Probability and Statistics
COSE112

College of Informatics
Korea University

Syllabus

- Official course name and code
  - 전산수학 II – COSE112
- Meeting time
  - Tue 2, Thu 2
- Instructor
  - Seung Jun Baek (sjbaek@korea.ac.kr)
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- Teaching Assistant
  - 김세현 (vnv73@korea.ac.kr)
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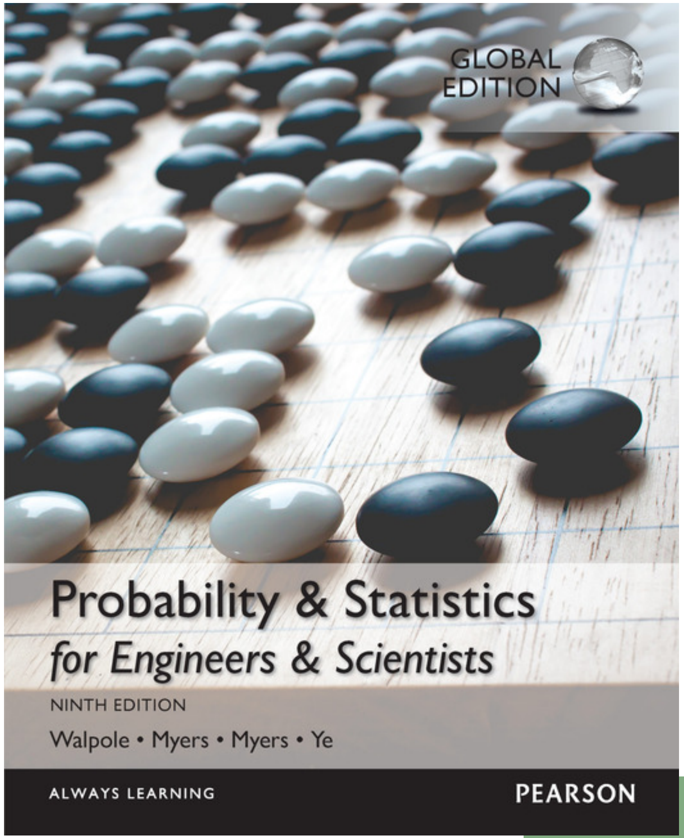
Textbooks

- Main textbook
  - "Probability and Statistics for Engineers" by Walpole et al.

  - Textbook is required
  - (problem exercises)



Blackboard

- Slides will be uploaded at BlackBoard without notice
  - Check BB regularly

- E-mail is primary method of communication
  - Important notification will be sent out through e-mail in Black Board system
  - Update your primary e-mail address at Black Board system

Prerequisite

- Prerequisites
  - basic calculus (differentiation, integration)
  - vector/multivariate calculus
    - Covered later
  - linear algebra

Prerequisite and Assignments

- Assignments
  - problem exercises
- Submission
  - online submission through BB
  - Late HW will **NOT** be accepted
  - grading: TA will choose 3 problems to grade
    - 1 point for submission + 1 point each for correct answer
    - total 4 points maximum
    - 0 points if not submitted or late

Grading

- Exams
  - One midterm, one final (all in-class)
  - Exams will be **OFFLINE**
    - **IF YOU CAN'T MAKE OFFLINE EXAM, LET ME KNOW NOW.**
  - All exams are closed book, closed notes

Exam policy

- **Exam policies:** All exams will be administered in class. Absence at the exams will be pardoned only under excruciating circumstances. If you are absent with permission, the exam score will be the lower of class average and the other exam score. If you are absent without permission, simply the score is zero.

Grading

- HW 15%, Midterm+Final 85%

- **Attendance points: Pass/Fail**

  - **There are NO attendance points except P/F**

Attendance

- **Attendance points: Pass/Fail**
  - You have to meet minimum requirement to pass
    - attend OVER 2/3 of total class
    - **Fail with absence of 1/3 or more**
- **online class attendance** at Blackboard
- **IMPORTANT:**
  - **BLACKBOARD ON-LINE ATTENDANCE IS THE ONLY WAY TO CHECK YOUR ATTENDENCE**
  - **CHECK-IN YOUR ATTENDANCE WITHIN 5 MINUTES BEFORE OR AFTER CLASS STARTS**

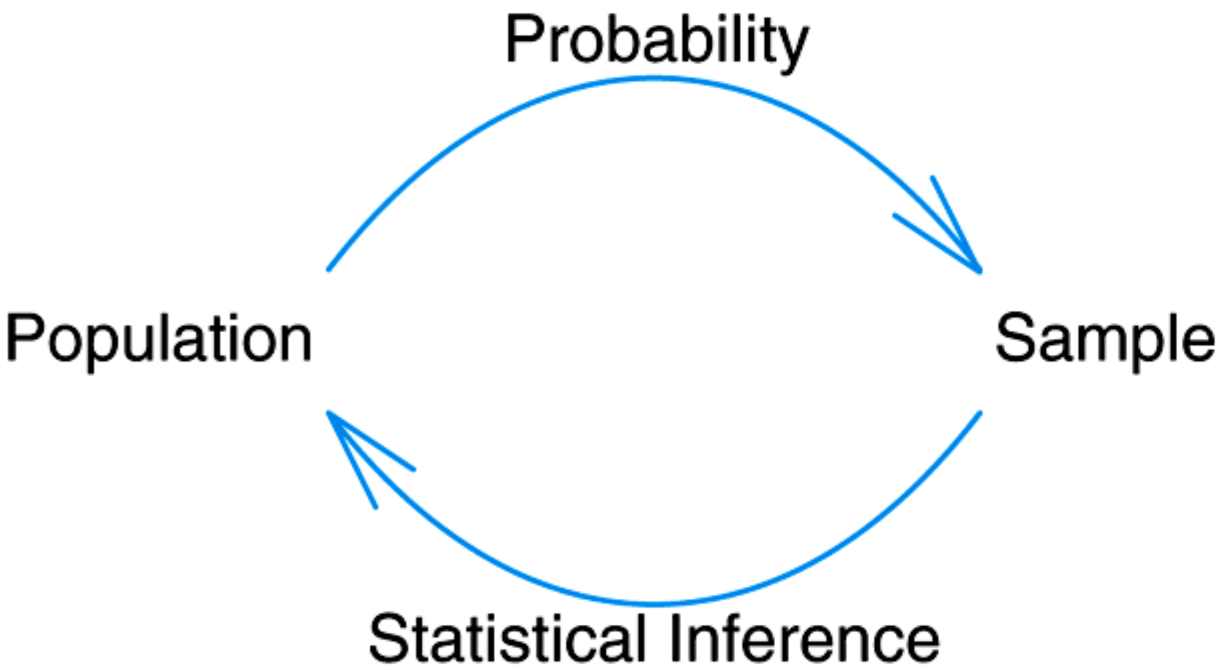
Course topics

- probability
- random variable
- probability distribution
- expectation
- discrete random variable
- continuous random variable
- function of random variables
- estimation
- hypothesis test
- linear regression

why study probability/statistics?

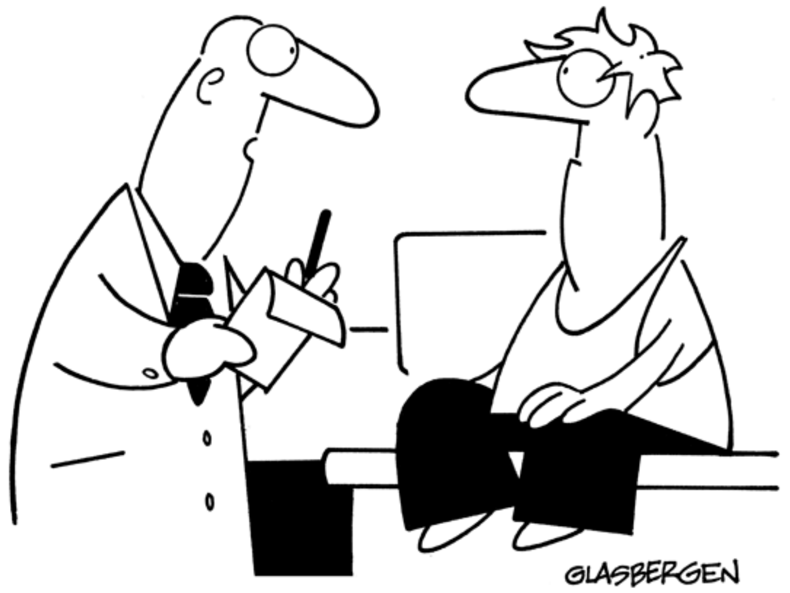
- **many fields of science and industry uses probabilistic/statistical methods**
  - **manufacturing, finance, medicine, computer engineering, insurance, physics…**
- **scientific experiments and observations**
- **record of the results: data**
- **how to interpret data, and draw informed conclusions**

Fundamental relationship between probability and statistics



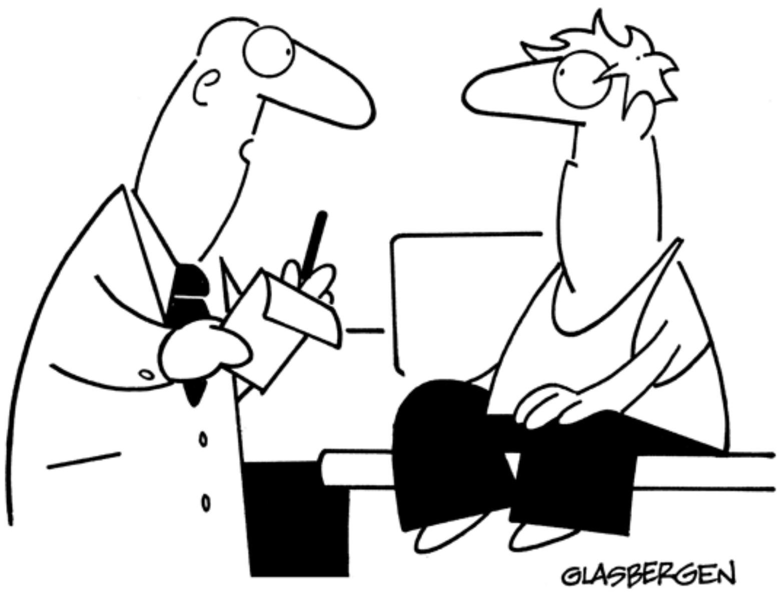
- **we first study the concepts of probability**

probability around us (1)



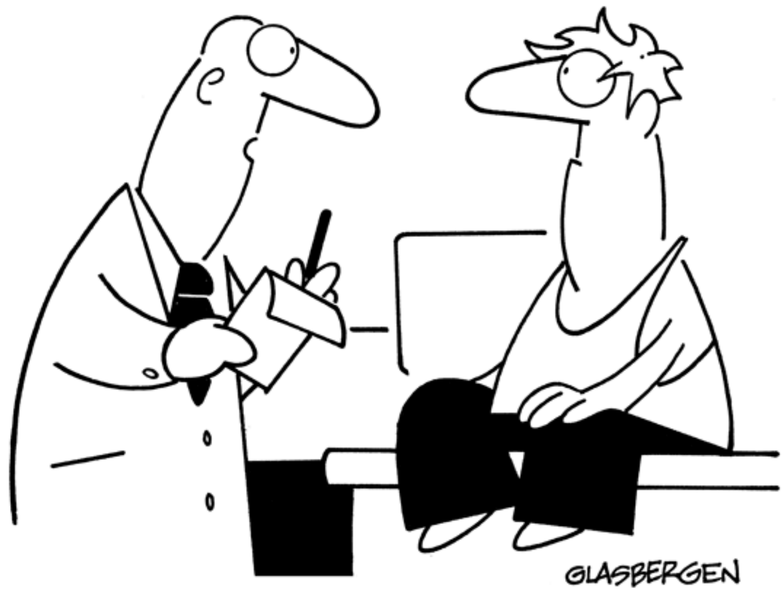
"Doctor, what is the chance of survival of this disease?"
"I'm sorry to say, but it is 20%."

probability around us (1)



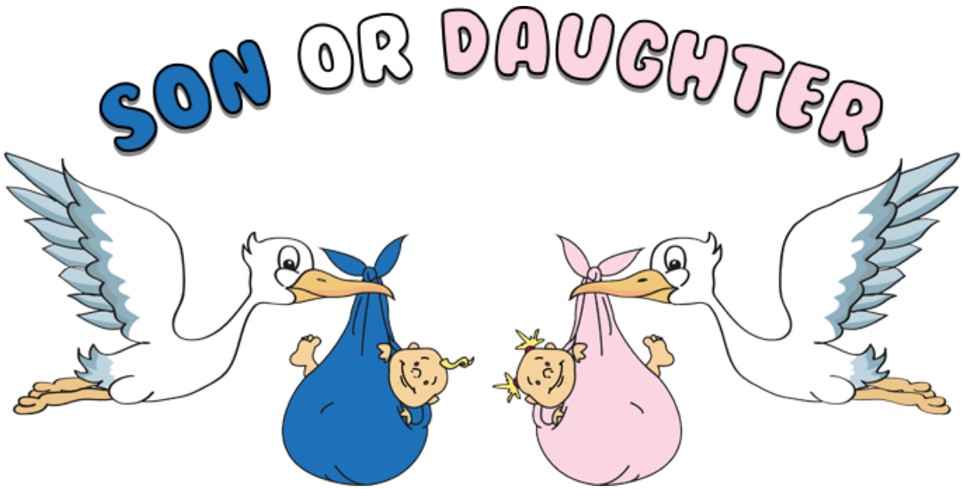
“How many patients with this disease have you treated?”
“4”
“How many have recovered?”
“Sorry, but none of them made it.”

probability around us (1)



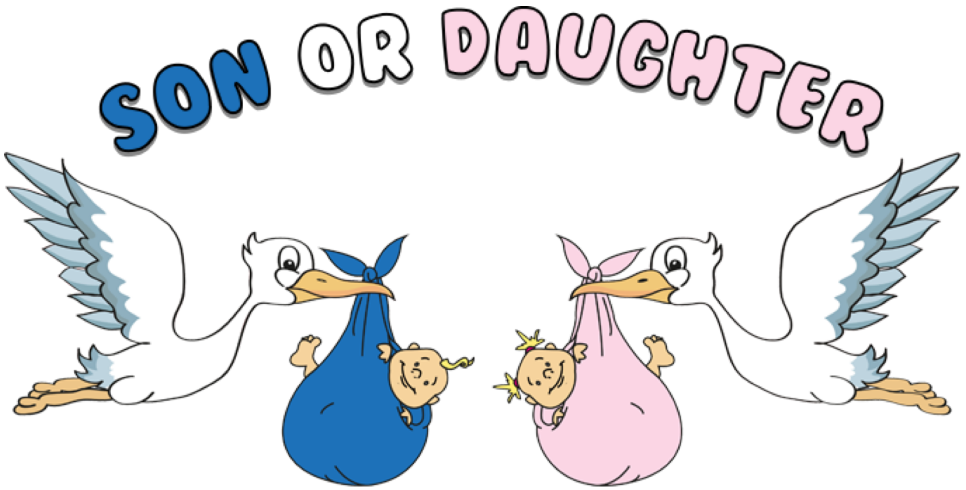
“Yay! This means that I will surely recover!”

probability around us (2)



- Suppose a couple had 3 sons. They plan to have 4^{th} child. What is the probability of the child is a daughter?

probability around us (2)



- 1. It is more likely that it is a daughter.
- 2. There is 50-50 chance.
- 3. It is more likely that it is a son.

probability around us (3)

- Should we buy lottery?
- The expected payoff is 50% of the lottery ticket price

probability around us (3)

- 50% seems to low! What if a fair lottery?
  - buy \$1 lottery: you get \$2 return or nothing with 50-50 chance
- You can play as many times as you want. How many times would you play?

probability around us (4)

- consider more favorable investment
  - 50% chance, will return a profit of 900%
    - your money X => 9X
  - 50% chance, will have loss of 90%
    - your money X => 0.1X

probability around us (4)

- on average, my money will be 4.55X after single investment
- What if I repeat this investment many times?
- Pitfall of “averages”!

Chapter 2

Probability

Section 2.1

SAMPLE SPACE

Definition 2.1

The set of all possible outcomes of a statistical experiment is called the **sample space** and is represented by the symbol S.

Figure 2.1 Tree diagram for Example 2.2

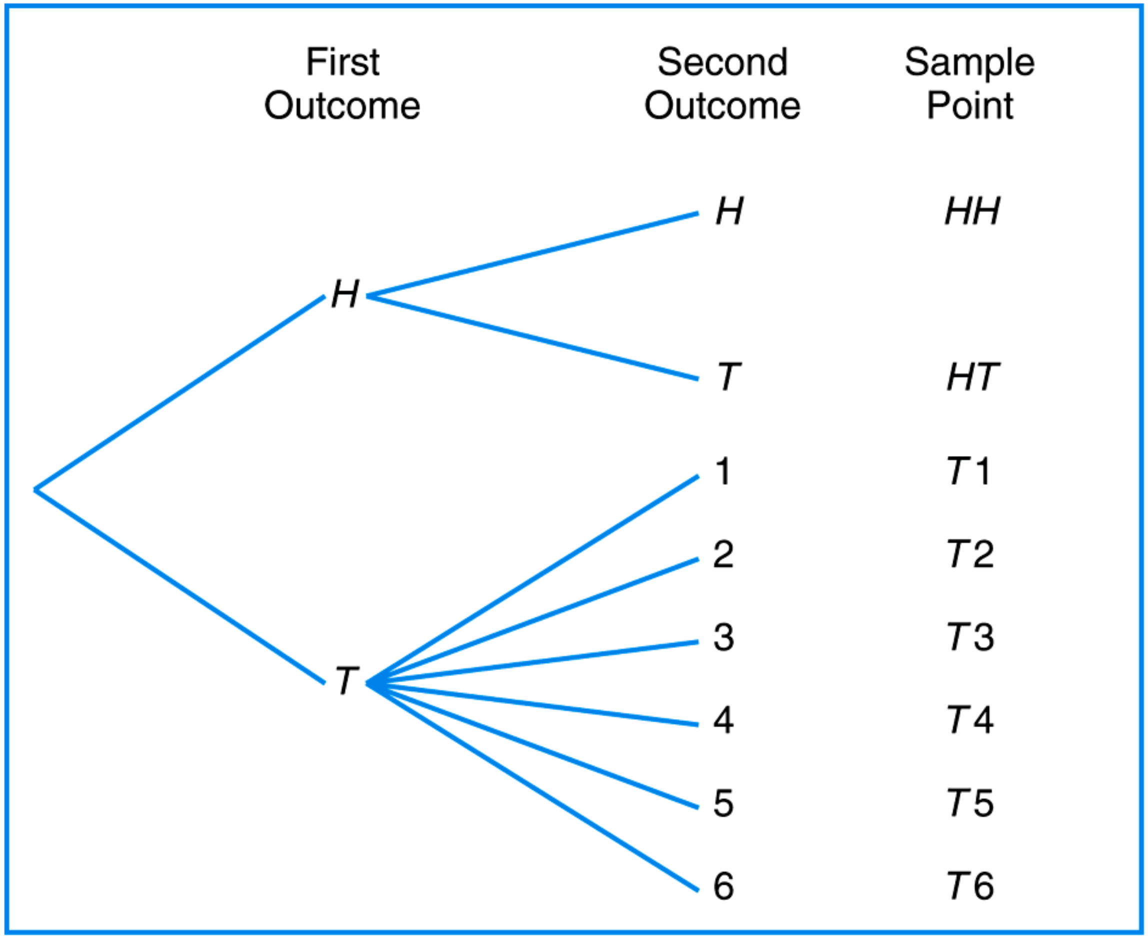
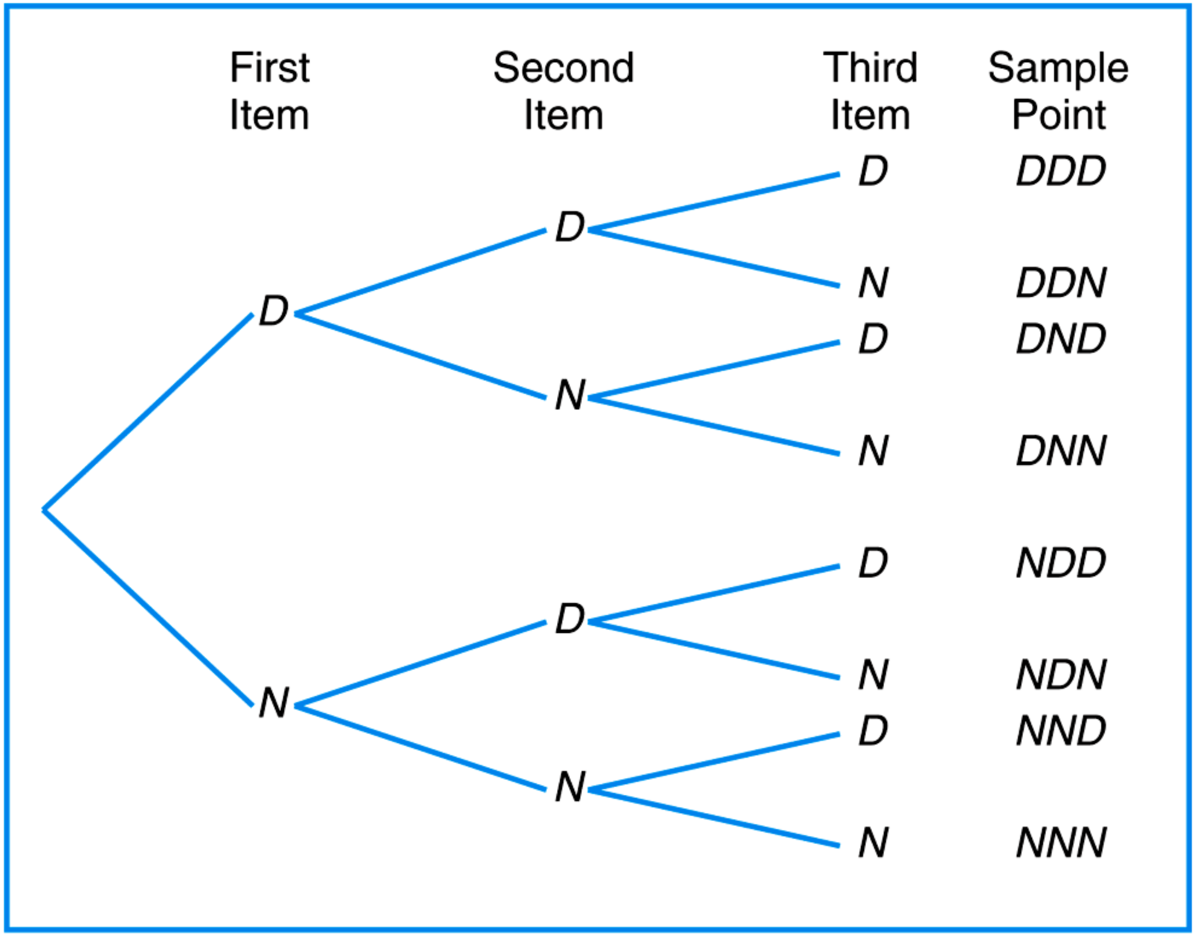


Figure 2.2 Tree Diagram for Example 2.3



Chapter 2

Probability

Section 2.2

EVENTS

Definition 2.2

An **event** is a subset of a sample space.

Definition 2.3

The **complement** of an event A with respect to S is the subset of all elements of S that are not in A. We denote the complement of A by the symbol A'.

Definition 2.4

The **intersection** of two events A and B, denoted by the symbol $A \cap B$, is the event containing all elements that are common to A and B.

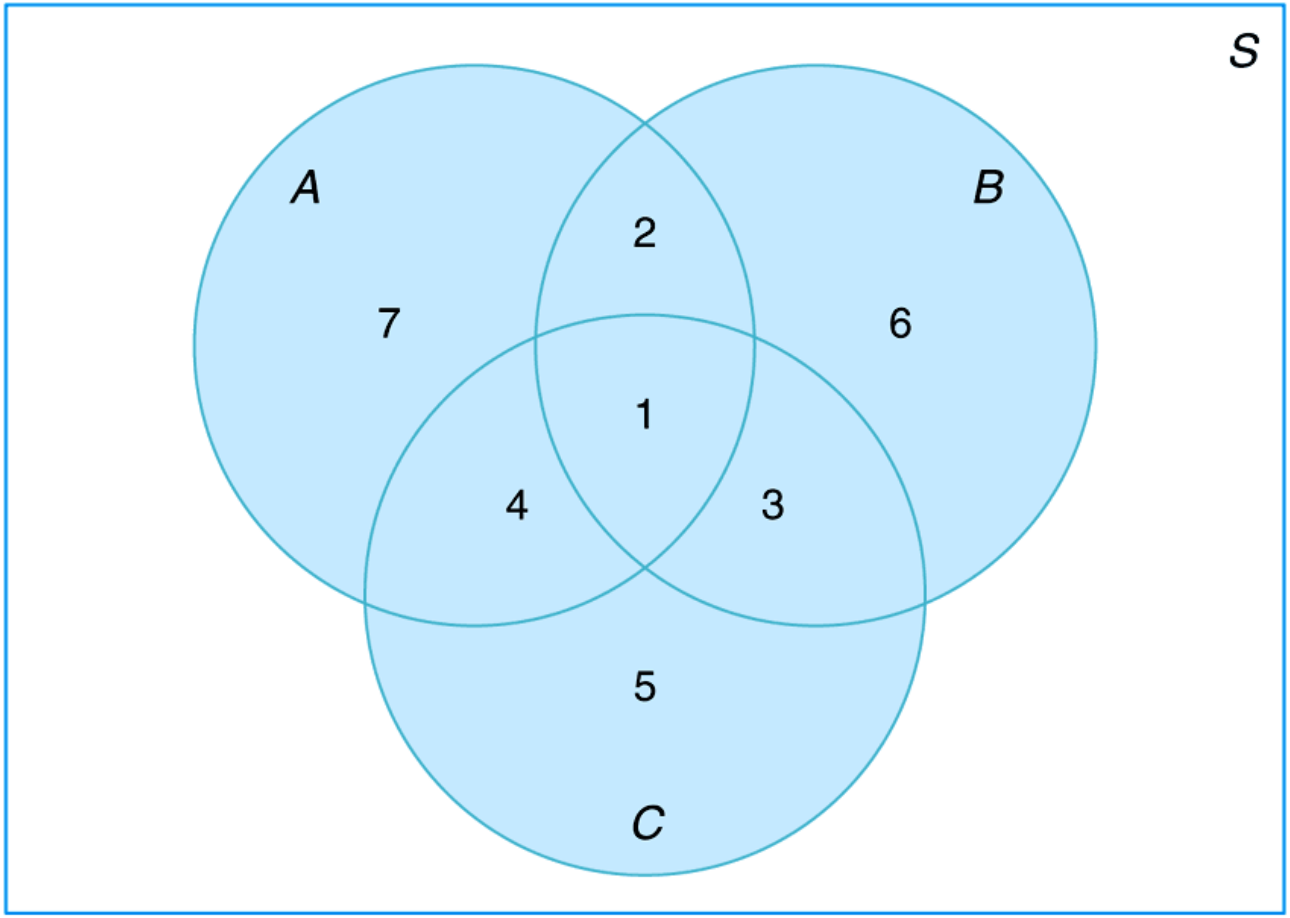
Definition 2.5

Two events A and B are **mutually exclusive**, or **disjoint**, if $A \cap B = \phi$, that is, if A and B have no elements in common.

Definition 2.6

The **union** of the two events A and B, denoted by the symbol $A \cup B$, is the event containing all the elements that belong to A or B or both.

Figure 2.3 Events represented by various regions



$A \cup C$:

$B' \cap A$:

$A \cap B \cap C$:

$(A \cup B) \cap C'$:

Figure 2.4 Events of the sample space S

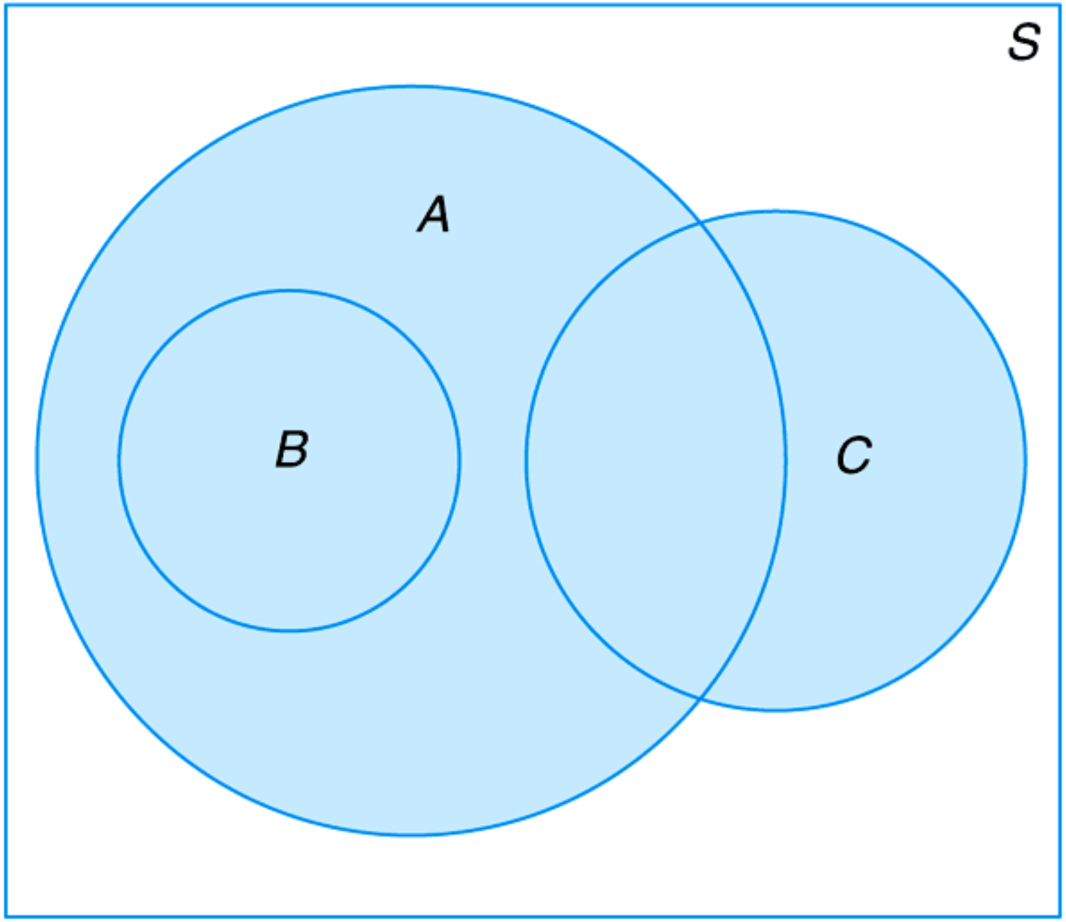
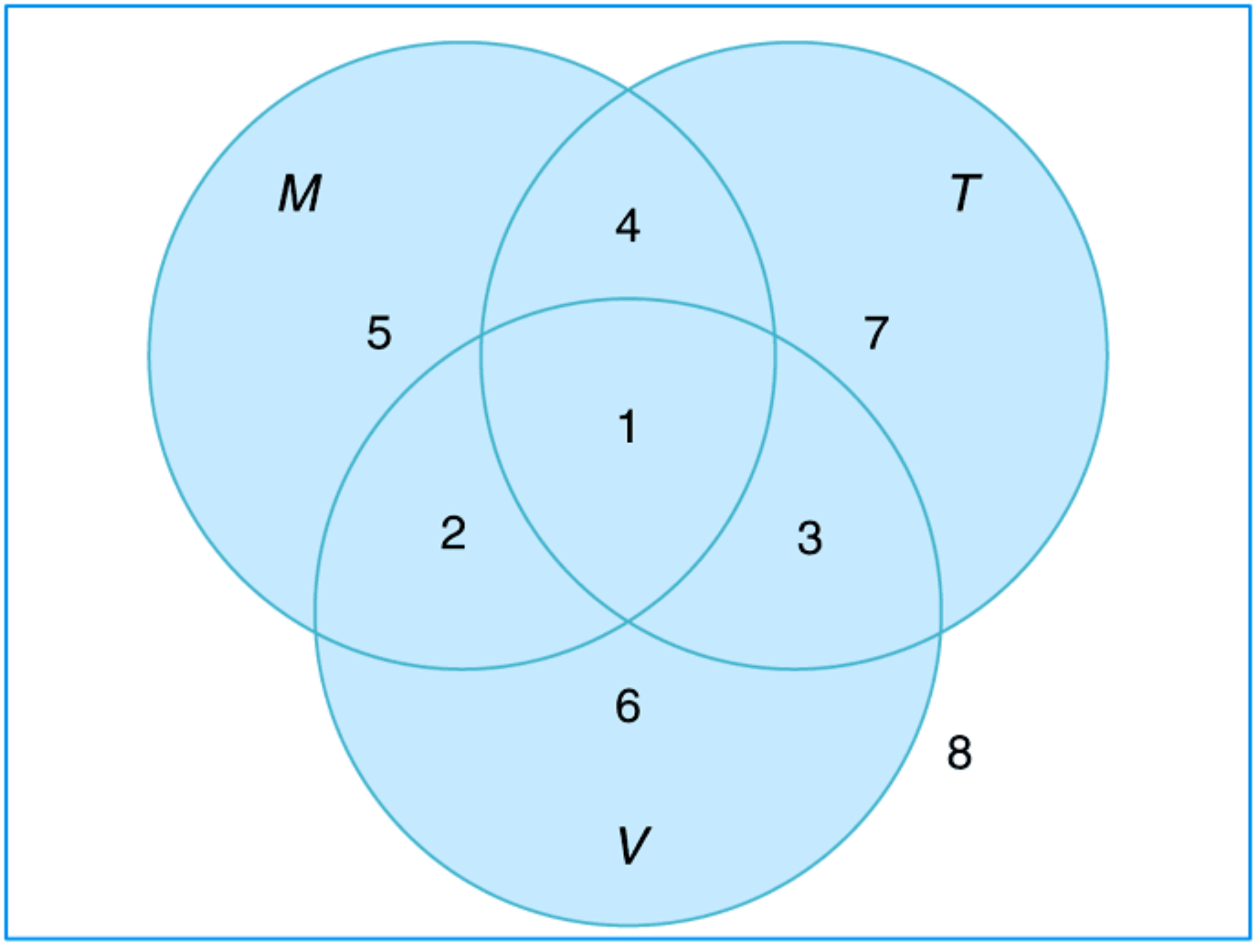


Figure 2.5 Venn diagram for Exercises 2.19 and 2.20



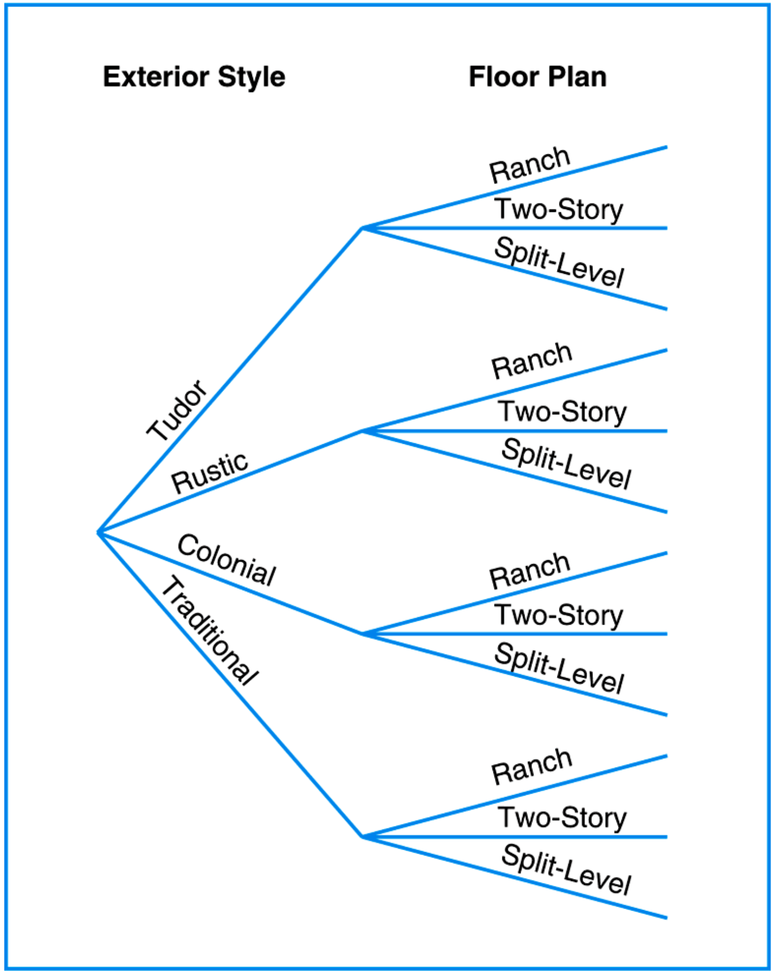
Section 2.3

Counting Sample Points

Rule 2.1

If an operation can be performed in n_1 ways, and if for each of these ways a second operation can be performed in n_2 ways, then the two operations can be performed together in $n_1 n_2$ ways.

Figure 2.6 Tree diagram for Example 2.14



Rule 2.2

If an operation can be performed in n_1 ways, and if for each of these a second operation can be performed in n_2 ways, and for each of the first two a third operation can be performed in n_3 ways, and so forth, then the sequence of k operations can be performed in $n_1 n_2 \cdots n_k$ ways.

Rule 2.2

If an operation can be performed in n_1 ways, and if for each of these a second operation can be performed in n_2 ways, and for each of the first two a third operation can be performed in n_3 ways, and so forth, then the sequence of k operations can be performed in $n_1 n_2 \cdots n_k$ ways.

Example 2.16: Sam is going to assemble a computer by himself. He has the choice of chips from two brands, a hard drive from four, memory from three, and an accessory bundle from five local stores. How many different ways can Sam order the parts?

Definition 2.7

$\boxed{\text{A } \textbf{permutation} \text{ is an arrangement of all or part of a set of objects.}}$

Definition 2.8

For any non-negative integer n, $n!$, called "n factorial," is defined as

$$n! = n(n-1)\cdots(2)(1),$$

with special case $0! = 1$.

Theorem 2.1

$$\boxed{\text{The number of permutations of } n \text{ objects is } n!.}$$

Theorem 2.2

The number of permutations of n distinct objects taken r at a time is

$$ {}_nP_r = \frac{n!}{(n-r)!}. $$

Theorem 2.2

The number of permutations of n distinct objects taken r at a time is

$$_nP_r = \frac{n!}{(n-r)!}.$$

In a contest, there are 10 participants. There are three –gold, silver, and bronze prizes for the participants. What is the total number of cases of prize reception?

Theorem 2.3

$\boxed{\text{The number of permutations of } n \text{ objects arranged in a circle is } (n-1)!.}$

Theorem 2.4

The number of distinct permutations of n things of which n_1 are of one kind, n_2 of a second kind, $\ldots$, n_k of a kth kind is

$$\frac{n!}{n_1!n_2!\cdots n_k!}.$$

Theorem 2.5

The number of ways of partitioning a set of n objects into r cells with n_1 elements in the first cell, n_2 elements in the second, and so forth, is

$$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1!n_2!\cdots n_r!},$$

where $n_1 + n_2 + \cdots + n_r = n$.

Theorem 2.6

The number of combinations of n distinct objects taken r at a time is

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

Section 2.4

Probability of an Event

Definition 2.9

The **probability** of an event A is the sum of the weights of all sample points in A. Therefore,

$$0 \leq P(A) \leq 1, \quad P(\phi) = 0, \quad \text{and} \quad P(S) = 1.$$

Furthermore, if $A_1, A_2, A_3, \ldots$ is a sequence of mutually exclusive events, then

$$P(A_1 \cup A_2 \cup A_3 \cup \cdots) = P(A_1) + P(A_2) + P(A_3) + \cdots .$$

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Example 2.24: A coin is tossed twice. What is the probability that at least 1 head occurs?

Rule 2.3

If an experiment can result in any one of N different equally likely outcomes, and if exactly n of these outcomes correspond to event A, then the probability of event A is

$$P(A) = \frac{n}{N}.$$

Example 2.27

Example 2.27: A statistics class for engineers consists of 25 industrial, 10 mechanical, 10 electrical, and 8 civil engineering students. If a person is randomly selected by the instructor to answer a question, find the probability that the student chosen is (a) an industrial engineering major and (b) a civil engineering or an electrical engineering major.

Example 2.28

Example 2.28: In a poker hand consisting of 5 cards, find the probability of holding 2 aces and 3 jacks.

Section 2.5

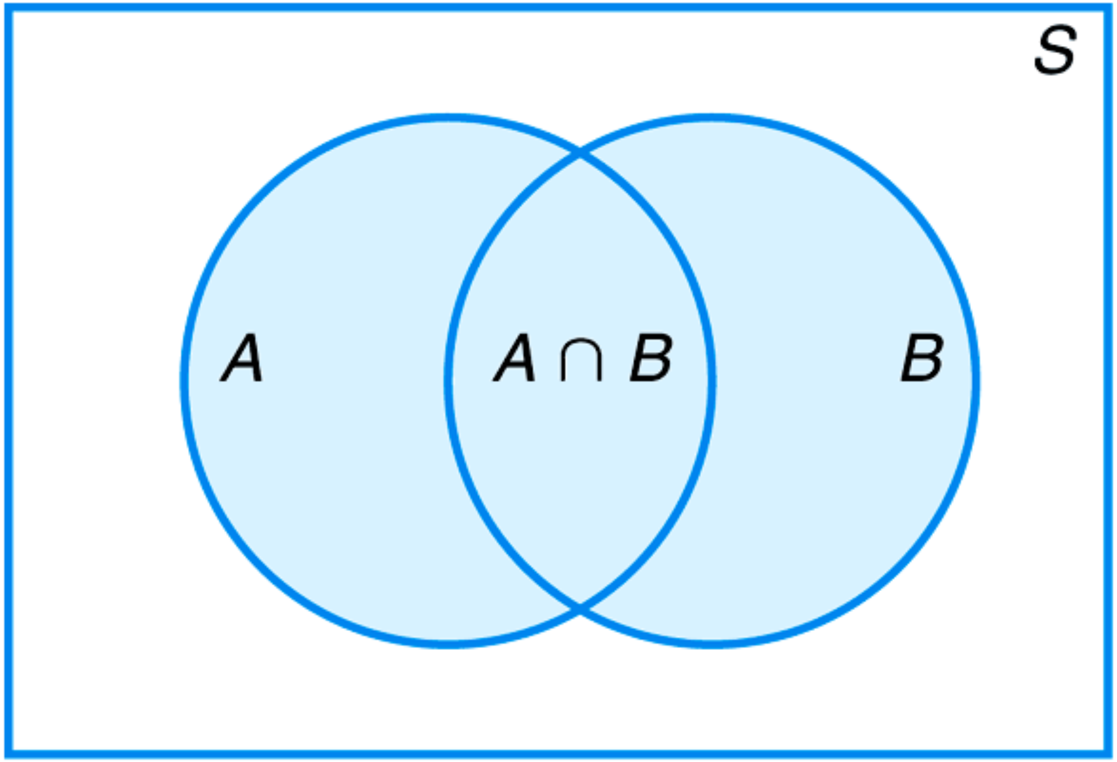
Additive Rules

Theorem 2.7

If A and B are two events, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Figure 2.7 Additive rule of probability



Corollary 2.1

If A and B are mutually exclusive, then

$$P(A \cup B) = P(A) + P(B).$$

Corollary 2.2

If $A_1,\ A_2, \ldots, A_n$ are mutually exclusive, then

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = P(A_1) + P(A_2) + \cdots + P(A_n).$$

Corollary 2.3

If $A_1,\ A_2, \ldots, A_n$ is a partition of sample space S, then

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = P(A_1) + P(A_2) + \cdots + P(A_n) = P(S) = 1.$$

Theorem 2.8

For three events A, B, and C,

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C).$$

Example 2.30

Example 2.30: What is the probability of getting a total of 7 or 11 when a pair of fair dice is tossed?

Theorem 2.9

If A and A' are complementary events, then

$$P(A) + P(A') = 1.$$

Example 2.32

Example 2.32: If the probabilities that an automobile mechanic will service 3, 4, 5, 6, 7, or 8 or more cars on any given workday are, respectively, 0.12, 0.19, 0.28, 0.24, 0.10, and 0.07, what is the probability that he will service at least 5 cars on his next day at work?

Section 2.6

Conditional Probability, Independence, and the Product Rule

Definition 2.10

The conditional probability of B, given A, denoted by $P(B|A)$, is defined by

$$P(B|A) = \frac{P(A \cap B)}{P(A)}, \quad \text{provided} \quad P(A) > 0.$$

- **Conditional probability is related to prior information**

Table 2.1 Categorization of the Adults in a Small Town

	Employed	Unemployed	Total
Male	460	40	500
Female	140	260	400
Total	600	300	900

- **Let E denote the event of employed and M denote the event of being male.**
- **What is the probability of a person being Male given that the person is Employed?**
  - What is the probability of a person both being male and employed?

Example 2.35

Example 2.35: The concept of conditional probability has countless uses in both industrial and biomedical applications. Consider an industrial process in the textile industry in which strips of a particular type of cloth are being produced. These strips can be defective in two ways, length and nature of texture. For the case of the latter, the process of identification is very complicated. It is known from historical information on the process that 10% of strips fail the length test, 5% fail the texture test, and only 0.8% fail both tests. If a strip is selected randomly from the process and a quick measurement identifies it as failing the length test, what is the probability that it is texture defective?

Definition 2.11

Two events A and B are **independent** if and only if

$$P(B|A) = P(B) \quad \text{or} \quad P(A|B) = P(A),$$

assuming the existences of the conditional probabilities. Otherwise, A and B are **dependent**.

Theorem 2.10

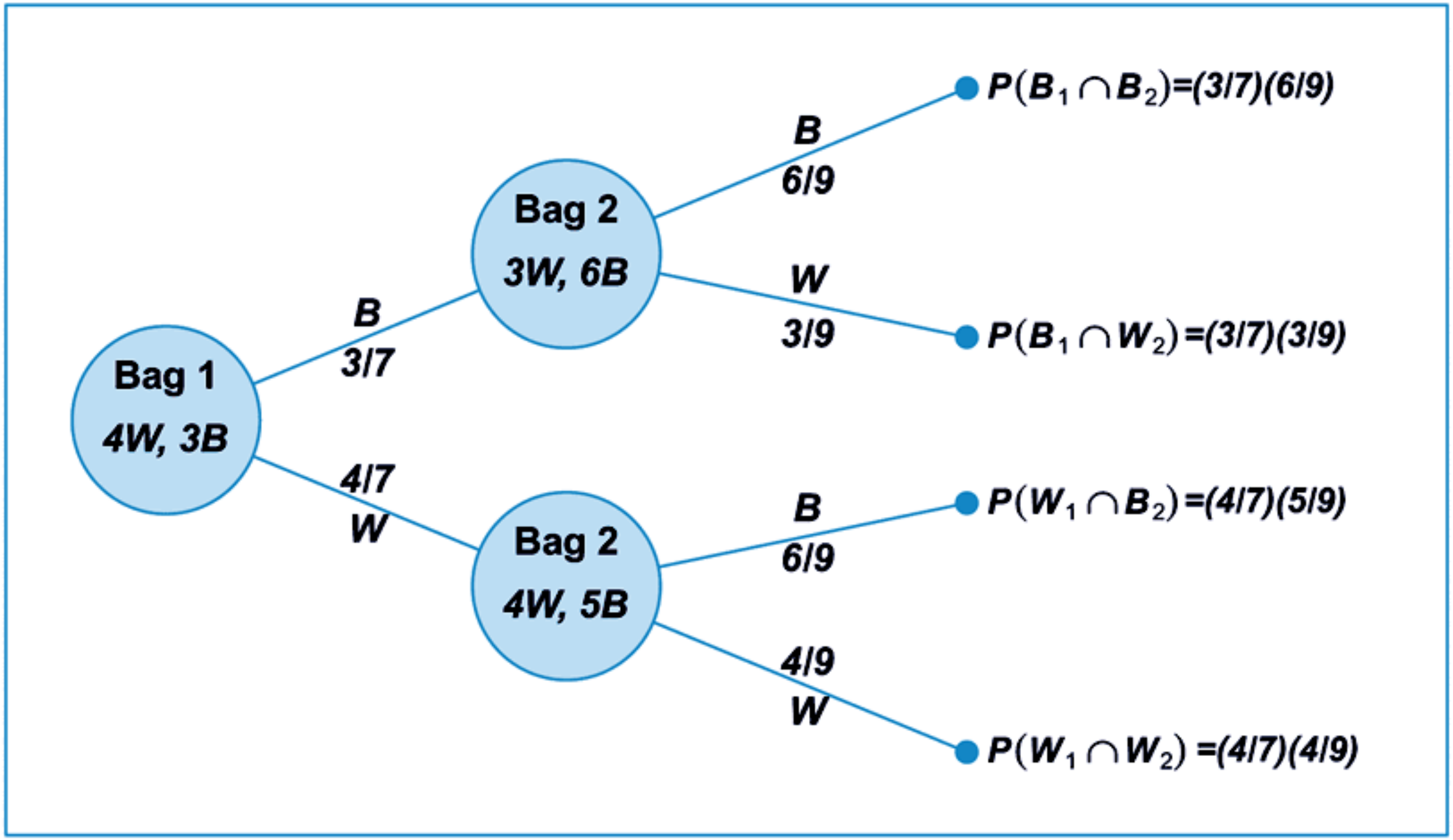
If in an experiment the events A and B can both occur, then

$$P(A \cap B) = P(A)P(B|A), \text{ provided } P(A) > 0.$$

Example 2.37

Example 2.37: One bag contains 4 white balls and 3 black balls, and a second bag contains 3 white balls and 5 black balls. One ball is drawn from the first bag and placed unseen in the second bag. What is the probability that a ball now drawn from the second bag is black?

Figure 2.8 Tree diagram for Example 2.37



Theorem 2.11

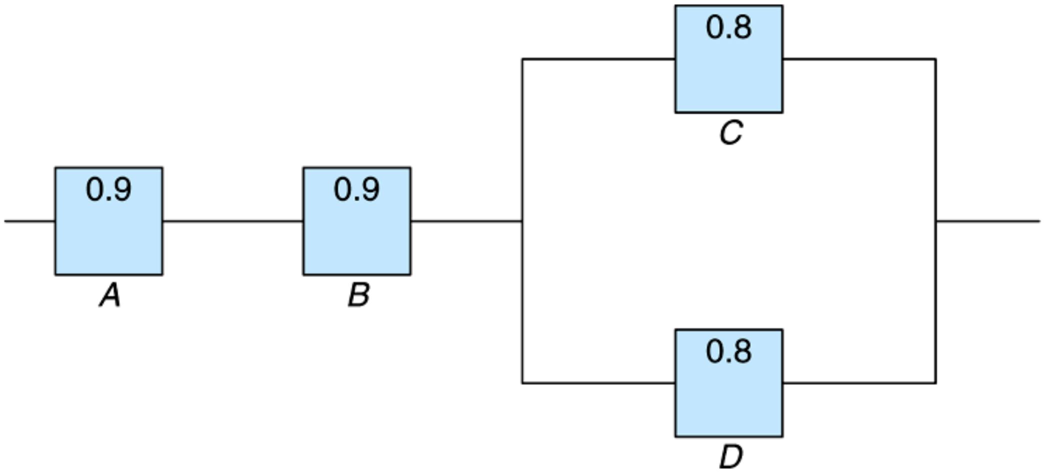
Two events A and B are independent if and only if

$$P(A \cap B) = P(A)P(B).$$

Therefore, to obtain the probability that two independent events will both occur, we simply find the product of their individual probabilities.

Figure 2.9 An electrical system for Example 2.39

Example 2.39: An electrical system consists of four components as illustrated in Figure 2.9. The system works if components A and B work and either of the components C or D works. The reliability (probability of working) of each component is also shown in Figure 2.9. Find the probability that (a) the entire system works and (b) the component C does not work, given that the entire system works. Assume that the four components work independently.



Theorem 2.12

If, in an experiment, the events $A_1, A_2, \ldots, A_k$ can occur, then

$$P(A_1 \cap A_2 \cap \cdots \cap A_k)$$
$$= P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2) \cdots P(A_k|A_1 \cap A_2 \cap \cdots \cap A_{k-1}).$$

If the events $A_1, A_2, \ldots, A_k$ are independent, then

$$P(A_1 \cap A_2 \cap \cdots \cap A_k) = P(A_1)P(A_2) \cdots P(A_k).$$

Example 2.40

Example 2.40: Three cards are drawn in succession, without replacement, from an ordinary deck of playing cards. Find the probability that the event $A_1 \cap A_2 \cap A_3$ occurs, where A_1 is the event that the first card is a red ace, A_2 is the event that the second card is a 10 or a jack, and A_3 is the event that the third card is greater than 3 but less than 7.

Definition 2.12

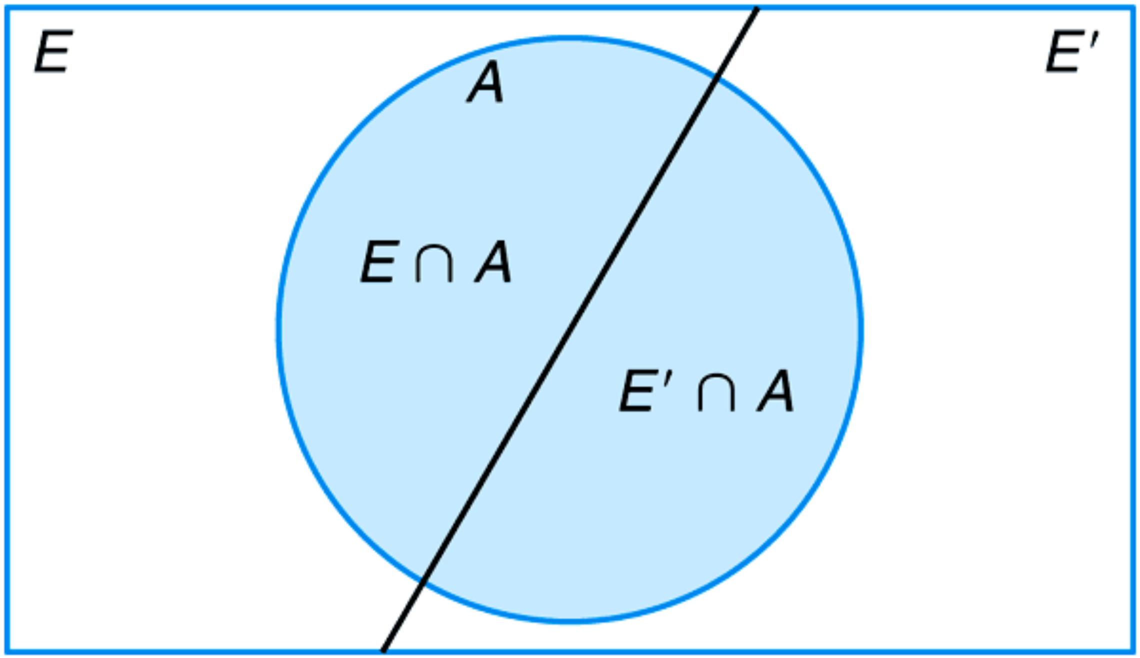
A collection of events $\mathcal{A} = \{A_1, \ldots, A_n\}$ are mutually independent if for any subset of $\mathcal{A}$, $A_{i_1}, \ldots, A_{i_k}$, for $k \leq n$, we have

$$P(A_{i_1} \cap \cdots \cap A_{i_k}) = P(A_{i_1}) \cdots P(A_{i_k}).$$

Section 2.7

Bayes' Rule

Figure 2.12 Venn diagram for the events *A*, *E* and *E*′



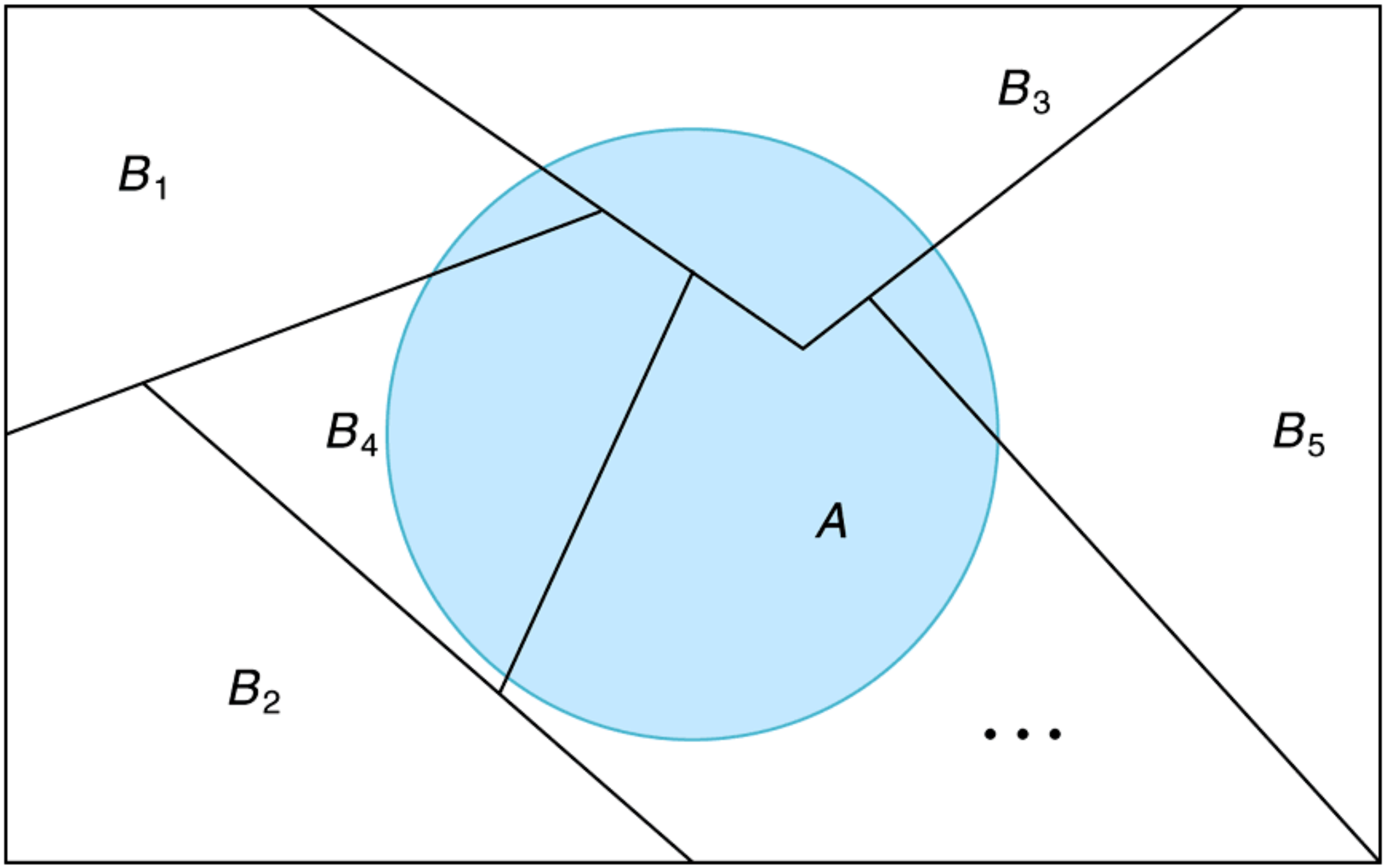
$$P(A) = P[(E \cap A) \cup (E' \cap A)] = P(E \cap A) + P(E' \cap A)$$
$$= P(E)P(A|E) + P(E')P(A|E').$$

Theorem 2.13

If the events $B_1, B_2, \ldots, B_k$ constitute a partition of the sample space S such that $P(B_i) \neq 0$ for $i = 1, 2, \ldots, k$, then for any event A of S,

$$P(A) = \sum_{i=1}^{k} P(B_i \cap A) = \sum_{i=1}^{k} P(B_i)P(A|B_i).$$

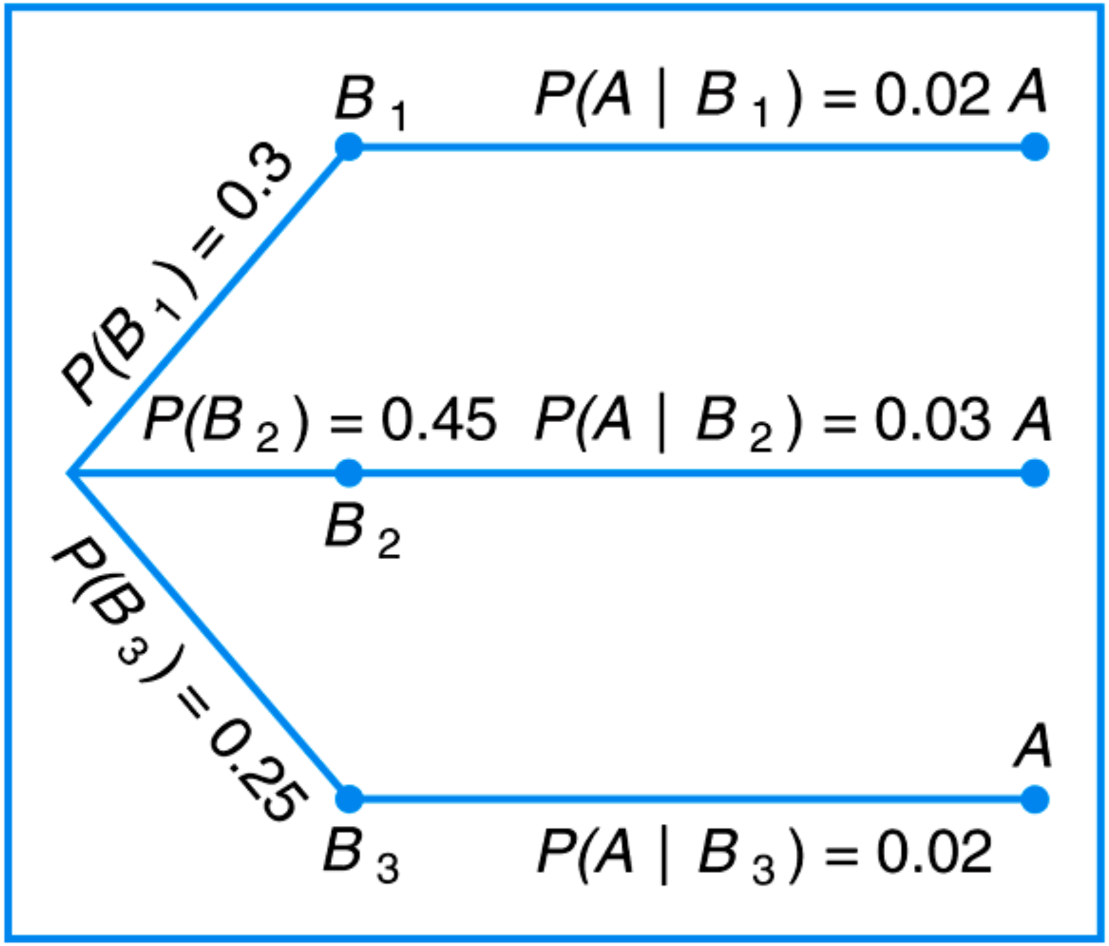
Figure 2.14 Partitioning the sample space S



Example 2.41

Example 2.41: In a certain assembly plant, three machines, B_1, B_2, and B_3, make 30%, 45%, and 25%, respectively, of the products. It is known from past experience that 2%, 3%, and 2% of the products made by each machine, respectively, are defective. Now, suppose that a finished product is randomly selected. What is the probability that it is defective?

Figure 2.15 Tree diagram for Example 2.41



Theorem 2.14

(Bayes' Rule) If the events $B_1, B_2, \ldots, B_k$ constitute a partition of the sample space S such that $P(B_i) \neq 0$ for $i = 1, 2, \ldots, k$, then for any event A in S such that $P(A) \neq 0$,

$$P(B_r|A) = \frac{P(B_r \cap A)}{\sum_{i=1}^{k} P(B_i \cap A)} = \frac{P(B_r)P(A|B_r)}{\sum_{i=1}^{k} P(B_i)P(A|B_i)} \quad \text{for } r = 1, 2, \ldots, k.$$

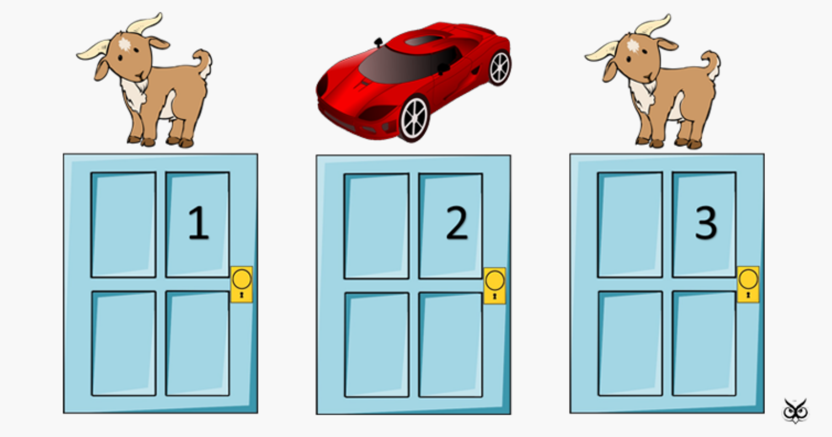
Example 2.43

Example 2.43: A manufacturing firm employs three analytical plans for the design and development of a particular product. For cost reasons, all three are used at varying times. In fact, plans 1, 2, and 3 are used for 30%, 20%, and 50% of the products, respectively. The defect rate is different for the three procedures as follows:

$$P(D|P_1) = 0.01, \qquad P(D|P_2) = 0.03, \qquad P(D|P_3) = 0.02,$$

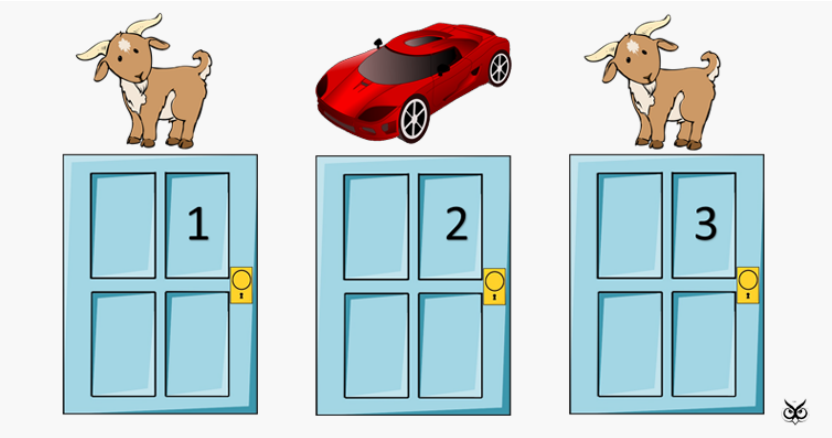
where $P(D|P_j)$ is the probability of a defective product, given plan j. If a random product was observed and found to be defective, which plan was most likely used and thus responsible?

Fun stuff- Monty Hall problem



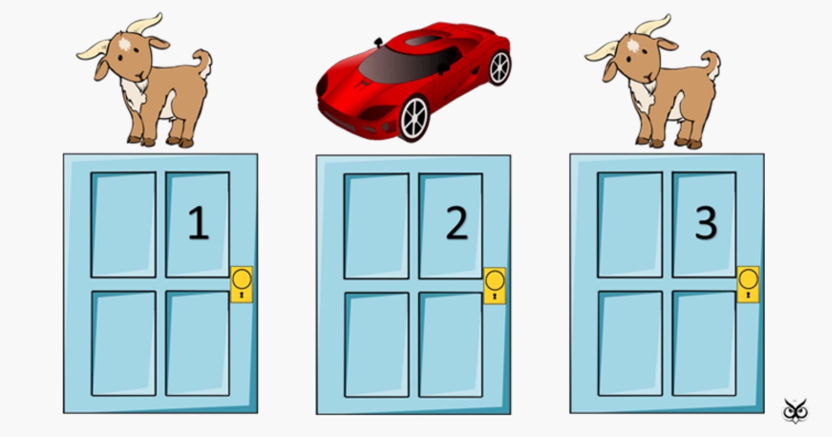
- There are three doors. Behind one with car, two with goats

Monty Hall problem



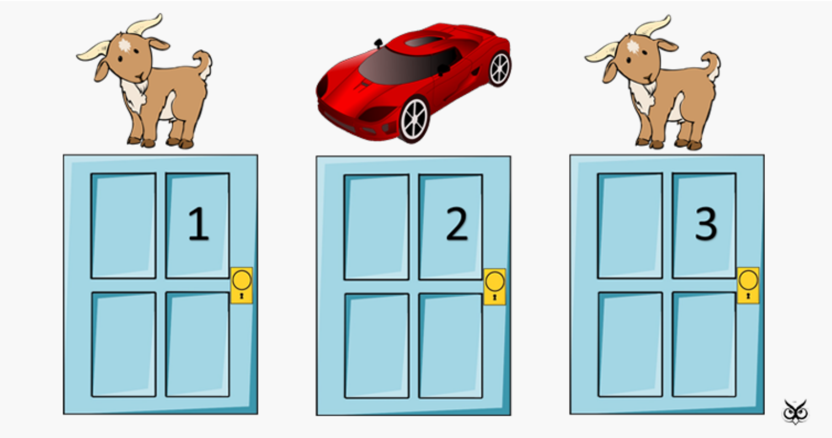
- You first pick one door.
- Before you open it, the show host opens another door with a goat.
- You now have chance to switch door. Will you?

Monty Hall problem



- three strategies
- 1. stay with your first choice
- 2. randomly choose between the remaining two doors
- 3. always switch to the other door
- what is the best strategy?

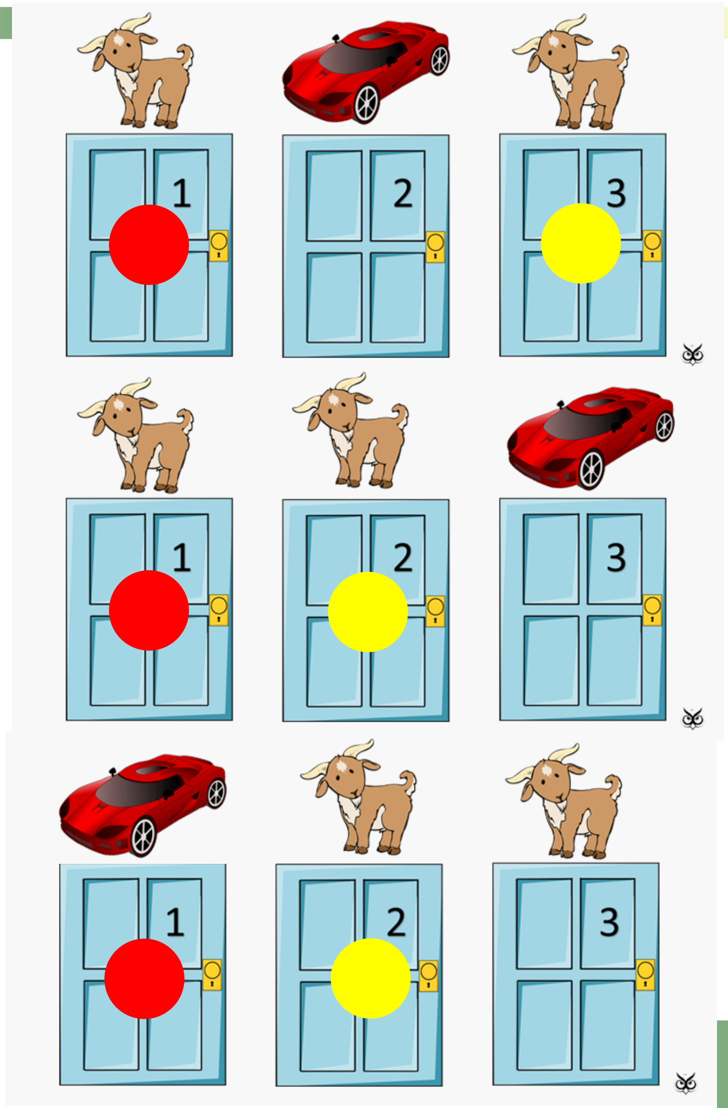
Monty Hall problem



- Your strategy is given by switching probability p
- Using conditional probability
- $P(W) = P(W \mid S) * p + P(W \mid S^c) * (1-p)$

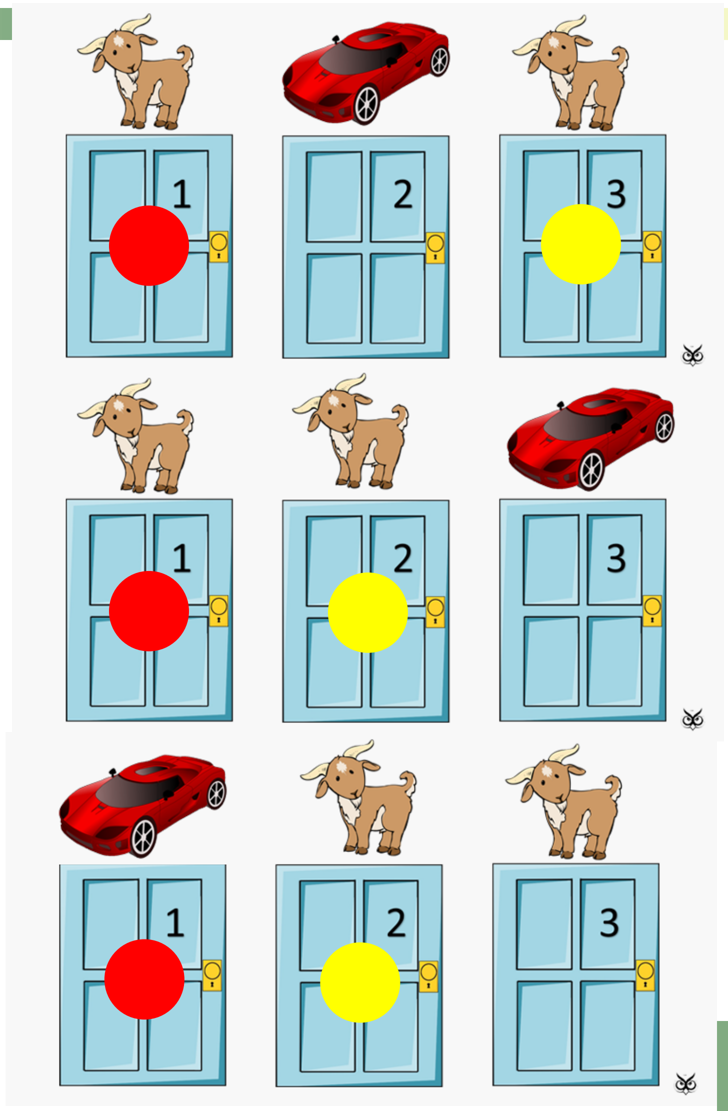
Monty Hall problem

- What is P(W | S)?

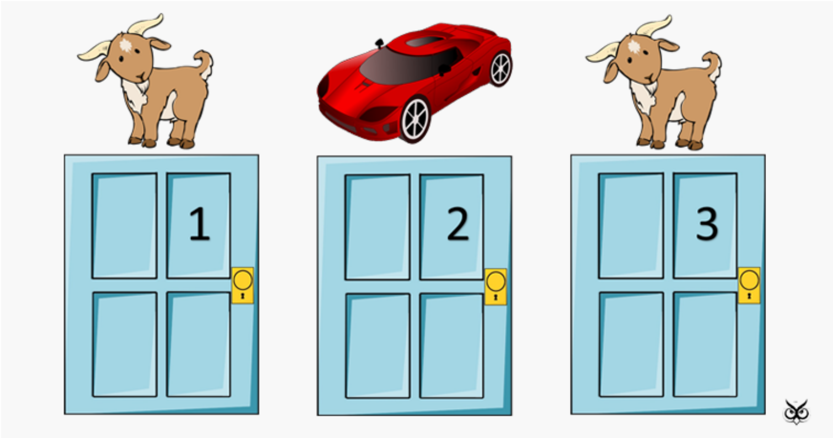


Monty Hall problem

- What is $P(W \mid S^c)$?

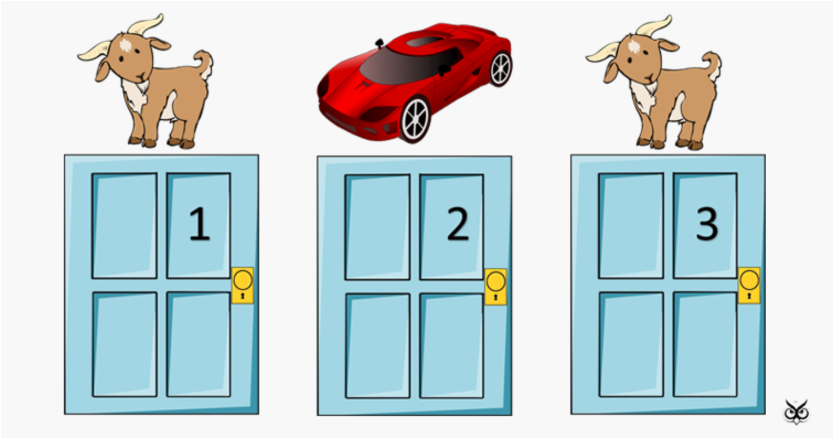


Monty Hall problem



- Your winning probability
- $P(W) = 2/3 * p + 1/3 * (1-p) = 1/3 + 1/3*p$

Monty Hall problem



- $P(W) = 2/3 * p + 1/3 * (1-p) = 1/3 + 1/3*p$
- Best strategy: $p=1$, $P(W)=2/3$ (always switch!)
- Worst strategy: $p=0$, $P(W)=1/3$ (never switch!)
- Random strategy: $p=1/2$, $P(W)=1/2$ (people confused..)