

ON CRITICALITY AND ADDITIVITY OF THE PSEUDOACHROMATIC NUMBER UNDER JOIN

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ABSTRACT. A vertex coloring of a graph is said to be pseudocomplete if, for any two distinct colors, there exists at least one edge with those two colors as its end vertices. The pseudoachromatic number of a graph is the greatest number of colors possible used in a pseudocomplete coloring. This paper studies properties relating to additivity of the pseudoachromatic number under the join. Errors from the literature are corrected and the notion of weakly critical is introduced in order to study the problem.

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1. INTRODUCTION

Some of the oldest problems in graph theory study colorings subject to various constraints. A coloring of the vertices of a graph is called *pseudocomplete* if every pair of disjoint colors are adjacent via at least one edge. The maximum possible number of colors used in a pseudocomplete coloring of a graph G is called the *pseudoachromatic number*,

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$\Psi(G)$. Though not studied here, requiring the colorings be proper gives rise to the well studied achromatic number, *e.g.*, [12].

The pseudoachromatic number was first used by Harary *et al.*, [13], and Gupta, [11]. See [16] for additional comments on the origin of the name. For some of the history of work on Ψ , see the following: [15], [8], [6], [21], [10], [17], [18], [9], [19], [20], [14], [3], [4], [1], [2], and [5].

Initial definitions and background results are found in Section §2, including the definition of *criticality*, Definition 2.5, which plays a role in some aspects of Ψ being additive under join.

Section §3 examines additivity of Ψ and preservation of criticality under the join. On this front, for graphs G and H , there is an error in the literature that claims that G and H are critical if and only if

$$(1.1) \quad \Psi(G \nabla H) = \Psi(G) + \Psi(H).$$

Though sufficient (Theorem 3.3), in fact, the converse is not true, see Remark 3.4. Similarly (Theorem 3.5), when G and H are critical, so is $G \nabla H$, though the converse is also false, see Remark 3.6. Finally apriori bounds and structural results are given in Theorems 3.1 and 3.7.

Section §4 introduces the notion of *weakly critical* in Definition 2.5. A graph invariant equivalence is given in Theorem 4.3 and a structural equivalence is given in Theorem 4.5. One of the main results of this paper, Theorem 4.7, shows that Equation 1.1 implies that either G or H is weakly critical.

Section §5 studies the above topics in the context of $\nabla^k G$. For $k \geq 2$, Theorem 5.3 shows that $\nabla^k G$ is critical precisely when $k(\omega(G) + |V|)$ is even. More generally, Theorem 5.5 shows that $\nabla^k G$ is always weakly critical. Finally, additivity or near-additivity of Ψ on $\nabla^k G$ is determined in Theorems 5.4 and 5.6.

2. INITIAL DEFINITIONS AND BACKGROUND

We write \mathbb{N} for the nonnegative integers and \mathbb{Z}^+ for the positive integers. In this paper, $G = (V, E)$ is a simple, finite graph. If multiple graphs may lead to ambiguity, we will write $G = (V_G, E_G)$. If the vertices of G are equipped with a labeling by some $C \subseteq \mathbb{Z}$, we generally write $\ell : V \rightarrow C$ for the coloring. Finally, we write ω for the *clique number* and ∇ for the graph theoretic *join*.

We begin with one of the central definitions of the paper.

Definition 2.1. A *pseudocomplete coloring* of a graph G is a coloring of the vertices of G , $\ell : V \rightarrow C$, so that, for any distinct $i, j \in C$, there

exists $uv \in E$ so that $\{i, j\} = \{\ell(u), \ell(v)\}$. Note that the coloring here need not be proper.

The *pseudoachromatic number* of a graph G , written $\Psi(G)$, is the greatest possible number of colors employed in a pseudocomplete coloring of G .

If K is a maximal clique of G , then coloring K with distinct colors and the rest of G arbitrarily gives a pseudocomplete coloring. In addition, the defintion of a pseudocomplete coloring requires at least $\binom{\Psi(G)}{2} \leq |E|$. As a result,

$$\omega(G) \leq \Psi(G) \leq \frac{1 + \sqrt{1 + 8|E|}}{2}.$$

Moreover importantly, the following upper bound is known.

Lemma 2.2 ([7], Lemma 2). *Let G be a graph. Then*

$$\Psi(G) \leq \left\lfloor \frac{\omega(G) + |V|}{2} \right\rfloor.$$

In addition, there is a lower bound for Ψ of the join of graphs.

Lemma 2.3 ([7], Corollary 4). *Let G and H be graphs. Then*

$$\Psi(G) + \Psi(H) \leq \Psi(G \nabla H).$$

Determining when the inequalities in Lemmas 2.2 and 2.2 become equalities leads to the following definitions.

Definition 2.4. Let G be a graph and $k \in \mathbb{N}$ with $k \leq |V|$. Define the *minimal psi-drop function* as

$$\text{mpd}_G(k) = \min\{\Psi(G) - \Psi(G \setminus X) \mid X \subseteq G, |X| = k\}.$$

If $X \subseteq G$ satisfies $|X| = k$ and $\text{mpd}_G(k) = \Psi(G) - \Psi(G \setminus X)$, we call X a *realizing subgraph* for $\text{mpd}_G(k)$.

Note that $0 \leq \text{mpd}_G(k) \leq k$ as Ψ can drop by at most one when removing a vertex.

The next definition is a reformulation of the one found in [7].

Definition 2.5. We say G is *critical* if

$$\text{mpd}_G(k) \geq \left\lceil \frac{k}{2} \right\rceil$$

for all $0 \leq k \leq |G|$.

There is a useful known equivalence for criticality in terms of maximizing the inequality in Lemma 2.2 without the floor function.

Lemma 2.6 ([7], Lemma 10). *Let G be a graph. Then G is critical if and only if*

$$\Psi(G) = \frac{\omega(G) + |V|}{2}.$$

3. CRITICALITY AND ADDITIVITY OF Ψ UNDER JOINS

We begin with an apriori bound on $\Psi(G \nabla H)$ between a minimal sum involving a clique number and vertex count and an average sum of clique numbers and vertex counts.

Theorem 3.1. *Let G and H be graphs. Let*

$$m = \min\{\omega(G) + |V_H|, \omega(H) + |V_G|\}.$$

Then

$$m \leq \Psi(G \nabla H) \leq \left\lfloor \frac{\omega(G) + \omega(H)}{2} + \frac{|V_G| + |V_H|}{2} \right\rfloor.$$

Proof. The upper bound comes from Lemma 2.2 and the additivity of ω under join.

For the lower bound, let K_G and K_H be maximal cliques of G and H , respectively. Color K_G and K_H with $\omega(G) + \omega(H)$ distinct colors. Let $X_G = G \setminus K_G$ and $X_H = H \setminus K_H$. After relabeling, we may assume that

$$|V_G| - \omega(G) = |X_G| \leq |X_H| = |V_H| - \omega(H).$$

Color both X_G and X_H with an additional identical $|V_G| - \omega(G)$ colors. It is straightforward to verify that this is a pseudocomplete coloring of $G \nabla H$ with

$$(\omega(G) + \omega(H)) + (|V_G| - \omega(G)) = \omega(H) + |V_G|$$

colors. After noting that $\omega(H) + |V_G| \leq \omega(G) + |V_H|$, we are done. \square

Next we give a criteria for Ψ to be additive over the join.

Theorem 3.2. *Let G, H be graphs. Then*

$$\Psi(G \nabla H) = \Psi(G) + \Psi(H)$$

if and only if

$$\text{mpd}_G(k) + \text{mpd}_H(k) \geq k$$

for all $0 \leq k \leq \min\{|G|, |H|\}$.

Moreover, in that case, there exists a k , $1 \leq k \leq \min\{|G|, |H|\}$, so that

$$\text{mpd}_G(k) + \text{mpd}_H(k) = k.$$

Proof. Let ℓ be a pseudocomplete coloring of $G \nabla H$ with $\Psi(G \nabla H)$ colors. Let $C = \ell(V_G) \cap \ell(V_H)$. By recoloring if necessary, we may assume that every color in C appears exactly once in G and once in H . Define induced subgraphs of G and H by $X_G = \ell^{-1}(C) \cap V_G$ and $X_H = \ell^{-1}(C) \cap V_H$, respectively. We see that $|C| = |X_G| = |X_H|$. Note that ℓ restricted to $G \setminus X_G$ is pseudocomplete and satisfies $|\ell(G \setminus X_G)| = \Psi(G \setminus X_G)$ (and similar for $H \setminus X_H$). By construction, it follows that

$$\Psi(G \nabla H) = \Psi(G \setminus X_G) + |C| + \Psi(H \setminus X_H).$$

From the observations that $\Psi(G) \geq \Psi(G \setminus X_G) + \text{mpd}_G(|C|)$ and $\Psi(H) \geq \Psi(H \setminus X_H) + \text{mpd}_H(|C|)$, it follows that

$$\Psi(G) + \Psi(H) \geq \Psi(G \setminus X_G) + \Psi(H \setminus X_H) + \text{mpd}_H(|C|) + \text{mpd}_G(|C|)$$

so that

$$\Psi(G) + \Psi(H) \geq \Psi(G \nabla H) + (\text{mpd}_H(|C|) + \text{mpd}_G(|C|) - |C|).$$

From this, we see that the condition $\text{mpd}_G(k) + \text{mpd}_H(k) \geq k$ for all k forces $\Psi(G) + \Psi(H) \geq \Psi(G \nabla H)$, with equality for $k = |C|$, and, by Lemma 2.3, additivity of Ψ over the join.

Conversely, suppose that $\text{mpd}_G(k_0) + \text{mpd}_H(k_0) < k_0$ for some $1 \leq k_0 \leq \min\{|G|, |H|\}$. Choose realizing subgraphs for $\text{mpd}_G(k_0)$ and $\text{mpd}_H(k_0)$, X_G and X_H , respectively. It follows that $\text{mpd}_G(k_0) = \Psi(G) - \Psi(G \setminus X_G)$, $\text{mpd}_H(k_0) = \Psi(H) - \Psi(H \setminus X_H)$, and $|X_G| = |X_H| = k_0$. Then

$$\Psi(G) + \Psi(H) < \Psi(G \setminus X_G) + \Psi(H \setminus X_H) + k_0.$$

We can pseudocompletely color $G \nabla H$ as follows: color $G \setminus X_G$ pseudocompletely with $\Psi(G \setminus X_G)$ colors, color $H \setminus X_H$ pseudocompletely with $\Psi(H \setminus X_H)$ additional colors, and color both X_G and X_H with k_0 further colors. Then we have

$$\Psi(G \setminus X_G) + \Psi(H \setminus X_H) + k_0 \leq \Psi(G \nabla H)$$

and so

$$\Psi(G) + \Psi(H) < \Psi(G \nabla H).$$

□

From Theorem 3.2, we immediately get the following result.

Theorem 3.3. *Let G and H be critical graphs. Then*

$$\Psi(G \nabla H) = \Psi(G) + \Psi(H).$$

Remark 3.4. Though erroneously claimed in [7] Corollary 6, the converse of Theorem 3.3 is not true which invalidates some of the results and proofs in [7]. The failure of the converse can be seen with P_3 and C_8 . Using Lemma 2.6 and the straightforward facts that $\Psi(P_3) = 2$ and $\Psi(C_8) = 4$, neither P_3 nor C_8 is critical. However, a labeling of the join using $\{1, 2, 3\}$ and $\{1, 4, 5, 6, 4, 4, 4, 4\}$ and Lemma 2.2 show that $\Psi(P_3 \nabla C_8) = 6$ so that $\Psi(P_3 \nabla C_8) = \Psi(P_3) + \Psi(C_8)$. See Theorem 4.7 below for as close to a converse as possible.

Theorem 3.3 allows us to show that criticality is preserved under join.

Theorem 3.5. *If G and H are critical graphs, then $G \nabla H$ is also critical.*

Proof. Use Lemma 2.6 and Theorem 3.3 to see that $G \nabla H$ is critical if and only if

$$2(\Psi(G) + \Psi(H)) = (\omega(G) + \omega(H)) + (|V_G| + |W_H|)$$

and we are done. \square

Remark 3.6. The converse of Theorem 3.5 is false. To see this, recall that P_3 is not critical. However, a labeling of $P_3 \nabla P_3$ with $\{1, 2, 3\}$ and $\{1, 4, 5\}$ on each P_3 shows that $\Psi(P_3 \nabla P_3) \geq 5$. Equality and the fact that $P_3 \nabla P_3$ is critical then follow from Lemma 2.2 and Lemma 2.6. Note that this shows that the partial converse to Theorem 3.5 is false even when restricting to the case of $G = H$. Of note, see the comment after Equation 5.1 and Theorem 5.3.

We end this section with a general constraint on critical graphs and their pseudocomplete coloring.

Theorem 3.7. *If G is critical, then there is a pseudocomplete coloring of G in which $\omega(G)$ colors are used exactly once on a maximal clique and $\frac{|V| - \omega}{2}$ colors are used exactly twice on the complement of the maximal clique. Moreover,*

$$|E| \geq \frac{(|V| + \omega)(|V| + \omega - 2)}{8}.$$

Proof. Let ℓ be a pseudocomplete coloring of G with the colors $S = \{1, 2, \dots, \Psi(G)\}$. For $k \in \mathbb{Z}^+$, let

$$m_k = \{i \in S \mid |\ell^{-1}(i)| = k\}$$

and $n_k = |m_k|$. As ℓ is pseudocomplete, $K = \bigcup_{i \in m_1} \ell^{-1}(i)$ is a clique of size n_1 .

Moreover,

$$\begin{aligned}
|V| &= \sum_{k \in \mathbb{Z}^+} kn_k \\
&= n_1 + 2 \sum_{k \geq 2} n_k + \sum_{k \geq 3} (k-2)n_k \\
&= n_1 + 2(\Psi(G) - n_1) + \sum_{k \geq 3} (k-2)n_k \\
&= 2\Psi(G) - n_1 + \sum_{k \geq 3} (k-2)n_k \\
&\geq 2\Psi(G) - n_1 \\
&\geq 2\Psi(G) - \omega(G).
\end{aligned}$$

By Lemma 2.6, G is critical if and only if $|V| = 2\Psi(G) - \omega(G)$. From the equations above, we see G is critical if and only if we have equality at each step. In particular, if and only if $m_k = \emptyset$ for $k \geq 3$ and $n_1 = \omega(G)$.

As a result, K is a maximal clique and its colors are used exactly once. In addition, the colors of $X = G \setminus K$ are used exactly twice. Therefore X may be broken into two sets of equal size, X_1 and X_2 , with the colors of X_1 used exactly once in X_1 and exactly once in X_2 .

Finally, write \tilde{X} for the contraction of G that identifies each vertex in X_1 with the vertex in X_2 of the same color. As ℓ is a pseudocomplete coloring, we see that \tilde{X} is the complete graph on $\omega(G) + \frac{|V| - \omega(G)}{2}$ vertices and we are done. \square

4. WEAK CRITICALITY

As demonstrated in Remark 3.4, criticality is not necessary for additivity of Ψ over the join. Indeed, Theorem 3.2 shows that additivity of Ψ for a graph pair depends on subtle structures in both graphs, rendering a simple characterization of additive pairs out of the question. Despite that, in this section we demonstrate that additivity of Ψ requires at least one graph to have a weak form of criticality.

The following is a subtle tweak of Definition 2.5. Note the change from the ceiling function to the floor function.

Definition 4.1. We say G is *weakly critical* if

$$\text{mpd}_G(k) \geq \left\lfloor \frac{k}{2} \right\rfloor$$

for all $0 \leq k \leq |G|$.

Note that Definitions 2.5 and 4.1 imply that every critical graph is weakly critical. The converse to this is not true as P_3 is weakly critical but not critical.

We immediately get the following by applying Theorem 3.2 and noting that $\lfloor \frac{k}{2} \rfloor + \lceil \frac{k}{2} \rceil = k$ for all k .

Corollary 4.2. *Let G and H be graphs with G critical and H weakly critical. Then*

$$\Psi(G \nabla H) = \Psi(G) + \Psi(H).$$

The following results concerning weak criticality are analogues of various results for criticality. The first is the analogue of Lemma 2.6 for being weakly critical and avoids the parity constraint of criticality.

Theorem 4.3. *Let G be a graph. Then G is weakly critical if and only if*

$$\Psi(G) = \left\lfloor \frac{\omega(G) + |V|}{2} \right\rfloor.$$

Proof. By Lemma 2.2, it suffices to prove that

$$(4.1) \quad \Psi(G) < \left\lfloor \frac{\omega(G) + |V|}{2} \right\rfloor$$

happens if and only if G is not weakly critical.

First suppose that Equation 4.1 holds. Choose $K \subseteq G$ to be a maximal clique and let $X = G \setminus K$. Then

$$\begin{aligned} \Psi(G) &< \left\lfloor \frac{\omega(G) + |V|}{2} \right\rfloor \\ &= \omega(G) + \left\lfloor \frac{|V| - \omega(G)}{2} \right\rfloor \\ &= \Psi(G \setminus X) + \left\lfloor \frac{|X|}{2} \right\rfloor. \end{aligned}$$

Therefore $\text{mpd}_G(|X|) \leq \Psi(G) - \Psi(G \setminus X) < \left\lfloor \frac{|X|}{2} \right\rfloor$ and G is not weakly critical.

Conversely, suppose there exists G is not weakly critical. Then there exists $k \leq |G|$ with $\text{mpd}_G(k) < \left\lfloor \frac{|X|}{2} \right\rfloor$. Let X be a realizing subgraph

for $\text{mpd}_G(k)$. Then, using Lemma 2.2,

$$\begin{aligned}\Psi(G) &< \Psi(G \setminus X) + \left\lfloor \frac{|X|}{2} \right\rfloor \\ &\leq \left\lfloor \frac{\omega(G \setminus X) + (|V| - |X|)}{2} \right\rfloor + \left\lfloor \frac{|X|}{2} \right\rfloor \\ &\leq \left\lfloor \frac{\omega(G) + |V| - |X|}{2} \right\rfloor + \left\lfloor \frac{|X|}{2} \right\rfloor \\ &\leq \left\lfloor \frac{\omega(G) + |V|}{2} \right\rfloor\end{aligned}$$

and we are done. \square

From Theorem 4.3 and Lemma 2.6, observe that if $\omega(G) + |V|$ is even, then G is critical if and only if it is weakly critical. However, when $\omega(G) + |V|$ is odd, G is not critical, but G may still be weakly critical. The graph P_3 is such an example.

We now give the analogue of Theorem 3.7 for weakly critical.

Theorem 4.4. *Let G be weakly critical but not critical. Then there is a pseudocomplete coloring of G so that either*

- (1) $\omega(G)$ colors are used exactly once on a maximal clique, a single color is used exactly three times in the complement of the clique, and $\frac{|V|-\omega-3}{2}$ colors are used exactly twice on remaining vertices or
- (2) $\omega(G)-1$ colors are used exactly once on a clique of size $\omega(G)-1$ and $\frac{|V|-\omega+1}{2}$ colors are used exactly twice on complement of the clique.

Moreover,

$$|E| \geq \frac{(|V| + \omega - 1)(|V| + \omega - 3)}{8}.$$

Proof. This result follows from the proof of Theorem 3.7. As Theorem 4.3 shows that weakly critical in this setting is the same as requiring

$$|V| + \omega(G) - 2\Psi(G) - 1 = 0,$$

the proof of Theorem 3.7 shows that $n_k = 0$ for $k \geq 4$. Moreover, either $n_1 = \omega$ and $m_3 = 1$ or $n_1 = \omega(G) - 1$ and $m_3 = 0$. \square

The remainder of this section is devoted to demonstrating that weak criticality is required of at least one graph in a Ψ -additive graph pair.

The following result will be an important technical tool.

Theorem 4.5. *Let G be a graph. Then G is not weakly critical if and only if there exists induced subgraphs $M_1 \subseteq M_2 \subseteq G$ so that*

- (1) $|V_{M_2 \setminus M_1}| = 2$,
- (2) $\Psi(M_1) = \Psi(M_2)$,
- (3) $\Psi(G) = \Psi(M_2) + \left\lfloor \frac{|V_{G \setminus M_2}|}{2} \right\rfloor$.

Moreover, if ℓ is a maximal pseudocomplete coloring of G , there exists $C \subseteq V_G$ with $|C| = \left\lfloor \frac{|V_{G \setminus M_1}|}{2} \right\rfloor + 1 \geq 2$ and $\ell(G) = \ell(G \setminus C)$.

Proof. Suppose G is not weakly critical and, by Definition 4.1, choose M_1 to be a maximal subgraph of G satisfying

$$\Psi(G) \leq \Psi(M_1) + \left\lfloor \frac{|V_{G \setminus M_1}|}{2} \right\rfloor - 1.$$

As $\Psi(G) \geq \Psi(M_1)$, $|V_{G \setminus M_1}| \geq 2$. Choose M_2 to be any induced subgraph of G satisfying $M_1 \subseteq M_2$ with $|V_{M_2 \setminus M_1}| = 2$.

By maximality, we get

$$\begin{aligned} \Psi(G) &\geq \Psi(M_2) + \left\lfloor \frac{|V_{G \setminus M_2}|}{2} \right\rfloor \\ &= \Psi(M_2) + \left\lfloor \frac{|V_{G \setminus M_1}|}{2} \right\rfloor - 1 \\ &\geq \Psi(M_1) + \left\lfloor \frac{|V_{G \setminus M_1}|}{2} \right\rfloor - 1 \\ &\geq \Psi(G). \end{aligned}$$

Therefore $\Psi(M_1) = \Psi(M_2)$ and $\Psi(G) = \Psi(M_2) + \left\lfloor \frac{|V_{G \setminus M_2}|}{2} \right\rfloor$.

For the converse, observe that we would have

$$\begin{aligned} \Psi(G) &= \Psi(M_2) + \left\lfloor \frac{|V_{G \setminus M_2}|}{2} \right\rfloor \\ &= \Psi(M_1) + \left\lfloor \frac{|V_{G \setminus M_1}|}{2} \right\rfloor - 1 \end{aligned}$$

so that $\text{mpd}_G(|M_1|) \leq \Psi(G) - \Psi(M_1) < \left\lfloor \frac{|V_{G \setminus M_1}|}{2} \right\rfloor$, and therefore G is not weakly critical.

For the final statement, write $\xi = \left\lfloor \frac{|V_{G \setminus M_1}|}{2} \right\rfloor$. Observe that

$$\Psi(G) = \Psi(M_1) + \xi - 1 \leq |M_1| + \xi - 1$$

and

$$|V| = |M_1| + |V_{G \setminus M_1}| \geq |M_1| + 2\xi.$$

If one vertex of each color is selected from G , we see that leaves at least $\xi + 1$ vertices which may be chosen as C . \square

It is worth pointing out that an analogous argument establishes the following characterization of critical graphs.

Theorem 4.6. *Let G be a graph. Then G is not critical if and only if there exists induced subgraphs $M_1 \subseteq M_2 \subseteq G$ so that*

- (1) $0 < |V_{M_2 \setminus M_1}| \leq 2$,
- (2) $\Psi(M_1) = \Psi(M_2)$,
- (3) $\Psi(G) = \Psi(M_2) + \left\lceil \frac{|V_{G \setminus M_2}|}{2} \right\rceil$.

Moreover, if ℓ is a maximal pseudocomplete coloring of G , there exists $C \subseteq V_G$ with $|C| = \left\lceil \frac{|V_{G \setminus M_2}|}{2} \right\rceil + |V_{M_2 \setminus M_1}| - 1$ and $\ell(G) = \ell(G \setminus C)$.

As a consequence of Theorem 4.5, we are now able to prove the following, which is as close to a converse of Corollary 3.3 as possible.

Theorem 4.7. *Let G and H be graphs. If*

$$\Psi(G) + \Psi(H) = \Psi(G \nabla H),$$

then at least one of G or H is weakly critical. Moreover, if one of G or H is weakly critical and has a coloring of form (1) as in Theorem 4.4, then the other is weakly critical and does not have such a coloring.

Proof. Let G and H be graphs with $\psi(G \nabla H) = \Psi(G) + \Psi(H)$.

Suppose that both G and H are not weakly critical. Using Theorem 4.5, choose induced subgraphs $M_1 \subseteq M_2 \subseteq G$ and $N_1 \subseteq N_2 \subseteq H$. After possibly relabeling, suppose $\left\lceil \frac{|V_{G \setminus M_1}|}{2} \right\rceil \leq \left\lceil \frac{|V_{H \setminus N_1}|}{2} \right\rceil$.

Color H with a pseudocomplete coloring with $\Psi(H)$ colors. Choose $C \subseteq H$ with $|C| = \left\lceil \frac{|V_{G \setminus M_1}|}{2} \right\rceil$ so that all $\Psi(H)$ colors are still represented in $H \setminus C$.

Write $V_{G \setminus M_1} = P_0 \amalg P_1$ with $|P_0| = \left\lfloor \frac{|V_{G \setminus M_1}|}{2} \right\rfloor \leq |P_1|$. Color $V_{G \setminus M_1}$ with $\left\lceil \frac{|V_{G \setminus M_1}|}{2} \right\rceil$ new colors such that each color appears in both P_0 and P_1 .

Finally, swap the colors in P_0 with those in C and color M_1 with a pseudocomplete coloring consisting of $\Psi(M_2)$ additional new colors.

By construction, this gives a pseudocomplete coloring with

$$\Psi(H) + \Psi(M_2) + \left\lceil \frac{|V_{G \setminus M_2}|}{2} \right\rceil + 1 = \Psi(H) + \Psi(G) + 1$$

colors. As this is a lower bound for $\Psi(G \nabla H)$, we get $\Psi(G) + \Psi(H) > \Psi(G \nabla H)$ which contradicts additivity of Ψ .

Thus one of G or H must be weakly critical.

Now, assume that G is weakly critical and has a pseudocomplete coloring of type (1). Let ℓ be such a coloring, i.e. a coloring using $\Psi(G) = \left\lfloor \frac{\omega(G) + |V_G|}{2} \right\rfloor$ many colors with vertices v_0, v_1 , and v_2 with $\ell(v_0) = \ell(v_1) = \ell(v_2)$. Note that ℓ uses all $\Psi(G)$ colors on $G \setminus \{v_0, v_1\}$.

Assume that H is either not weakly critical or weakly critical with a coloring of type (1). In either case, we will show that there are distinct $c_0, c_1 \in V_H$ and a pseudocomplete coloring ℓ' of H so that $\Psi(H)$ many colors appear in the complement of $\{c_0, c_1\}$.

If H is not weakly critical, by Theorem 4.5, there is a pseudocomplete coloring ℓ' of H using $\Psi(H)$ many colors and a subset C of V_H with $|C| = \left\lfloor \frac{|V_H \setminus N_1|}{2} \right\rfloor + 1 \geq 2$ such that each of the $\Psi(H)$ colors appears in the complement of C . Let $\{c_0, c_1\} \subseteq C$.

If H is weakly critical with a coloring of type (1), take ℓ' to be one such coloring and let c_0, c_1 , and c_2 be the three vertices that share a color. Clearly ℓ' uses all $\Psi(H)$ colors on $H \setminus \{c_0, c_1\}$.

Now, color $G \nabla H$ as follows: color $G \setminus \{v_0, v_1\}$ according to ℓ ; color $H \setminus \{c_0, c_1\}$ according to ℓ' ; color c_0 with $\ell(v_0)$; v_0 with $\ell'(c_0)$; and use one additional color to color both v_1 and c_1 . By construction, this is a pseudocomplete coloring consisting of $\Psi(G) + \Psi(H) + 1 > \Psi(G) + \Psi(H)$ many colors, establishing that $\Psi(G \nabla H) > \Psi(G) + \Psi(H)$, contradicting additivity of Ψ . Thus if G is weakly critical and has a pseudocomplete coloring of type (1), then H is weakly critical and does not have a pseudocomplete coloring of type (1). \square

5. REMARKS ON $\nabla^k G$

If we restrict to the case that $G = H$, we can get a partial converse to Theorem 3.3 with a stronger conclusion than that of Theorem 4.7.

Theorem 5.1. *Let G be a graph. Then G is critical if and only if*

$$\Psi(G \nabla G) = 2\Psi(G).$$

Proof. Theorem 3.2 shows that $\Psi(G \nabla G) = 2\Psi(G)$ if and only if for all $k \leq |G|$, $\text{mpd}_G(k) + \text{mpd}_G(k) \geq k$. Since $\text{mpd}_G(k)$ is an integer for all k , this happens if and only if $\text{mpd}_G(k) \geq \lceil \frac{k}{2} \rceil$ for all $k \leq |G|$, i.e., G is critical. \square

Remark 5.2. As noted in Remark 3.6, the converse of Theorem 3.5 fails. Indeed the converse fails even under the assumption that $G = H$. Theorem 3.1 immediately implies that

$$(5.1) \quad \Psi(G \nabla G) = \omega(G) + |V_G|$$

for all G , so by Lemma 2.2, $G \nabla G$ is always critical.

In fact, more is true.

Theorem 5.3. *Let $k \in \mathbb{Z}$ with $k \geq 2$. Then $\nabla^k G$ is critical if and only if*

$$k(\omega(G) + |V|)$$

is even.

Proof. If k is even, apply Equation 5.1 with G replaced by $\nabla^{k/2} G$ and then use Lemma 2.2 to get criticality.

If $\omega(G) + |V|$ is even with $k \geq 3$, begin with a maximal clique, K , of G and let $X = G \setminus K$. As $\omega(G) + |V|$ is even, so is $|X| = |V| - \omega(G)$. Divide X into sets X_1 and X_2 of equal order, $q = \frac{|V| - \omega(G)}{2}$.

In $\nabla^k G$, color $\nabla^k K$ with $k\omega(G)$ distinct colors. Next color $\nabla^k X_1$ with an additional kq colors. Finally, color $\nabla^k X_2$ with a shift of the colors used in $\nabla^k X_1$. More precisely, color the $(i+1)$ st copy of X_2 in $\nabla^k X_2$, i viewed as an element of \mathbb{Z}_k , with the same colors used on the i th copy of X_1 in $\nabla^k X_1$. This ensures that the colors used in each copy of X in $\nabla^k X$ are used in two different copies of X . It is straightforward to verify that this results in a pseudocomplete coloring of $\nabla^k G$. As

$$k \left(\omega(G) + \frac{|V| - \omega(G)}{2} \right) = k \frac{\omega(G) + |V|}{2},$$

Lemma 2.2 again gives criticality.

Finally, if $k(\omega(G) + |V|)$ is odd, Lemma 2.2 shows that $\nabla^k G$ is not critical. \square

Note that, as criticality of G requires $\omega(G) + |V|$ to be even, Theorem 3.1 says that $\nabla^k G$ is critical whenever parity makes criticality possible. See Theorem 5.5 below when parity makes criticality impossible.

Theorem 5.3 allows us to generalize Theorem 5.1. See Theorem 5.6 for the analogue when $k(\omega(G) + |V|)$ is odd. Note that this, along with Theorem 5.6 below, gives a correct proof of [7] Corollary 11.

Theorem 5.4. *Let $k \in \mathbb{Z}$ with $k \geq 2$ and $k(\omega(G) + |V|)$ even. Then G is critical if and only if*

$$k\Psi(G) = \Psi(\nabla^k G).$$

Proof. As Theorem 5.3 shows that $\nabla^k G$ is critical, it follows from Lemma 2.6 that $\Psi(\nabla^k G) = k \frac{\omega(G) + |V|}{2}$. Therefore, $k\Psi(G) = \Psi(\nabla^k G)$ if and only if $\Psi(G) = \frac{\omega(G) + |V|}{2}$ if and only if G is critical. \square

Recall that Theorem 5.3 showed that, for $k \geq 2$, $\nabla^k G$ is critical if and only if $k(\omega(G) + |V|)$ is even. We now show that weakly critical fills in the rest of the range.

Theorem 5.5. *Let $k \in \mathbb{Z}$ with $k \geq 2$. Then $\nabla^k G$ is weakly critical.*

Proof. By Theorem 5.3 and the comment following Theorem 4.3, it remains to show that $\nabla^k G$ is weakly critical when $k(\omega(G) + |V|)$ is odd with $k \geq 3$.

Begin with a maximal clique, K , of G and let $X = G \setminus K$. As $\omega(G) + |V|$ is odd, so is $|X| = |V| - \omega(G)$. Divide X into sets X_1 and X_2 of equal order, $q = \frac{|V| - \omega(G) - 1}{2}$, and a singleton vertex, v_0 .

In $\nabla^k G$, as in Theorem 5.3, color $\nabla^k K$ with $k\omega(G)$ distinct colors, $\nabla^k X_1$ with an additional kq colors, and $\nabla^k X_2$ with a shifted coloring of $\nabla^k X_1$. Finally color the copies of v_0 in $\nabla^k G$ with an additional $\frac{k-1}{2}$ colors. It is straightforward to verify that this generates a pseudocomplete coloring of $\nabla^k G$. As

$$\begin{aligned} k \left(\omega(G) + \frac{|V| - \omega(G) - 1}{2} \right) + \frac{k-1}{2} &= k \frac{\omega(G) + |V|}{2} - \frac{1}{2} \\ &= \left\lfloor \frac{k(\omega(G) + |V|)}{2} \right\rfloor, \end{aligned}$$

Theorem 4.3 gives weak criticality. \square

Theorem 5.5 now allows us to address the analogue of Theorem 5.4 when $k(\omega(G) + |V|)$ is odd.

Theorem 5.6. *Let $k \in \mathbb{Z}$ with $k \geq 3$ and $k(\omega(G) + |V|)$ odd. Then G is not critical. However, G is weakly critical if and only if*

$$k\Psi(G) + \left\lfloor \frac{k}{2} \right\rfloor = \Psi(\nabla^k G).$$

Proof. The fact that G is not critical follows by parity from Lemma 2.6. As Theorem 5.5 shows that $\nabla^k G$ is weakly critical, it follows from Theorem 4.3 that $\Psi(\nabla^k G) = \left\lfloor \frac{k(\omega(G) + |V|)}{2} \right\rfloor$. However, we know that G is weakly critical iff and only if $\Psi(G) = \left\lfloor \frac{\omega(G) + |V|}{2} \right\rfloor$. As this is equivalent to

$$\begin{aligned} k\Psi(G) &= k \left\lfloor \frac{\omega(G) + |V|}{2} \right\rfloor \\ &= \frac{k(\omega(G) + |V|)}{2} - \frac{k}{2} \\ &= \left\lfloor \frac{k(\omega(G) + |V|)}{2} \right\rfloor - \left\lfloor \frac{k}{2} \right\rfloor, \end{aligned}$$

we are done. \square

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