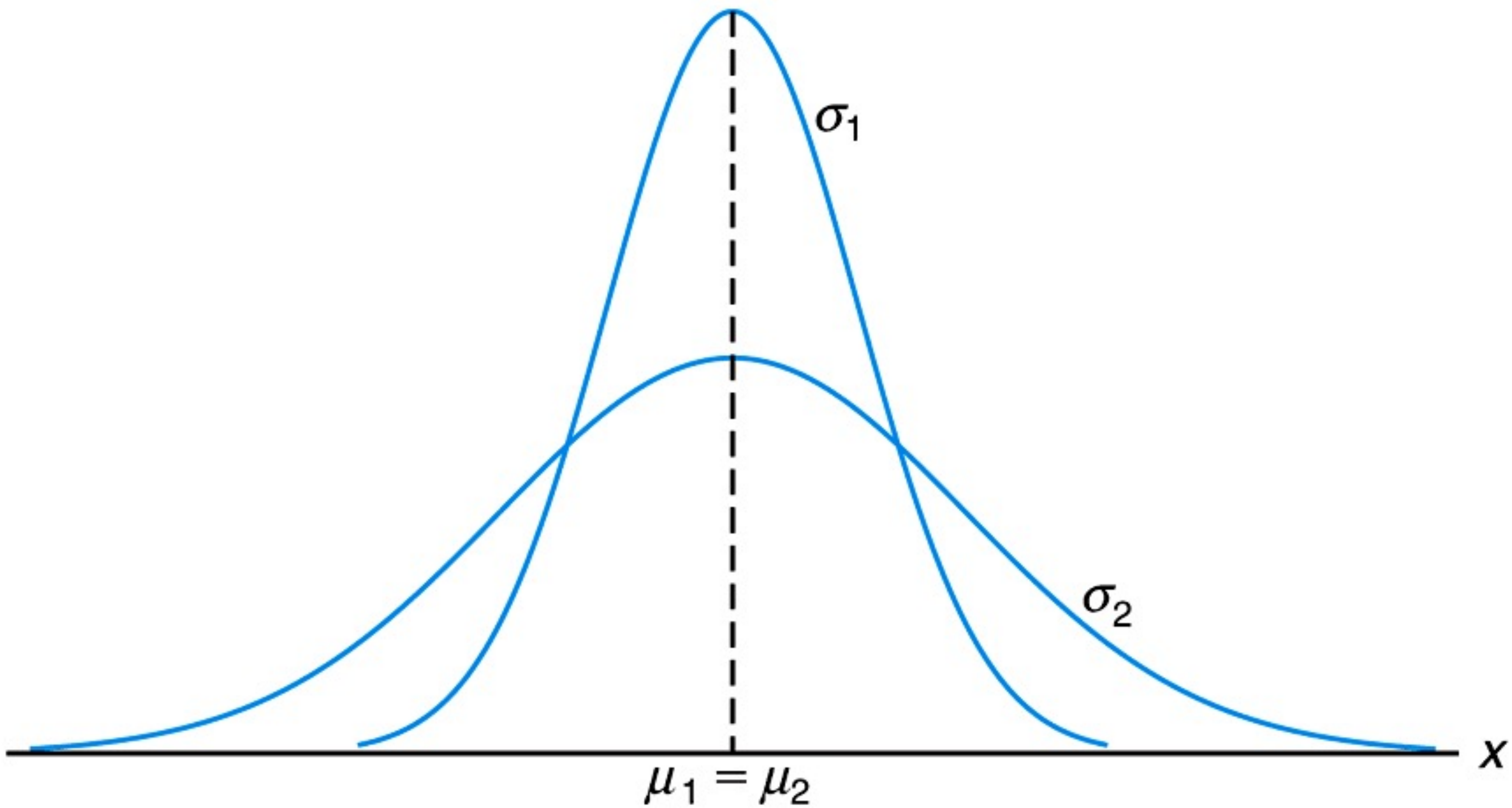
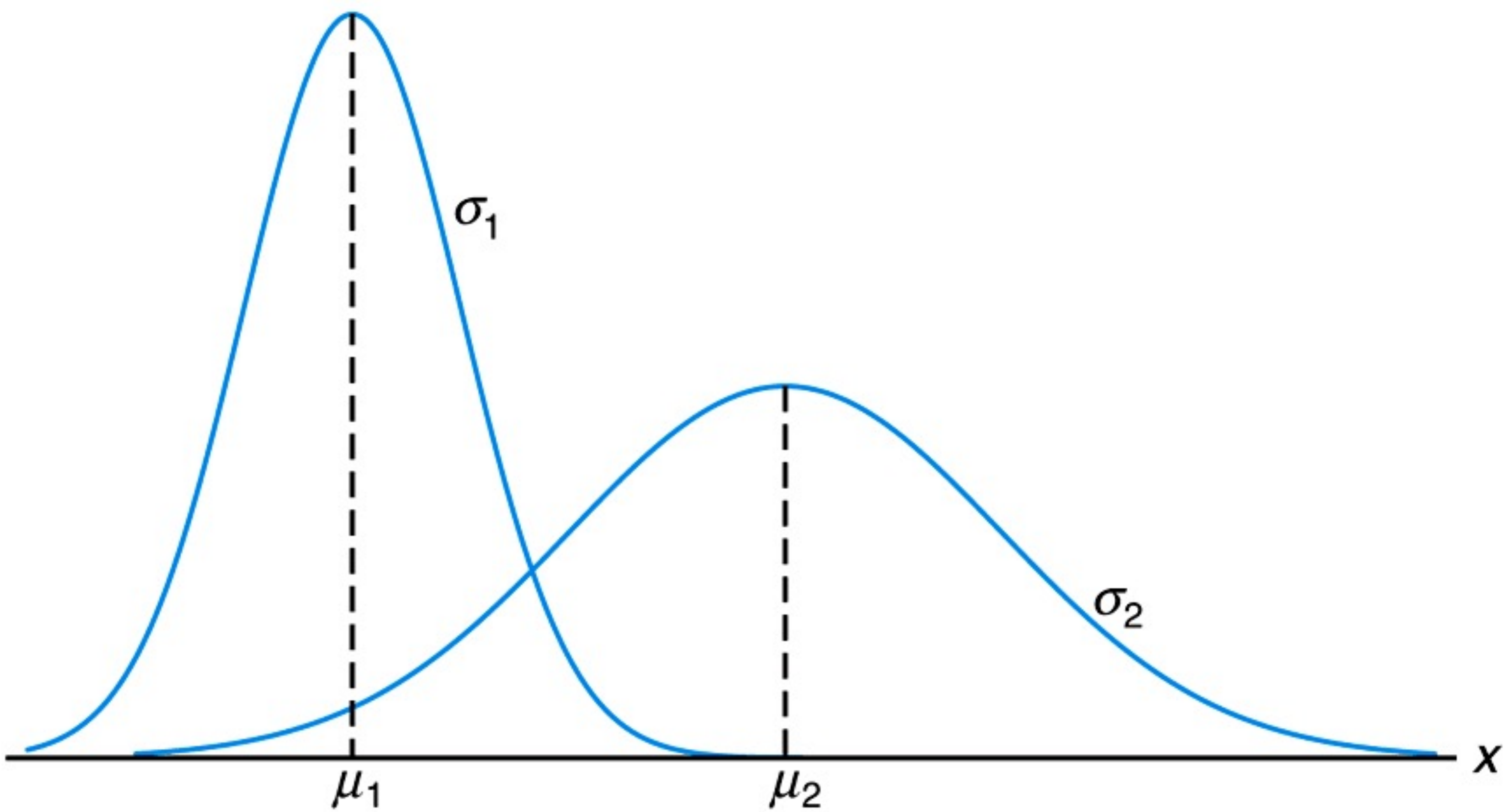


Figure 6.5 Normal curves with $\mu_1 < \mu_2$ and $\sigma_1 < \sigma_2$



Theorem 6.2

The mean and variance of $n(x; \mu, \sigma)$ are μ and σ^2, respectively. Hence, the standard deviation is σ.

Section 6.3

Areas under the
Normal Curve

Figure 6.6 $P(x_1 < X < x_2)$ = area of the shaded region

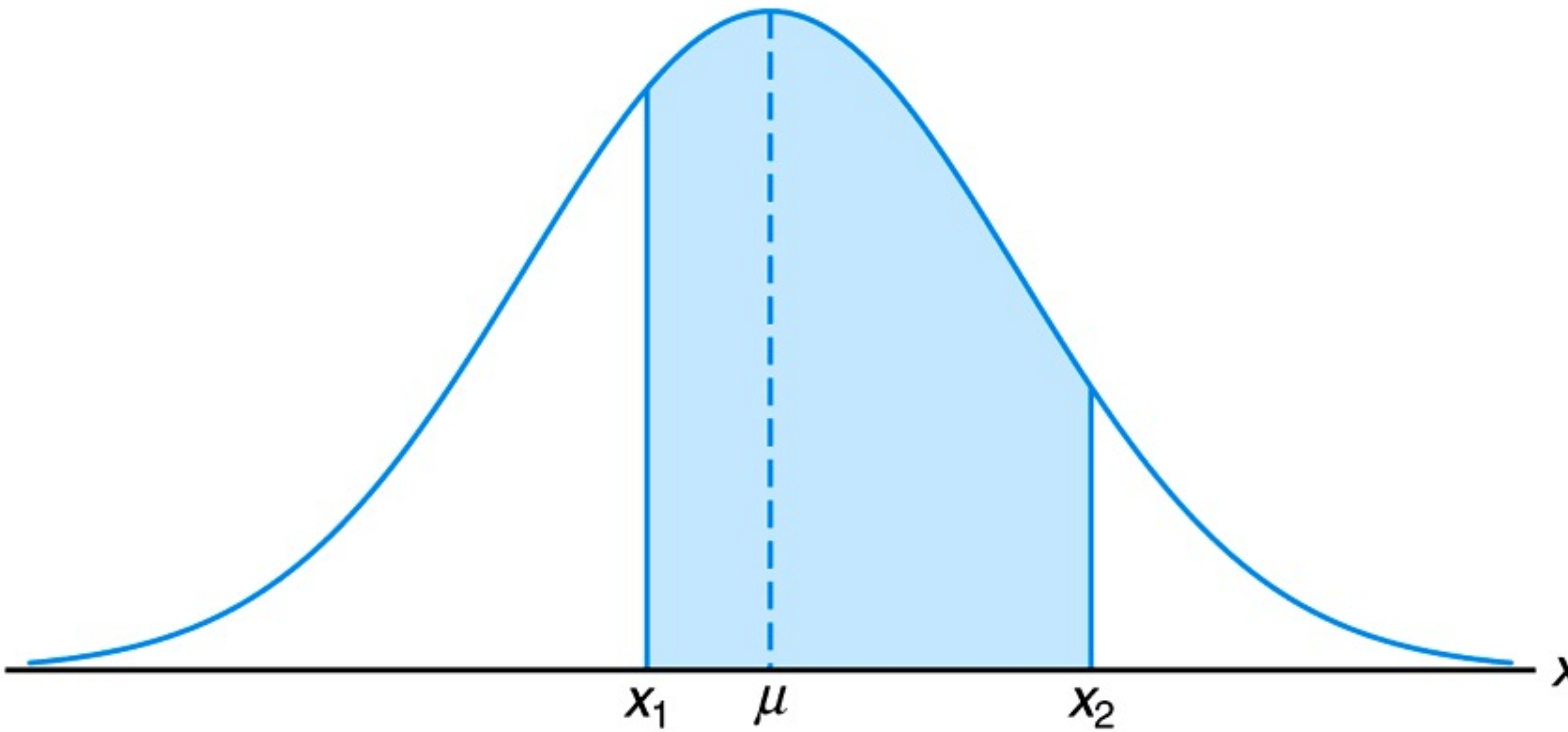
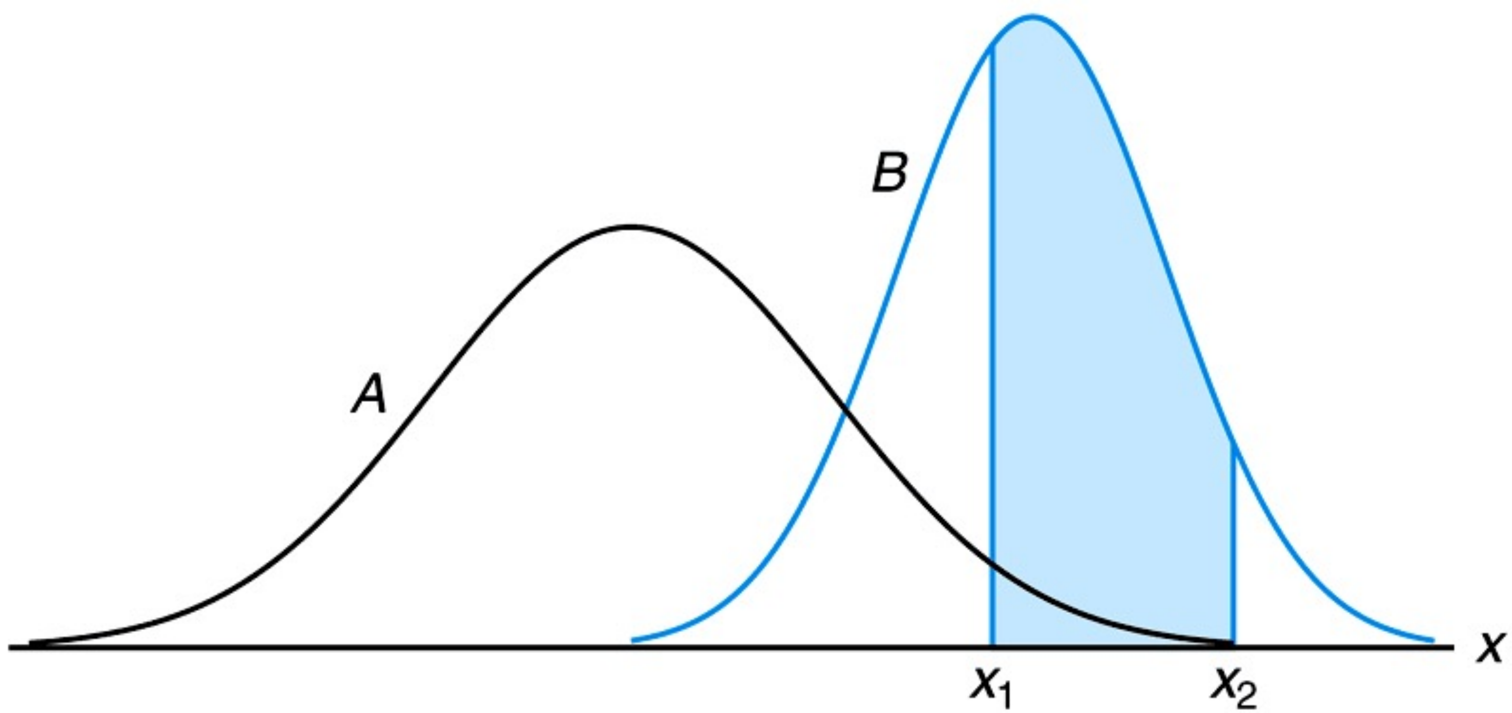


Figure 6.7 $P(x_1 < X < x_2)$ for different normal curves



Definition 6.1

The distribution of a normal random variable with mean 0 and variance 1 is called a **standard normal distribution**.

Standardization

- Suppose $X \sim N(\mu, \sigma^2)$ Then random variable Z given by transformation

$$Z = \frac{X - \mu}{\sigma}.$$

- Then $Z \sim N(0,1)$. Z is called the standard normal RV.

Standardization

- So if $X \sim N(\mu, \sigma^2)$ Then let Z be

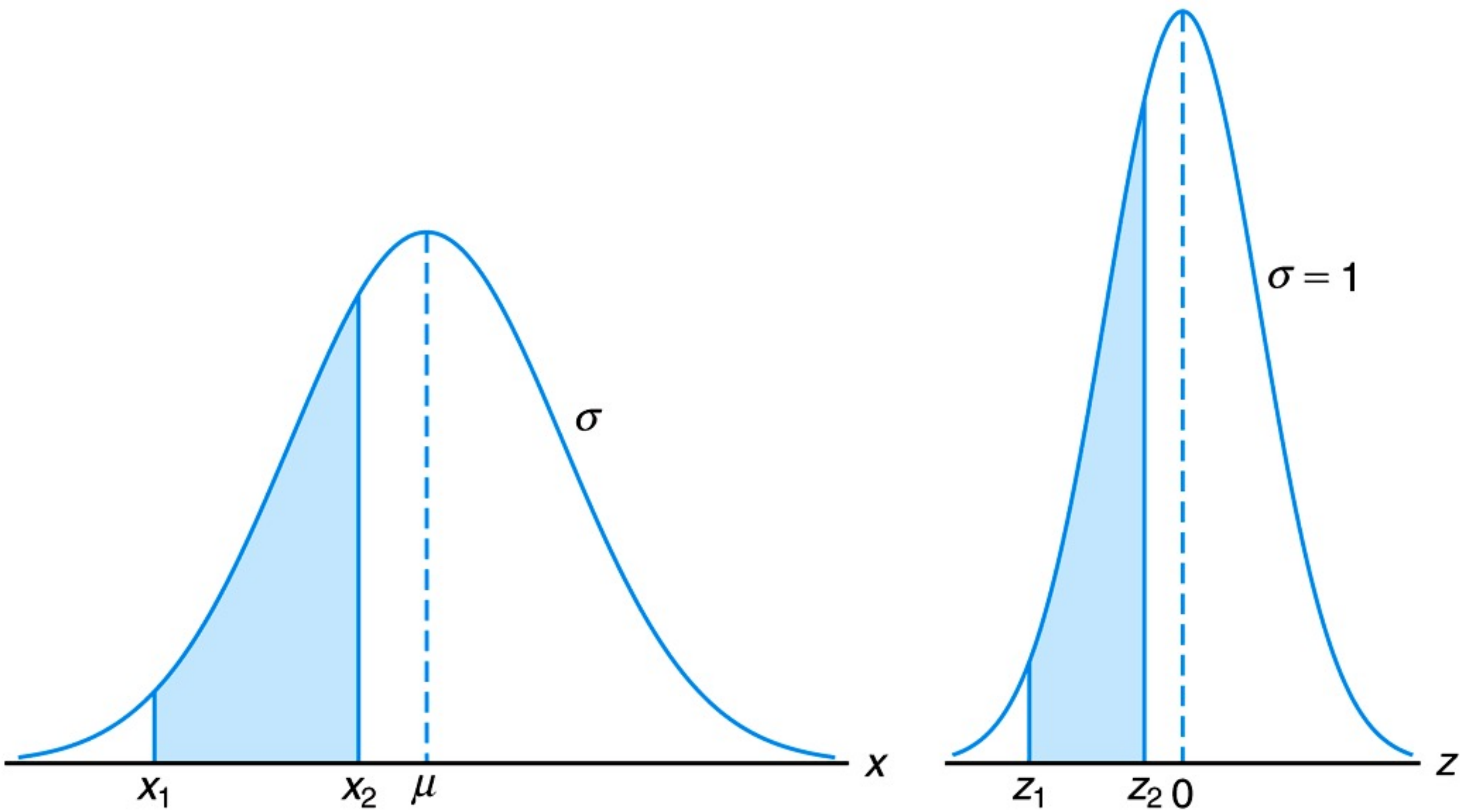
$$Z = \frac{X - \mu}{\sigma}.$$

- Then $Z \sim N(0,1)$.

$$P(x_1 \leq X \leq x_2) = P(z_1 \leq Z \leq z_2)$$

$$z_1 = \frac{x_1 - \mu}{\sigma}, z_2 = \frac{x_2 - \mu}{\sigma}$$

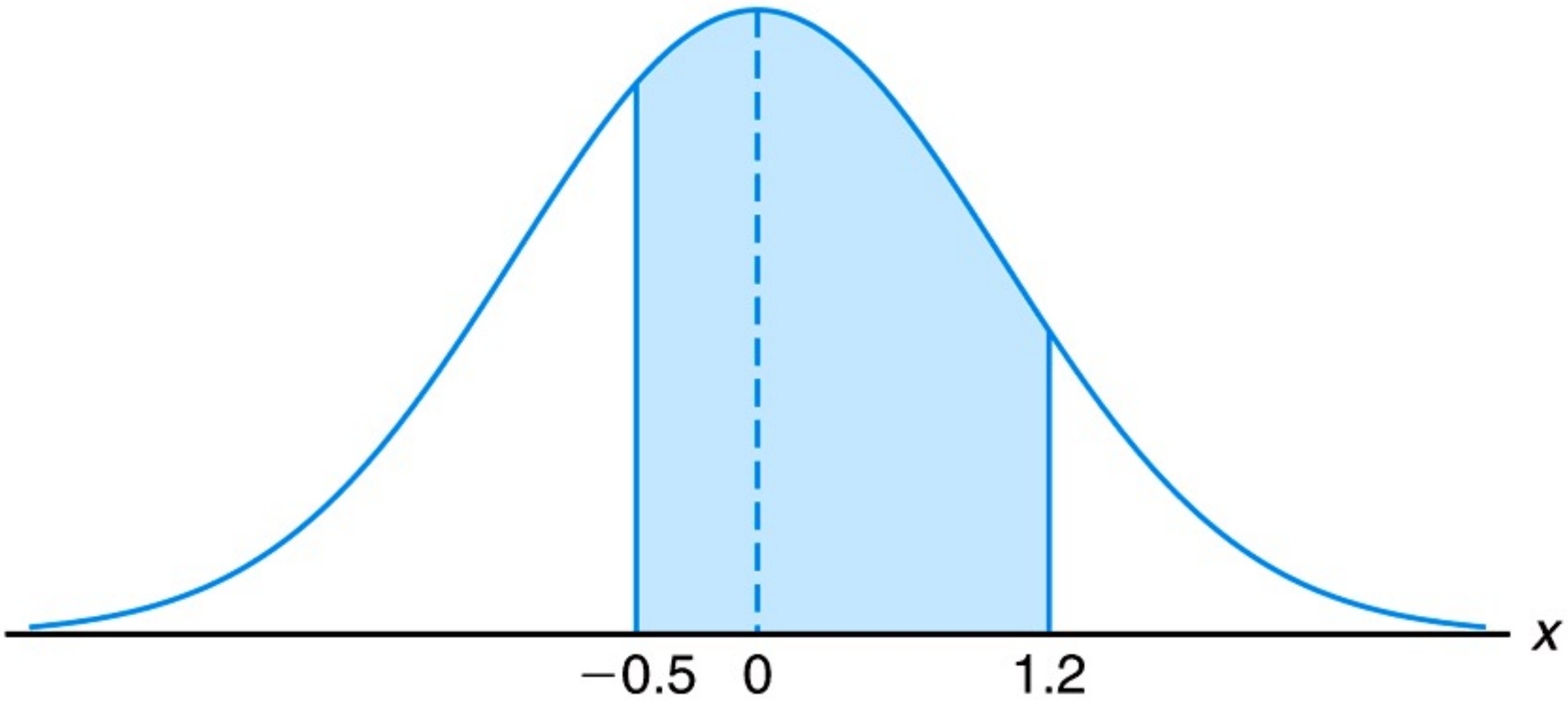
Figure 6.8 The original and transformed normal distributions



example 6.4

Example 6.4: Given a random variable X having a normal distribution with $\mu = 50$ and $\sigma = 10$, find the probability that X assumes a value between 45 and 62.

Figure 6.11 Area for Example 6.4



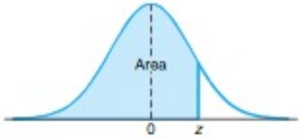


Table A.3 Areas under the Normal Curve

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
−3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
−3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
−3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
−3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
−2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
−2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
−2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
−2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
−2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
−2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
−2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
−2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
−2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
−2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
−1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
−1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
−1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
−1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
−1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
−1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
−1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
−1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
−1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
−1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
−0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
−0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
−0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
−0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
−0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
−0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
−0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
−0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
−0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
−0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641

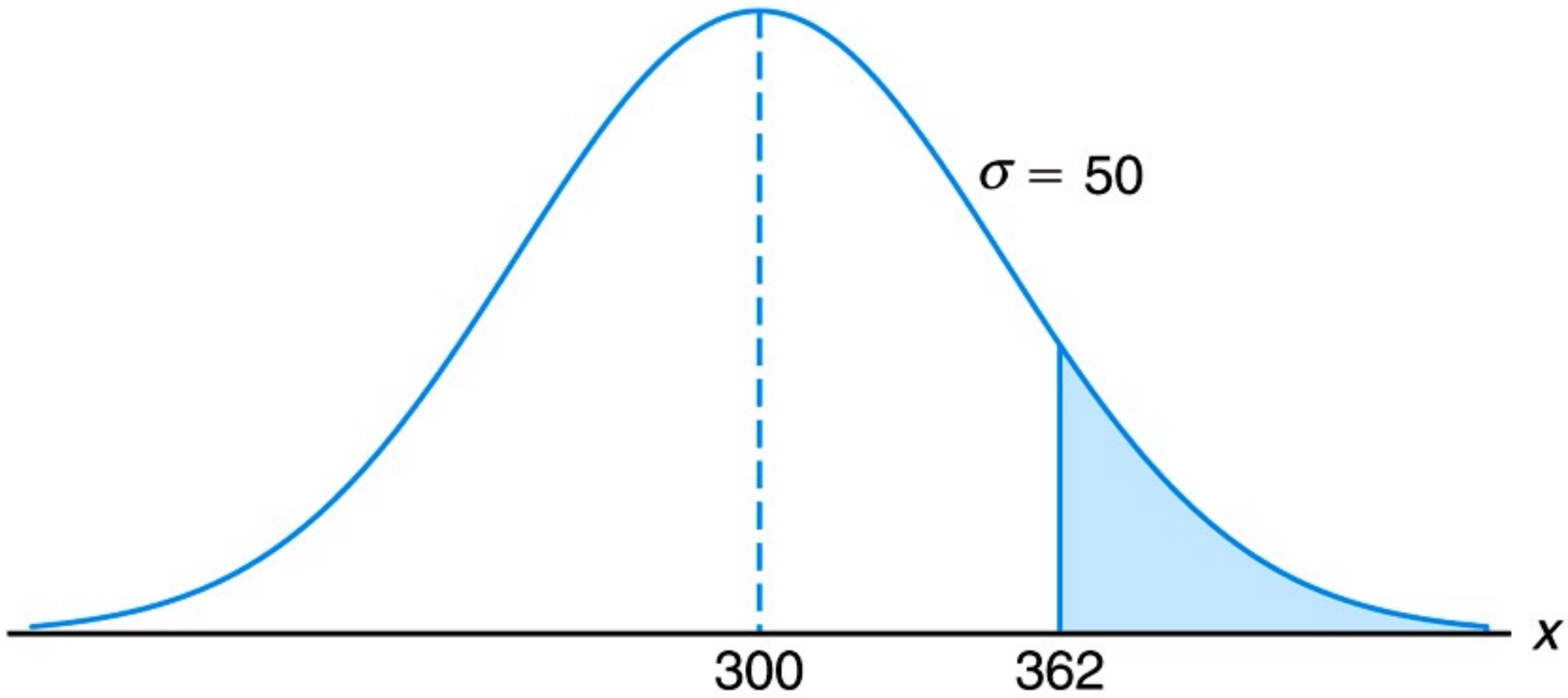
Table A.3 (continued) Areas under the Normal Curve

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

Example 6.5

Example 6.5: Given that X has a normal distribution with $\mu = 300$ and $\sigma = 50$, find the probability that X assumes a value greater than 362.

Figure 6.12 Area for Example 6.5



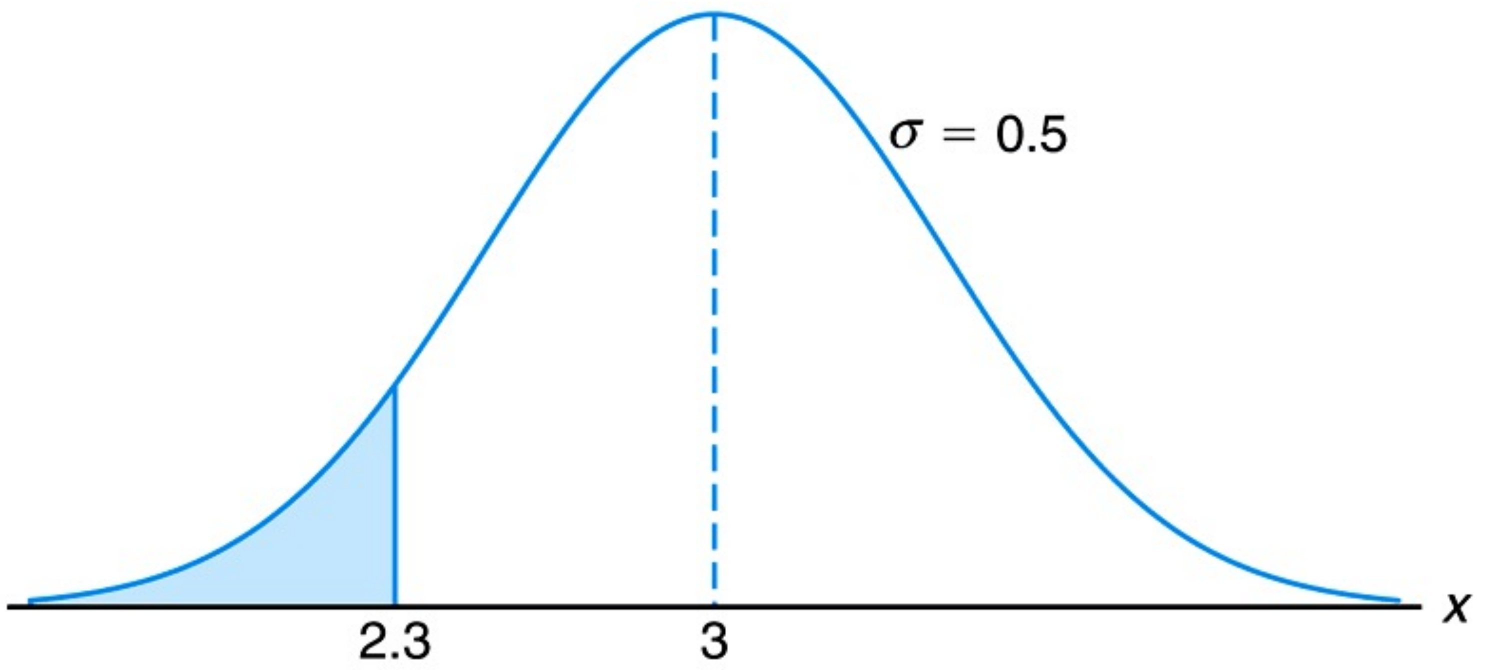
Section 6.4

Applications of the Normal Distribution

Example 6.7

Example 6.7: A certain type of storage battery lasts, on average, 3.0 years with a standard deviation of 0.5 year. Assuming that battery life is normally distributed, find the probability that a given battery will last less than 2.3 years.

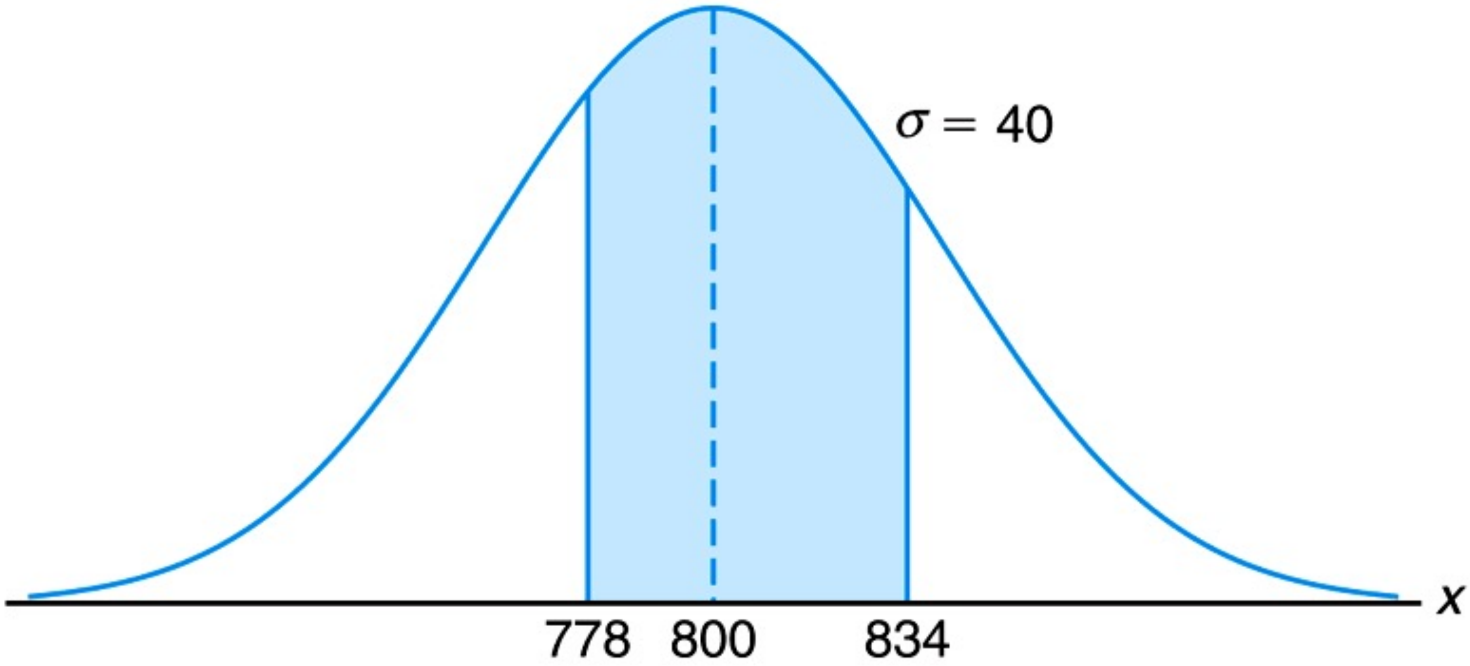
Figure 6.14 Area for Example 6.7



Example 6.8

Example 6.8: An electrical firm manufactures light bulbs that have a life, before burn-out, that is normally distributed with mean equal to 800 hours and a standard deviation of 40 hours. Find the probability that a bulb burns between 778 and 834 hours.

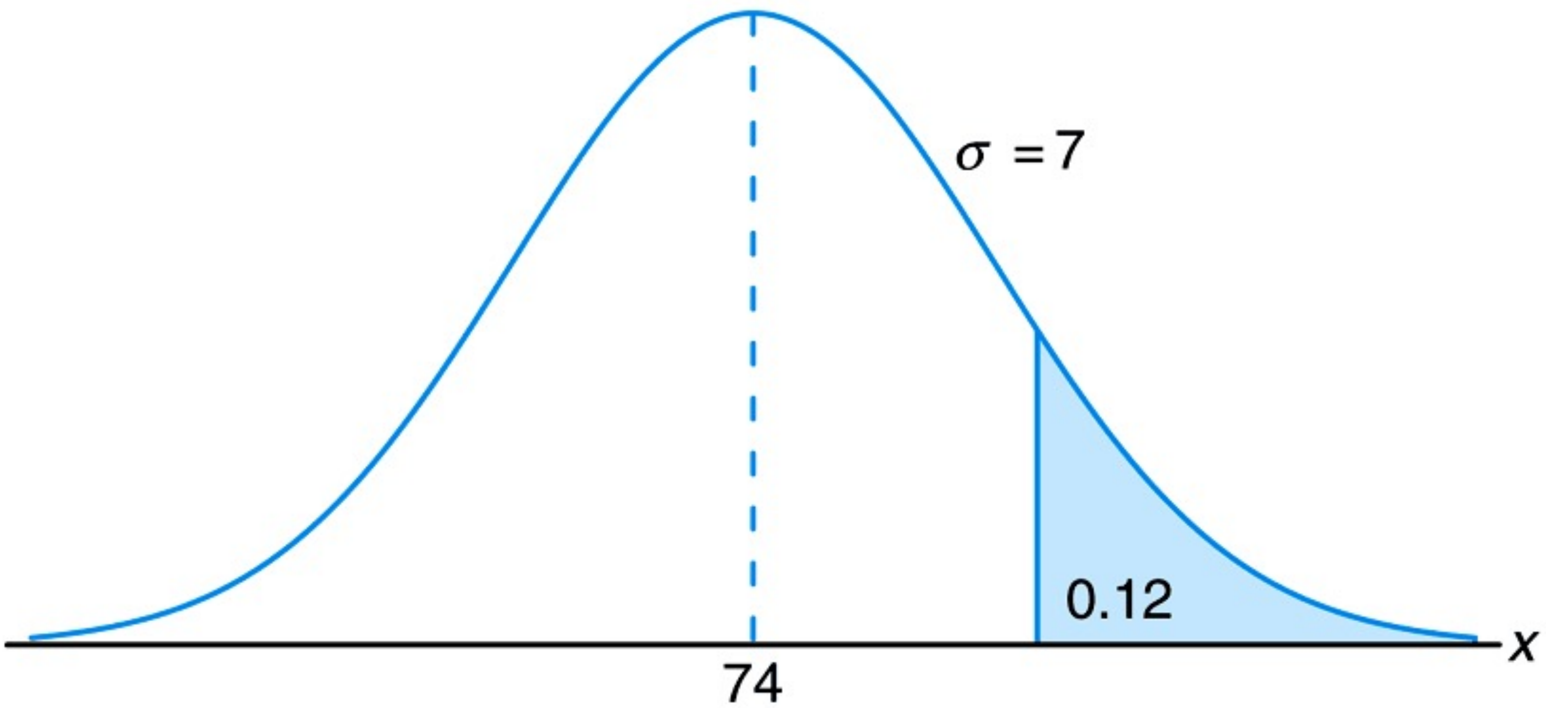
Figure 6.15 Area for Example 6.8



Example 6.13

Example 6.13: The average grade for an exam is 74, and the standard deviation is 7. If 12% of the class is given As, and the grades are curved to follow a normal distribution, what is the lowest possible A and the highest possible B?

Figure 6.20 Area for Example 6.13



Section 6.5

Normal Approximation to the Binomial

Theorem 6.3

If X is a binomial random variable with mean $\mu = np$ and variance $\sigma^2 = npq$, then the limiting form of the distribution of

$$Z = \frac{X - np}{\sqrt{npq}},$$

as $n \to \infty$, is the standard normal distribution $n(z; 0, 1)$.

Figure 6.22 Normal approximation of $b(x; 15, 0.4)$

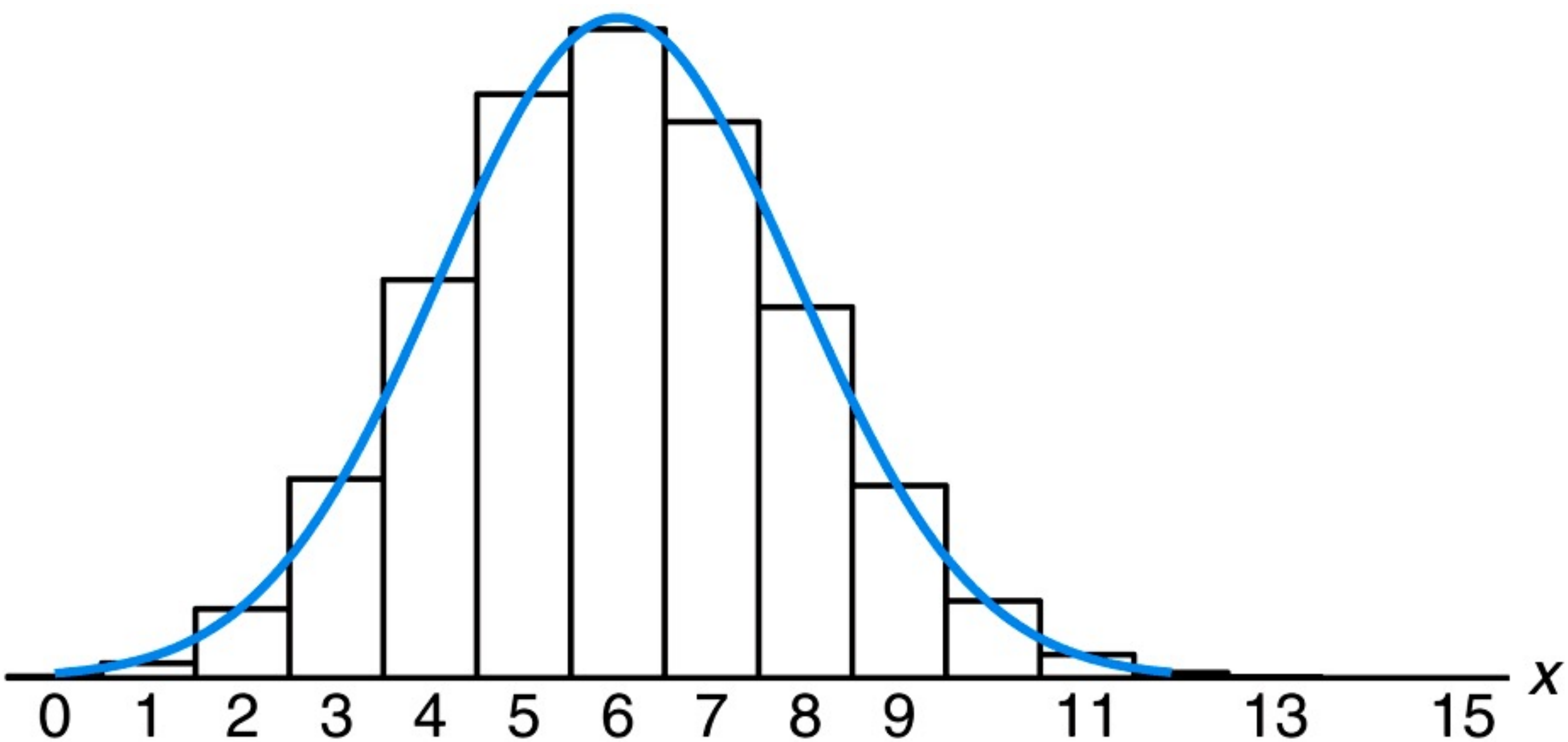


Figure 6.23 Normal approximation of $b(x; 15, 0.4)$ and $\sum_{x=7}^{9} b(x; 15, 0.4)$

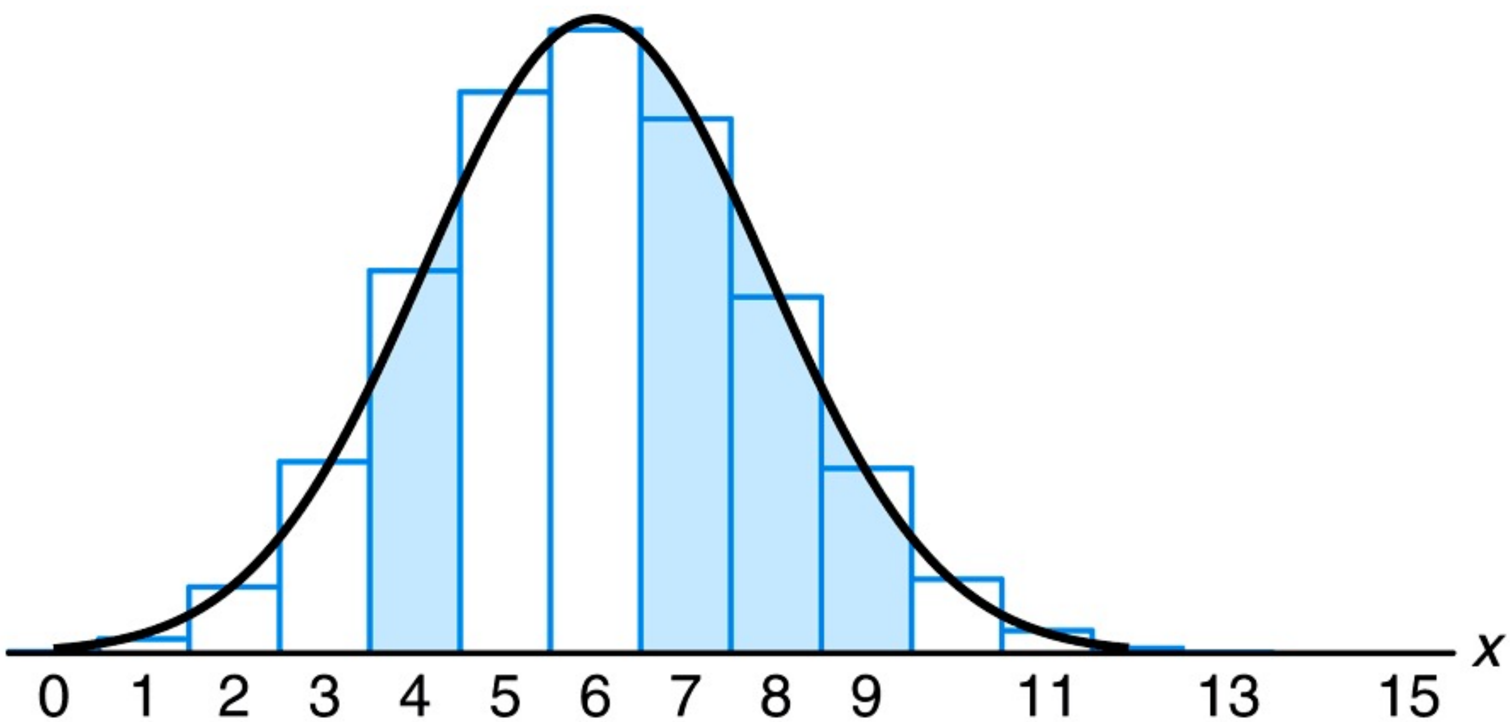


Figure 6.24 Histogram for $b(x; 6, 0.2)$

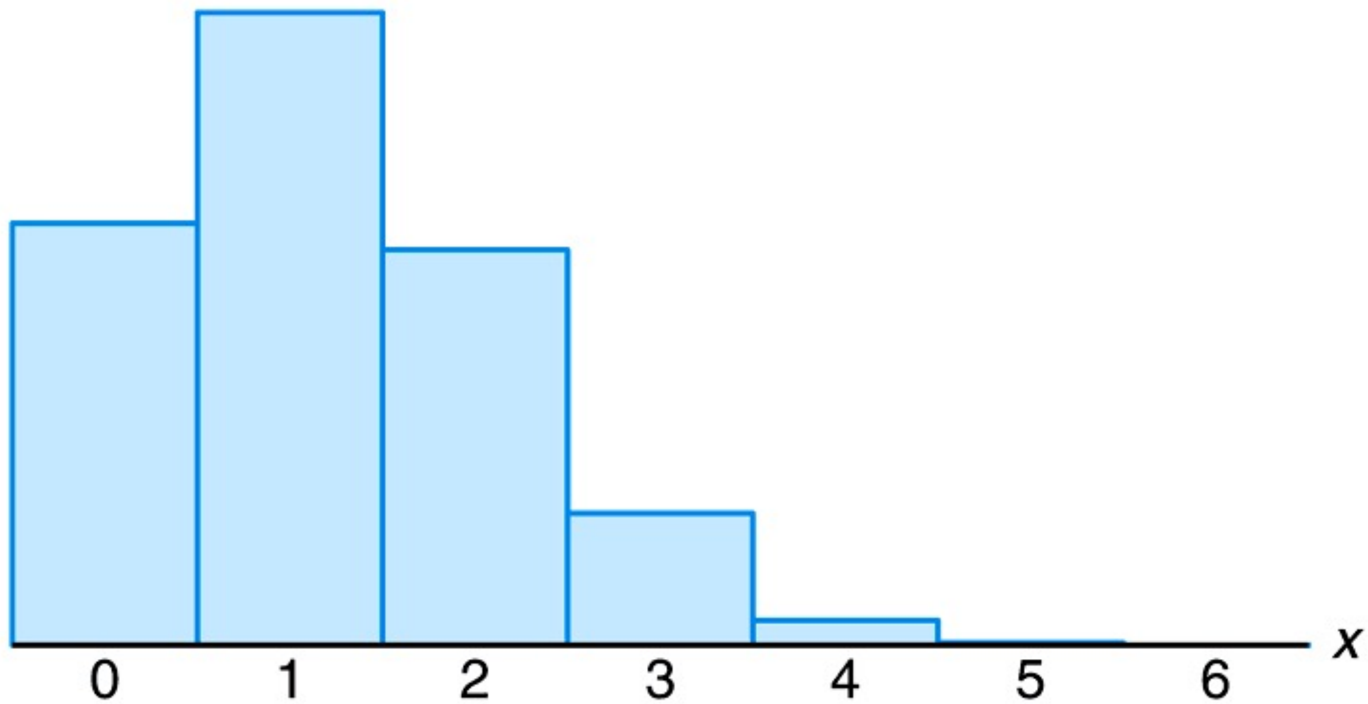
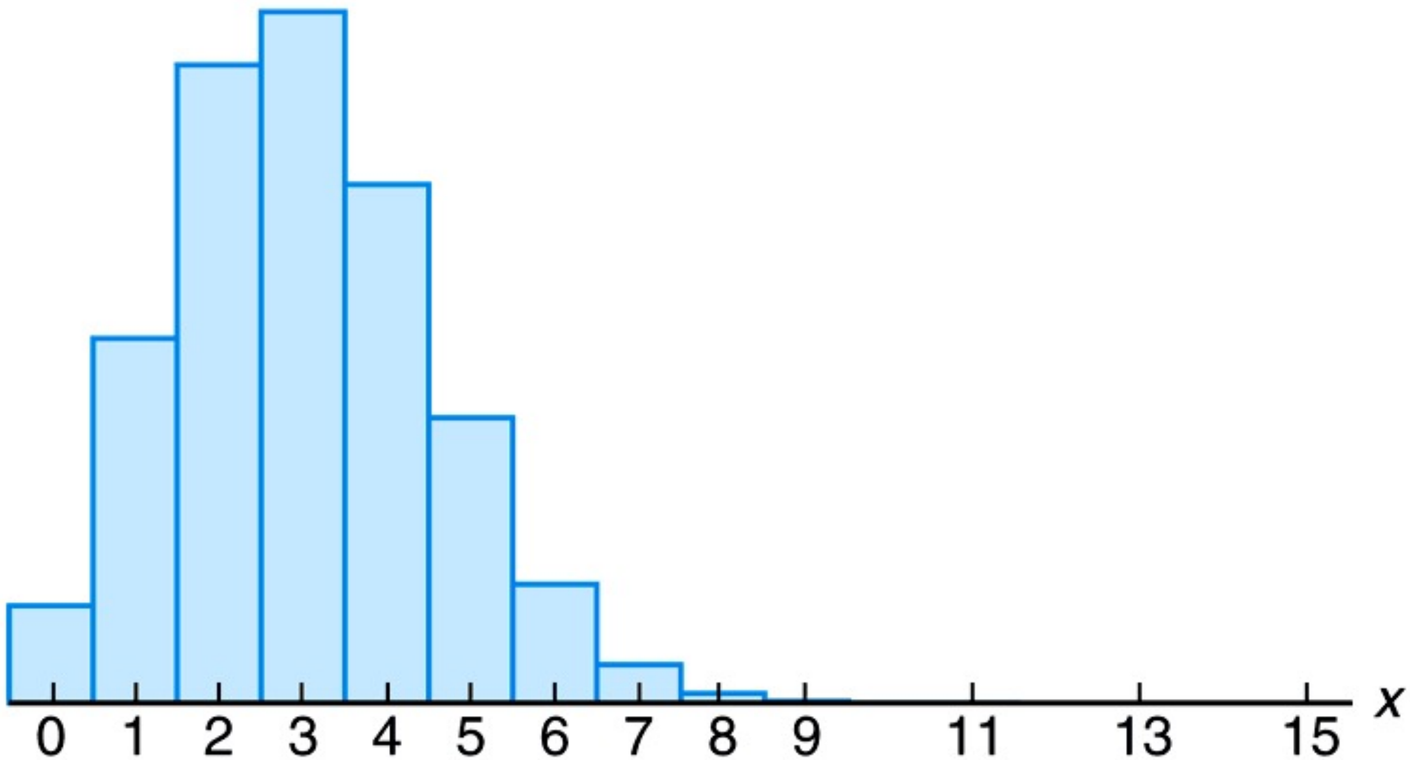


Figure 6.25 Histogram for $b(x; 15, 0.2)$



Approximating by Normal RV

$$P(X \leq x) \approx P(Z \leq \frac{x + 0.5 - np}{\sqrt{npq}})$$

- Why 0.5?
  - when x is integer,

$$P(X \leq x) = P(X < x + 1)$$

- so we take 'halfway' $P(X \leq x + 0.5)$
  - this is due to approximating discrete by continuous RV

Table 6.1 Normal Approximation and True Cumulative Binomial Probabilities

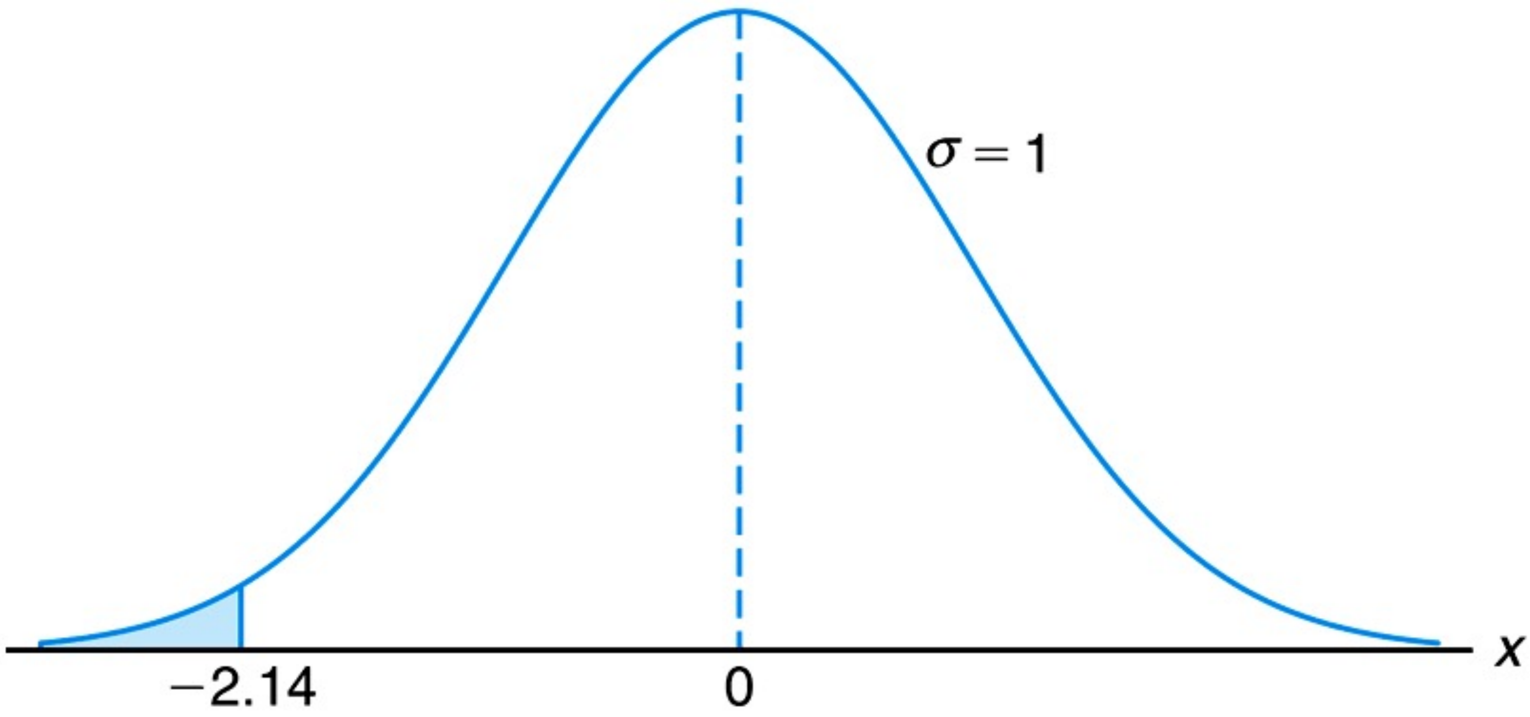
	$p = 0.05,\ n = 10$		$p = 0.10,\ n = 10$		$p = 0.50,\ n = 10$	
r	**Binomial**	**Normal**	**Binomial**	**Normal**	**Binomial**	**Normal**
0	0.5987	0.5000	0.3487	0.2981	0.0010	0.0022
1	0.9139	0.9265	0.7361	0.7019	0.0107	0.0136
2	0.9885	0.9981	0.9298	0.9429	0.0547	0.0571
3	0.9990	1.0000	0.9872	0.9959	0.1719	0.1711
4	1.0000	1.0000	0.9984	0.9999	0.3770	0.3745
5			1.0000	1.0000	0.6230	0.6255
6					0.8281	0.8289
7					0.9453	0.9429
8					0.9893	0.9864
9					0.9990	0.9978
10					1.0000	0.9997

	$p = 0.05$					
	$n = 20$		$n = 50$		$n = 100$	
r	**Binomial**	**Normal**	**Binomial**	**Normal**	**Binomial**	**Normal**
0	0.3585	0.3015	0.0769	0.0968	0.0059	0.0197
1	0.7358	0.6985	0.2794	0.2578	0.0371	0.0537
2	0.9245	0.9382	0.5405	0.5000	0.1183	0.1251
3	0.9841	0.9948	0.7604	0.7422	0.2578	0.2451
4	0.9974	0.9998	0.8964	0.9032	0.4360	0.4090
5	0.9997	1.0000	0.9622	0.9744	0.6160	0.5910
6	1.0000	1.0000	0.9882	0.9953	0.7660	0.7549
7			0.9968	0.9994	0.8720	0.8749
8			0.9992	0.9999	0.9369	0.9463
9			0.9998	1.0000	0.9718	0.9803
10			1.0000	1.0000	0.9885	0.9941

Example 6.15

Example 6.15: The probability that a patient recovers from a rare blood disease is 0.4. If 100 people are known to have contracted this disease, what is the probability that fewer than 30 survive?

Figure 6.26 Area for Example 6.15



Section 6.6

Gamma and
Exponential
Distributions

Definition 6.2

The **gamma function** is defined by

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\, dx, \qquad \text{for } \alpha > 0.$$

$\Gamma(n) = (n-1)!$ for a positive integer n.

$\Gamma(1) = 1$.

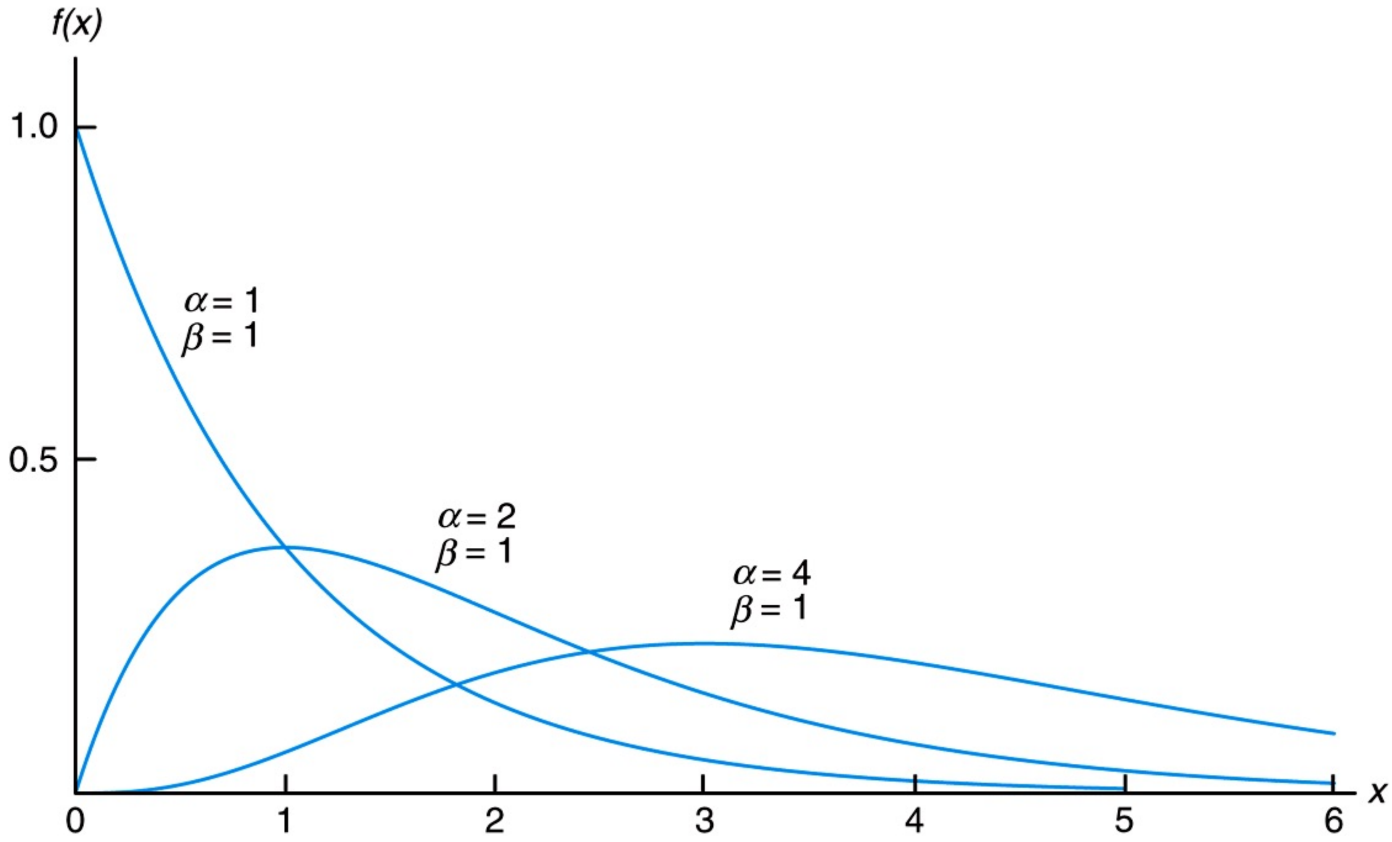
Gamma distribution

Gamma Distribution The continuous random variable X has a **gamma distribution**, with parameters α and β, if its density function is given by

$$f(x;\alpha,\beta) = \begin{cases} \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}, & x > 0, \\ 0, & \text{elsewhere}, \end{cases}$$

where $\alpha > 0$ and $\beta > 0$.

Figure 6.28 Gamma distributions



Theorem 6.4

The mean and variance of the gamma distribution are

$$\mu = \alpha\beta \text{ and } \sigma^2 = \alpha\beta^2.$$

Exponential distribution

Exponential Distribution The continuous random variable X has an **exponential distribution**, with parameter β, if its density function is given by

$$f(x;\beta) = \begin{cases} \frac{1}{\beta} e^{-x/\beta}, & x > 0, \\ 0, & \text{elsewhere}, \end{cases}$$

where $\beta > 0$.

Corollary 6.1

The mean and variance of the exponential distribution are

$$\mu = \beta \text{ and } \sigma^2 = \beta^2.$$

Relationship to Poisson RV

- Suppose events in unit time occur with ~ Poisson(λ)
- What is the probability of no event occur in a time interval of t?

$$p(0; \lambda t) = \frac{e^{-\lambda t}(\lambda t)^0}{0!} = e^{-\lambda t}$$

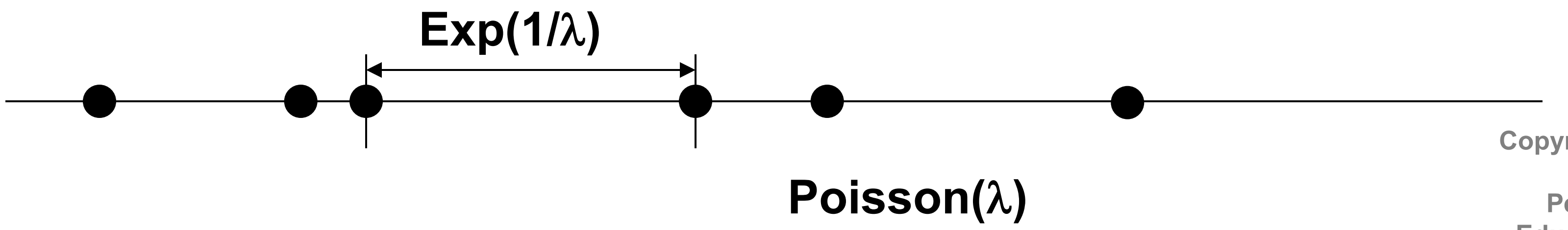
- This is equivalent to probability that, the time to next event is greater than t

Relationship to Poisson RV

- compare this with, when X is exponential RV with parameter 1/λ

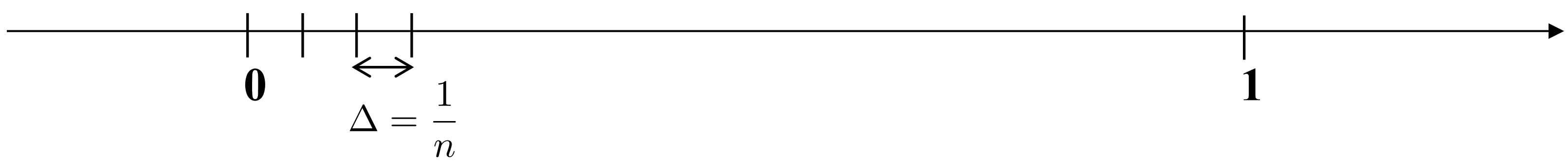
$$P(X > t) = e^{-\lambda t}$$

- distribution of the inter-arrival times of events occurring with Poisson(λ)
  - exponentially distributed with mean 1/λ



Relationship to Poisson RV

- we can also use limiting distribution

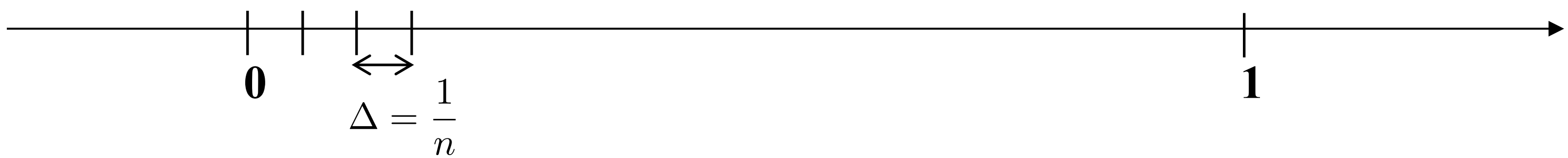


- probability that the first event occurs at time t?
- Geometric distribution!

$$x = nt, \ p = \frac{\lambda}{n}$$

Relationship to Poisson RV

- we can also use limiting distribution



- probability that the first event occurs at time t is given by

$$\left(1-\frac{\lambda}{n}\right)^{nt-1}\frac{\lambda}{n} \to \lambda \exp(-\lambda t)dt$$

example 6.17

Example 6.17: Suppose that a system contains a certain type of component whose time, in years, to failure is given by T. The random variable T is modeled nicely by the exponential distribution with mean time to failure $\beta = 5$. If 5 of these components are installed in different systems, what is the probability that at least 2 are still functioning at the end of 8 years?

Memoryless property of Exp RVs

- important property of Exp RVs: can be used to model "memoryless" RVs
- Suppose X models lifetime of a product and X~Exp(a). Then what is

$$P(X > t + s | X > s)$$

Memoryless property of Exp RVs

- we have

$$P(X > t + s | X > s) = \frac{P(X > t + s)}{P(X > s)} = P(X > t)$$

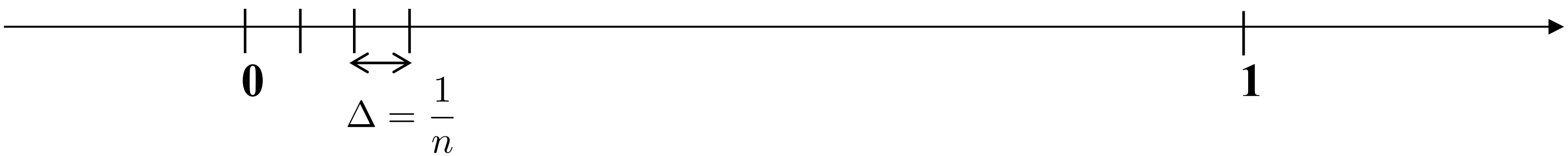
- that is, knowing 'history' of lifetime does not help predicting it!
  - A lightbulb that has lasted 10 yrs vs 1 day
  - if ~ Exp, the probability that it will burnt out tomorrow is the same!

Example

- Suppose that the mean lifetime of a lightbulb is known to be exponentially distributed with mean 5 years. Consider a lightbulb that has been used for 10 years. What is the probability that the bulb will last 2 more years?
  - $P(X>12|X>10)=P(X>2)=\exp(-0.4)$

Relationship to Poisson RV

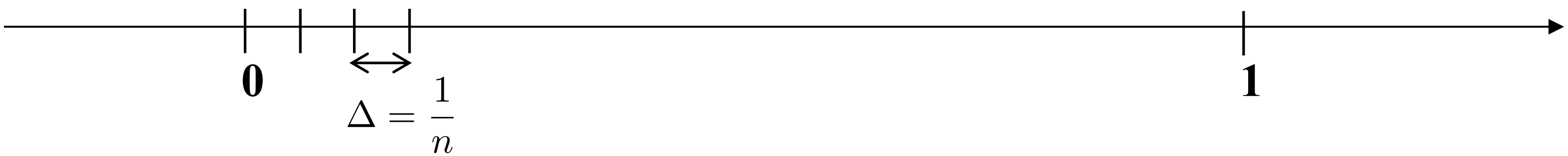
- we can also use limiting distribution



- What is k-th event occurs at time t?
- Negative binomial! b*(nt; k, p)

Relationship to Poisson RV

- we can also use limiting distribution



- What is k-th event occurs at time t?

$$\binom{nt-1}{k-1}\left(1-\frac{\lambda}{n}\right)^{nt-k}\left(\frac{\lambda}{n}\right)^{k} \to \frac{t^{k-1}\lambda^{k}}{(k-1)!}\exp(-\lambda t)dt$$

- This is pdf of Gamma distribution with α=k

Cont. RVs as Limiting case of disc. RVs

- Exponential distribution
  - ‘time until first occurrence’
  - limiting case of Geometric distribution
- Gamma distribution
  - ‘time until k-th occurrence’
  - limiting case of Neg. Binomial distribution
- Both are related to time
  - cf) Poisson is limiting case of Binomial: both are counting

Relation between RVs

- Let $X_i \sim \text{Exp}(\lambda)$, $i=1,\ldots,k$ be independent exponential RVs.

$$Y = X_1 + X_2 + \ldots + X_k$$

- Then $Y \sim \text{Gamma}(k, \lambda)$

- Recall the relation between Geo(p) and NB(k,p)!

Relation between RVs

- Let $X_i \sim \text{Exp}(\lambda)$, $i=1,\ldots,k$ be independent exponential RVs.

$$Y = X_1 + X_2 + \ldots + X_k$$

- Then $Y \sim \text{Gamma}(k, \lambda)$
- This also means that, $Z = X + Y$ with $X \sim \text{Gamma}(m,b)$, $Y \sim \text{Gamma}(n,b)$ and X, Y independent
  - $Z \sim \text{Gamma}(m+n,b)$
  - Sum of Gamma is also Gamma

example 6.18

Example 6.18: Suppose that telephone calls arriving at a particular switchboard follow a Poisson process with an average of 5 calls coming per minute. What is the probability that up to a minute will elapse by the time 2 calls have come in to the switchboard?

gamma distribution with α=2, β=1/5!

Section 6.7

Chi-Squared Distributions

Definition

Chi-Squared Distribution The continuous random variable X has a **chi-squared distribution**, with v **degrees of freedom**, if its density function is given by

$$f(x;v) = \begin{cases} \frac{1}{2^{v/2}\Gamma(v/2)} x^{v/2-1} e^{-x/2}, & x > 0, \\ 0, & \text{elsewhere,} \end{cases}$$

where v is a positive integer.

Relation between RVs

- χ^2-distribution, or Chi-square distribution is a special case of Gamma distribution
- Chi-square with k degrees of freedom (or χ^2 (k)) Gamma(a,b) with a = k/2 and b=2

Relation between RVs

- χ^2-distribution, or Chi-square distribution is a special case of Gamma distribution
- Chi-square with k degrees of freedom (or $\chi^2(k)$) Gamma(a,b) with a = k/2 and b=2
- Let $X_i \sim \chi^2(n_i)$, i=1,…,k be independent then.

$$Y = X_1 + X_2 + \ldots + X_k$$

- $Y \sim \chi^2(\Sigma_i n_i)$
- Sum of χ^2 is also χ^2 with d.o.f added

Theorem 6.5

The mean and variance of the chi-squared distribution are

$$\mu = v \text{ and } \sigma^2 = 2v.$$

Normal and Chi-square

- Let $X \sim N(0,1)$ and

$$Y = X^2$$

- Then $Y \sim \chi^2(1)$
- We show this later
- Let $Z_i \sim N(0,1)$

$$Y = Z_1^2 + Z_2^2 + \cdots + Z_k^2$$

- Then $Y \sim \chi^2(k)$, chi-square with k d.o.f

Section 6.9

Lognormal
Distribution

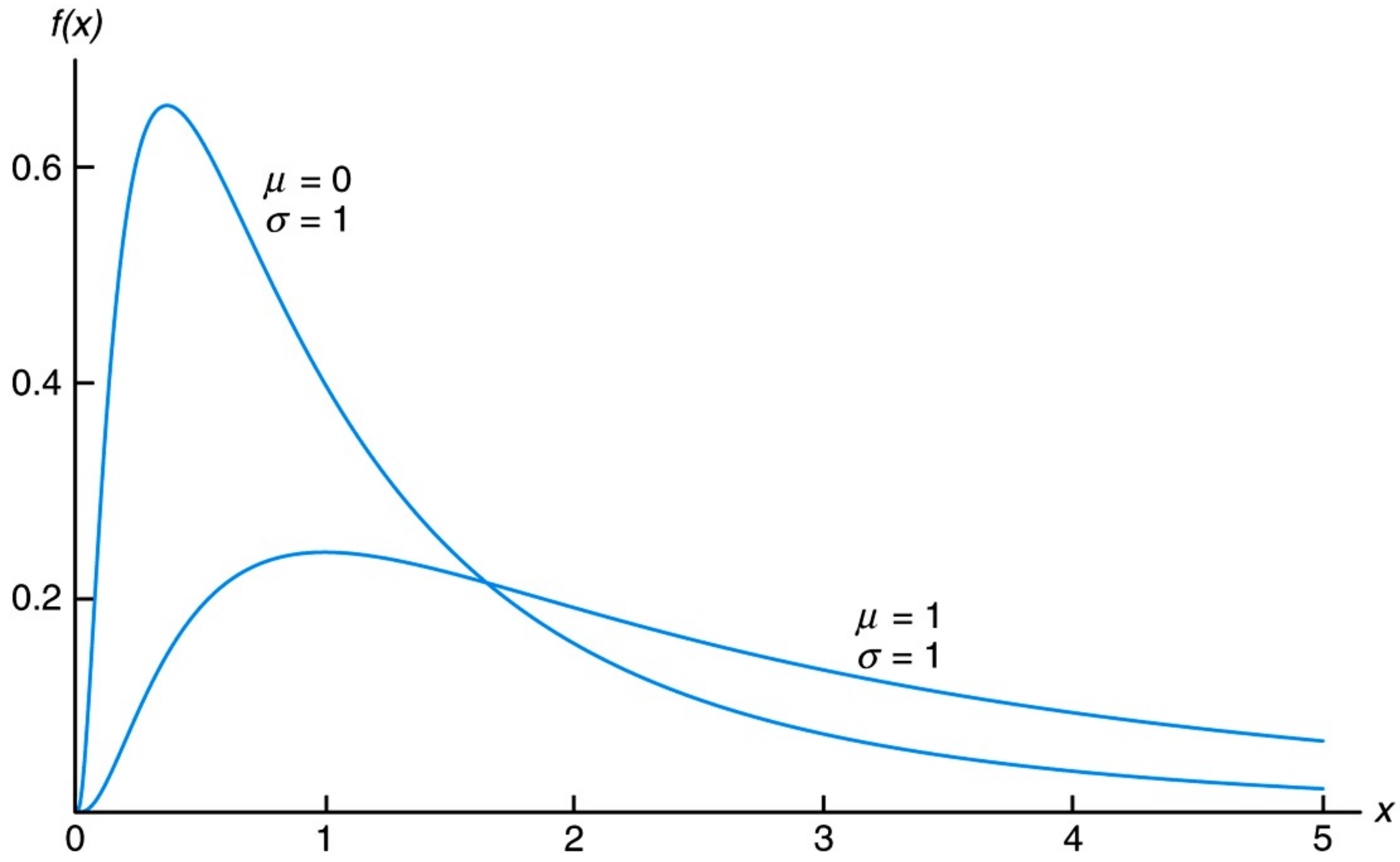
Definition

Lognormal Distribution

The continuous random variable X has a **lognormal distribution** if the random variable $Y = \ln(X)$ has a normal distribution with mean μ and standard deviation σ. The resulting density function of X is

$$f(x;\mu,\sigma) = \begin{cases} \frac{1}{\sqrt{2\pi}\sigma x} e^{-\frac{1}{2\sigma^2}[\ln(x)-\mu]^2}, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

Figure 6.29 Lognormal distributions



Theorem 6.7

The mean and variance of the lognormal distribution are

$$\mu = e^{\mu+\sigma^2/2} \text{ and } \sigma^2 = e^{2\mu+\sigma^2}(e^{\sigma^2} - 1).$$

example 6.22

Example 6.22: Concentrations of pollutants produced by chemical plants historically are known to exhibit behavior that resembles a lognormal distribution. This is important when one considers issues regarding compliance with government regulations. Suppose it is assumed that the concentration of a certain pollutant, in parts per million, has a lognormal distribution with parameters $\mu = 3.2$ and $\sigma = 1$. What is the probability that the concentration exceeds 8 parts per million?

Section 6.8

Beta Distribution

Definition 6.3

A **beta function** is defined by

$$B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \text{ for } \alpha, \beta > 0,$$

where $\Gamma(\alpha)$ is the gamma function.

Derivation of Normalization

$$
\begin{aligned}
\Gamma(a)\Gamma(b) &= \int_0^\infty x^{a-1}e^{-x}dx \int_0^\infty y^{b-1}e^{-y}dy \\
&= \int_0^\infty \int_0^\infty x^{a-1}y^{b-1}e^{-x-y}dxdy \\
&= \int_0^\infty \int_x^\infty x^{a-1}(z-x)^{b-1}e^{-z}dzdx \\
&= \int_0^z \int_0^\infty (x/z)^{a-1}(1-x/z)^{b-1}z^{a+b-2}e^{-z}dzdx \\
&= \int_0^1 \int_0^\infty (u)^{a-1}(1-u)^{b-1}z^{a+b-1}e^{-z}dzdu \\
&= \int_0^1 (u)^{a-1}(1-u)^{b-1}du \int_0^\infty z^{a+b-1}e^{-z}dz \\
&= B(a,b)\Gamma(a+b)
\end{aligned}
$$

Definition

Beta Distribution The continuous random variable X has a **beta distribution** with parameters $\alpha > 0$ and $\beta > 0$ if its density function is given by

$$f(x) = \begin{cases} \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}, & 0 < x < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

What is the use of Beta dist.?

- Suppose we have a biased coin
- We don't know p, prob of H
- Coin is flipped 10 times, 7 H and 3 T

- What is the probability that $p > 0.5$, given the experiment result?
  – Asking the probability on the probability value

What is the use of Beta dist.?

- Using Bayes' rule

$$P(p|E) = \frac{P(E|p)P(p)}{P(E)}$$

- E: experiment result (data)
- P(p): unconditional prob of H
- P(E|p) : ${}_{10}C_7\, p^7(1\text{-}p)^3$
- Want to know: P(p|E)
  - Called posterior probability

What is the use of Beta dist.?

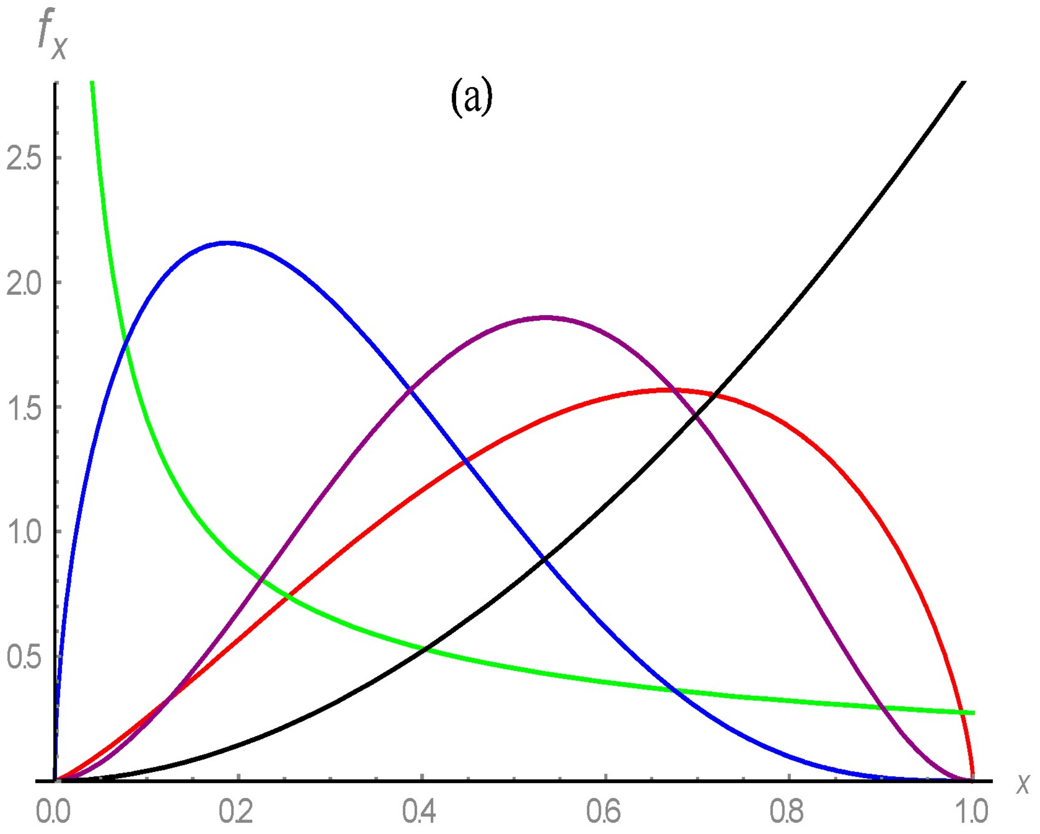
- We have

$$P(p|E) = (\text{const}) \cdot p^7 (1-p)^3$$

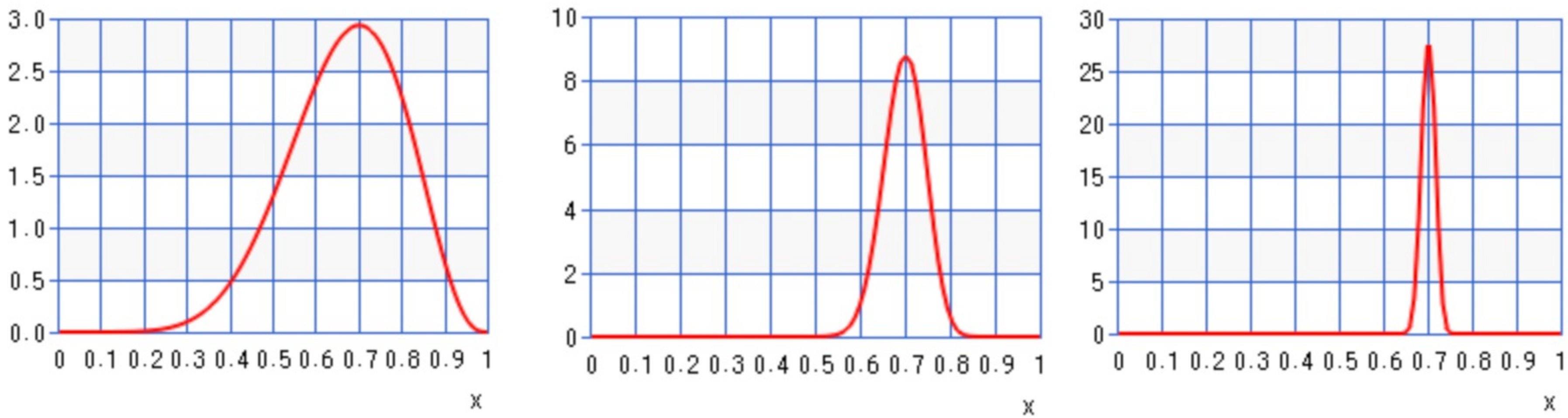
- Beta distribution with $\alpha=8$, $\beta=4$ (for some historical reason)

- (const) must be $B(8,4)$ for normalization

Plots of Beta Distribution



Plots of Beta Distribution



Theorem 6.6

The mean and variance of a beta distribution with parameters α and β are

$$\mu = \frac{\alpha}{\alpha + \beta} \text{ and } \sigma^2 = \frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)},$$

respectively.