Chapter 9

One-Sample Estimation Problems

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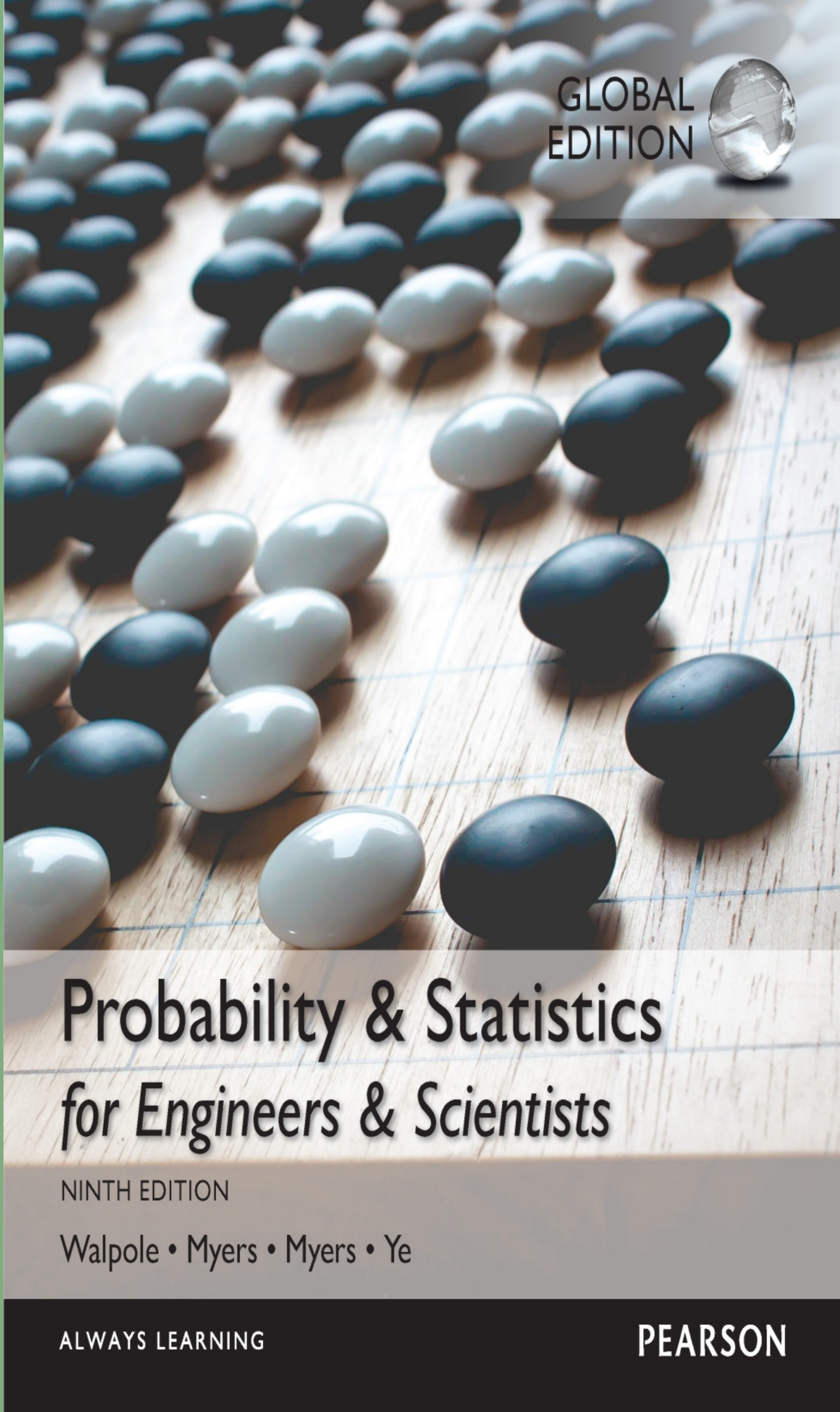

Section 9.3

Classical Methods of Estimation

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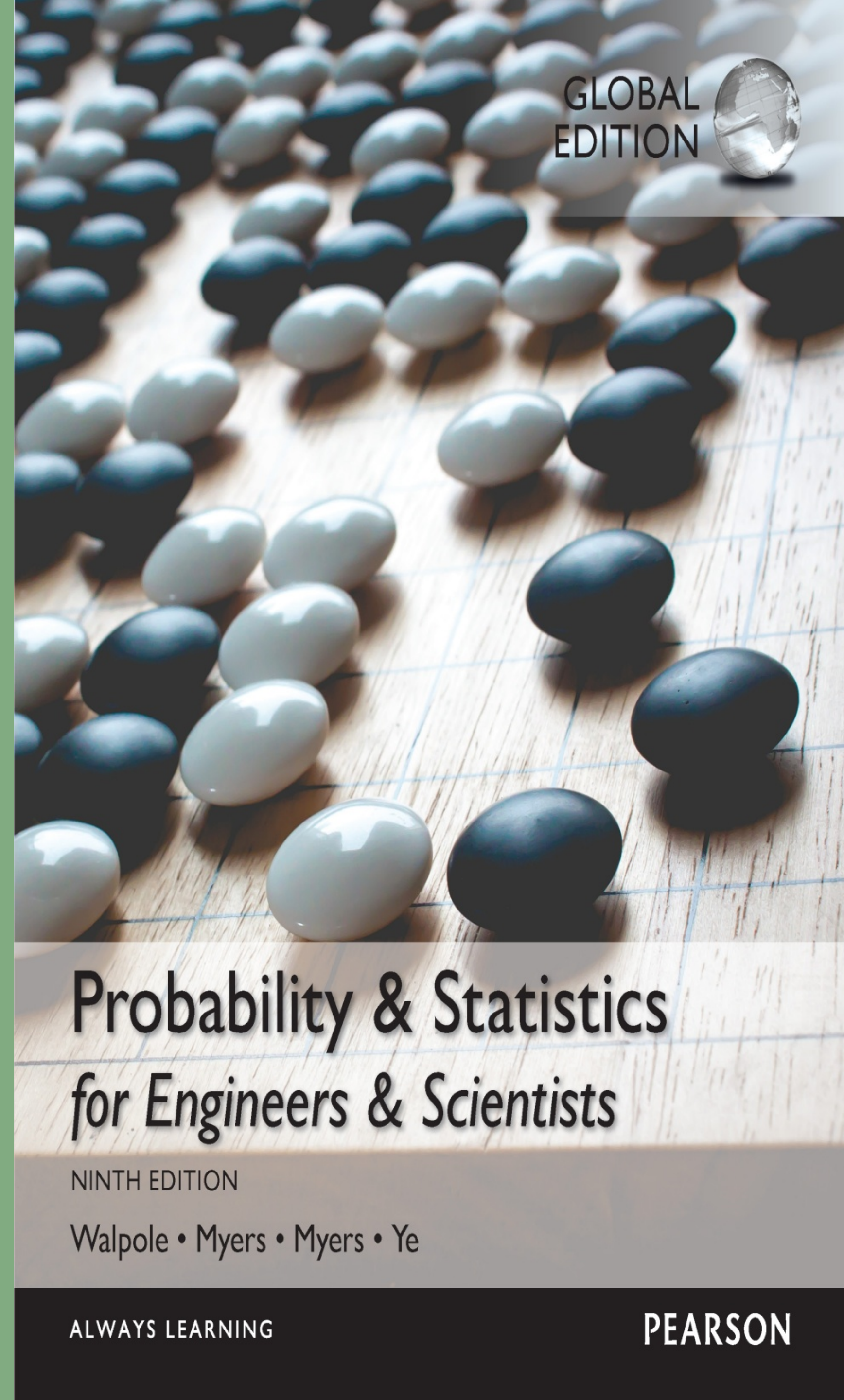

Definition 9.1

A statistic $\hat{\Theta}$ is said to be an **unbiased estimator** of the parameter θ if

$$\mu_{\hat{\Theta}} = E(\hat{\Theta}) = \theta.$$

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Definition 9.1

- Sample mean

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

is an unbiased estimator of μ

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Definition 9.1

- Sample variance

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

is an unbiased estimator of σ^2

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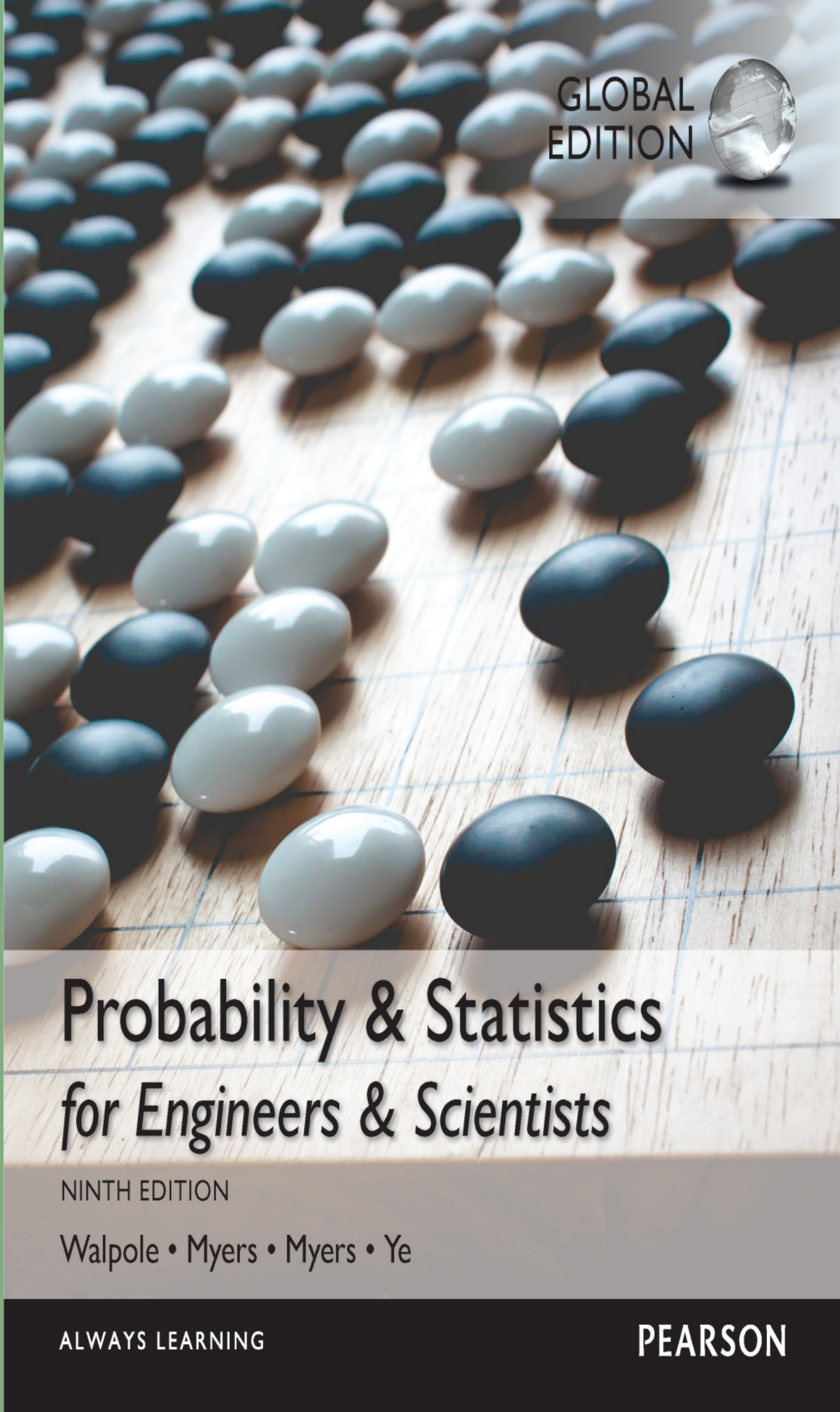
Section 9.4

Single Sample: Estimating the Mean



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Confidence interval

- Motivation
  - Suppose we draw n random samples
  - assume we know variance σ but don't know mean μ
  - what can be say about true mean μ, by using our sample mean?
  - assume n is sufficiently big, then

$$Z = \frac{\bar{X} - \mu}{\sigma / \sqrt{n}} \sim \; N(0, 1)$$

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Confidence interval

Confidence Interval on μ, σ^2 Known

If $\bar{x}$ is the mean of a random sample of size n from a population with known variance σ^2, a $100(1-\alpha)\%$ confidence interval for μ is given by

$$\bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}},$$

where $z_{\alpha/2}$ is the z-value leaving an area of $\alpha/2$ to the right.

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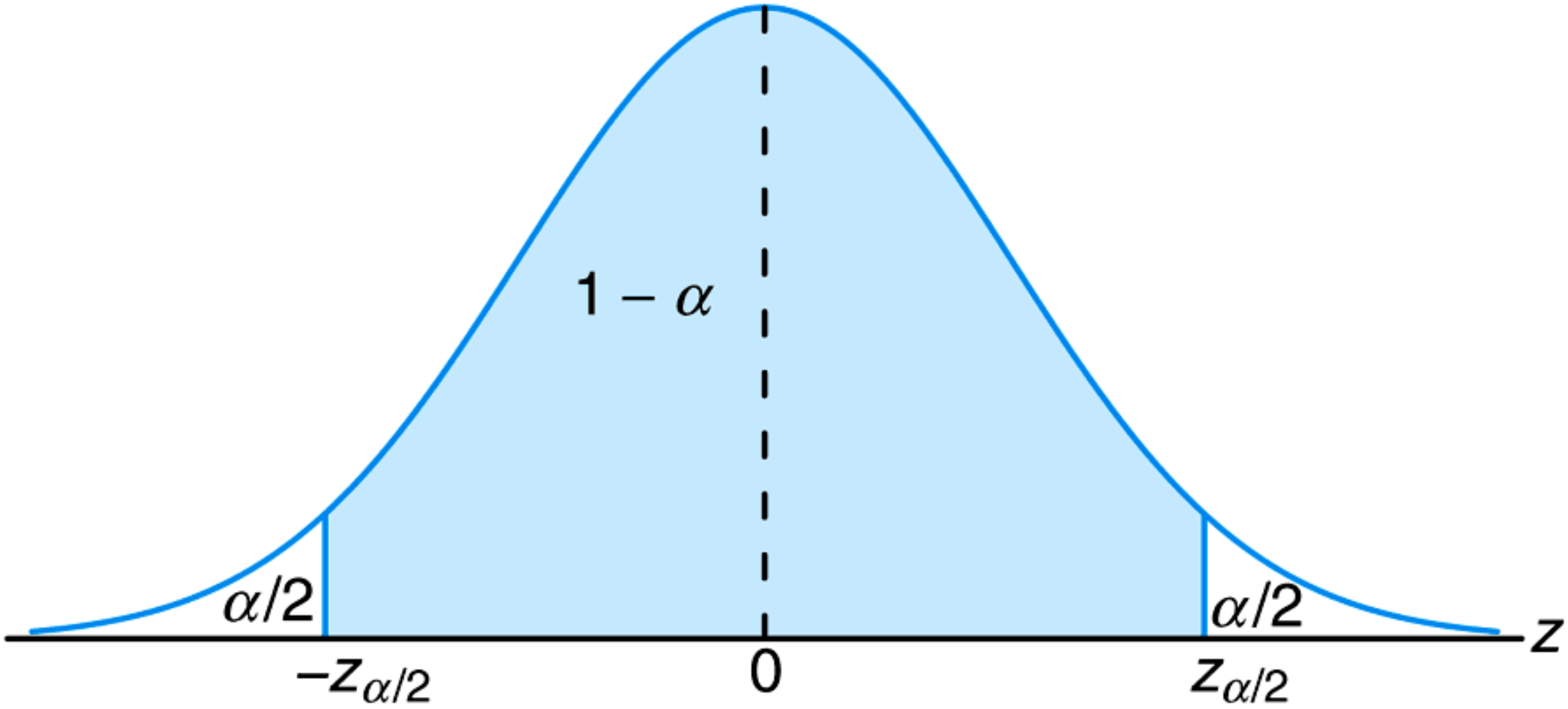
Figure 9.2 $P(-z_{\alpha/2} < Z < z_{\alpha/2}) = 1-\alpha$



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Theorem 9.1

If $\bar{x}$ is used as an estimate of μ, we can be $100(1-\alpha)\%$ confident that the error will not exceed $z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$.

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example 9.2

Example 9.2: The average zinc concentration recovered from a sample of measurements taken in 36 different locations in a river is found to be 2.6 grams per milliliter. Find the 95% and 99% confidence intervals for the mean zinc concentration in the river. Assume that the population standard deviation is 0.3 gram per milliliter.

the 95% confidence interval is

$$2.6 - (1.96)\left(\frac{0.3}{\sqrt{36}}\right) < \mu < 2.6 + (1.96)\left(\frac{0.3}{\sqrt{36}}\right),$$

which reduces to $2.50 < \mu < 2.70$.

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Theorem 9.2

If $\bar{x}$ is used as an estimate of μ, we can be $100(1-\alpha)\%$ confident that the error will not exceed a specified amount e when the sample size is

$$n = \left(\frac{z_{\alpha/2}\sigma}{e}\right)^2.$$

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example 9.3

Example 9.3: How large a sample is required if we want to be 95% confident that our estimate of μ in Example 9.2 is off by less than 0.05?

***Solution*:** The population standard deviation is $\sigma = 0.3$. Then, by Theorem 9.2,

$$n = \left[\frac{(1.96)(0.3)}{0.05}\right]^2 = 138.3.$$

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Stdev unknown for normal samples

Confidence Interval on μ, σ^2 Unknown	If $\bar{x}$ and s are the mean and standard deviation of a random sample from a normal population with unknown variance σ^2, a $100(1-\alpha)\%$ confidence interval for μ is $$\bar{x} - t_{\alpha/2}\frac{s}{\sqrt{n}} < \mu < \bar{x} + t_{\alpha/2}\frac{s}{\sqrt{n}},$$ where $t_{\alpha/2}$ is the t-value with $v = n-1$ degrees of freedom, leaving an area of $\alpha/2$ to the right.

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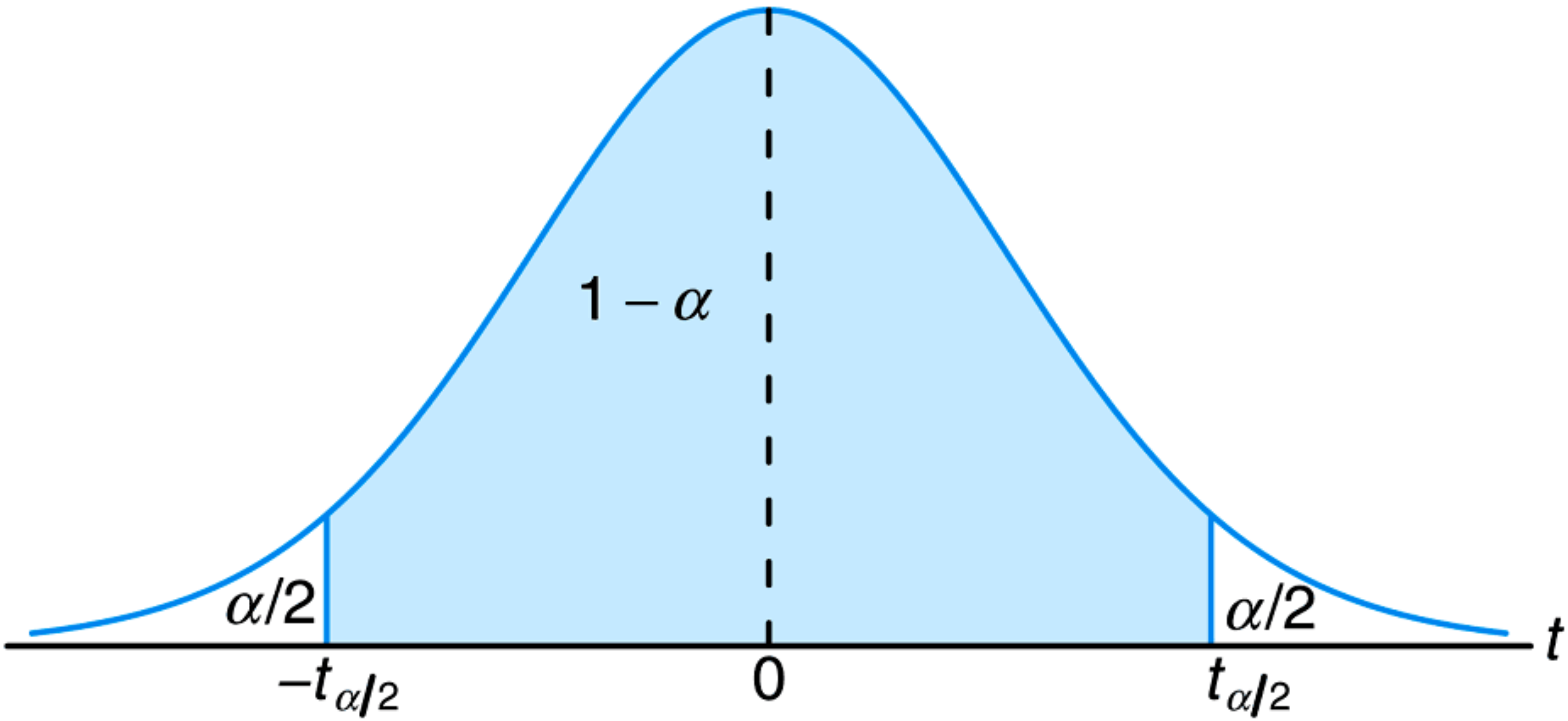
Figure 9.5

$P(-t_{\alpha/2} < T < t_{\alpha/2}) = 1-\alpha$



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example 9.5

Example 9.5: The contents of seven similar containers of sulfuric acid are 9.8, 10.2, 10.4, 9.8, 10.0, 10.2, and 9.6 liters. Find a 95% confidence interval for the mean contents of all such containers, assuming an approximately normal distribution.

The sample mean and standard deviation for the given data are

$$\bar{x} = 10.0 \qquad \text{and} \qquad s = 0.283.$$

95% confidence interval for μ is

$$10.0 - (2.447)\left(\frac{0.283}{\sqrt{7}}\right) < \mu < 10.0 + (2.447)\left(\frac{0.283}{\sqrt{7}}\right),$$

which reduces to $9.74 < \mu < 10.26$.

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Chapter 10

One- and Two-Sample Tests of Hypotheses

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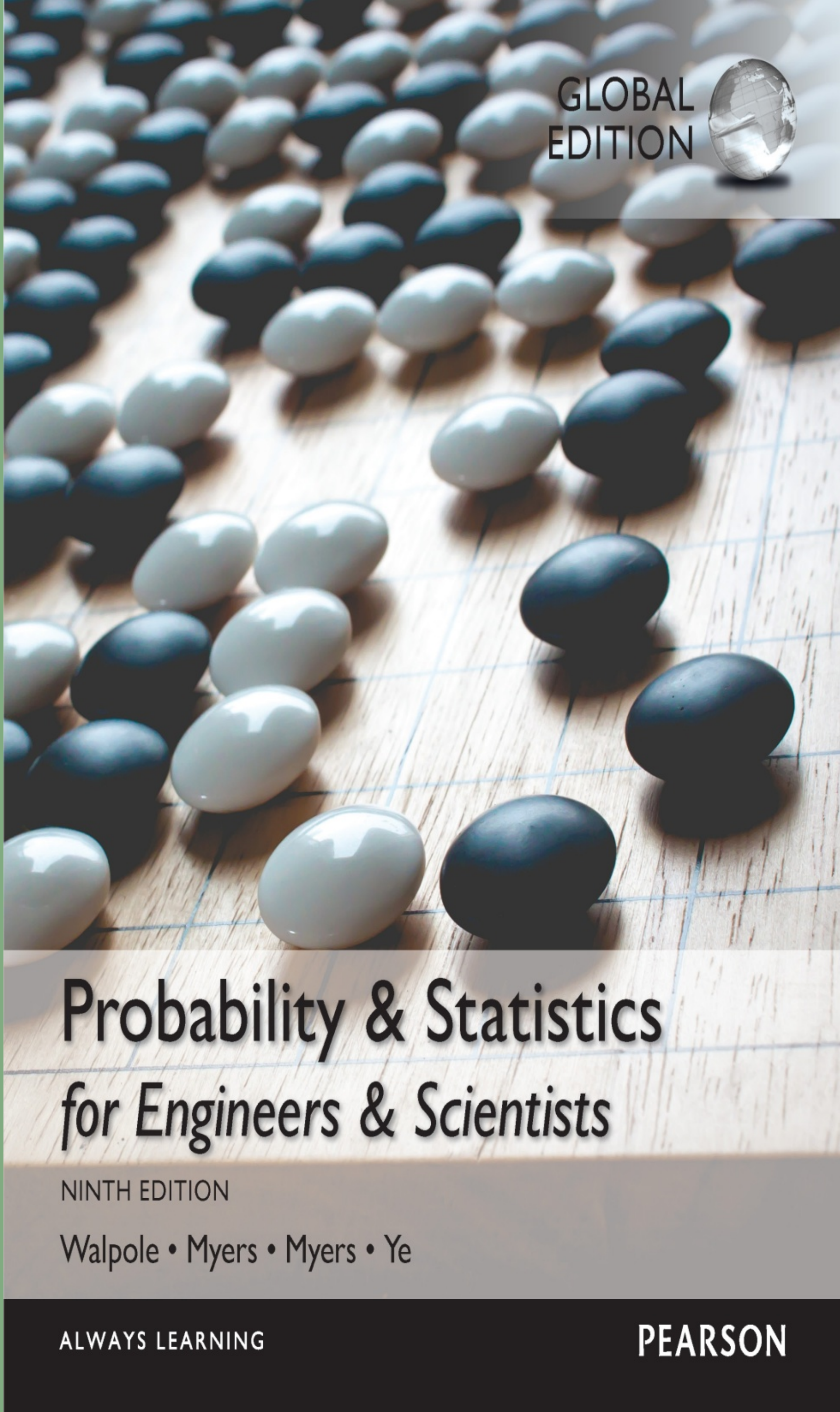
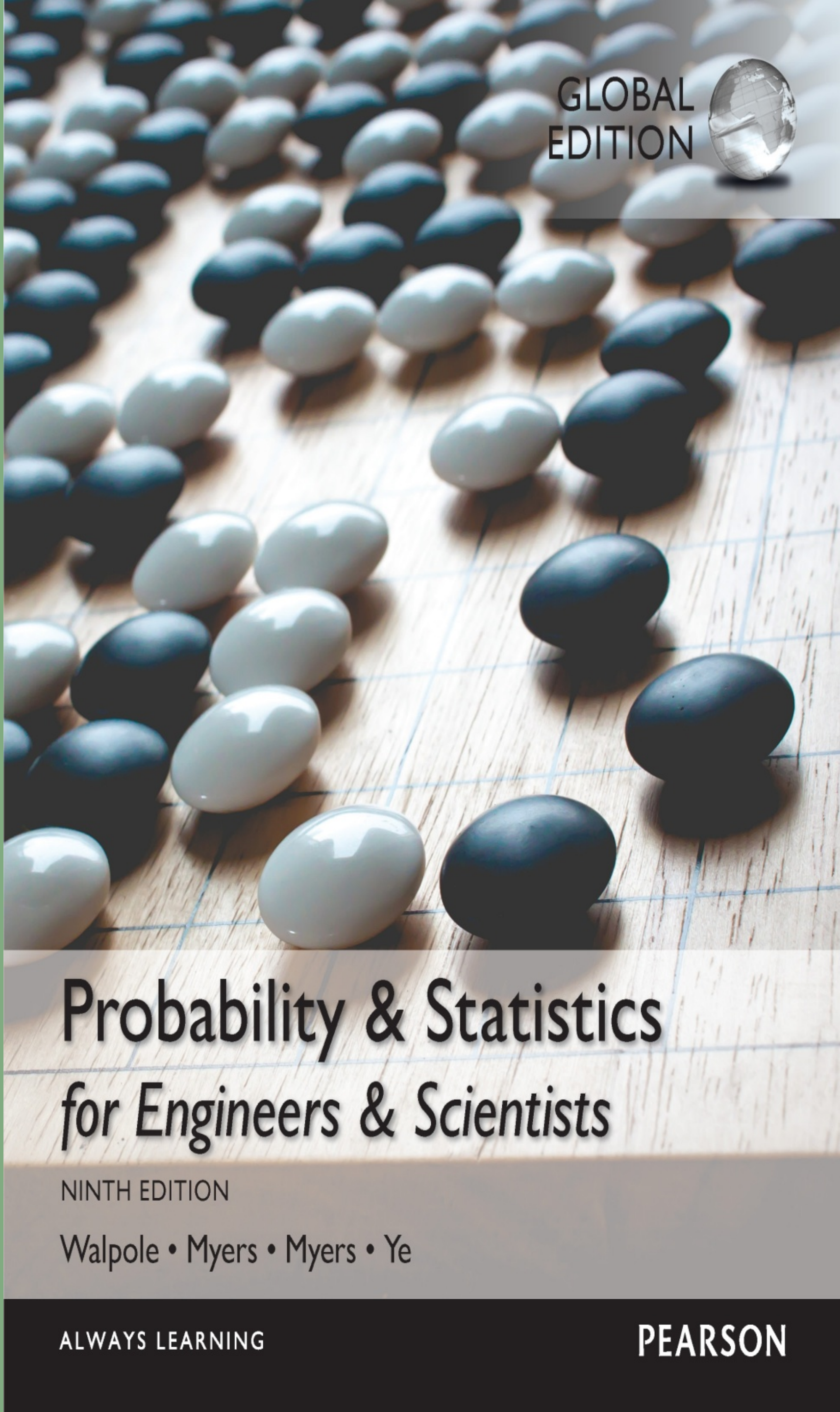


Section 10.2

Testing a Statistical Hypothesis



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Definition 10.1

A **statistical hypothesis** is an assertion or conjecture concerning one or more populations.

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Definition 10.1

- Null Hypothesis, H_0
  - Hypothesis that the population is in default or unchanged state (status quo)
- Suppose in certain area, the size of apples was reduced by 10% after measuring random samples. On that year, there was a long draught period.
- **Q: Did the reduction of size happen by chance?**

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Definition 10.1

- Null Hypothesis, H_0
  - **There was no change in the size of apples** (population maintains the default state)
  - So the observation of reduction in the sample mean happen **by chance** (b/c it is RV)
- How likely this happened by chance?
  - affected by the actual value of sample mean, sample size, variance, etc.

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Definition 10.1

- We consider two hypotheses
  - Null Hypothesis, H_0
  - Alternative Hypothesis, H_1
  - H_1 means H_0 is rejected
    - that is, the change in the default state is significant, and did not happen by chance
- H_1 typically stands for question to be answered or theory to be tested

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Definition 10.1

- Look at the sample mean, and consider the probability that the sample mean obtained by chance
  - if the probability is significantly low, reject H_0
  - otherwise do not reject H_0
- Careful: NOT rejecting $H_0 \neq$ ACCEPT H_0!
- Not rejecting H_0
  - don’t have sufficient evidence to rule out H_0

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Definition 10.1

- Typical null and alternative hypothesis
  - H_0: defendant is innocent
  - H_1: defendant is guilty
- By default, defendant is innocent, unless there is 'undoubtable' evidence (not by chance) against defendant
- Of course, if an evidence happened by chance, it means that we cannot reject H_0 – but not saying that defendant is innocent

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Definition 10.2

Rejection of the null hypothesis when it is true is called a **type I error**.

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Definition 10.3

Nonrejection of the null hypothesis when it is false is called a **type II error**.

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Table 10.1 Possible Situations for Testing a Statistical Hypothesis

	H_0 is true	**H_0 is false**
Do not reject H_0	Correct decision	Type II error
Reject H_0	Type I error	Correct decision

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Example

Consider the null hypothesis that
the average weight of male students in a certain college is 68 kilograms against the alternative hypothesis that it is unequal to 68.

$$H_0\colon\ \mu = 68,$$
$$H_1\colon\ \mu \neq 68.$$

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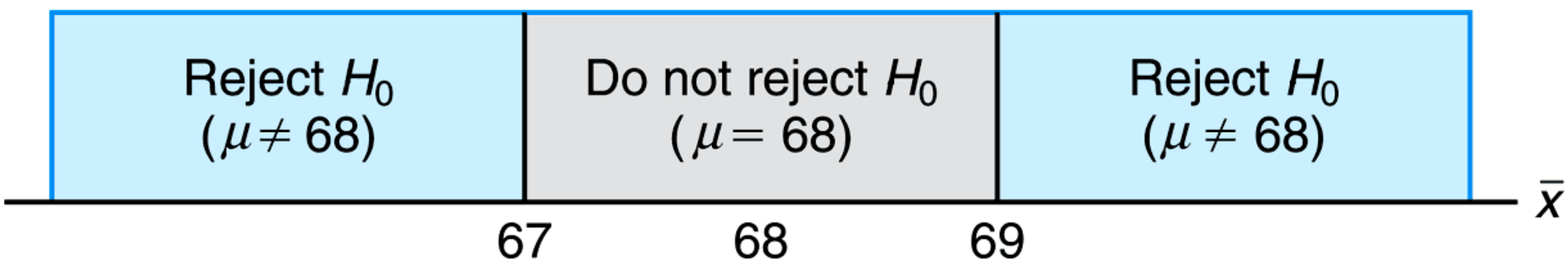
Figure 10.4 Critical region (in blue)



Suppose we reject H_0 if sample mean <67 or >69

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Figure 10.5 Critical Region for testing $\mu = 68$ versus $\mu \neq 68$

assume σ=3.6, n=36
then prob of false rejection
~0.095

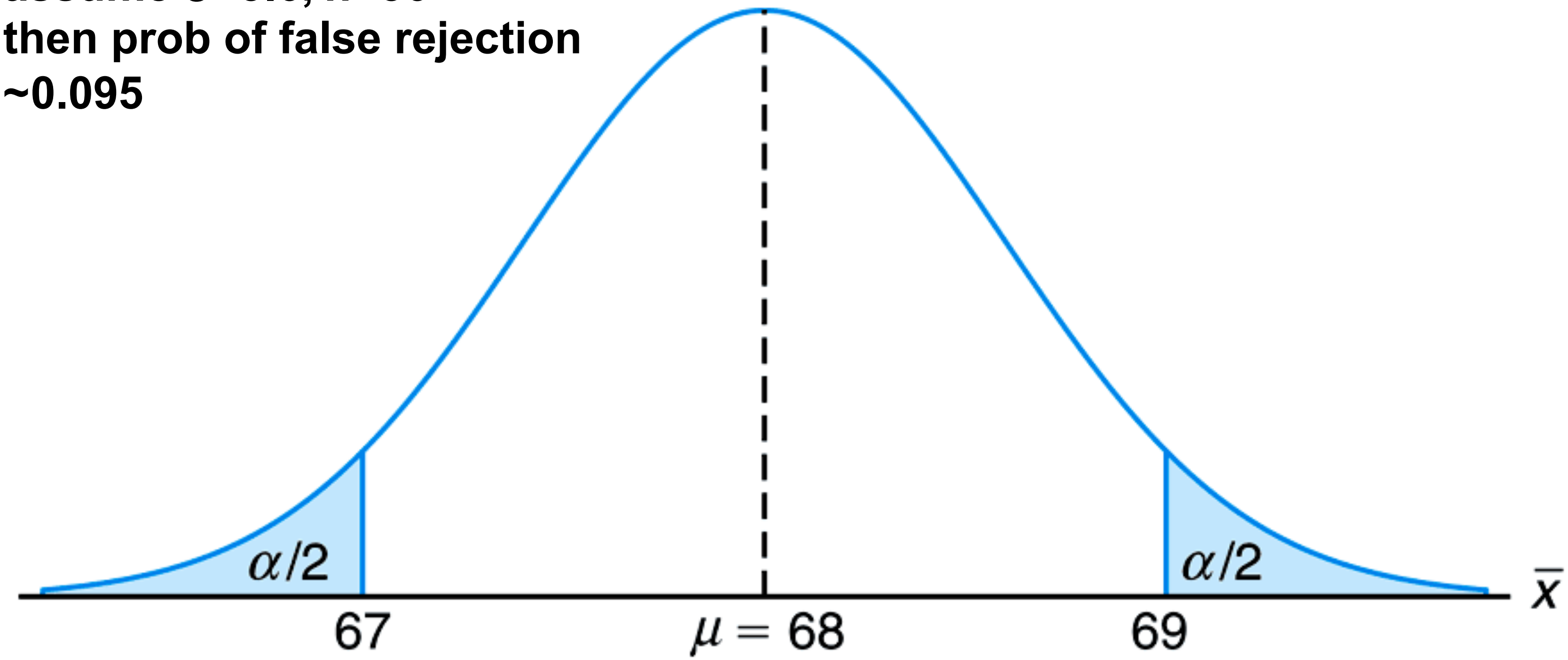

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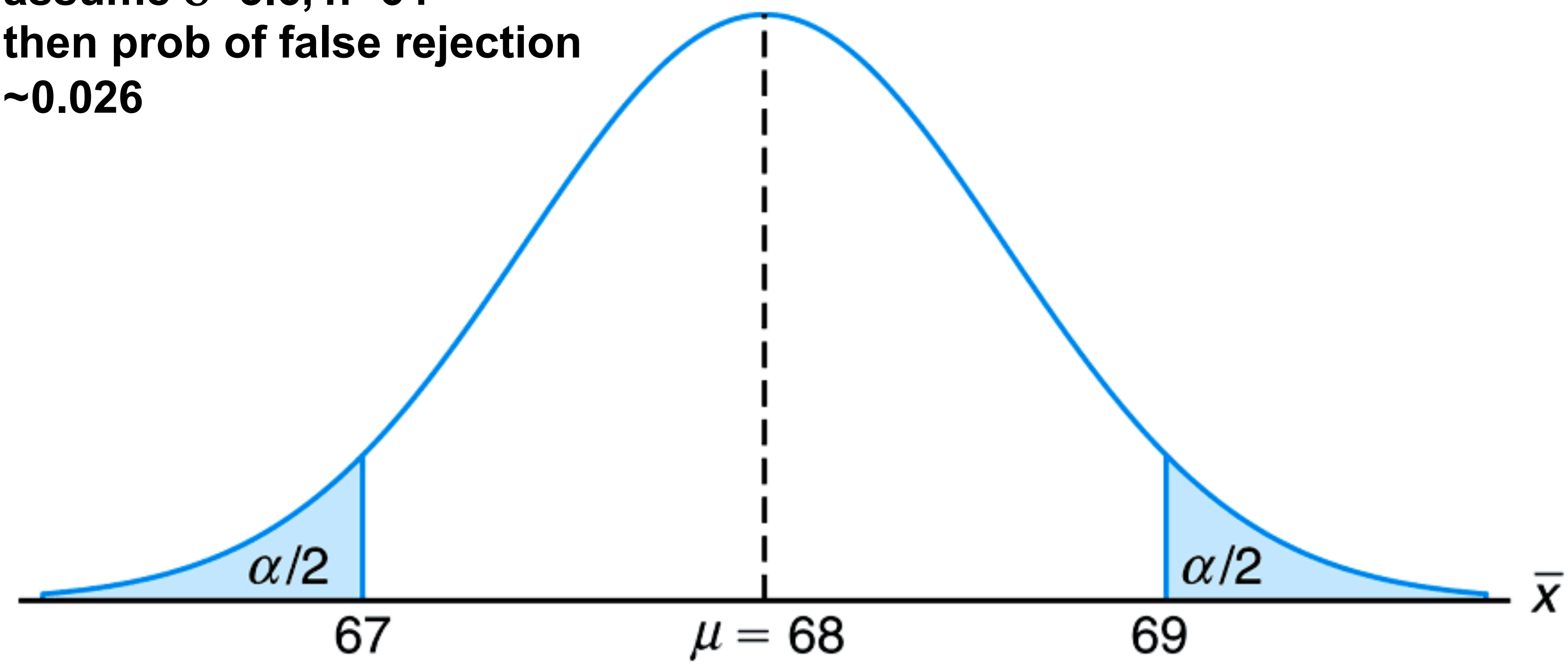
Figure 10.5 Critical Region for testing $\mu = 68$ versus $\mu \neq 68$

assume σ=3.6, n=64
then prob of false rejection
~0.026



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- one-sided test

$$H_0\colon\ \theta = \theta_0,$$
$$H_1\colon\ \theta > \theta_0$$

or

$$H_0\colon\ \theta = \theta_0,$$
$$H_1\colon\ \theta < \theta_0,$$

- two-sided test

$$H_0\colon\ \theta = \theta_0,$$
$$H_1\colon\ \theta \neq \theta_0,$$

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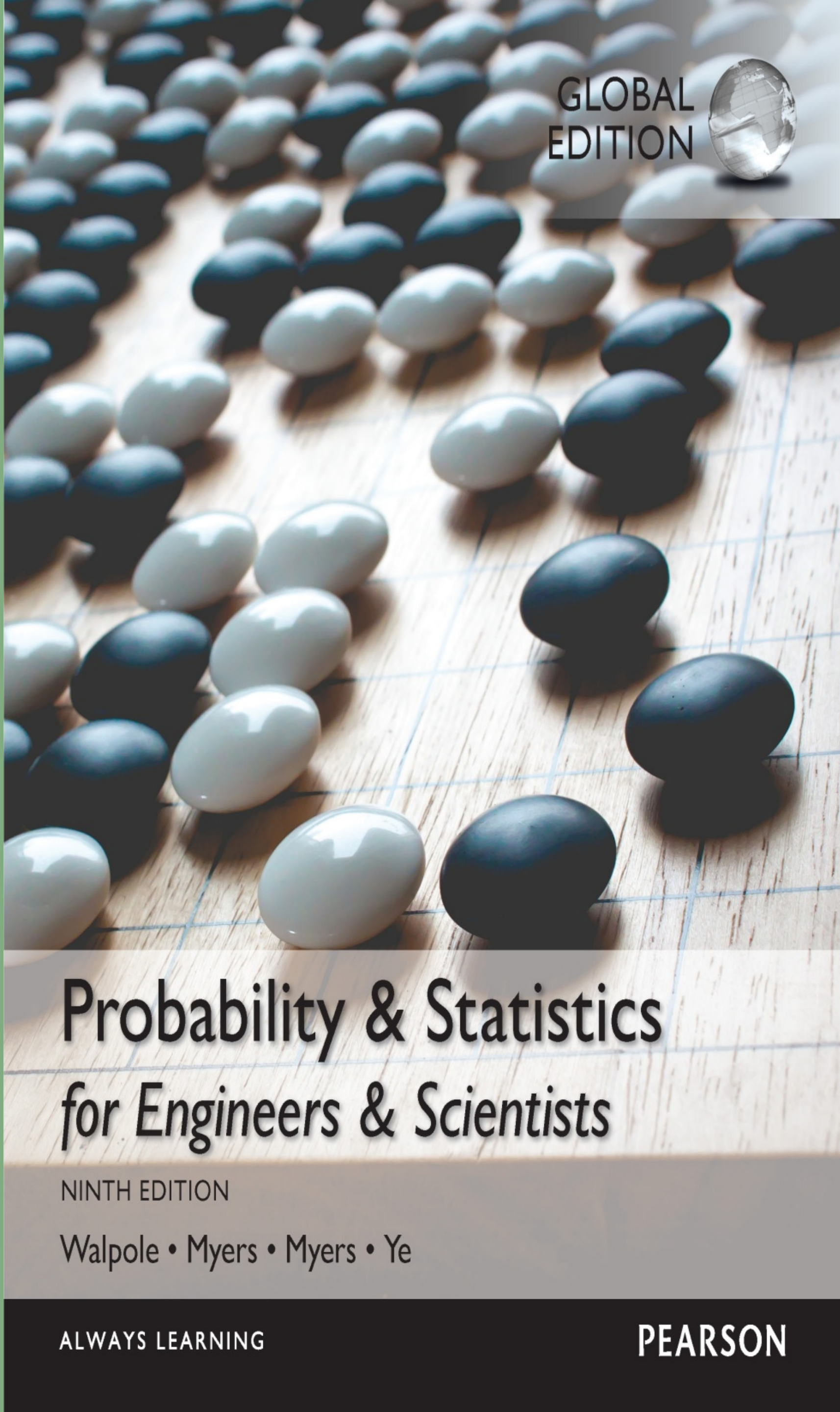
Section 10.3

The Use of *P*-Values for Decision Making in Testing Hypotheses



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Definition 10.5

A ***P*-value** is the lowest level (of significance) at which the observed value of the test statistic is significant.

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P-value

- For given test statistic, probability of committing false rejection if the statistic corresponded to the critical value.

$$H_0\text{: } \mu \text{ is } 10$$
$$H_1\text{: } \mu \text{ is not } 10$$

- suppose z=1.87. If z were critical value, prob. of false rejection is

$$P = 2\ P(z>1.87) = 0.0614$$

This is called P-value

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P-value

- significance level of testing
- If the probability of random chance (false rejection) is below this level, we think the given statistic represent ‘rare’ event, and considers significant
- Typical significance level: 5%, 1%

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P-value

- Typical significance level: 5%, 1%
- If P value is less than the significance level, we reject H_0.
- In the example, P=0.614 > 5%, so we cannot reject H_0
    - Probability of random chance is too high!
- P-value is widely used

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Section 10.4

Single Sample: Tests Concerning a Single Mean

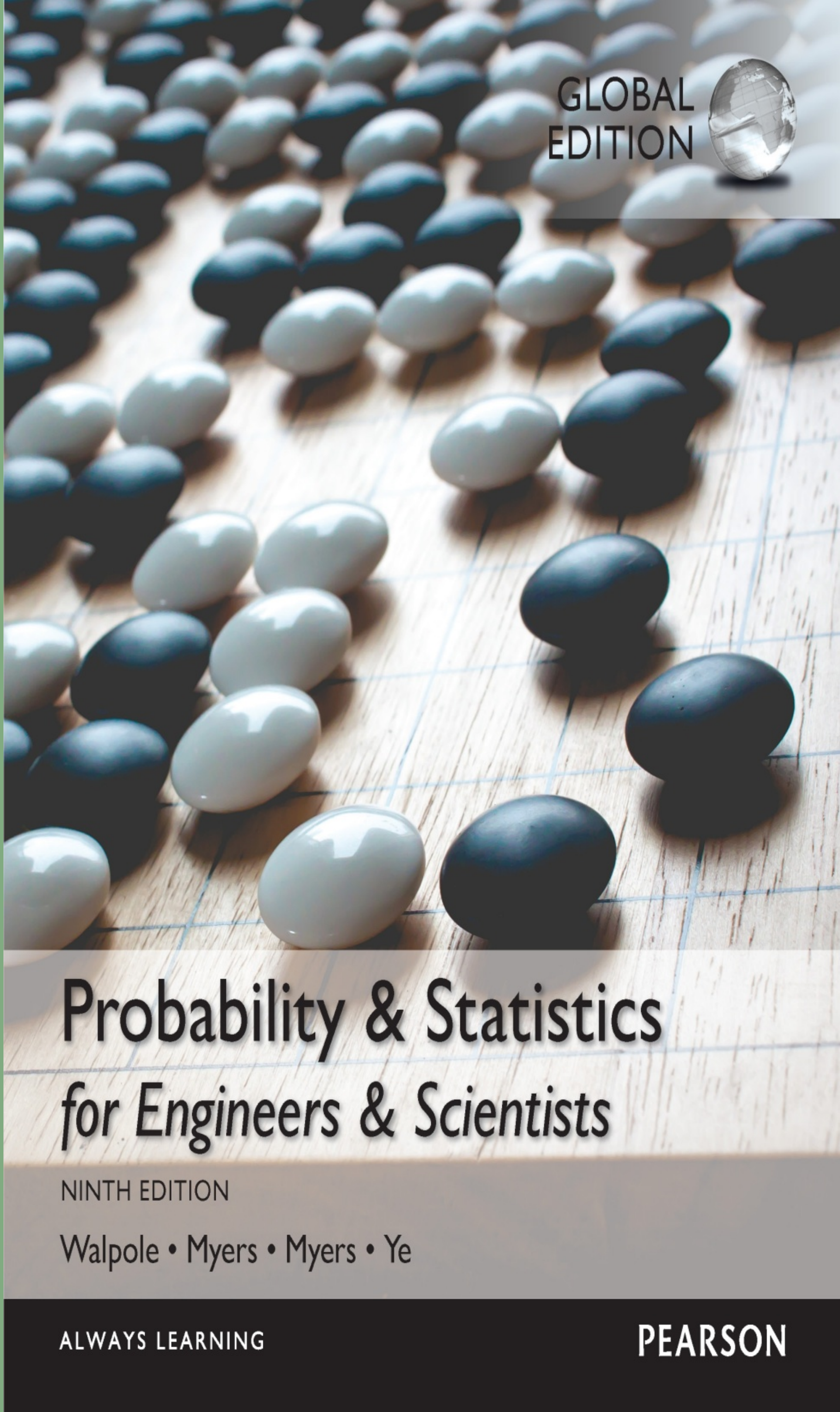

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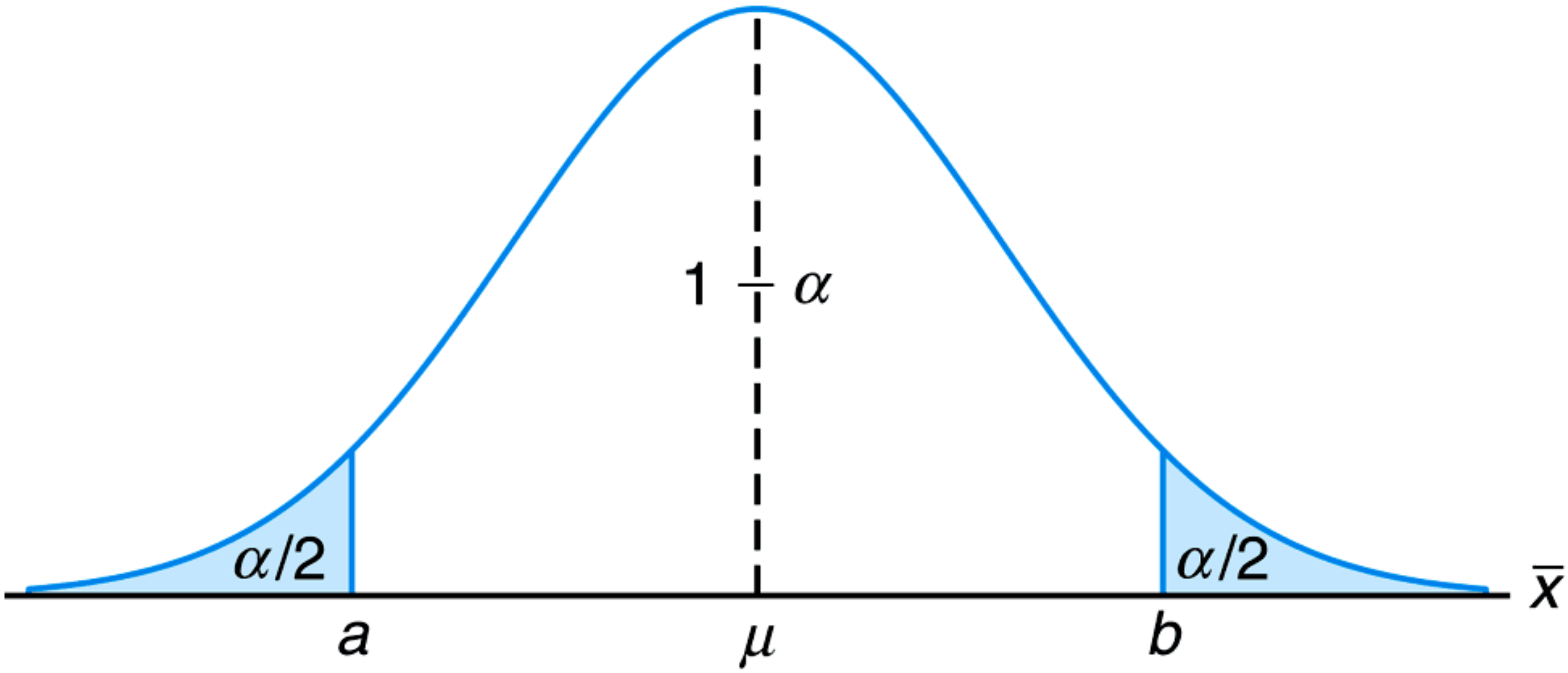
Figure 10.9 Critical region for the alternative hypothesis $\mu \neq \mu_o$



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example 10.3

Example 10.3: A random sample of 100 recorded deaths in the United States during the past year showed an average life span of 71.8 years. Assuming a population standard deviation of 8.9 years, does this seem to indicate that the mean life span today is greater than 70 years? Use a 0.05 level of significance.

***Solution*:**

1. H_0: $\mu = 70$ years.
2. H_1: $\mu > 70$ years.
3. $\alpha = 0.05$.
4. Critical region: $z > 1.645$, where $z = \frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}}$.
5. Computations: $\bar{x} = 71.8$ years, $\sigma = 8.9$ years, and hence $z = \frac{71.8-70}{8.9/\sqrt{100}} = 2.02$.
6. Decision: Reject H_0 and conclude that the mean life span today is greater than 70 years.

The P-value corresponding to $z = 2.02$ is given by the area of the shaded region in Figure 10.10.

Using Table A.3, we have

$$P = P(Z > 2.02) = 0.0217.$$

As a result, the evidence in favor of H_1 is even stronger than that suggested by a 0.05 level of significance.

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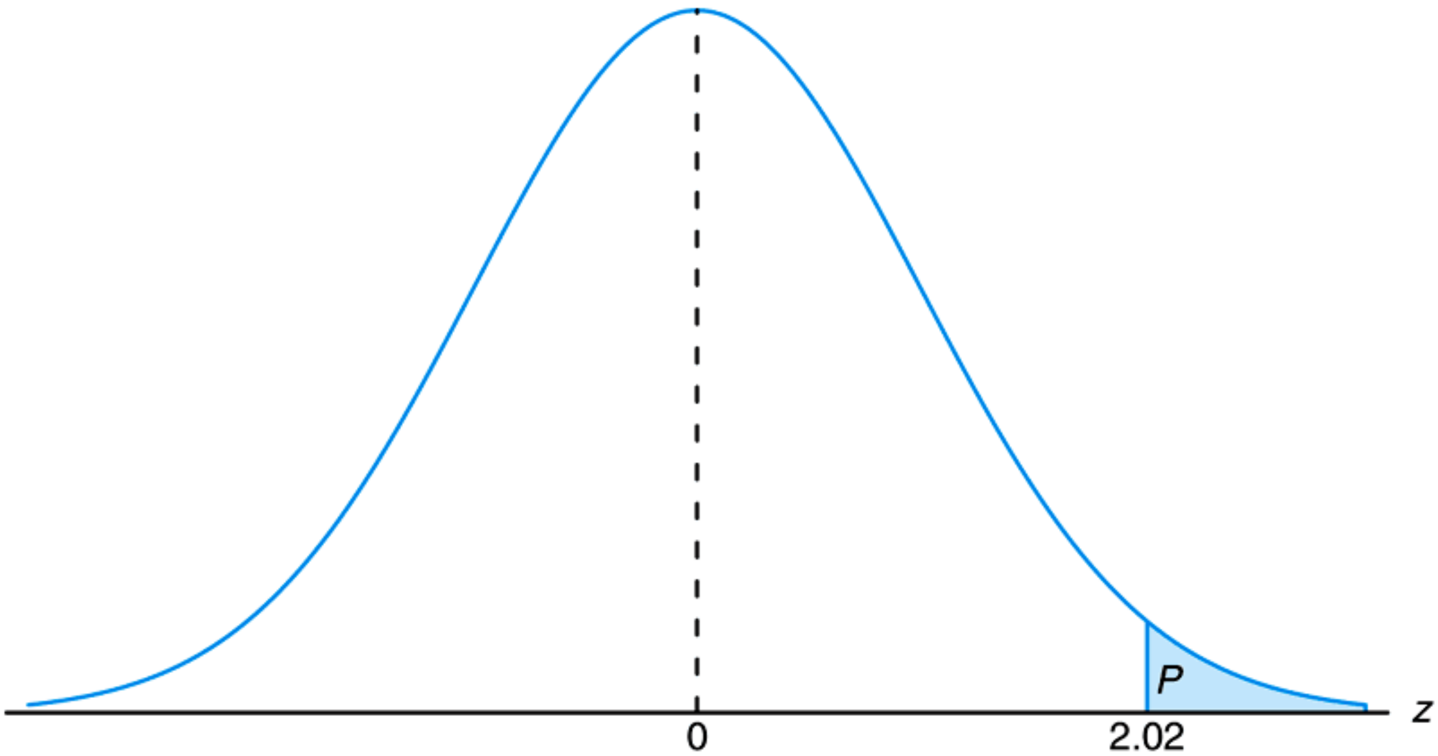
Figure 10.10 P-value for Example 10.3



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example 10.4

Example 10.4: A manufacturer of sports equipment has developed a new synthetic fishing line that the company claims has a mean breaking strength of 8 kilograms with a standard deviation of 0.5 kilogram. Test the hypothesis that $\mu = 8$ kilograms against the alternative that $\mu \neq 8$ kilograms if a random sample of 50 lines is tested and found to have a mean breaking strength of 7.8 kilograms. Use a 0.01 level of significance.

Solution:

1. H_0: $\mu = 8$ kilograms.
2. H_1: $\mu \neq 8$ kilograms.
3. $\alpha = 0.01$.
4. Critical region: $z < -2.575$ and $z > 2.575$, where $z = \frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}}$.
5. Computations: $\bar{x} = 7.8$ kilograms, $n = 50$, and hence $z = \frac{7.8-8}{0.5/\sqrt{50}} = -2.83$.
6. Decision: Reject H_0 and conclude that the average breaking strength is not equal to 8 but is, in fact, less than 8 kilograms.

Since the test in this example is two tailed, the desired P-value is twice the area of the shaded region in Figure 10.11 to the left of $z = -2.83$. Therefore, using Table A.3, we have

$$P = P(|Z| > 2.83) = 2P(Z < -2.83) = 0.0046,$$

which allows us to reject the null hypothesis that $\mu = 8$ kilograms at a level of significance smaller than 0.01.

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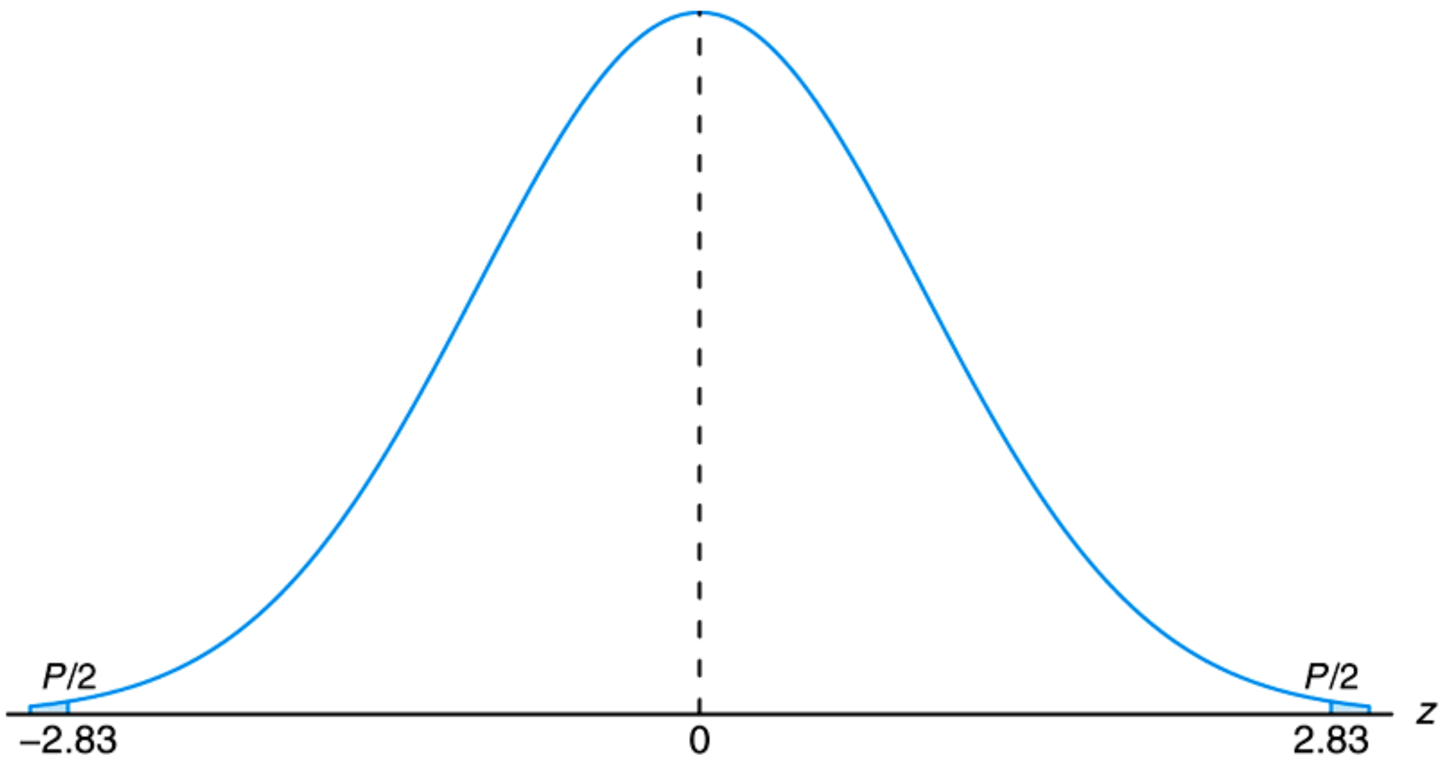
Figure 10.11 P-value for Example 10.4



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