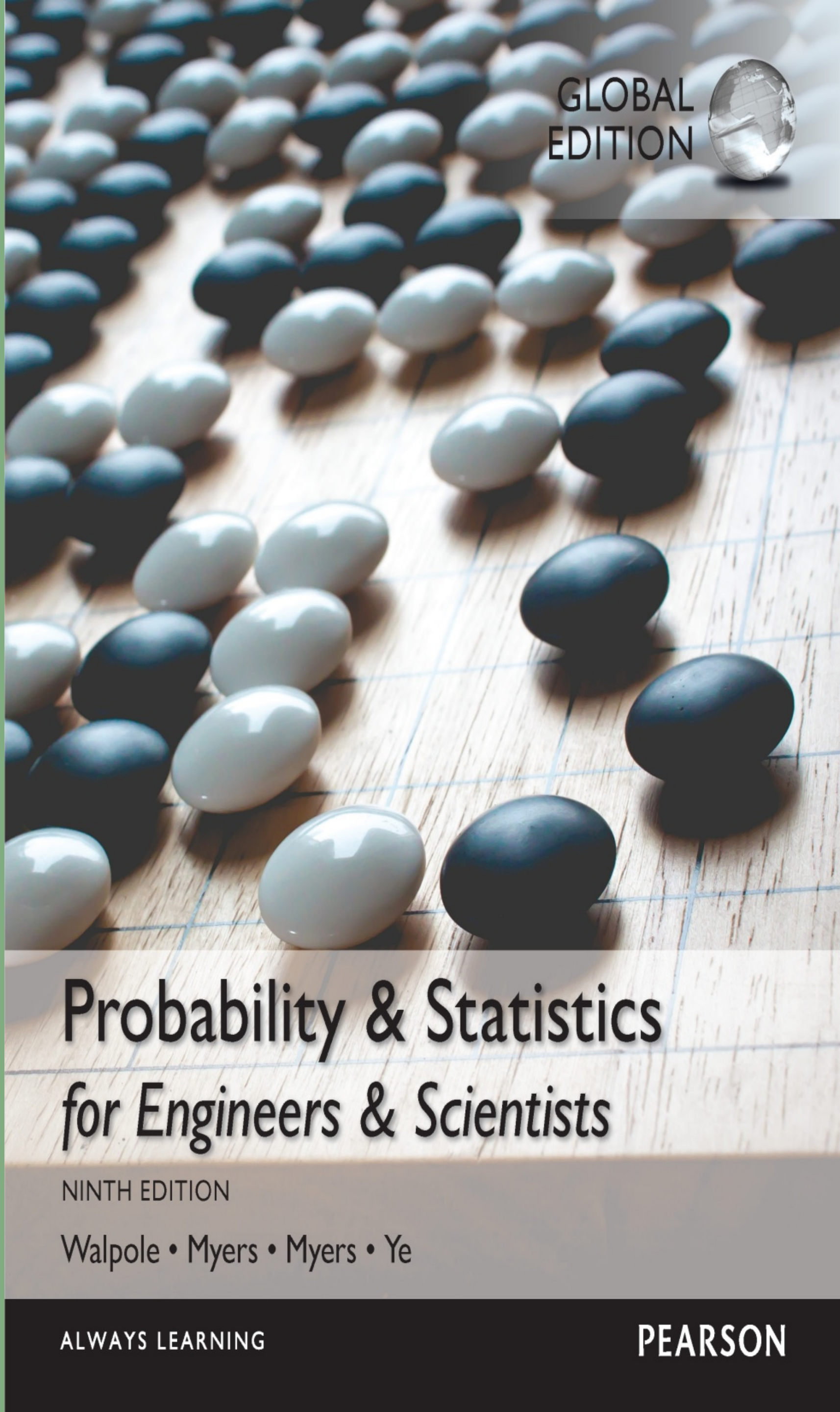
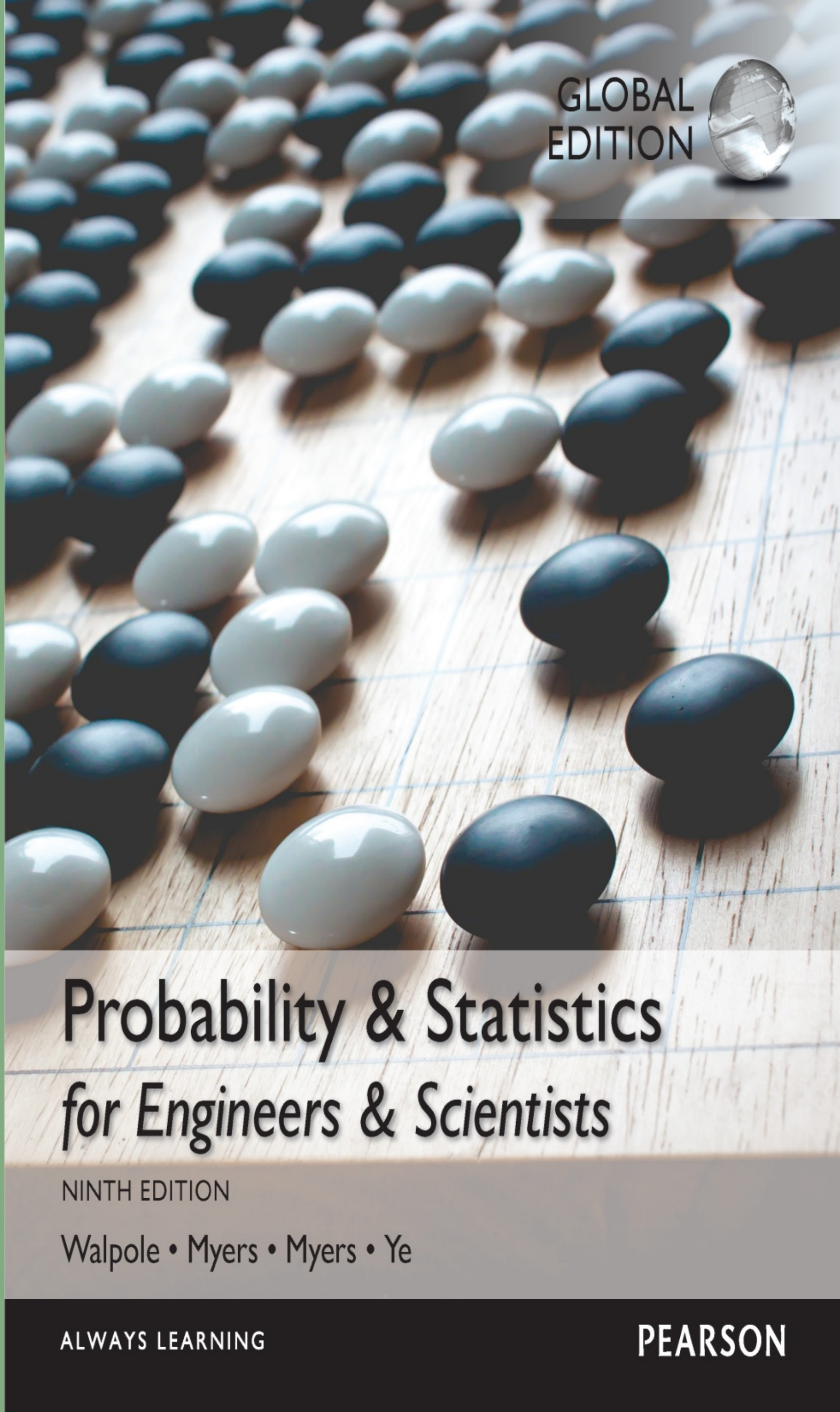
Chapter 5

Some Discrete Probability Distributions



Section 5.2

Binomial and Multinomial Distributions



Bernoulli RV

"Counting"

- Def.: An RV X has the distribution Bernoulli(p) if

$$\mathbb{P}(X = 1) = p$$
$$\mathbb{P}(X = 0) = 1 - p$$

1

整数

- X takes values from {0,1}

1 2 3 4 . . .

countably infinite

- Analogy: Toss of a 'biased' coin
  - Probability of 'H' is p
  - 'H' is considered to be 1, 'T' is considered to be 0

1

p

[0,1]

0

1-p

Bernoulli RV

– Mean and variance of Bernoulli RV

$$\mathbb{E}[X] = p,\ \sigma_X^2 = p(1-p)$$

– Can be seen as the number of seeing an outcome (H) for 1 trial of experiment

– Multiple, independent trials: Binomial RV

Binomial RV

counting

- Def.: An RV X has the distribution Binomial(n,p) with distribution b(k;n,p) if

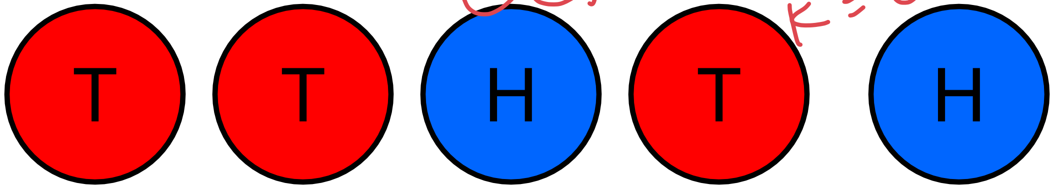
$$\mathbb{P}(X=k)=\binom{n}{k}p^k(1-p)^{n-k}$$

B(n,p)

  - X takes values from {0,1,…,n}

- Analogy: Total number of ‘H’ out of n tosses
  - Related to ‘counting’ of occurrences of an event
  - What is the probability that X=2 when n=5?

$$E(X)=\sum_{k=0}^{n} k\,P(X=k)$$



Binomial RV

- Def.: If Y~Binomial(n,p) , then

$$Y = X_1 + X_2 + \ldots + X_n$$

$E[Y] = E[X_1] + E[X_2] + E[X_n]$

$= \text{sum of RV}$

p

$= np$

- X_i~Bernoulli(p), X_i are independent
- Useful for calculating mean/variance

$f = f_1 + f_2 + \ldots f_n$

Theorem 5.1

mutually independent

$E(X^2)$ $E(X(X-1))$

The mean and variance of the binomial distribution $b(x; n, p)$ are

$$\mu = np \text{ and } \sigma^2 = npq.$$

$x_1, \ldots, x_n$

$np(1-p)$

$Var(Y) = Var(X_1) + \cdots + Var(X_n)$

$= np(1-p)$

$Ber(p)$

Example 5.2

a) $P(X \geq 10) = \sum_{k=10}^{15} \binom{15}{k} 0.4^{k} 0.6^{15-k}$

Example 5.2: The probability that a patient recovers from a rare blood disease is 0.4. If 15 people are known to have contracted this disease, what is the probability that (a) at least 10 survive, (b) from 3 to 8 survive, and (c) exactly 5 survive?

b) $P(3 \leq X \leq 8) = \sum_{k=3}^{8} \binom{15}{k} 0.4^{k} 0.6^{15-k}$

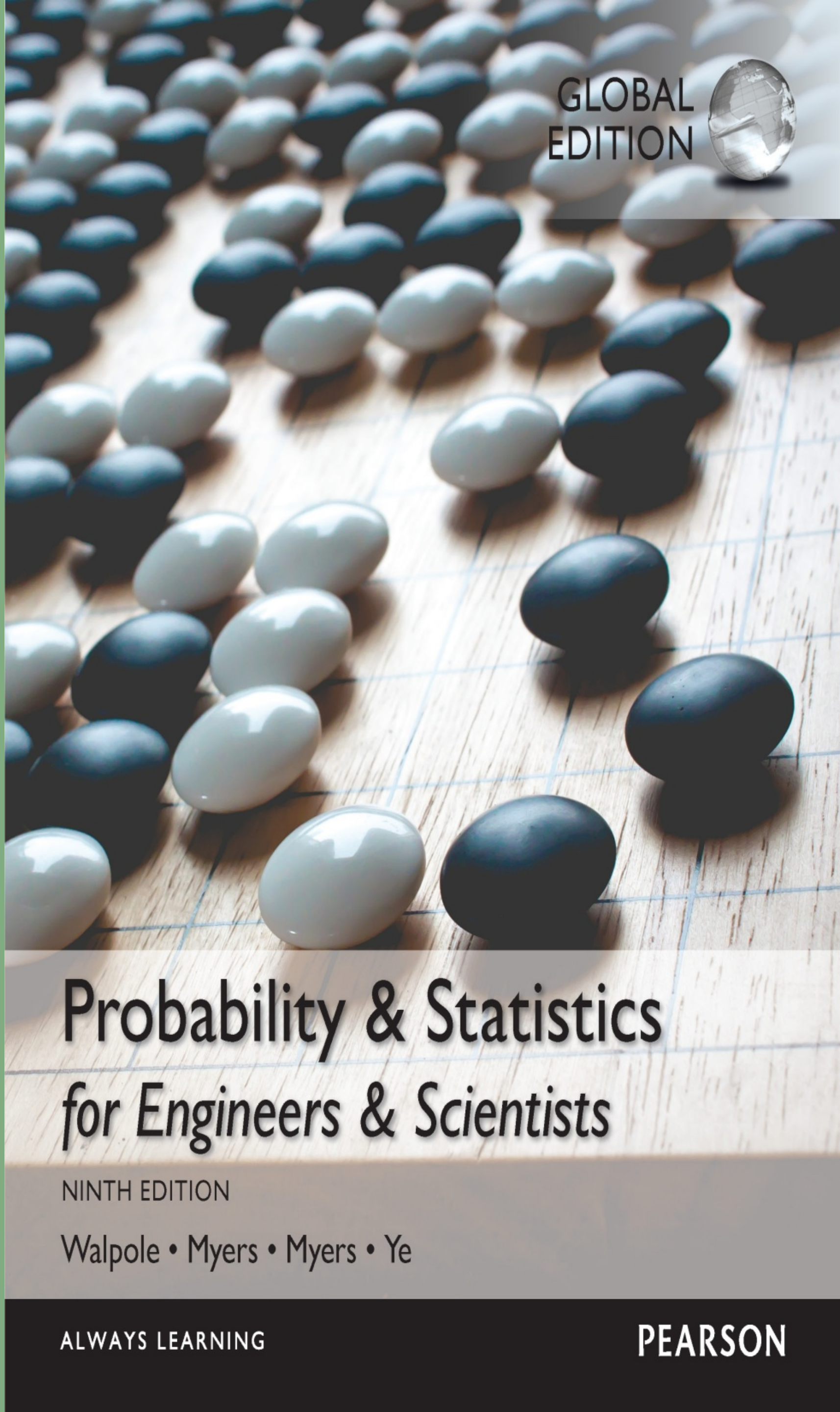
(how many recovered out of 15)

$X \sim B(\underset{n}{15}, \underset{p}{0.4})$,

c) $P(X=5) = \binom{15}{5} 0.4^{5} 0.6^{10}$

Section 5.4

Negative Binomial and Geometric Distributions



Geometric RV

k

k-1

- Def.: An RV X has the distribution Geometric(p) if

$$\mathbb{P}(X = k) = p(1-p)^{k-1}$$

  - X takes values from {1,2,3,…}

time

$p \times (1-p)^{k-1}$

- Analogy: Total number of coin tosses until we see the first ‘H’
  - Or ‘time until the first success’
  - What is the probability that X=4?

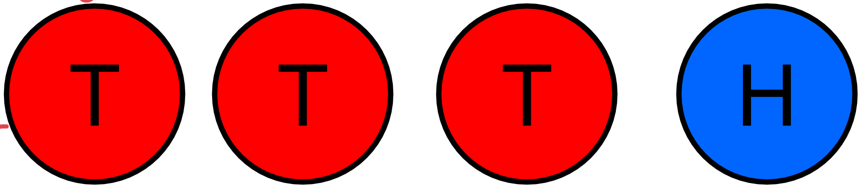
"H"

p

T

$(1-p)$

$(1-p)^3 \times p$



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Theorem 5.3

$= E[X(X-1)] = \frac{2(1-p)}{p^2}$

$E[X^2] - \frac{1}{p} = \frac{2(1-p)}{p^2} = \frac{2}{p^2} - \frac{1}{p}$

$pE[X(X-1)] = 2p(1-p) + 4p(1-p)^2 + 6p(1-p)^3 + 8$

The mean and variance of a random variable following the geometric distribution are

$$\mu = \frac{1}{p} \text{ and } \sigma^2 = \frac{1-p}{p^2}.$$

$\sigma^2 = E[X^2] - \mu^2$

$E[X^2] - E[X]^2$

$= 2(1-p)\left[\sum_{k=1}^{\infty} kp(1-p)^{k-1}\right] \frac{1}{p}$

Example 5.15

Example 5.15: For a certain manufacturing process, it is known that, on the average, 1 in every 100 items is defective. What is the probability that the fifth item inspected is the first defective item found?

$$X \sim Geo(0.01)$$

$$p(1-p)^4$$

$$\Rightarrow P(X=5) = 0.99^4 \; 0.01$$

Negative Binomial RV

- Def: X is said to have negative Binomial RV ~NB(k,p) with distribution b*(x;k,p)

$$b^*(x;k,p) = \binom{x-1}{k-1} p^k (1-p)^{x-k}$$

X~NB(k,p)

X = k, k+1, k+2, ...

- Meaning : # of experiments until see k successes
- Geometric RV is special case, k=1

T T T H T T T T T H T H

Negative Binomial RV

- Negative binomial ~~extension~~ expansion

$$(1+t)^{-k} = 1 + \frac{-k}{1!}t + \frac{(-k)(-k-1)}{2!}t^2 + \dots$$

$$= \sum_{x=0}^{\infty} (-1)^x \frac{(k+x-1)(k+x-2)\dots(k)}{x!} t^x$$

Taylor series

$$= \sum_{x=0}^{\infty} \binom{x+k-1}{k-1} (-t)^x$$

$$= \sum_{x=k}^{\infty} \binom{x-1}{k-1} (-t)^{x-k}$$

- Setting t=-p shows that b*(x;k,p) is a pmf

Negative Binomial RV

- Def.: If Y~NB(k,p), then

k indep

$$Y = X_1 + X_2 + \ldots + X_k$$

- X_i~Geometric(p), X_i are independent

k=3

p

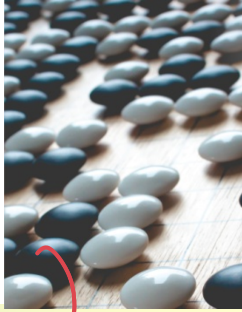
- Useful for calculating mean/variance

2 4 3

T H T T T H T T H

1 2 3

Negative Binomial RV



$\mathrm{Var}(X) = \mathrm{Var}(X_1) + \dots + \mathrm{Var}(X_k)$

- Mean and variance of NB RV X is given by

$$\mathbb{E}[X] = \frac{k}{p} = k\left(\frac{1-p}{p^2}\right)$$

$$\sigma_X^2 = k\frac{1-p}{p^2}$$

$\frac{1}{p}$ $\frac{1}{p}$

$E[X] = E[X_1] + E[X_2] + \dots + E[X_k]$ $\frac{1}{p}$

- Try to show these!

$X_i \sim \mathrm{Geo}(p)$

Example 5.14

$X \sim NB(k=5, \frac{1}{2})$

$P(X=4) + P(X=5) + P(X=6) + P(X=7)$

$P(X \geq 20)$

$\binom{6-1}{4-1} 0.55^4 \, 0.45^2$

$x=6 \quad k=4$

Example 5.14: In an NBA (National Basketball Association) championship series, the team that wins four games out of seven is the winner. Suppose that teams A and B face each other in the championship games and that team A has probability 0.55 of winning a game over team B.

0.45

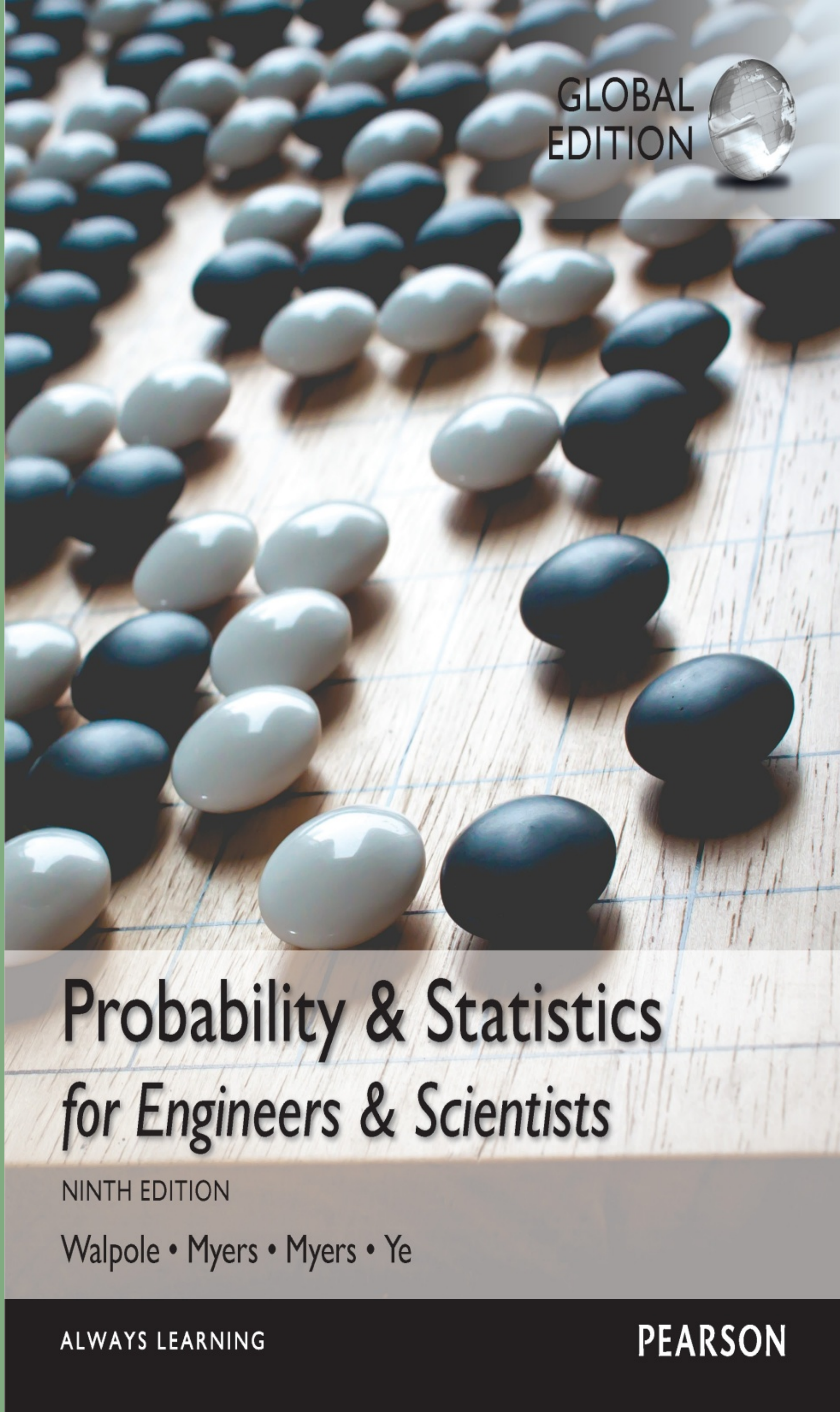
(a) What is the probability that team A will win the series in 6 games?

(b) What is the probability that team A will win the series?

(c) If teams A and B were facing each other in a regional playoff series, which is decided by winning three out of five games, what is the probability that team A would win the series?

Section 5.5

Poisson Distribution and the Poisson Process



Poisson RV

- Def.: An RV X has the distribution Poisson(λ) if its

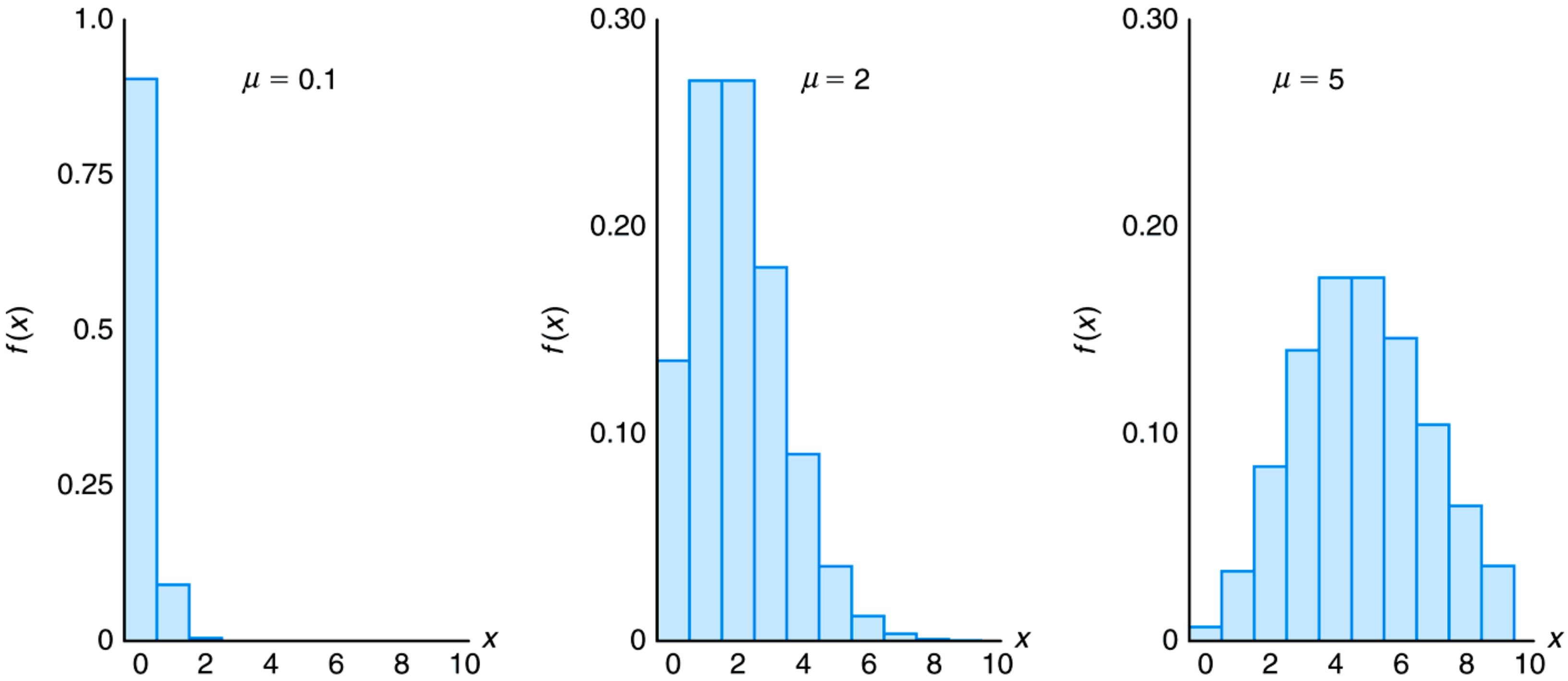
$$\mathbb{P}(X = k) = \frac{\lambda^k}{k!} e^{-\lambda}$$

  - X takes values from {0,1,2,…}
- **The mean of X is λ**
- **Analogy:** ***Total number of 'H' out of many tosses with small probability of success***
  - ***Models the total number of successes during a time period***
  - ***"Limiting" version of binomial RVs***

Theorem 5.4

$$\boxed{\text{Both the mean and variance of } X \sim \text{Poisson}(\lambda) \text{ is } \lambda}$$

Figure 5.1 Poisson density functions for different means



Theorem 5.5

Let X be a binomial random variable with probability distribution $b(x; n, p)$. When $n \to \infty$, $p \to 0$, and $np \overset{n\to\infty}{\longrightarrow} \mu$ remains constant,

$$b(x; n, p) \overset{n\to\infty}{\longrightarrow} p(x; \mu).$$

Limiting case of Binomial RV

- Let us divide unit time into *n* timeslots
- Suppose we repeat independent coin tossing at every timeslot
- The probability of flipping 'H' is

$$p = \frac{\lambda}{n}$$

- Note as n goes to infinity,
  - The number of trials goes to infinity
  - Probability of flipping H goes to 0

Theorem 5.4

Suppose # of occurrence of an event during unit time follows Poisson(λ)

$$\boxed{\text{The \# of occurrence of the event during time interval} \sim \text{Poisson}(\lambda t)}$$

Ex 5.18

- (assume Poisson distribution)

Example 5.18: Ten is the average number of oil tankers arriving each day at a certain port. The facilities at the port can handle at most 15 tankers per day. What is the probability that on a given day tankers have to be turned away?