1. X, Y two discrete random variables with a joint probability distribution

$$f(x,y) = \begin{cases} \frac{(x+2y)}{27}, & x, y = 0, 1, 2 \\ 0, & \text{otherwise} \end{cases}$$

$Z = X + Y \qquad P(Z) = ?$

$Z = 0, 1, 2, 3, 4$

when $Z = 0 \quad f(x=0, y=0) = \frac{0+2\cdot 0}{27} = \boxed{0}$

$Z = 1 \quad f(x=0, y=1) + f(x=1, y=0) = \frac{0+2\cdot 1}{27} + \frac{1+2\cdot 0}{27} = \frac{2}{27} + \frac{1}{27} = \frac{3}{27} = \boxed{\frac{1}{9}}$

$Z = 2 \quad f(x=0, y=2) + f(x=1, y=1) + f(x=2, y=0) = \frac{0+2\cdot(2)}{27} + \frac{1+2(1)}{27} + \frac{2+2(0)}{27} = \frac{4+3+2}{27} = \frac{9}{27} = \boxed{\frac{1}{3}}$

$Z = 3 \quad f(x=1, y=2) + f(x=2, y=1) = \frac{1+2(2)}{27} + \frac{2+2(1)}{27} = \frac{5+4}{27} = \frac{9}{27} = \boxed{\frac{1}{3}}$

$Z = 4 \quad f(x=2, y=2) = \frac{2+2(2)}{27} = \frac{6}{27} = \boxed{\frac{2}{9}}$

$f(x,y)$	x = 0	1	2
0	0	$\frac{1}{27}$	$\frac{2}{27}$
1	$\frac{2}{27}$	$\frac{3}{27}$	$\frac{4}{27}$
2	$\frac{4}{27}$	$\frac{5}{27}$	$\frac{6}{27}$

2.

$$f(x) = \begin{cases} e^{-x}, & 0 < x < \infty \\ 0, & \text{elsewhere} \end{cases}$$

$Y = -3X + 5 \qquad x = w(y) \longrightarrow 3x = 5 - Y \quad x = \frac{(5-Y)}{3}, \quad 0 < \frac{5-Y}{3} < \infty, \; 0 < 5 - Y < \infty, \; -5 < -Y < \infty$

$5 > Y > -\infty$

$x = \frac{5-Y}{3}$ for $-\infty < Y < 5 \qquad |J| = w'(y) = \left| -\frac{1}{3} \right| = \frac{1}{3}$

$f(w(Y))|J| = \begin{cases} e^{-\frac{(5-Y)}{3}} \frac{1}{3} \end{cases}$ for $-\infty < Y < 5$

$g(y) = f(w(y)) = \boxed{\frac{1}{3} e^{-\frac{(5-Y)}{3}}, \quad -\infty < y < 5}$

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3. $f(x) = \begin{cases} e^{-x}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$

$Y_1 = x_1 + x_2 \qquad x_1 = Y_1 Y_2$

$Y_2 = \frac{x_1}{(x_1 + x_2)} \qquad x_2 = Y_1 - Y_1 Y_2 = Y_1(1 - Y_2)$

$Y_1 Y_2 > 0 \rightarrow Y_1 > 0, \; Y_2 > 0 \qquad \therefore 0 < Y_1, \quad 0 < Y_2 < 1$

$Y_1(1 - Y_2) > 0 \rightarrow 1 > Y_2$

$x_1 = Y_1 Y_2 \quad x_2 = Y_1(1-Y_2)$

$$J = \begin{vmatrix} Y_2 & Y_1 \\ 1-Y_2 & -Y_1 \end{vmatrix} = -Y_1Y_2 - Y_1(1-Y_2) = -\cancel{Y_1Y_2} - Y_1 + \cancel{Y_1Y_2} = -Y_1 \quad |-Y_1| = Y_1$$

$f(x_1, x_2) = f(x_1) f(x_2)$ it x_1 and x_2 are independent each others

$$= e^{-x_1} \times e^{-x_2} = e^{-(x_1+x_2)}$$

$$g(Y_1, Y_2) = f(Y_1Y_2, Y_1(1-Y_2))|J| = Y_1 e^{-(Y_1Y_2 + Y_1(1-Y_2))} = Y_1 e^{-Y_1} \text{ for } Y_1 > 0 \text{ and } 0 < Y_2 < 1$$

$$g(Y_1) = \int_0^1 Y_1 e^{-Y_1} dY_2 \quad Y_1 > 0 = Y_1 e^{-Y_1}, \ Y_1 > 0$$

Y_1 range

$$g(Y_2) = \int_0^\infty Y_1 e^{-Y_1} dY_1 = \int_0^\infty Y_1^{2-1} e^{-Y_1} dY_1 \Rightarrow \Gamma(2) = 1 \quad \text{for } 0 < Y_2 < 1$$

gamma function

$$1 \times Y_1 e^{-Y_1} = Y_1 e^{-Y_1} \quad \therefore \boxed{\text{they are independent}}$$

4.

$$f(x) = \begin{cases} \frac{1+x}{2}, & -1 < x < 1 \\ 0, & \text{elsewhere} \end{cases}$$

$Y = x^2 \quad x = \sqrt{Y} \text{ or } \sqrt{Y}$ range: $-1 < \sqrt{Y} < 1 \rightarrow 0 < \sqrt{Y} < 1 \rightarrow 0 < Y < 1$

when $x = \sqrt{Y}$

$$|J| = (Y^{\frac{1}{2}})' = \frac{1}{2} Y^{-\frac{1}{2}} = \frac{1}{2\sqrt{Y}}$$

$$g(Y) = f(w(Y)) = \frac{1+\sqrt{Y}}{2} \frac{1}{2\sqrt{Y}} = \frac{1+\sqrt{Y}}{4\sqrt{Y}} \quad \text{for } 0 < Y < 1$$

when $x = -\sqrt{Y}$

$$g(Y) = f(w(Y)) = \frac{1-\sqrt{Y}}{2} \cdot \frac{1}{2\sqrt{Y}} = \frac{(-\sqrt{Y})}{4\sqrt{Y}}$$

$$g(Y) = g(\sqrt{Y}) + g(\sqrt{Y}) = \frac{1+\sqrt{Y}}{4\sqrt{Y}} + \frac{(-\sqrt{Y})}{4\sqrt{Y}} = \frac{\cancel{2}}{\cancel{4}\sqrt{Y}} = \frac{1}{2\sqrt{Y}} \quad \text{for } 0 < Y < 1$$

$$\boxed{\therefore \frac{1}{2\sqrt{Y}} \text{ for } 0 < Y < 1}$$

$$\sum_{n=0}^{\infty} n^x = \frac{1}{1-n}$$

5. $g(x;p) = pq^{x-1}$ for $x = 1,2,3$

$$M_x(t) = E(e^{tx}) = \sum_{x=1}^{\infty} e^{tx}\, pq^{x-1} = p\sum_{x=1}^{\infty} e^{tx}\cdot\frac{q^x}{q} = \frac{p}{q}\sum_{x=1}^{\infty}(e^t q)^x = \boxed{\frac{pe^t}{1-qe^t}}$$

$$\mu = M'_x(0) = \text{As a quotient rule} = \frac{(1-qe^t)pe^t + pqe^{2t}}{(1-qe^t)^2}\Bigg|_{t=0} = \frac{(1-q)p+pq}{(1-q)^2} = \boxed{\frac{1}{p}}$$

$$\mu' = M''_x(0) = \frac{(1-qe^t)^2 pe^t + 2pqe^{2t}(1-qe^t)}{(1-q\beta^t)^4}\Bigg|_{t=0} = \frac{2-p}{p^2}$$

$$\mu_2' = \frac{d^2 M(x)^t}{dt^2}\Bigg|_{t=0} = \frac{d}{dt}\left(\frac{pe^t}{(1-qe^t)^2}\right)\Bigg|_{t=0} = \frac{pe^t\cdot(1-qe^t)^2 - pe^t\cdot 2(1-qe^t)\cdot(-qe^t)}{(1-qe^t)^4}\Bigg|_{t=0} = \frac{pe^t - pq^2e^t}{(1-qe^t)^4}\Bigg|_{t=0}$$

$$= \frac{p-pq^2}{(1-q)^4} = \frac{p(1-q)(1+q)}{p^3(1-q)} = \frac{p^2(1+q)}{p^4} = \frac{1+q}{p^2}$$

$$\sigma^2 = \mu_2' - \mu^2 = \frac{1+q}{p^2} - \left(\frac{1}{p}\right)^2 = \boxed{\frac{q}{p^2}}$$

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6. $p(x;\mu) = \dfrac{e^{-\mu}\mu^x}{x!}$ $x = 0,1,2,\ldots$

$$M_x(t) = E[e^{tx}] = \sum_{x=0}^{\infty} e^{tx}\left(\frac{e^{-\mu}\mu^x}{x!}\right) = \sum_{x=0}^{\infty}\frac{e^{tx}e^{-\mu}\mu x}{x!} = e^{-\mu}\sum_{x=0}^{\infty}\frac{e^{tx}\mu^x}{x!}$$

$$= e^{-\mu}\boxed{\sum_{x=0}^{\infty}\frac{(\mu e^t)^x}{x!}} = e^{-\mu}\cdot e^{\mu e^t} = \boxed{e^{\mu(e^t-1)}}$$

$-\mu + \mu e^t = \mu(e^t-1)$

$\boxed{e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}}$

$$M_x(t)' = \left(e^{\mu(e^t-1)}\right)' = e^{\mu(e^t-1)} \times (\underbrace{\mu(e^t-1)}_{\mu e^t})' = e^{\mu(e^t-1)} \times \mu e^t = \mu e^{\mu(e^t-1)+t}$$

when $t=0$ $\mu e^0 = \mu\times 1 = \boxed{\mu}$, $E[x]$

$$M_x(t)'' = \left[\mu e^{\mu(e^t-1)+t}\right]' = \mu e^{\mu(e^t-1)+t}\cdot(\underbrace{\mu(e^t-1)+t}_{\mu e^t+1})' = \mu e^{(\mu(e^t-1)+t)}\cdot(\mu e^t+1)$$

$= \text{when } t=0, \quad E[x^2] = \mu \times (\mu+1) = \mu(\mu+1) = E[x^2]$

$\sigma^2 = E[x^2] - (E[x])^2 = \mu^2 + \mu - \mu^2 = \boxed{\mu}$

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7. Independently geometric distribution

$P(X=r) = P(Y=r) = pq^r, \quad r = 0, 1, 2, 3 \dots$

(a) $U = X+Y$ find pdf

when $x=1, y=0$ or when $x=0, y=1$

$U = X+Y$ so the range of $U = 0, 1, 2, 3, \dots \infty$

$x+y$ when $x=0, y=0$

$\therefore U = x+y$

$P(U) = P(x+y) = \sum_0^\infty P(x)\, P(y) = \sum_0^\infty P(x)\, P(u-x)$

$= \sum_{x=0}^{u} (pq^x)(pq^{u-x}) = \sum_{x=0}^{u} p^2 q^{x+u-x} = p^2 \sum_{x=0}^{u} p^u = \boxed{(u+1)p^u p^2}$

(b) $P(X=x / U=X+Y) = \frac{P(X=x, Y=u-x)}{P(U=X+Y)} = \frac{q^x p \times q^{u-x} p)}{(u+1)p^u p^2} = \frac{q^u p^2}{(u+1) q^u p^2}$

$= \boxed{\frac{1}{(u+1)}} \quad x = 0, 1, 2, 3, \dots, u$

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8. $E[X] = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \mu$

$Y = C^T X$

$C = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$E[Y] = C^T \mu = [-2 \;\; 1] \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \boxed{0}$ mean

$-2+2$

$1 \times 2 \times 2 \times 2 \quad 1 \times 2$

(b) Variance

$\sigma^2 y = C^T x \Sigma x c = [-2 \;\; 1] \times \begin{bmatrix} 1 & 0.2 \\ 0.2 & 1 \end{bmatrix} \times \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \boxed{4.2}$

$[-1.8 \;\; 0.6] \times \begin{bmatrix} -2 \\ 1 \end{bmatrix} = 4.2$

$-2+0.2=-1.8$ $3.6+0.6$

$-0.4+1=0.6$

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9. $E[XY]=0.6$

$E[X]=0,\ E[Y]=0$

$\sigma_X^2=1,\ \sigma_Y^2=1$

(a)

$$\sigma_{XY}=E[XY]-(E[X]\times E[Y])^2=0.6-0\times 0=\boxed{0.6}=\rho$$

$$\frac{1}{2\pi\sqrt{(1-\rho^2)}}\exp\left[-\frac{1}{2(1-\rho^2)}(x^2-2\rho xy+y^2)\right]$$

$$\boxed{\frac{1}{2\pi\sqrt{1-(0.6)^2}}\exp\left(-\frac{1}{2(1-0.6^2)}(x^2-2(0.6)xy+y^2)\right)}$$

$(0.6)^2=0.36$ 1.28

$1-0.36=0.64$

$\sqrt{0.64}=0.8$

$$\frac{1}{1.6\pi}\exp\left(\frac{1}{1.28}(x^2-1.2xy+y^2)\right)=\boxed{\frac{1}{1.6\pi}\exp(0.78125\times(x^2-1.2xy+y^2))}$$

(b) $$P(X>1.1\,/\,Y=0.5)=\frac{P(X>1.1,Y=0.5)}{P(Y=0.5)}=$$

$$X|Y=y\sim N(\rho y,1-\rho^2)$$
$$=y\sim N(0.6y,0.64)=y\sim N(0.3,0.64)$$

Normal distribution

$$z\geq\frac{1.1-0.3}{\sqrt{0.64}}=\frac{0.8}{0.8}=1\qquad P(z\geq 1)=1-0.8413$$
$$=\boxed{0.1587}$$