

Extremal minimal bipartite matching covered graphs *

Amit Kumar Mallik

Indian Institute of Technology Bombay
amit.km287@gmail.com

Ajit A. Diwan

Indian Institute of Technology Bombay
aad@cse.iitb.ac.in

Nishad Kothari

Indian Institute of Technology Madras
nishadkothari@gmail.com

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The first and last authors dedicate this work to their coauthor,
the late Ajit A. Diwan

Abstract

A connected graph, on four or more vertices, is *matching covered* (aka 1-*extendable*) if every edge is present in some perfect matching. An ear decomposition theorem (similar to the one for 2-connected graphs) exists for bipartite matching covered graphs due to Heteyi. From the results and proofs of Lovász and Plummer [*Matching Theory*, Annals of Discrete Math. 29, 1986], that rely on Heteyi's Theorem, one may deduce that any minimal bipartite matching covered graph has at least $2(m - n + 2)$ vertices of degree two (where *minimal* means that deleting any edge results in a graph that is not matching covered); such a graph is *extremal* if it attains the stated bound.

In this paper, we provide a complete characterization of the class of extremal minimal bipartite matching covered graphs. In particular, we prove that every such graph G is obtained from two copies of a tree devoid of degree two vertices, say T and T' , by adding edges — each of which joins a leaf of T with the corresponding leaf of T' .

Apart from the aforementioned bound, there are four other bounds that appear in, or may be deduced from, the work of Lovász and Plummer. Each of these bounds leads to a notion of extremality. In this paper, we obtain a complete characterization of all of these extremal classes and also establish relationships between them. Two of our characterizations are in the same spirit as the one stated above. For the remaining two extremal classes, we reduce each of them to one of the already characterized extremal classes using standard matching theoretic operations.

A connected graph is *k-extendable* if it has a matching of cardinality k and each such matching extends to a perfect matching. We also discuss bounds proved by Lou [*On the structure of minimally n-extendable bipartite graphs*, Discrete Math. 202 (1), 1999] for minimal *k*-extendable bipartite graphs (where *minimal* means that deleting any edge results in a graph that is not *k*-extendable). We conjecture stronger bounds and provide

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evidence for our conjectures by constructing tight examples that are straightforward generalizations of the ones that appear in the 1-extendable case.

1 Introduction and summary

All graphs considered here are loopless; however, we allow parallel/multiple edges. For notation and terminology, we largely follow Bondy and Murty [1]. For a graph $G := (V, E)$, its *order* is the number of vertices denoted by n , and its *size* is the number of edges denoted by m . We use the notation $G[A, B]$ to denote a bipartite graph with specified color classes A and B .

A graph is *matchable* if it has a perfect matching and an edge is *matchable* if it belongs to some perfect matching. A subgraph H of a graph G is *conformal* if $G - H$ is matchable. A connected graph of order four or more is *matching covered* if every edge is matchable; these graphs are also referred to as *1-extendable* in the literature since each edge extends to a perfect matching; see Lovász and Plummer [11]. There is an elegant ear decomposition theory for the class of bipartite matching covered graphs, due to Heteyi [6], that may be viewed as a refinement of the more well-known ear decomposition theory for the larger class of 2-connected graphs due to Whitney [16]. We describe this below.

An *ear* of H in G is an odd path of G whose ends are in H but is otherwise disjoint from H . For a bipartite graph G , a sequence of subgraphs (G_0, G_1, \dots, G_r) is an *ear decomposition* of G if: (i) $C := G_0$ is an (even) cycle, (ii) $G_{i+1} = G_i \cup P_{i+1}$ where P_{i+1} is an ear of G_i for each $i \in \{0, 1, \dots, r-1\}$, and (iii) $G_r = G$. It is easy to observe that each subgraph G_i is a conformal matching covered subgraph of G , and that $r = m - n$. We refer to (C, P_1, \dots, P_r) as the *associated ear sequence*, or simply an *ear sequence* of G . Figure 1 shows a bipartite matching covered graph and an ear sequence for the same.



Figure 1: a bipartite matching covered graph and its ear decomposition

The following is the aforementioned result by Heteyi that we will find useful in order to establish that certain bipartite graphs are indeed matching covered.

Theorem 1.1. [EAR DECOMPOSITION THEOREM]

A bipartite graph is matching covered if and only if it admits an ear decomposition.

There is also a generalization of the above theorem for nonbipartite matching covered graphs due to Lovász and Plummer [11]. However, we do not describe this here since our work focuses on bipartite matching covered graphs.

1.1 Minimality, bounds and corresponding notions of extremality

A matching covered graph is *minimal* if deletion of any edge results in a graph that is not matching covered. We use \mathcal{H} to denote the class of minimal bipartite matching covered graphs. Using the ear decomposition theory, Lovász and Plummer [11] proved that each member of \mathcal{H} has at least $m - n + 2$ pairwise nonadjacent 2-edges — where a *2-edge* is an edge whose each end has degree two. Thus, such a graph has at least $2(m - n + 2)$ degree two vertices. Furthermore, for each of these invariants, one may also deduce lower bounds solely in terms of n ; in particular, that each member of \mathcal{H} has at least $\frac{n+10}{6}$ 2-edges and at least $\frac{n}{2} + 2$ vertices of degree two; see Corollaries 7.9 and 7.1. They also proved an upper bound — namely, that each member of \mathcal{H} , distinct from C_4 , has at most $\frac{3n-6}{2}$ edges.

Each of the bounds stated in the above paragraph leads to a notion of extremality for minimal bipartite matching covered graphs. As we have five different bounds, we have five notions of extremality as defined in Table 1. For instance, \mathcal{H}_2 denotes the class of *2-vertex extremal* minimal bipartite matching covered graphs — that is, those members of \mathcal{H} that satisfy $|V_2| = 2(m - n + 2)$ where V_2 is the set of degree two vertices. Likewise, we use E_2 to denote the set of 2-edges.

Class	Property	Notation
<i>2-edge extremal</i>	$ E_2 = m - n + 2$	\mathcal{H}_0
<i>2-edge n-extremal</i>	$ E_2 = \frac{n+10}{6}$	\mathcal{H}_1
<i>2-vertex extremal</i>	$ V_2 = 2(m - n + 2)$	\mathcal{H}_2
<i>2-vertex n-extremal</i>	$ V_2 = \frac{n}{2} + 2$	\mathcal{H}_3
<i>edge extremal</i>	$ E = \frac{3n-6}{2}$	\mathcal{H}_4

Table 1: Definitions of different notions of extremality within \mathcal{H}

In this paper, we completely characterize all of the five classes of “extremal” minimal bipartite matching covered graphs in terms of special trees. Our characterizations are reminiscent of similar characterization(s) of “extremal” minimal 2-connected graphs — where minimality is defined with respect to edge deletion; we briefly describe some of these results below before stating our characterizations.

Dirac [3] proved that a minimal 2-connected graph has at least $\frac{n+4}{3}$ vertices of degree two. A minimal 2-connected graph is *extremal* if it satisfies this lower bound with equality. Oxley [13] gave a generation theorem for extremal minimal 2-connected graphs. Karpov [8] then characterized this class of graphs in terms of special trees. In particular, they proved that a graph is extremal minimal 2-connected if and only if it can be obtained from two copies of a tree, each of whose non-leaves is of degree three, by identifying the corresponding leaves as per some fixed isomorphism; see Figure 2 for an example.

A tree is said to be a *Halin tree* if all of its non-leaves have degree at least three, or equivalently, if it has no vertices of degree two; see Figure 3 for an example. We call them halin trees because a *Halin graph* is a (planar) graph that is obtained from a planar embedding of such a tree by adding a cycle each of whose edges joins two leaves that appear consecutively in the cyclic order (as per the planar embedding). Halin graphs, first introduced

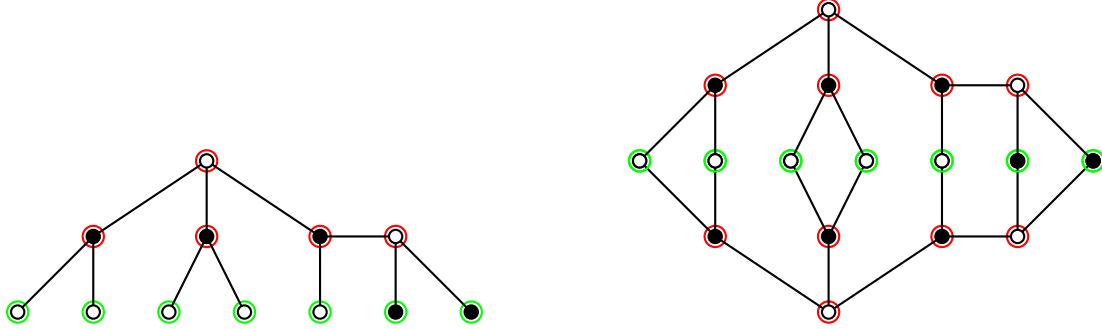


Figure 2: illustration of Karpov’s characterization of extremal minimal 2-connected graphs

by Halin [5], as examples of minimal 3-connected graphs, have been studied extensively in the literature. Halin trees are precisely the “homeomorphically irreducible” trees; finding such spanning trees in a cubic graph is a well-studied problem; see [7]. A Halin tree is *cubic* if the corresponding Halin graph is cubic. To put it differently, a tree is a cubic Halin tree if all of its non-leaves have degree exactly three; see Figure 2 for an example. Thus, the aforementioned Karpov’s characterization of extremal minimal 2-connected graphs draws a bijection between this graph class and cubic halin trees.

1.2 Characterizations of \mathcal{H}_2 , \mathcal{H}_3 and \mathcal{H}_4 using halin trees

We are now ready to state our characterizations of “extremal” minimal bipartite matching covered graphs — that are similar to Karpov’s characterization. However, in our case, identifying the corresponding leaves is not the correct operation.

For two disjoint copies of a Halin tree, say T and T' , let H be the (bipartite) graph obtained from $T \cup T'$ by adding a matching each of whose edges joins a leaf of T with the corresponding leaf of T' as per some fixed isomorphism between T and T' . We say that H is obtained from T and T' by *isomorphic leaf matching*, or simply, that H is obtained from T by isomorphic leaf matching.

Note that, since K_2 is the smallest Halin tree, C_4 is the smallest graph that may be obtained using the operation defined above. The following observation is easily proved and it implies that, for every other graph obtained using this operation, one may recover the two copies of the Halin tree by simply deleting the set of 2-edges.

Proposition 1.2. *If a graph H , distinct from C_4 , is obtained from a Halin tree T by isomorphic leaf matching, then $H - E_2$ has precisely two components, each of which is isomorphic to T .* \square

We first focus on the 2-vertex extremal class \mathcal{H}_2 (see Table 1) and obtain the following characterization establishing a bijection between 2-vertex extremal bipartite matching covered graphs and Halin trees.

Theorem 1.3. [MAIN THEOREM: CHARACTERIZATION OF \mathcal{H}_2]

A graph is a 2-vertex extremal minimal bipartite matching covered graph if and only if it is obtained from a Halin tree by isomorphic leaf matching.

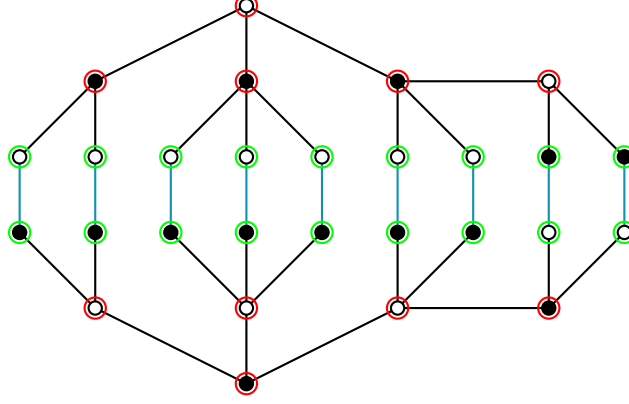


Figure 3: a member of \mathcal{H}_2 — deleting the 2-edges results in two copies of a Halin tree

Figure 3 shows an illustration of the above theorem. In this figure, and relevant figures henceforth, we adopt the following color conventions: the red vertices have degree three or more, the green vertices are of degree two, and the cyan edges are precisely the 2-edges.

Thereafter, we will use our Main Theorem (1.3) to derive characterizations of the extremal classes \mathcal{H}_3 and \mathcal{H}_4 as stated below; see Figure 4 for an example of each.

Theorem 1.4. [CHARACTERIZATION OF \mathcal{H}_3]

A graph is a 2-vertex n -extremal minimal bipartite matching covered graph if and only if it is obtained from a cubic Halin tree by isomorphic leaf matching.

By a *star*, we mean $K_{1,p}$ where $p \geq 2$. Observe that these, except $K_{1,2}$, comprise a restricted subclass of Halin trees; they play a crucial role in our characterization of \mathcal{H}_4 stated below.

Theorem 1.5. [CHARACTERIZATION OF \mathcal{H}_4]

A graph is an edge extremal minimal bipartite matching covered graph if and only if it is obtained from a star by isomorphic leaf matching.

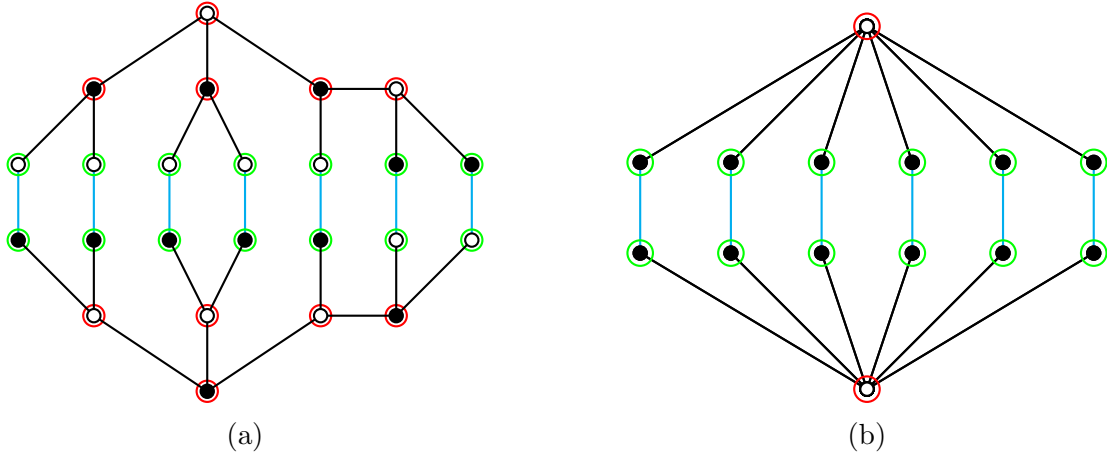


Figure 4: (a) a member of \mathcal{H}_3 : deleting the 2-edges results in two copies of a cubic Halin tree
(b) a member of \mathcal{H}_4 : deleting the 2-edges results in two copies of a star

In the next subsection, we prove the easier implications of the characterizations stated above (that is, Theorems 1.3, 1.4 and 1.5).

1.3 Proofs of easier implications

We begin by proving that any graph obtained from a nontrivial tree by isomorphic leaf matching is a minimal bipartite matching covered graph. We shall find ear decompositions useful to establish the matching covered property; however, for minimality, we need an easy observation pertaining to cuts.

Given a graph $G := (V, E)$, for any $W \subseteq V$, the *cut* comprising the edges joining W and its complement $\overline{W} := V - W$ is denoted by $\partial(W)$. A cut $\partial(W)$ is *trivial* if either W or \overline{W} comprises at most one vertex, and *nontrivial* otherwise. A k -*cut* refers to a cut of cardinality k . We use abbreviated notations $\partial(v) := \partial(\{v\})$ for a vertex v , and $\partial(H) := \partial(V(H))$ for a subgraph H .

An edge e of a matching covered graph G is *removable* if $G - e$ is also matching covered. Thus, a matching covered graph is minimal if and only if it has no removable edges. Since matching covered graphs are 2-connected, we immediately observe the following.

Lemma 1.6. *In a matching covered graph, any edge that participates in a 2-cut is not removable.* \square

The above lemma is the aforementioned observation that helps us in establishing minimality in the proof of the following result; we shall find it useful later as well.

Proposition 1.7. *Any graph obtained from a nontrivial tree by isomorphic leaf matching is a minimal bipartite matching covered graph.*

Proof. Let G be a graph obtained from two copies of a nontrivial tree, say T and T' , by isomorphic leaf matching. We will first prove, by induction on the order of T , that G is a bipartite matching covered graph. If T is a path then G is a even cycle, and we are done.

Now, let T be a tree that is not a path and let v denote a leaf. Let P be the maximal path starting at v , each of whose internal vertices has degree two in T . Let u be the end of P distinct from v . Observe that $d_T(u) \geq 3$. Let $v'P'u'$ be the path corresponding to vPu in T' . Let $H := G - V(P - u) - V(P' - u')$. Observe that H is obtained from the tree $T - V(P - u)$ by isomorphic leaf matching. By the induction hypothesis, H is a bipartite matching covered graph. Note that G may be obtained from H by adding the ear $uPvv'P'u'$. By the ear decomposition theorem (1.1), G is a bipartite matching covered graph.

Now, we prove minimality. Observe that any edge e of G , whose each end has degree at least three, either belongs to T or belongs to T' . Adjust notation so that $e \in T$. Let e' be the copy of e in T' . Note that $\{e, e'\}$ is a 2-cut of G . Thus, by Lemma 1.6, e is not removable. We thus infer that G is minimal. \square

We note that the converse of the above proposition does not hold. The graph G , shown in Figure 5, is a minimal bipartite matching covered graph that can not be obtained from a tree

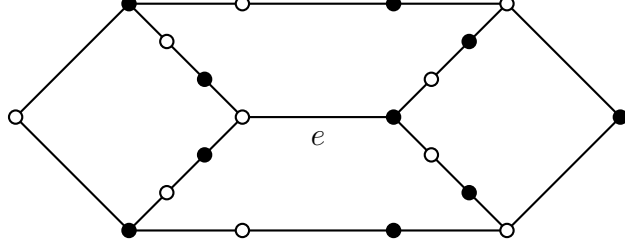


Figure 5: a counterexample to the converse of Proposition 1.7

by isomorphic leaf matching. To see this, suppose to the contrary that G is obtained from two copies of a tree, say T and T' , by isomorphic leaf matching. Then, as per the proof of Proposition 1.7, the labeled edge e belongs either to T or to T' as both ends of e have degree greater than two; furthermore, e belongs to a 2-cut of G . However, $G - e$ is 2-connected. This is a contradiction. We shall revisit this example in Section 4.1.

In light of Proposition 1.7, in order to prove the easier implications of Theorems 1.3, 1.4 and 1.5, we simply need to argue that if we choose a tree T from the corresponding class (as in that theorem's statement), then the resulting minimal bipartite matching covered graph G (obtained by isomorphic leaf matching) satisfies the desired extremality notion. We first note that if T is the star $K_{1,2}$ then $G = C_6$ is an edge extremal minimal bipartite matching covered graph. All of the remaining cases are handled by the following theorem — which we prove using straightforward counting arguments.

Theorem 1.8. *Any graph G obtained from a Halin tree T by isomorphic leaf matching is a minimal bipartite matching covered graph that is 2-vertex extremal. Furthermore:*

- (i) *if T is a cubic Halin tree then G is also 2-vertex n -extremal, whereas*
- (ii) *if T is a star then G is also edge extremal.*

Proof. Let $G := (V, E)$ be a graph obtained from two copies of a Halin tree, say T and T' , by isomorphic leaf matching. By Proposition 1.7, G is a minimal bipartite matching covered graph. It remains to prove “extremality”. We use V_2 and E_2 to denote $V_2(G)$ and $E_2(G)$, respectively.

Since T is a Halin tree, the non-leaves of T and T' comprise those vertices of G that have degree at least three, whereas the leaves of T and T' comprise the remaining vertices of G (that is, its vertices of degree two). In particular, there are no 2-edges in $T \cup T'$, and $T \cup T' = G - E_2$.

Now, $|E(T)| = |V(T)| - 1$. So, $|E(T \cup T')| = n - 2$. On the other hand, $|E(T \cup T')| = |E| - |E_2| = m - |E_2|$. Thus, $|E_2| = m - n + 2$. Since E_2 is a matching of G , and the corresponding matched vertices comprise the set V_2 , we get $|V_2| = 2|E_2| = 2(m - n + 2)$. Hence, G is 2-vertex extremal. It remains to prove statements (i) and (ii).

First, suppose that T is a cubic Halin tree. So, $T \cup T'$ has $|V_2|$ leaves and $n - |V_2|$ vertices of degree three. By handshaking lemma, $3(n - |V_2|) + |V_2| = 2|E(T \cup T')| = 2(n - 2)$. By simplifying, we get $|V_2| = \frac{n}{2} + 2$; whence G is 2-vertex n -extremal.

Now, suppose that T is a star $K_{1,p}$, where $p \geq 3$. Then, $n = 2(p + 1)$. Observe that $m = 3p = \frac{3n-6}{2}$. Ergo, G is edge extremal. \square

We now switch our attention to the remaining “extremal” classes \mathcal{H}_0 and \mathcal{H}_1 .

1.4 Characterizations of \mathcal{H}_0 and \mathcal{H}_1

In general, members of \mathcal{H}_0 and \mathcal{H}_1 may not be obtained from a tree by isomorphic leaf matching, unlike the members of $\mathcal{H}_2, \mathcal{H}_3$ and \mathcal{H}_4 . To see this, we discuss a couple of examples that are shown in Figure 6; using the ear decomposition theorem (1.1) and Lemma 1.6, the reader may easily verify that both graphs belong to \mathcal{H} . By counting, one may infer these graphs belong to \mathcal{H}_0 and \mathcal{H}_1 , respectively. For the graph shown in Figure 6a, deleting all of the 2-edges (displayed in blue) results in precisely two components that are trees but nonisomorphic. On the other hand, for the graph shown in Figure 6b, deleting all of the 2-edges results in two isomorphic trees; however, the resultant graph is not obtained by isomorphic leaf matching operation. Interestingly, we are able to reduce members of \mathcal{H}_0 and \mathcal{H}_1 to members of \mathcal{H}_2 and \mathcal{H}_4 , respectively, using the notion of “partial retract” that we describe next.

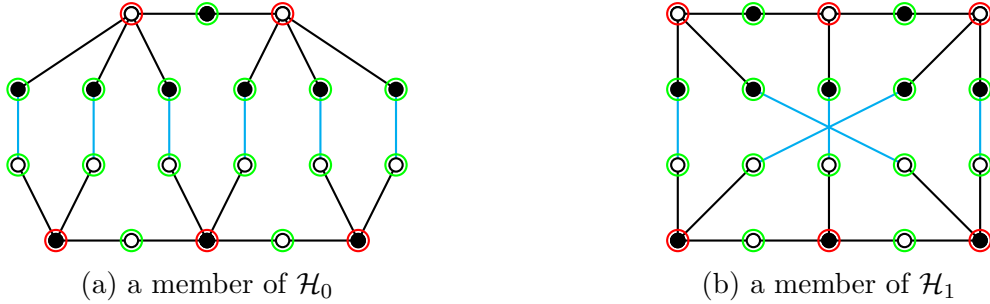


Figure 6: extremal graphs that may not be obtained from a tree by isomorphic leaf matching

Let G be a graph and v be a vertex of degree two that has two distinct neighbours, say v_1 and v_2 . Let G' denote the graph obtained from G by contracting the two edges vv_1 and vv_2 . We say that G' is obtained from G by *bicontraction* (of v), and we denote it by $G' := G/v$; see Figure 15. Furthermore, if each of v_1 and v_2 has degree at least three in G , we say that G' is obtained from G by *restricted bicontraction* (of v); see Figure 16. Note that restricted bicontraction may be performed if and only if the subgraph of G induced by its degree two vertices has an isolated vertex.

We define the *partial retract* of G as the graph \hat{G} obtained by repeatedly applying the restricted bicontraction operation until the resulting graph has the property that the subgraph induced by its degree two vertices has no isolated vertex. Observe that given a graph G , its partial retract \hat{G} is unique; Figure 7 shows an example.

Using the notion of partial retract, we are able to relate the extremal class \mathcal{H}_0 with the extremal class \mathcal{H}_2 , as stated below.

Theorem 1.9. [CHARACTERIZATION OF \mathcal{H}_0]

A graph G , distinct from C_4 , is a 2-edge extremal minimal bipartite matching covered graph if and only if its partial retract \hat{G} is a 2-vertex extremal minimal bipartite matching covered graph.

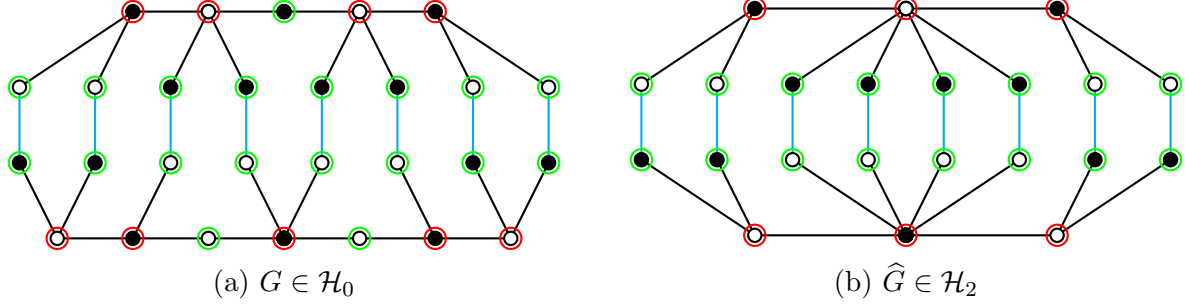


Figure 7: an illustration of Theorem 1.9 that relates \mathcal{H}_0 and \mathcal{H}_2

To see an example of the above theorem, consider the graph G shown in Figure 7a and its partial retract \hat{G} shown in Figure 7b. The reader may verify using Theorem 1.1 and Lemma 1.6 that G and \hat{G} belong to \mathcal{H} . Note that $G \in \mathcal{H}_0$ as $|E_2(G)| = m - n + 2$. On the other hand, $\hat{G} \in \mathcal{H}_2$ as $|V_2(\hat{G})| = 2(m - n + 2)$.

Likewise, using the notion of partial retract, we are able to relate the extremal class \mathcal{H}_1 with the extremal class \mathcal{H}_4 , as stated below.

Theorem 1.10. [CHARACTERIZATION OF \mathcal{H}_1]

A graph G is a 2-edge n -extremal minimal bipartite matching covered graph if and only if its partial retract \hat{G} is an edge extremal minimal bipartite matching covered graph and $\Delta(G) = 3$.



Figure 8: an illustration of Theorem 1.10 that relates \mathcal{H}_1 and \mathcal{H}_4

To see an example of the above theorem, consider the graph G shown in Figure 8a and its partial retract \hat{G} shown in Figure 8b. The reader may verify using Theorem 1.1 and Lemma 1.6 that G and \hat{G} belong to \mathcal{H} . Note that $G \in \mathcal{H}_1$ as $|E_2(G)| = \frac{n+10}{6}$. On the other hand, $\hat{G} \in \mathcal{H}_4$ as $|E(\hat{G})| = \frac{3n-6}{2}$.

We have thus stated our characterizations for all of the extremal classes shown in Table 1. We now proceed to discuss further relations between them.

1.5 Containment poset for the extremal classes

In order to discuss the containment relationships between different extremal classes, we shall find it convenient to exclude the cycle graphs. Following Lucchesi and Murty [12], for an element t and a set S , we use $S - t$ to denote the set obtained from S by deleting t . It follows from the definitions of the extremal classes (see Table 1) that neither of \mathcal{H}_0 and \mathcal{H}_1 contains any cycle graph, that C_4 is the only cycle in \mathcal{H}_2 as well as in \mathcal{H}_3 , whereas C_6 is the only cycle in \mathcal{H}_4 . In light of this, we let $\mathcal{H}_2^* := \mathcal{H}_2 - C_4$, $\mathcal{H}_3^* := \mathcal{H}_3 - C_4$ and $\mathcal{H}_4^* := \mathcal{H}_4 - C_6$.

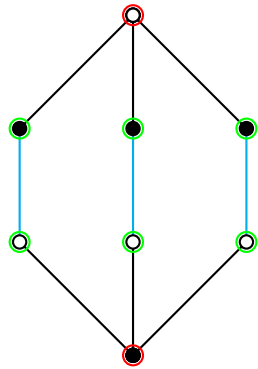


Figure 9: The Θ graph

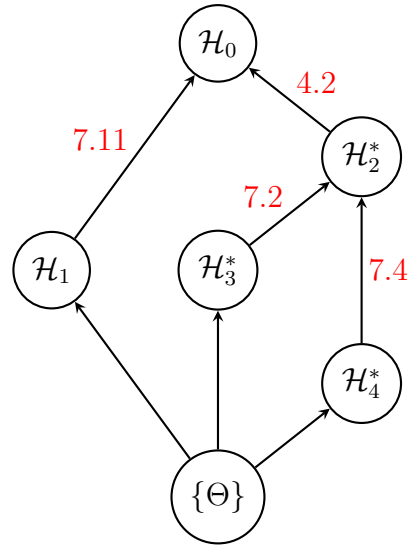


Figure 10: Containment poset

Figure 10 shows the containment poset for all of the extremal classes. For instance, the arrow from \mathcal{H}_1 to \mathcal{H}_0 , labelled as 7.11, is to be read as follows: Corollary 7.11 states that $\mathcal{H}_1 \subset \mathcal{H}_0$. Additionally, our characterizations imply that $\mathcal{H}_1 \cap \mathcal{H}_2 = \mathcal{H}_3 \cap \mathcal{H}_4 = \{\Theta\}$ where Θ is the graph shown in Figure 9; see Corollaries 7.5 and 7.12.

1.6 Organization of the rest of this paper

In the rest of the paper (except the last section), we will prove the difficult directions of Theorems 1.3, 1.4 and 1.5, and we will also prove Theorems 1.9 and 1.10.

We first focus on the class \mathcal{H}_2 . In order to prove the Main Theorem (1.3), we need some known properties of bipartite matching covered graphs and minimal bipartite matching covered graphs; these are discussed in Sections 2 and 3. In particular, in Section 3, we state and prove the lower bound on the number of 2-edges due to Lovász and Plummer [11], and we deduce an easy lower bound on the number of degree two vertices. In Section 4, we enforce equality in its proof to infer some useful properties of members of \mathcal{H}_2 — one of which is the existence of “balanced” 2-cuts. We then develop an induction tool using balanced 2-cuts in Section 5. In Section 6, we prove the Main Theorem (1.3).

In Section 7, we address the other notions of extremality. In Subsections 7.1 and 7.2, we first state and prove the bounds corresponding to the classes \mathcal{H}_3 and \mathcal{H}_4 , respectively; thereafter,

we enforce equality in their proofs to deduce their characterizations (Theorems 1.4 and 1.5) using the Main Theorem (1.3). In Subsection 7.3, we prove our characterization of \mathcal{H}_0 (Theorem 1.9). Lastly, in Subsection 7.4, we prove our characterization of \mathcal{H}_1 (Theorem 1.10) using the characterization of \mathcal{H}_0 .

Finally, in Section 8, we consider problems similar to those solved by Corollaries 3.5, 7.1 and 7.3 in the context of “minimal k -extendable” bipartite graphs. We first discuss the bounds established by Lou [10] and then proceed to conjecture stronger bounds; see Conjectures 8.9, 8.10 and 8.11. In the rest of that section, we provide evidence for our conjectures by constructing tight examples that are straightforward generalizations of the ones that appear in the 1-extendable case.

2 Ear decomposition theory

Recall that a subgraph H of a graph G is conformal if $G - H$ is matchable. It is easy to see that, in a matching covered graph, there is a conformal cycle containing any two adjacent edges. As an immediate consequence, we get the following.

Proposition 2.1. *Let G be a matching covered graph, let u be a vertex of degree at least three and let $e \in \partial(u)$. Then there is a conformal cycle C such that $u \in C$ but $e \notin C$. \square*

In fact, Little [9] proved the stronger statement that any two edges (not necessarily adjacent) lie in a conformal cycle; we will not require this.

Recall that an ear of a subgraph H in a graph G is an odd path of G whose ends are in H but is otherwise disjoint from H . We will now discuss the proof of the Ear Decomposition Theorem (1.1) as it appears in Lovász and Plummer [11, Theorem 4.1.6]. In particular, we extract a couple of lemma statements that are implicit in their proof. The first of these shows that “adding an ear” preserves the matching covered property. For a subgraph H and an ear P of H , the subgraph $H \cup P$ is said to be obtained by adding the ear P to H .

Lemma 2.2. *Let G be a bipartite graph, H be a matching covered subgraph and P be an ear of H in G . Then $H \cup P$ is matching covered.*

The second lemma establishes the existence of an ear of any conformal matching covered proper subgraph H with the additional constraint that any specified edge “sticking out of H ” is included in the ear.

Lemma 2.3. *Let G be a bipartite matching covered graph, H be any conformal matching covered subgraph and $e \notin H$ denote an edge that has at least one end in H . Then there exists an ear of H , say P_e , containing e such that $H \cup P_e$ is a conformal matching covered subgraph of G .*

The reader may verify that the above two lemmas imply the following stronger version of the Ear Decomposition Theorem (1.1) that is also stated in Lovász and Plummer [11, pg 126].

Theorem 2.4. *For any conformal matching covered subgraph H of a bipartite matching*

covered graph G , there exists an ear decomposition of G in which H appears as one of the subgraphs. \square

In the next section, we discuss the proofs of the lower bounds on the number of 2-edges due to Lovász and Plummer [11, Theorem 4.2.7]. These proofs rely heavily on the ear decomposition theory, and we will need some of their details to prove our results.

3 Lower bounds established by Lovász and Plummer

Recall that, for a matching covered graph G , an edge e is removable if $G - e$ is matching covered. Thus, G is minimal if and only if it has no removable edges. It follows from Theorem 1.1 that, given any ear decomposition of a bipartite matching covered graph G , every trivial ear (that is, an ear comprising a single edge) is a removable edge in G . This observation, coupled with Theorem 2.4, yields the following; see Lovász and Plummer [11, Theorem 4.2.1].

Lemma 3.1. *Every conformal matching covered subgraph of a minimal bipartite matching covered graph is an induced subgraph.* \square

In order to establish their lower bound on the number of 2-edges in a minimal bipartite matching covered graph, Lovász and Plummer proved a lower bound for each element (the starting conformal cycle and each ear) of an ear sequence. The following is their lemma for ears.

Lemma 3.2. *Let G be a minimal bipartite matching covered graph and let $(C, P_1, P_2, \dots, P_r)$ be any ear sequence. Then each ear P_i contains a 2-edge of G .*

Note that the above lemma (see Lovász and Plummer [11, Lemma 4.2.4]) refers to degrees in the original graph G . Adding a nontrivial ear P_i clearly creates a 2-edge in the resultant subgraph. However, one may add ears later thereby increasing the degrees of internal vertices of P_i and possibly destroying all 2-edges of P_i in the final graph G . The above lemma states that at least one such 2-edge survives in G . We now switch our attention to conformal cycles.

Let G be a minimal bipartite matching covered graph that is not a cycle. Using Lemma 3.2 and some standard ear decomposition tricks, one may infer that each conformal cycle C of G contains a pair of 2-edges that are separated in C by vertices that have degree at least three (in G). Interestingly, the same holds even for cycles that are not conformal; see Lovász and Plummer [11, Corollary 4.2.5].

A matching M is *induced* if the subgraph induced by $V(M)$ contains precisely the edges of M . The facts stated in the above paragraph, along with Lemma 3.1, imply the following (that is easily verified in the case in which the graph itself is a cycle of order six or more).

Corollary 3.3. *In a minimal bipartite matching covered graph, distinct from C_4 , each conformal cycle contains a pair of 2-edges that comprise an induced matching.*

We now invoke Lemma 3.2 and Corollary 3.3, and the fact that the length of any ear sequence is $m - n + 1$, to deduce the following lower bound result that is a mild strengthening

of what is stated in Lovász and Plummer; see [11, Theorem 4.2.7].

Theorem 3.4. *A minimal bipartite matching covered graph, distinct from C_4 , has an induced matching of size at least $m - n + 2$, each of whose members is a 2-edge.*

Proof. Let G be a minimal bipartite matching covered graph and (C, P_1, \dots, P_r) be any ear sequence. By Corollary 3.3, the conformal cycle C has a pair of 2-edges, say e and e' , that comprise an induced matching of G . By Lemma 3.2, each ear P_i has a 2-edge, say e_i . Observe that each end of e and e' has both its neighbours in C . Likewise, each end of e_i has both its neighbours in P_i . Furthermore, the ends of each ear P_i have degree at least three in G . All of these observations imply that the set $F := \{e, e', e_1, e_2, \dots, e_r\}$ is an induced matching of G and each of its members is a 2-edge. Also, $r = m - n$. Thus, $|F| = m - n + 2$. \square

The following lower bound on the number of vertices of degree two comes for free.

Corollary 3.5. *In a minimal bipartite matching covered graph, $|V_2| \geq 2(m - n + 2)$.*

Proof. The statement clearly holds for C_4 . Now, let G be a minimal bipartite matching covered graph, distinct from C_4 , and let F denote a maximum induced matching, each of whose members is a 2-edge. Clearly, $|V_2| \geq 2|F|$. By Theorem 3.4, $|F| \geq m - n + 2$. Thus, $|V_2| \geq 2(m - n + 2)$. \square

Throughout the next three sections, for the sake of brevity, we shall use the term *extremal* to refer to 2-vertex extremal — that is, those minimal bipartite matching covered graphs that satisfy the bound in Corollary 3.5 with equality.

4 Properties of extremal graphs

In this section, we establish some useful properties of this class including the “balanced 2-cut” property that is crucial for our inductive proof of Theorem 1.3 that characterizes this class. To this end, we shall enforce equality in the proofs of Corollary 3.5 and Theorem 3.4.

Lemma 4.1. *Every extremal minimal bipartite matching covered graph, distinct from C_4 , satisfies the following:*

- (i) E_2 is a perfect matching of $G[V_2]$ and $|E_2| = m - n + 2$,
- (ii) each ear in any ear decomposition has exactly one 2-edge, and
- (iii) each conformal cycle has precisely two 2-edges.

Proof. Let G be an extremal minimal bipartite matching covered graph, distinct from C_4 , and let F denote an induced matching of maximum cardinality, each of whose members is a 2-edge, as in the proof of Corollary 3.5. Since, by definition, $|V_2| = 2(m - n + 2)$, equality holds everywhere in the proof of Corollary 3.5. In particular, $|F| = m - n + 2$ and $|V_2| = 2|F|$; thus $V_2 = V(F)$. Observe that, since F is an induced matching, $E_2 = F$. Consequently, E_2 is a perfect matching of $G[V_2]$ and $|E_2| = m - n + 2$. This proves (i).

Now, let (C, P_1, \dots, P_r) be any ear sequence of G . Since $|F| = m - n + 2$, equality holds everywhere in the proof of Theorem 3.4. Furthermore, using the fact that $|E_2| = m - n + 2$, we infer that the conformal cycle C contributes exactly two 2-edges, and each ear P_i contributes precisely one 2-edge, to the set E_2 . This proves (ii) and (iii). \square

Using Lemma 4.1 (i), we immediately deduce the following containment.

Corollary 4.2. $\mathcal{H}_2 - C_4 \subset \mathcal{H}_0$. \square

We now derive, using the above lemma and simple counting arguments, a few additional properties of this extremal class. In order to do so, we need some notation and terminology that we define next.

Following Lovász and Plummer [11], we call an edge (of a graph G) a *3-edge* if each of its ends has degree at least three, and we use $E_3(G)$ to denote the set comprising these edges. In the same spirit, we use $V_3(G)$ to denote the set of vertices of degree at least three. Technically, it should be denoted by $V_{\geq 3}(G)$. However, throughout this paper, we distinguish between vertices of degree two and the rest — that is, vertices of degree three or more. So, for the sake of brevity, we drop “ \geq ” from the notation. Finally, we use $E_{3,2}(G)$ to denote the set of those edges that join a vertex of degree two with a vertex of degree at least three. As usual, if the graph G is clear from the context, we shall drop G from these notations. Note that $E_{3,2} = \partial(V_3) = \partial(V_2)$.

Now, let G be an extremal minimal bipartite matching covered graph distinct from C_4 . By definition, $|V_3| = n - |V_2| = 3n - 2m - 4$. By Lemma 4.1, each vertex of degree two has exactly one neighbour of degree at least three; consequently, $|E_{3,2}| = |V_2| = 2(m - n + 2)$. Lastly, $|E_3| = m - |E_2| - |E_{3,2}| = 3n - 2m - 6$. Also, since $|V_3| = 3n - 2m - 4$, it follows that $|E_3| = |V_3| - 2$. This proves the following.

Corollary 4.3. *Every extremal minimal bipartite matching covered graph, distinct from C_4 , satisfies the following:*

- (i) $|V_3| = 3n - 2m - 4$,
- (ii) $|E_{3,2}| = 2m - 2n + 4$, and
- (iii) $|E_3| = 3n - 2m - 6 = |V_3| - 2$. \square

We shall now use Lemma 4.1 to prove the “balanced 2-cut” property of extremal minimal bipartite matching covered graphs that was alluded to at the beginning of this section.

4.1 Balanced 2-cut property

For a cut $\partial(W)$ of a bipartite graph $G[A, B]$, we classify the edges of $\partial(W)$ into two types, depending upon the color classes of their ends in W . We use $\partial^A(W)$ to denote those edges of $\partial(W)$ whose end in W belongs to A . The set $\partial^B(W)$ is defined analogously. The cut $\partial(W)$ is *balanced* if $|\partial^A(W)| = |\partial^B(W)|$. In our work, balanced 2-cuts play a crucial role. Note that a 2-cut $\partial(W)$ is balanced if $|\partial^A(W)| = |\partial^B(W)| = 1$. We are now ready to prove the balanced 2-cut property stated below.

Theorem 4.4. [BALANCED 2-CUT PROPERTY]

In an extremal minimal bipartite matching covered graph, each 3-edge participates in a balanced 2-cut with some other 3-edge.

Proof. Let $G[A, B]$ denote an extremal minimal bipartite matching covered graph and let $e := ab$ denote any 3-edge so that $a \in A$. We intend to show the existence of another 3-edge, say f , such that $\{e, f\}$ is a balanced 2-cut.

By Proposition 2.1, there is a conformal cycle C such that $a \in C$ and $e \notin C$. Now, let H be a maximal conformal matching covered subgraph of G such that $a \in H$ and $e \notin H$. By Lemma 3.1, H is induced; thus $b \notin H$. In particular, $e \in \partial^A(H)$. We shall demonstrate that $\partial(H)$ is the desired balanced 2-cut. In order to do so, we state and prove specific claims; within their proofs, we shall invoke ear decomposition results (namely, Lemma 2.3 and Theorem 2.4) several times without mentioning it explicitly.

4.4.1. $\partial^A(H) = \{e\}$.

Proof. Suppose not; then there is an edge $f \neq e$ in $\partial^A(H)$. Now, there is an ear P_f of H containing f . Observe that $e \notin P_f$ since P_f is odd. Thus, $H' := H \cup P_f$ is a larger conformal matching covered subgraph of G such that $a \in H'$ and $e \notin H'$; this contradicts the maximality of H . This proves that $\partial^A(H) = \{e\}$. \square

In the proofs of the next two claims, every instance of 2-edge refers to the condition of being a 2-edge in the graph G similar to our earlier discussion (after the proof of Lemma 3.2).

4.4.2. $|\partial^B(H)| = 1$.

Proof. Since G is 2-connected, $|\partial^B(H)| \geq 1$. Now suppose that f_1 and f_2 are distinct edges in $\partial^B(H)$ and let b_1 and b_2 denote their (not necessarily distinct) ends in H , respectively. Now, there is an ear P' of H containing f_1 (which must also contain e). Furthermore, there is an ear P_2 of $H \cup P'$ containing f_2 . Let $x \in A$ be the end of P_2 distinct from b_2 (see Figure 11a). Observe that $x \notin H$ since $\partial^A(H) = \{e\}$ and $e \in P'$. Thus, x is an internal vertex of P' . Consequently, x splits P' into two paths, say $P := aP'x$ and $P_1 := xP'b_1$. Note that P_2 is an ear of $H \cup (P + P_1) = H \cup P'$; likewise, P_1 is an ear of $H \cup (P + P_2)$. By Lemma 4.1 (ii), each of P' , P_1 and P_2 has exactly one 2-edge. Since P_1 is a subpath of P' , we conclude that P is free of 2-edges. Furthermore, by Lemma 4.1 (i), each vertex of P has degree at least three in G .

Now, let y be the neighbour of x in P . As discussed above, $d_G(y) \geq 3$. So, there is an ear P_y of $H \cup (P + P_1) \cup P_2$ starting from y . Let z be the other end of P_y . Observe that $z \notin H$ as $\partial^A(H) = \{e\}$.

Now, suppose that $z \in P$ (see Figure 11b). Note that the conformal matching covered subgraph $H \cup (P + P_1) \cup P_y$ can also be obtained from H by adding the ears $aPzP_yxP_1b_1$ and yPz in that order. However, yPz has no 2-edges since, as noted earlier, each vertex of P has degree at least three in G . This contradicts Lemma 4.1 (ii). Thus, $z \notin P$.

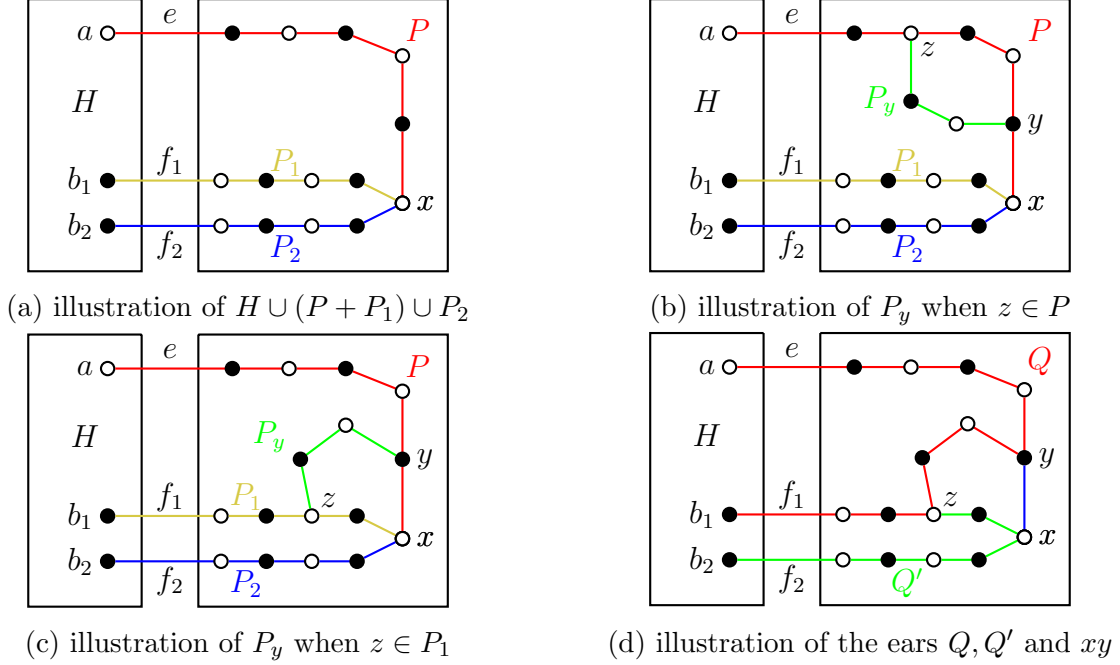


Figure 11: illustrations for the proof of statement 4.4.2

Thus, $z \in P_1 \cup P_2$. Adjust notation so that $z \in P_1$ (see Figure 11c). Let $Q := aPyP_yzP_1b_1$ and $Q' := zP_1xP_2b_2$ (see Figure 11d). Observe that the conformal matching covered subgraph $H \cup (P + P_1) \cup P_2 \cup P_y$ can also be obtained from H by adding the ears Q, Q' and xy in that order. In particular, $H \cup Q \cup Q'$ is a conformal matching covered subgraph of G that is not induced. This contradicts Lemma 3.1. Thus, $|\partial^B(H)| = 1$. \square

Let f denote the unique edge of $\partial^B(H)$. Note that $\partial(H) = \{e, f\}$ is indeed a balanced 2-cut. It remains to prove the following claim.

4.4.3. f is a 3-edge.

Proof. We let $f := uv$ where $v \in V(H) \cap B$. As H is matching covered, $d_H(v) \geq 2$; thus $d_G(v) \geq 3$. Suppose to the contrary that $d_G(u) = 2$. By Lemma 4.1 (i), the neighbour of u distinct from v , say w , has degree two in G . Let P be an ear of H containing f . Observe that $uw, e \in P$. By Lemma 4.1 (ii), uw is the only 2-edge of P . Furthermore, by Lemma 4.1 (i), each internal vertex of P , distinct from u and w , has degree at least three in G .

Since $d_G(b) \geq 3$, there is an ear P' of $H \cup P$ starting from b (see Figure 12a). Let $x \in A$ be the other end of P' . Observe that x is an internal vertex of P that is distinct from u . Let $Q := abP'xPv$ and $Q' := bPx$ (see Figure 12b). Observe that the conformal matching covered subgraph $H \cup P \cup P'$ can also be obtained from H by adding the ears Q and Q' in that order. It follows from the preceding paragraph that each vertex of Q' has degree at least three in G . Consequently, the ear Q' has no 2-edges. This contradicts Lemma 4.1 (ii). Thus, $d_G(u) \geq 3$ and f is indeed a 3-edge. \square

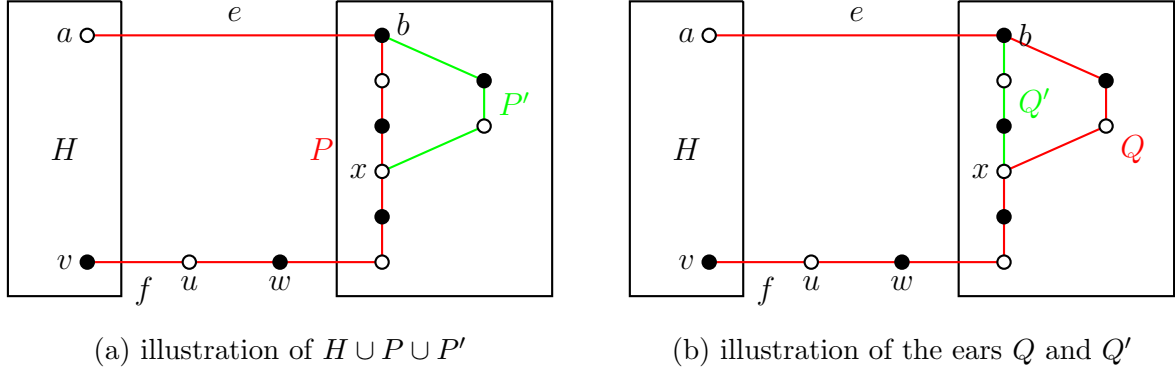


Figure 12: illustrations for the proof of statement 4.4.3

This completes the proof of Theorem 4.4. \square

Note that Theorem 4.4 does not hold for all minimal bipartite matching covered graphs — that is, it requires the extremality hypothesis. As we noted in Section 1, the graph G shown in Figure 5 is a minimal bipartite matching covered graph. Observe that e is a 3-edge that is not contained in any 2-cut since $G - e$ is 2-connected.

In the next section, we establish an induction tool based on balanced 2-cuts. This tool, coupled with Theorem 4.4, will be used in Section 6 to prove the Main Theorem (1.3).

5 An induction tool using balanced 2-cuts

We begin this section by defining an operation that “breaks” a given bipartite graph, that has a balanced 2-cut, into “smaller” bipartite graphs, and its converse operation.

Let $G[A, B]$ denote a bipartite graph that has a balanced 2-cut, say $F := \partial(X)$. Let $H_1 := G[X]$ and $H_2 := G[\overline{X}]$. As F is balanced, for each $i \in \{1, 2\}$, let $a_i \in A$ and $b_i \in B$ denote the ends of the edges of F in H_i , as shown in Figure 13a. Let G_1 be the graph obtained from H_1 by adding the ear $a_1v_1u_1b_1$ (of length three), as shown in Figure 13b. The graph G_2 is obtained from H_2 analogously. We say that the (bipartite) graphs G_1 and G_2 are obtained from G by a *balanced 2-cut decomposition*, or simply by a *2-cut decomposition*, across the 2-cut F . Note that u_1v_1 is a 2-edge in G_1 ; likewise, u_2v_2 is a 2-edge in G_2 .

Conversely, let G_1 and G_2 be disjoint bipartite graphs, each of which has a 2-edge, say u_1v_1 and u_2v_2 , respectively. Let a_1 denote the neighbour of v_1 that is distinct from u_1 , and let b_1 denote the neighbour of u_1 that is distinct from v_1 , as shown in Figure 13b. The vertices a_2 and b_2 are defined analogously. Let G be the graph obtained from $G_1 \cup G_2$ by deleting the vertices u_1, v_1, u_2 and v_2 , and adding the edges a_1b_2 and a_2b_1 , as shown in Figure 13a. We say that the (bipartite) graph G is obtained from G_1 and G_2 by *2-edge splicing* across the pair of 2-edges $\{u_1v_1, u_2v_2\}$. Note that $\{a_1b_2, a_2b_1\}$ is a balanced 2-cut of G .

Observe that the 2-cut decomposition operation is the “inverse” of the 2-edge splicing operation — that is, G_1 and G_2 are obtained from G by 2-cut decomposition across a

balanced 2-cut if and only if G is obtained from G_1 and G_2 by 2-edge splicing across a pair of 2-edges. We now show that these operations preserve the matching covered, minimality and extremality properties.

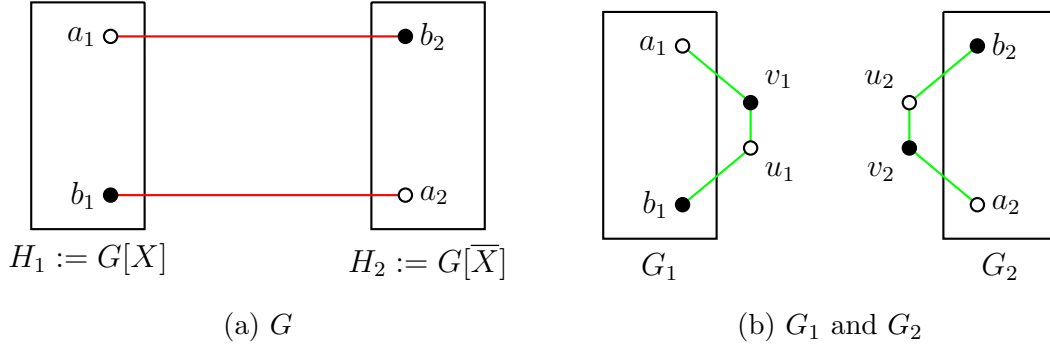


Figure 13: illustration of balanced 2-cut decomposition and 2-edge splicing

Theorem 5.1. [BALANCED 2-CUT INDUCTION TOOL]

Let G be a bipartite graph that has a balanced 2-cut F , and let G_1 and G_2 be the (bipartite) graphs obtained from G by a 2-cut decomposition across F . Then the following statements hold:

- (i) G is matching covered if and only if G_1 and G_2 are both matching covered;
- (ii) furthermore, G is minimal if and only if G_1 and G_2 are both minimal; and
- (iii) finally, G is extremal if and only if G_1 and G_2 are both extremal.

Proof. We adopt all of the notations from the above definition of 2-cut decomposition, as shown in Figure 13. We will prove the three statements one by one.

5.1.1. G is matching covered if and only if G_1 and G_2 are both matching covered.

Proof. First suppose that G is matching covered. Let C denote a conformal cycle containing a_1b_2 . Note that $a_2b_1 \in C$ as well. Let (C, P_1, \dots, P_r) be an ear sequence of G ; its existence is guaranteed by Theorem 2.4. Observe that each ear P_i is a subgraph of exactly one of H_1 and H_2 . Let $P_{11}, P_{12}, \dots, P_{1r_1}$ denote those ears that are subgraphs of H_1 and appear in that order in the ear sequence (C, P_1, \dots, P_r) . Let $C_1 := (C \cap H_1) + a_1v_1u_1b_1$. Observe that $(C_1, P_{11}, P_{12}, \dots, P_{1r_1})$ is an ear sequence of G_1 . Thus, by the Ear Decomposition Theorem (1.1), the bipartite graph G_1 is matching covered. An analogous argument proves that G_2 is matching covered.

Conversely, suppose that G_1 and G_2 are matching covered. Let C_i be a conformal cycle of G_i containing the 2-edge $u_i v_i$ for $i \in \{1, 2\}$. Let $(C_1, P_{11}, P_{12}, \dots, P_{1r_1})$ and $(C_2, P_{21}, P_{22}, \dots, P_{2r_2})$ be ear sequences of G_1 and G_2 , respectively; these exist due to Theorem 2.4. Let $C := (C_1 - u_1 - v_1) + (C_2 - u_2 - v_2) + a_1b_2 + a_2b_1$. Observe that $(C, P_{11}, P_{12}, \dots, P_{1r_1}, P_{21}, P_{22}, \dots, P_{2r_2})$ is an ear sequence of G . Thus, by the Ear Decomposition Theorem (1.1), G is matching covered. \square

Henceforth, we assume that G is matching covered, or equivalently that G_1 and G_2 are both matching covered.

5.1.2. *G is minimal if and only if G_1 and G_2 are both minimal.*

Proof. First suppose that G is minimal. We claim that G_1 is also minimal. Suppose not, and let e be a removable edge of G_1 . Note that $e \in H_1$ since each of the remaining three edges has an end of degree two. Observe that $G_1 - e$ and G_2 are obtained by a 2-cut decomposition of $G - e$. Consequently, by 5.1.1, the graph $G - e$ is matching covered; this contradicts the minimality of G . Thus, G_1 is minimal. Likewise, G_2 is minimal.

Conversely, suppose that G_1 and G_2 are both minimal. We claim that G is minimal. Suppose not, and let e be a removable edge of G . Note that, since F is 2-cut, e belongs to exactly one of H_1 and H_2 . Adjust notation so that $e \in H_1$. As in the preceding paragraph, $G_1 - e$ and G_2 are obtained by a 2-cut decomposition of $G - e$. Thus, by 5.1.1, the graph $G_1 - e$ is matching covered; this contradicts the minimality of G_1 . Hence, G is minimal. \square

Henceforth, we assume that G is a minimal bipartite matching covered graph, or equivalently that G_1 and G_2 are both minimal bipartite matching covered graphs.

5.1.3. *G is extremal if and only if G_1 and G_2 are both extremal.*

Proof. In order to argue extremality, we find it convenient to use $\lambda := |V_2(G)| - 2(m - n + 2)$ to denote the “surplus” vertices of degree two in G . Analogously, we define λ_1 and λ_2 for the graphs G_1 and G_2 , respectively. By Corollary 3.5, since all three graphs are minimal bipartite matching covered graphs, the quantities λ , λ_1 and λ_2 are non-negative. Note that G is extremal if and only if $\lambda = 0$. Likewise, for $i \in \{1, 2\}$, the graph G_i is extremal if and only if $\lambda_i = 0$. Thus, it suffices to prove that $\lambda = 0$ if and only if $\lambda_1 = 0$ and $\lambda_2 = 0$.

For $i \in \{1, 2\}$, we let $n_i := |V(G_i)|$ and $m_i := |E(G_i)|$. The reader may find it useful to look at Figure 13. Note that $n_1 + n_2 = n + 4$ and $m_1 + m_2 = m + 4$. Observe that each vertex w of G corresponds to a vertex of G_1 or of G_2 (that is distinct from u_1, v_1, u_2 and v_2) whose degree is the same as that of w . This implies that $|V_2(G_1)| + |V_2(G_2)| = |V_2(G)| + 4$. Using these three equalities, and the definitions of λ , λ_1 and λ_2 :

$$\begin{aligned} \lambda_1 + \lambda_2 &= |V_2(G_1)| + |V_2(G_2)| - 2(m_1 + m_2 - n_1 - n_2 + 4) \\ &= |V_2(G)| + 4 - 2(m - n + 4) \\ &= |V_2(G)| - 2(m - n + 2) = \lambda \end{aligned}$$

Thus, since $\lambda, \lambda_1, \lambda_2$ are non-negative, $\lambda = 0$ if and only if $\lambda_1 = 0$ as well as $\lambda_2 = 0$. \square

This completes the proof of Theorem 5.1. \square

6 Main Theorem: Characterization of \mathcal{H}_2

In this section, we prove our Main Theorem (1.3) that provides a characterization of extremal minimal bipartite matching covered graphs. Before doing so, we prove a lemma that will be used in the base case of our inductive proof.

Lemma 6.1. *Every extremal minimal bipartite matching covered graph, that has precisely two vertices of degree at least three, is obtained from a star by isomorphic leaf matching.*

Proof. Let G be an extremal minimal bipartite matching covered graph with precisely two vertices of degree at least three, say a and b . Let (C, P_1, \dots, P_r) be an ear sequence. Clearly, G is not a cycle; thus $r \geq 1$. Also, each end of each ear P_i has degree at least three in G . Thus, every ear is an ab -path. Consequently, $a, b \in C$. By Lemma 4.1 (ii), each P_i has length three. Furthermore, by Lemma 4.1 (iii), the vertices a and b split C into two paths — each of length three. In particular, C is a 6-cycle. Observe that G is indeed obtained from the star $K_{1,r+2}$ by isomorphic leaf matching. \square

We will now use the balanced 2-cut property (Theorem 4.4) and the balanced 2-cut induction tool (Theorem 5.1), along with the above lemma, to prove our characterization of the class of extremal minimal bipartite matching covered graphs.

Theorem 1.3. [MAIN THEOREM: CHARACTERIZATION OF \mathcal{H}_2]

A graph G belongs to \mathcal{H}_2 if and only if it is obtained from a Halin tree by isomorphic leaf matching.

Proof. The reverse implication is simply the main statement of Theorem 1.8 that we have already proved.

For the forward direction, we let $G[A, B]$ denote a member of \mathcal{H}_2 , and we proceed by induction on the order of G . First suppose that $|V_3| \leq 2$. Since $|A| = |B|$, it follows that $|V_3| \in \{0, 2\}$. If $|V_3| = 0$, then G is a cycle and $|V| = |V_2| = 2(m - n + 2) = 4$; thus G is C_4 . On the other hand, if $|V_3| = 2$, we invoke Lemma 6.1. In either case, we conclude that G is obtained from a Halin tree by isomorphic leaf matching.

Now, suppose that $|V_3| \geq 3$. By Corollary 4.3 (iii), we note that $|E_3| = |V_3| - 2 \geq 1$, and we let a_1b_2 denote a 3-edge. By Theorem 4.4, there exists another 3-edge b_1a_2 such that $\{a_1b_2, b_1a_2\}$ is a balanced 2-cut, say $\partial(X)$, where $\{a_1, b_1\} \subseteq X \subset V(G)$. Now, let G_1 and G_2 be the graphs obtained by 2-cut decomposition across $\partial(X) = \{a_1b_2, b_1a_2\}$. In particular, G_1 is obtained from $G[X]$ by adding the ear $a_1v_1u_1b_1$; likewise, G_2 is obtained from $G[\overline{X}]$ by adding the ear $a_2v_2u_2b_2$. See Figure 13 for an illustration.

By Theorem 5.1, each of G_1 and G_2 belongs to \mathcal{H}_2 . Since each of a_1, b_1, a_2 and b_2 has degree at least three in G , it follows that each of X and \overline{X} has at least four vertices. This implies that $6 \leq |V(G_i)| < |V(G)|$ for each $i \in \{1, 2\}$. Thus, by the induction hypothesis, each of G_1 and G_2 is obtained from a Halin tree by isomorphic leaf matching; also, neither of them is isomorphic to C_4 . By Proposition 1.2, for each $i \in \{1, 2\}$, the graph $G_i - E_2(G_i)$ has precisely

two components that are isomorphic Halin trees, say T_i and T'_i . We let $\phi_i : V(T_i) \mapsto V(T'_i)$ denote the corresponding isomorphism as per the isomorphic leaf matching used to obtain G_i .

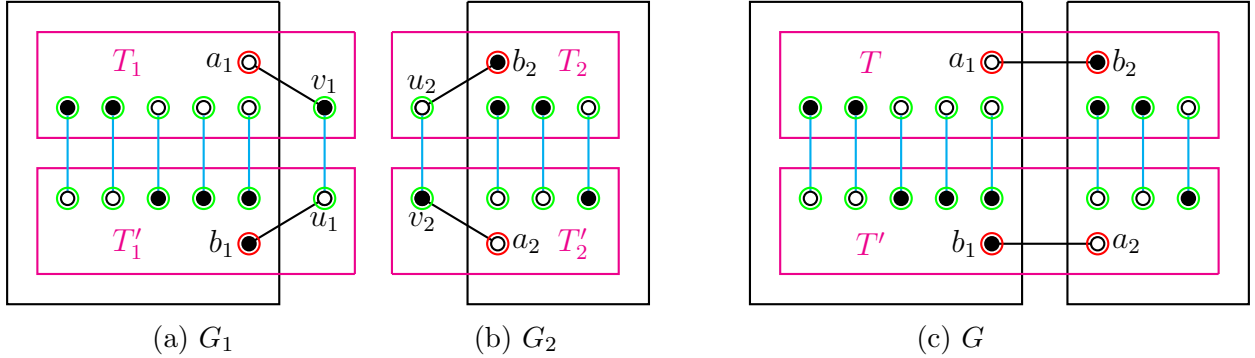


Figure 14: illustration for the proof of the Main Theorem

Since u_1v_1 is a 2-edge of G_1 , we may adjust notation so that $v_1, a_1 \in T_1$ and $u_1, b_1 \in T'_1$. See Figure 14. Likewise, we adjust notation so that $u_2, b_2 \in T_2$ and $v_2, a_2 \in T'_2$. Note that $\phi_1(v_1) = u_1$. Consequently, $\phi_1(a_1) = b_1$. Likewise, $\phi_2(b_2) = a_2$. We let $T := (T_1 - v_1) + (T_2 - u_2) + a_1b_2$ and $T' := (T'_1 - u_1) + (T'_2 - v_2) + a_2b_1$. Note that T and T' are Halin trees. Observe that $\phi : V(T) \mapsto V(T')$, as defined below, is an isomorphism between T and T' .

$$\phi(v) := \begin{cases} \phi_1(v) & \text{if } v \in V(T_1) \\ \phi_2(v) & \text{if } v \in V(T_2) \end{cases}$$

Finally, note that G is obtained from T and T' by isomorphic leaf matching as per the above defined isomorphism. This completes the proof of the Main Theorem. \square

7 Other notions of extremality

We now shift our focus to the remaining notions of extremality — that is, the extremal classes \mathcal{H}_3 , \mathcal{H}_4 , \mathcal{H}_0 and \mathcal{H}_1 — in that order.

7.1 Characterization of \mathcal{H}_3

We begin by deducing the lower bound on the number of degree two vertices in a minimal bipartite matching covered graph solely in terms of n , that was stated in Subsection 1.1, from the lower bound of Lovász and Plummer (Corollary 3.5).

Corollary 7.1. *In a minimal bipartite matching covered graph, $|V_2| \geq \frac{n}{2} + 2$.*

Proof. Using the fact that $n = |V_2| + |V_3|$, and the handshaking lemma:

$$m - n = \left(\sum_{v \in V_3} \frac{d(v)}{2} + |V_2| \right) - n \geq \left(\frac{3}{2}|V_3| + |V_2| \right) - n = \frac{1}{2}|V_3|$$

We invoke Corollary 3.5 to infer that $|V_2| \geq 2(m - n + 2) \geq |V_3| + 4 = n - |V_2| + 4$. Consequently, $|V_2| \geq \frac{n}{2} + 2$. \square

We are now ready to prove our characterization for \mathcal{H}_3 . To this end, we shall enforce equality in the proof of the above corollary.

Theorem 1.4. [CHARACTERIZATION OF \mathcal{H}_3]

A graph G belongs to \mathcal{H}_3 if and only if it is obtained from a cubic Halin tree by isomorphic leaf matching.

Proof. The reverse implication is simply statement (i) of Theorem 1.8 that we have already proved. It remains to prove the forward implication.

Let G be a member of \mathcal{H}_3 . That is, G is a minimal bipartite matching covered graph such that $|V_2| = \frac{n}{2} + 2$. Thus, G satisfies each inequality in the proof of Corollary 7.1 with equality. Firstly, $|V_2| = 2(m - n + 2)$; thus $G \in \mathcal{H}_2$. By the Main Theorem (1.3), G is obtained from a Halin tree T by isomorphic leaf matching. Secondly, each vertex in V_3 has degree precisely three in G . This implies that each non-leaf of T is cubic. Thus, T is a cubic Halin tree. \square

We end this section by noting the following proper containment that is established within the above proof.

Corollary 7.2. $\mathcal{H}_3 \subset \mathcal{H}_2$. \square

7.2 Characterization of \mathcal{H}_4

We will follow the same approach as in the previous subsection. We begin by deducing the upper bound $|E| \leq \frac{3n-6}{2}$ (see Lovász and Plummer [11, Theorem 4.2.3]), that was stated in Subsection 1.1, from the lower bound of Lovász and Plummer (Corollary 3.5).

Corollary 7.3. *A minimal bipartite matching covered graph, distinct from C_4 , has at most $\frac{3n-6}{2}$ edges.*

Proof. Let $G[A, B]$ denote a minimal bipartite matching covered graph distinct from C_4 . Note that $n \geq 6$ if and only if $n \leq \frac{3n-6}{2}$. Consequently, if G is a cycle, $m = n \leq \frac{3n-6}{2}$.

Now suppose that G is not a cycle; in other words, $|V_3| \geq 1$. Since $|A| = |B|$, we infer $|V_3| \geq 2$. Using Corollary 3.5 and the fact that $n = |V_2| + |V_3|$, we have $2(m - n + 2) \leq |V_2| = n - |V_3|$. Since $|V_3| \geq 2$, we get $2(m - n + 2) \leq n - 2$. On simplifying, we conclude $m \leq \frac{3n-6}{2}$. \square

We shall now enforce equality in the proof of the above corollary to prove our characterization of \mathcal{H}_4 .

Theorem 1.5. [CHARACTERIZATION OF \mathcal{H}_4]

A graph G belongs to \mathcal{H}_4 if and only if it is obtained from a star by isomorphic leaf matching.

Proof. The reverse implication is settled by statement (ii) of Theorem 1.8 for stars $K_{1,p}$ where $p \geq 3$, and the paragraph above it for $K_{1,2}$. It remains to prove the forward implication.

Let G be a member of \mathcal{H}_4 . That is, G is a minimal bipartite matching covered graph such that $|E| = \frac{3n-6}{2}$. If G is a cycle, then $n = 6$ and G is C_6 which is obtained from the star $K_{1,2}$ by isomorphic leaf matching.

Now, suppose that G is not a cycle. Consequently, G satisfies all of the inequalities in the last paragraph of the proof of Corollary 7.3 with equality. In particular, $|V_2| = 2(m - n + 2)$ and $|V_3| = 2$. In other words, G is a member of \mathcal{H}_2 that has precisely two vertices of degree at least three. Thus, by Lemma 6.1, we conclude that G is obtained from a star by isomorphic leaf matching. \square

We end this subsection with a couple of easy consequences. The first of them is the following proper containment that is already established within the above proof.

Corollary 7.4. $\mathcal{H}_4 - C_6 \subset \mathcal{H}_2$. \square

Note that Θ , shown in Figure 9, belongs to \mathcal{H}_3 as well as \mathcal{H}_4 . Conversely, let G be any graph in $\mathcal{H}_3 \cap \mathcal{H}_4$. Since $G \in \mathcal{H}_4$, by Theorem 1.5, G is obtained from a star $T := K_{1,p}$ by isomorphic leaf matching. On the other hand, since $G \in \mathcal{H}_3$, by Theorem 1.4, T is a cubic Halin tree. Thus $p = 3$; in other words, G is Θ . This proves the following.

Corollary 7.5. $\mathcal{H}_4 \cap \mathcal{H}_3 = \{\Theta\}$. \square

7.3 Characterization of \mathcal{H}_0

Recall the bicontraction and the restricted bicontraction operations as defined in Subsection 1.4. In this subsection, we will first show that restricted bicontraction preserves 2-edge extremality. Then, we will use this to deduce our characterization of \mathcal{H}_0 .

In order to achieve the above, we first define the inverse of restricted bicontraction as well as of bicontraction. Observe that the bicontraction operation (when applicable) results in a vertex of degree at least two, whereas the restricted bicontraction operation (when applicable) results in a vertex of degree at least four.

Let G' be a graph with a vertex v' of degree two or more. We partition the edges of $\partial(v')$ into two sets F_1 and F_2 each of size at least one. Let G be obtained from $G' - v'$ by (i) introducing three new vertices v, v_1 and v_2 , (ii) adding two edges vv_1 and vv_2 , and (iii) for each $i \in \{1, 2\}$, adding an edge joining u and v_i for each $uv \in F_i$. See Figure 15 for an illustration. We say that G is obtained from G' by *bisplitting* the vertex v' . Observe that, in G , the vertex v has degree two and each of its neighbours v_1 and v_2 has degree at least two.

Furthermore, if v' has degree at least four and each of the sets F_1 and F_2 has size at least two, we say that G is obtained from G' by *restricted bisplitting* of the vertex v' . In this case, note that the vertex v has degree two and each of its neighbours v_1 and v_2 has degree at least three. See Figure 16 for an illustration.

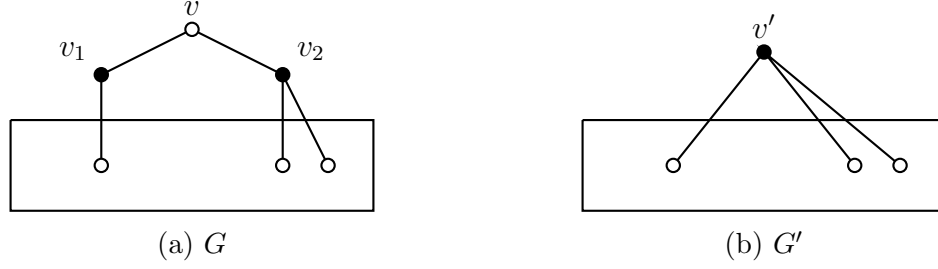


Figure 15: an illustration of bicontraction and bisplitting

Observe that G' is obtained from G by bicontraction if and only if G may be obtained from G' by bisplitting. Likewise, G' is obtained from G by restricted bicontraction if and only if G may be obtained from G' by restricted bisplitting. The following lemma pertaining to bicontraction/bisplitting is easily proved, and may also be deduced from [2, Propositions 2.1 and 2.3].

Lemma 7.6. [BICONTRACTION PRESERVES MATCHING COVERED PROPERTY]

Let G be a graph that has a vertex of degree two, say v , that has two distinct neighbours each of which has degree two or more, and let $G' := G/v$. Then G is matching covered if and only if G' is matching covered. \square

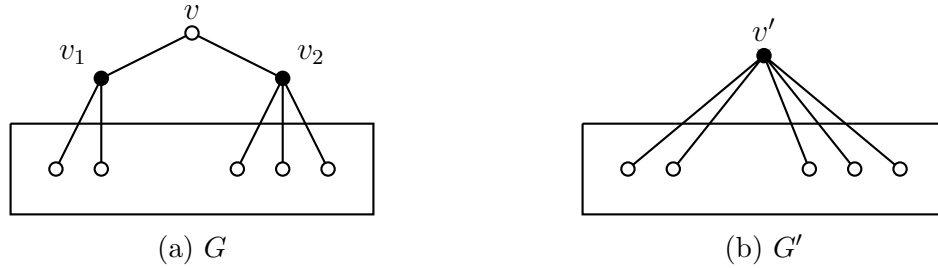


Figure 16: an illustration of restricted bicontraction and restricted bisplitting

However, bicontraction may not preserve minimality. For instance, the graph shown in Figure 3 belongs to \mathcal{H} , but any application of the bicontraction operation results in a graph that has a removable edge. On the other hand, the restricted bicontraction operation preserves minimality as well as 2-edge extremality, as proved below.

Theorem 7.7. [RESTRICTED BICONTRACTION PRESERVES 2-EDGE EXTREMALITY]

Let G be a bipartite graph that has a vertex of degree two, say v , each of whose neighbours has degree three or more, and let $G' := G/v$. Then the following statements hold:

- (i) G is matching covered if and only if G' is matching covered;
- (ii) furthermore, G is minimal if and only if G' is minimal; and
- (iii) finally, G belongs to \mathcal{H}_0 if and only if G' belongs to \mathcal{H}_0 .

Proof. We adopt notation from the definition of restricted bisplitting, as shown in Figure 16. Observe that (i) follows immediately from Lemma 7.6. Henceforth, we assume that G is

matching covered, or equivalently, that G' is matching covered. We will prove the other two statements one-by-one.

7.7.1. *G is minimal if and only if G' is minimal.*

Proof. First suppose that G' is not minimal, and let e be a removable edge. Note that $d_{G-e}(v_i) \geq d_G(v_i) - 1 \geq 2$ since the operation is a restricted bicontraction. Consequently, $G' - e$ is obtained from $G - e$ by bicontracting v . Thus, by Lemma 7.6, $G - e$ is matching covered. In other words, G is not minimal.

Conversely, suppose that G is not minimal, and let e be a removable edge. Note that $e \notin \partial(v)$ since $d(v) = 2$. Now, observe that $G' - e$ is obtained from $G - e$ by bicontracting v . Thus, by Lemma 7.6, $G' - e$ is matching covered. Consequently, G' is not minimal. \square

Henceforth, we assume that G is a minimal bipartite matching covered graph, or equivalently, that G' is a minimal bipartite matching covered graph. Recall that G belongs to \mathcal{H}_0 if $|E_2(G)| = m - n + 2$.

7.7.2. *G belongs to \mathcal{H}_0 if and only if G' belongs to \mathcal{H}_0 .*

Proof. Let $n' := |V(G')|$ and $m' := |E(G')|$. Note that $m' = m - 2$ and $n' = n - 2$. Thus, $m' - n' = m - n$. By definition of restricted bicontraction operation, $d_G(v_1) \geq 3$ and $d_G(v_2) \geq 3$; subsequently $d_{G'}(v') \geq 3$. These observations imply that $|E_2(G')| = |E_2(G)|$. \square

This completes the proof of Theorem 7.7 \square

Recall the definition of partial retract from Subsection 1.4. The following is an immediate consequence of the above theorem.

Corollary 7.8. *A graph G belongs to \mathcal{H}_0 if and only if its partial retract \widehat{G} belongs to \mathcal{H}_0 .* \square

We are now ready to prove our characterization of the extremal class \mathcal{H}_0 in terms of the extremal class \mathcal{H}_2 , as restated below.

Theorem 1.9. [CHARACTERIZATION OF \mathcal{H}_0]

A graph G , distinct from C_4 , belongs to \mathcal{H}_0 if and only if its partial retract \widehat{G} belongs to \mathcal{H}_2 .

Proof. First suppose that \widehat{G} belongs to \mathcal{H}_2 . Observe that either $\widehat{G} = G$ or \widehat{G} has a vertex of degree at least four. In either case, $\widehat{G} \neq C_4$. By the containment established in Corollary 4.2, we infer that $\widehat{G} \in \mathcal{H}_0$. Now, by Corollary 7.8, we conclude that $G \in \mathcal{H}_0$.

Now suppose that G belongs to \mathcal{H}_0 . We invoke Corollary 7.8 to infer that $\widehat{G} \in \mathcal{H}_0$. Thus, by definition, $|E_2(\widehat{G})| = \widehat{m} - \widehat{n} + 2$, where $\widehat{n} := |V(\widehat{G})|$ and $\widehat{m} := |E(\widehat{G})|$. By definition of partial retract, the subgraph of \widehat{G} induced by its degree two vertices has no isolated vertex. Applying the handshaking lemma to this subgraph gives us: $|V_2(\widehat{G})| \leq 2|E_2(\widehat{G})| = 2(\widehat{m} - \widehat{n} + 2)$. However, by the lower bound established in Corollary 3.5, $|V_2(\widehat{G})| \geq 2(\widehat{m} - \widehat{n} + 2)$. Thus, $|V_2(\widehat{G})| = 2(\widehat{m} - \widehat{n} + 2)$; by definition, $\widehat{G} \in \mathcal{H}_2$. \square

We now switch our attention to the only remaining extremal class.

7.4 Characterization of \mathcal{H}_1

We begin by deducing a lower bound on the number of 2-edges in a minimal bipartite matching covered graph solely in terms of n (see Lovász and Plummer [11, Lemma 4.2.4 (b)]), that was stated in Subsection 1.1, from the lower bound of Lovász and Plummer (Theorem 3.4).

Corollary 7.9. *In a minimal bipartite matching covered graph, $|E_2| \geq \frac{n+10}{6}$.*

Proof. Let G be a minimal bipartite matching covered graph. By Theorem 3.4:

$$|E_2| \geq m - n + 2 \quad (1)$$

Now, using the handshaking lemma, and the fact that $d(v) \geq 3$ for each $v \in V_3$, we have: $2m \geq 3|V_3| + 2|V_2| = 3n - |V_2|$. In particular:

$$2m \geq 3n - |V_2| \quad (2)$$

By counting the edges incident with vertices of degree two in two different ways, we have: $2|V_2| = 2|E_2| + |E_{23}| = |E_2| + m - |E_3|$. Now, using the fact that $|E_3| \geq 0$, we have:

$$|E_2| + m \geq 2|V_2| \quad (3)$$

Now, by scaling and adding the above inequalities as $(1) \times 5 + (2) \times 2 + (3) \times 1$, we arrive at the desired inequality: $6|E_2| \geq n + 10$. \square

Next, we invite the reader to make a simple observation pertaining to the restricted bicontraction operation.

Lemma 7.10. *If a graph G has a vertex of degree two, say v , each of whose neighbours has degree at least three, then $|E_3(G)| = |E_3(G/v)|$. Consequently, $|E_3(G)| = |E_3(\widehat{G})|$. \square*

We are now ready to prove our characterization of the extremal class \mathcal{H}_1 in terms of the extremal class \mathcal{H}_4 , as restated below.

Theorem 1.10. [CHARACTERIZATION OF \mathcal{H}_1]

A graph G belongs to \mathcal{H}_1 if and only if its partial retract \widehat{G} belongs to \mathcal{H}_4 and $\Delta(G) = 3$.

Proof. For the reverse direction, let G be a graph such that $\widehat{G} \in \mathcal{H}_4$ and $\Delta(G) = 3$. In what follows, we shall demonstrate that all of the inequalities that appear in the proof of Corollary 7.3 hold with equality. Firstly, by the characterization of \mathcal{H}_4 obtained in Theorem 1.5, we infer that $|E_3(\widehat{G})| = 0$. Thus, by Lemma 7.10, $|E_3(G)| = |E_3(\widehat{G})| = 0$. Secondly, by the containment established in Corollary 7.4, $\widehat{G} \in \mathcal{H}_2$. By Theorem 1.9, $G \in \mathcal{H}_0$. In particular, $|E_2(G)| = m - n + 2$. Lastly, as $\Delta(G) = 3$, clearly $d_G(v) = 3$ for each $v \in V_3$. The reader may thus verify that equality holds in all of the inequalities discussed in the proof of Corollary 7.9. Hence, $|E_2(G)| = \frac{n+10}{6}$ and $G \in \mathcal{H}_1$.

For the forward direction, let G be a graph in \mathcal{H}_1 . Consequently, G satisfies all of the inequalities that appear in the proof of Corollary 7.9 with equality. Firstly, $|E_2(G)| = m - n + 2$; thus $G \in \mathcal{H}_0$. Consequently, by Theorem 1.9, $\widehat{G} \in \mathcal{H}_2$. Secondly, $|E_3(G)| = 0$. Thus, by Lemma 7.10, $|E_3(\widehat{G})| = 0$. By Corollary 4.3 (iii), we deduce that $|V_3(\widehat{G})| = 2$. Hence, by Lemma 6.1 and Theorem 1.8 (ii), we conclude that $\widehat{G} \in \mathcal{H}_4$. Finally, $d_G(v) = 3$ for each $v \in V_3(G)$. Thus, $\Delta(G) \leq 3$. Note that if G is a cycle then $n = m = E_2 = \frac{n+10}{6}$ which implies that $n = 2$; a contradiction. Thus, G is not a cycle; whence $\Delta(G) = 3$. \square

We end this subsection, with a couple of easy consequences. The first of them is the following proper containment that is already established within the above proof.

Corollary 7.11. $\mathcal{H}_1 \subset \mathcal{H}_0$. \square

Note that Θ , shown in Figure 9, belongs to \mathcal{H}_1 as well as \mathcal{H}_2 . Conversely, let $G \in \mathcal{H}_1 \cap \mathcal{H}_2$. Since $G \in \mathcal{H}_2$, by Lemma 4.1 (i), E_2 is a perfect matching of $G[V_2]$ which implies that there is no vertex of degree two each of whose neighbours has degree three or more. Thus, $\widehat{G} = G$. On the other hand, since $G \in \mathcal{H}_1$, by Theorem 1.10, $G \in \mathcal{H}_4$. Furthermore, $\Delta(G) = 3$. Observe that Θ is the only graph in \mathcal{H}_4 with maximum degree three. Thus, $G = \Theta$; this proves the following.

Corollary 7.12. $\mathcal{H}_1 \cap \mathcal{H}_2 = \{\Theta\}$. \square

In the final section, we present conjectures that are natural generalizations of Theorems 1.3, 1.4 and 1.5 to k -extendable bipartite graphs — that we proceed to define.

8 Generalization to k -extendable bipartite graphs

For a positive integer k , a connected graph, of order least $2k + 2$, is k -extendable if it has a matching of size k and each such matching extends to a perfect matching. This notion was first introduced by Plummer [14], and has been studied extensively by various authors since then; we impose the additional technical condition of order at least $2k + 2$ since otherwise most of our results admit small counterexamples. Observe that a graph is k -extendable if and only if its underlying simple graph is k -extendable.

Note that matching covered graphs are precisely the 1-extendable graphs. For the sake of completeness, we also use the term 0-extendable graphs to refer to (not necessarily connected) matchable graphs. Plummer [15] proved the following regarding the connectedness of k -extendable graphs.

Theorem 8.1. *For $k \geq 1$, every k -extendable graph is $(k + 1)$ -connected.*

This immediately yields the following.

Corollary 8.2. *In a k -extendable graph, each vertex has degree at least $k + 1$.* \square

A k -extendable graph G is *minimal* if $G - e$ is not k -extendable for each edge e . We emphasize that the notion of minimality is tied with the value of k . For instance, it is easily

verified (using Theorem 8.3) that every 2-extendable bipartite cubic graph is a minimal 2-extendable graph, but is not minimal 1-extendable.

In light of Corollary 8.2, it is natural to ask whether there is a lower bound on the number of degree $k + 1$ vertices in a minimal k -extendable graph. Henceforth, we restrict our attention to bipartite graphs.

We begin by recalling the well-known Hall's Theorem which states that a bipartite graph $G[A, B]$, where $|A| = |B|$, is 0-extendable if and only if $|N(S)| \geq |S|$ for each $S \subseteq A$, where $N(S)$ denotes the neighbourhood of S . Using this, Plummer [15] established the following characterization of k -extendable bipartite graphs.

Theorem 8.3. [CHARACTERIZATION OF k -EXTENDABLE BIPARTITE GRAPHS]

For a bipartite graph $G[A, B]$ of order at least $2k + 2$, where $|A| = |B|$, the following are equivalent:

- (i) G is k -extendable,
- (ii) for every nonempty subset $S \subseteq A$, either $N(S) = B$ or $|N(S)| \geq |S| + k$, and
- (iii) $G - X - Y$ is matchable for all $X \subseteq A$ and $Y \subseteq B$ such that $|X| = |Y| = k$.

Note that statement (ii) in the above theorem is equivalent to enforcing $|N(S)| \geq |S| + k$ for every nonempty subset $S \subset A$ such that $|S| \leq |A| - k$; the latter is the more commonly used condition in Plummer's work. We also state an immediate corollary of statement (iii).

Corollary 8.4. *If $G[A, B]$ is a k -extendable bipartite graph, where $k \geq 1$, then $G - a - b$ is $(k - 1)$ -extendable for each pair $a \in A$ and $b \in B$. \square*

We now proceed to discuss a few bounds established by Lou [10], and our conjectures pertinent to these.

8.1 Lou's bounds and our conjectures

In the same spirit as in Section 4, with respect to a fixed value of k , for a k -extendable bipartite graph G , we let $V_{k+1}(G)$ denote the set of vertices that have degree precisely $k + 1$, and we let $V_{k+2}(G)$ denote the set of vertices that have degree at least $k + 2$. We drop G from the notation when the graph is clear from the context. For the sake of brevity, we use $E_{k+1} := E(G[V_{k+1}])$ and $E_{k+2} := E(G[V_{k+2}])$.

Note that a minimal 0-extendable bipartite graph is precisely a perfect matching. Thus, $|V_1| = n$ and $|E| = \frac{n}{2}$. Henceforth, we shall be interested in bounds for $k \geq 1$. Lou [10] proved the following.

Theorem 8.5. *In a minimal k -extendable bipartite graph G , the induced subgraph $G[V_{k+2}]$ is a forest.*

Using the above, Lou [10] deduced the following lower bound on the number of vertices of degree $k + 1$ in terms of n .

Corollary 8.6. *In a minimal k -extendable bipartite graph, $|V_{k+1}| \geq \frac{kn+2}{2k+1}$.*

We now use Lou's Theorem (8.5) to infer another lower bound in terms of m and n .

Corollary 8.7. *For $k \geq 1$, in a minimal k -extendable bipartite graph, $|V_{k+1}| \geq \frac{1}{k}(m - n + 1)$.*

Proof. Note the $(k+1)|V_{k+1}| = |\partial(V_{k+1})| + 2|E_{k+1}| \geq |\partial(V_{k+1})| + |E_{k+1}|$. Furthermore, by Theorem 8.5, $|E_{k+2}| \leq |V_{k+2}| - 1$. Combining these, $m = |E_{k+2}| + |E_{k+1}| + |\partial(V_{k+1})| \leq |V_{k+2}| - 1 + (k+1)|V_{k+1}| = n - 1 + k|V_{k+1}|$. By rearranging, $|V_{k+1}| \geq \frac{1}{k}(m - n + 1)$. \square

Using Theorem 8.5, Lou also deduced the following upper bound on the size.

Corollary 8.8. *In a minimal k -extendable bipartite graph, $|E| \leq (k+1)n - 1$.*

It is worth noting that Lou did not provide examples that satisfy any of the aforementioned bounds tightly. We are also unable to construct such examples. Thus, we conjecture stronger bounds that are generalizations of the corresponding bounds that appear in Table 1.

Conjecture 8.9. [MAIN CONJECTURE]

In a minimal k -extendable bipartite graph, $|V_{k+1}| \geq \frac{2}{2k-1}(m - n + 2k)$.

As we shall see soon, one may prove the following two conjectures assuming the above conjecture.

Conjecture 8.10. *In a minimal k -extendable bipartite graph, where $k \geq 1$, $|V_{k+1}| \geq \frac{n}{2} + 2$.*

For our final conjecture, we are able to construct small counterexamples, and thus impose a lower bound on the order. This is reminiscent of Corollary 7.3 wherein C_4 appears as the only counterexample.

Conjecture 8.11. *There exists an integer $N_k \leq 4k^2 + 2k$ such that every minimal k -extendable bipartite graph, on N_k or more vertices, satisfies $|E| \leq \frac{(2k+1)(n-2k)}{2}$.*

As evidence for the above conjecture, we mention the following result due to Fabres, Kothari and Carvalho [4] whose bound coincides exactly with our conjecture.

Theorem 8.12. *Every minimal 2-extendable bipartite graph, on twelve or more vertices, satisfies $|E| \leq \frac{5n-20}{2}$.*

For each of the above three conjectures, we shall provide tight examples that attain the corresponding bounds, in Subsection 8.3. Our examples may be viewed as straightforward generalizations of the characterizations of the corresponding extremal classes that appear in Theorems 1.3, 1.4 and 1.5. Below, we prove that our Main Conjecture (8.9) implies the other two.

Theorem 8.13. *Conjecture 8.9 implies Conjecture 8.10.*

Proof. Using the fact that $n = |V_{k+1}| + |V_{k+2}|$, and the handshaking lemma:

$$2m = \sum_{v \in V_{k+2}} d(v) + (k+1)|V_{k+1}| \geq (k+2)|V_{k+2}| + (k+1)|V_{k+1}| = (k+2)n - |V_{k+1}|$$

By substituting the above in Conjecture 8.9, we infer that $(2k-1)|V_{k+1}| \geq 2(m - n + 2k) \geq kn - |V_{k+1}| + 4k$. By rearranging, $|V_{k+1}| \geq \frac{n}{2} + 2$. \square

Note that Conjecture 8.11 already holds when $k = 1$ due to Corollary 7.3. Now, we prove Conjecture 8.11 assuming the Main Conjecture (8.9) for $k \geq 2$.

Theorem 8.14. *Conjecture 8.9 implies Conjecture 8.11.*

Proof. Let $G[A, B]$ denote a minimal k -extendable bipartite graph, where $k \geq 2$, that satisfies Conjecture 8.9. Using the fact that $n = |V_{k+1}| + |V_{k+2}|$, we deduce that $2(m - n + 2k) \leq (2k - 1)|V_{k+1}| = (2k - 1)(n - |V_{k+2}|)$. If $|V_{k+2}| \geq 2k$ then $2(m - n + 2k) \leq (2k - 1)(n - 2k)$, and by rearranging, we arrive at the desired conclusion: $m \leq \frac{2k+1}{2}(n - 2k)$. It remains to deal with the case: $|V_{k+2}| < 2k$. Before that, we make a few easy observations.

Note that $m = |E_{k+1}| + |\partial(V_{k+1})| + |E_{k+2}|$ and also $|\partial(V_{k+1})| = (k + 1)|V_{k+1}| - 2|E_{k+1}|$. We deduce that $m = (k + 1)|V_{k+1}| - |E_{k+1}| + |E_{k+2}|$. Now, by Theorem 8.5, $|E_{k+2}| \leq |V_{k+2}| - 1$. Combining these:

$$m \leq (k + 1)|V_{k+1}| - |E_{k+1}| + |V_{k+2}| - 1 \quad (4)$$

In what follows, we shall lower bound $|E_{k+1}|$ in order to obtain an upper bound on m . For convenience, we use V_{k+2}^A to denote the set $V_{k+2} \cap A$, and likewise for V_{k+1}^A and V_{k+1}^B .

Henceforth, we assume that $|V_{k+2}| < 2k$ and adjust notation so that $|V_{k+2}^A| \leq k - 1$. Consequently, each vertex in V_{k+1}^B has at least two neighbours in V_{k+1}^A . As a result, $|E_{k+1}| \geq 2|V_{k+1}^B| = 2(\frac{n}{2} - |V_{k+2}^B|) \geq 2(\frac{n}{2} - |V_{k+2}|) = n - 2|V_{k+2}|$. Combining this with Equation 4, and replacing $|V_{k+1}|$ by $n - |V_{k+2}|$:

$$m \leq (k + 1)|V_{k+1}| - n + 2|V_{k+2}| + |V_{k+2}| - 1 = kn - (k - 2)|V_{k+2}| - 1$$

Since $k \geq 2$, we conclude that $m \leq kn - 1 < kn$. Finally, observe that $kn \leq \frac{(2k+1)(n-2k)}{2}$ if and only if $n \geq 2k(2k + 1) = 4k^2 + 2k$; this completes the proof of Theorem 8.14. \square

We now proceed to establish a result on the connectedness of k -extendable bipartite graphs that will help us in deducing that the graphs yielded by our constructions, described in Subsection 8.3, are indeed minimal.

8.2 A result on connectedness of k -extendable bipartite graphs

As discussed earlier, Plummer [15] proved that k -extendable graphs are $(k + 1)$ -connected for $k \geq 1$; see Theorem 8.1. In this subsection, we prove another interesting property pertaining to the connectedness of k -extendable bipartite graphs. We first prove the following technical inequality that surprisingly shows up in our proof of Theorem 8.16.

Lemma 8.15. *Let p and q be nonnegative real numbers where $p < q$, let $D := [p, q]$, and let $f : D^2 \mapsto \mathbb{R}$ be the function $f(x, y) := y(p + q - x) + x(p + q - y)$. Then $f(x, y) \geq 2pq$ for each $(x, y) \in D^2$.*

Proof. Let $(x, y) \in D^2$. We begin by observing a couple of symmetries: $f(y, x) = f(x, y) = f(p + q - x, p + q - y)$. Consequently, we may adjust notation so that $y \leq \frac{p+q}{2}$. Note that $f(x_1, y) - f(x_2, y) = (x_1 - x_2)(p + q - 2y)$. Thus, $f(x, y) \geq f(p, y) \geq f(p, p) = 2pq$. \square

A graph of order four or more is *essentially r -edge-connected* if each nontrivial cut has size at least r .

Theorem 8.16. *Every k -extendable bipartite graph is essentially $2k$ -edge-connected.*

Proof. Let $\partial(W)$ be a nontrivial cut in a k -extendable bipartite graph $G[A, B]$; clearly, we may assume G to be simple. We let $W_A := W \cap A$, and define W_B, \overline{W}_A and \overline{W}_B analogously, and adjust notation so that W_A is a smallest set among the four. We now consider three cases based on $|W_A|$, and argue that $|\partial(W)| \geq 2k$ in each case.

First, suppose that $W_A = \emptyset$. By Corollary 8.2, $|\partial(W)| = \sum_{b \in W_B} d(b) \geq (k+1)|W_B|$. Now, $|W_B| = |W| \geq 2$ as $\partial(W)$ is nontrivial. Thus, $|\partial(W)| \geq 2k + 2 \geq 2k$.

Next, suppose that $|W_A| \geq k$; thus, $|\overline{W}_A| \geq k$. Consequently, $|W_A| = |A| - |\overline{W}_A| \leq |A| - k$. Thus, by Theorem 8.3, $|N(W_A)| \geq |W_A| + k$. An analogous argument proves that $|N(W_B)| \geq |W_B| + k$. Combining all of these,

$$|\partial(W)| \geq |N(W_A) - W_B| + |N(W_B) - W_A| \geq |N(W_A)| - |W_A| + |N(W_B)| - |W_B| \geq 2k$$

In order to deal with the remaining case, we let $x := |W_A|$ and $y := |W_B|$, and make some observations. Each vertex in W_B has at most x edges going to W_A . Consequently, by Corollary 8.2, each vertex in W_B has at least $(k+1-x)$ edges going to \overline{W}_A . Thus, $|\partial(W)| \geq y(k+1-x)$. Using analogous arguments, $|\partial(W)| \geq x(k+1-y)$. Since these inequalities are referring to disjoint sets of edges, $|\partial(W)| \geq y(k+1-x) + x(k+1-y)$.

Finally, suppose that $1 \leq x \leq k-1$. If $y \geq k$, then $|\partial(W)| \geq y(k+1-x) \geq 2k$. Otherwise $1 \leq y \leq k-1$. Since $x, y \in [1, k]$, we invoke Lemma 8.15 to conclude:

$$|\partial(W)| \geq y(k+1-x) + x(k+1-y) \geq 2k$$

This completes the proof of Theorem 8.16. \square

Following the terminology of [4], an edge e in a k -extendable graph G is *superfluous* if $G - e$ is also k -extendable. Note that, only for $k = 1$, superfluous edges are precisely the removable edges. However, for general k , we prefer to use the term superfluous since the notion of removability appears in the literature extensively; see Lucchesi and Murty [12].

Thus, to rephrase, a k -extendable graph is minimal if and only if it is free of superfluous edges. Finally, we record the following easy consequence of Corollary 8.2 and Theorem 8.16.

Corollary 8.17. *An edge e of a k -extendable bipartite graph is not superfluous if either of the following holds:*

- (i) *either an end of e has degree $k+1$,*
- (ii) *or e belongs to a nontrivial $2k$ -cut.*

\square

We shall find the above useful in establishing minimality of the k -extendable bipartite graphs obtained by our constructions.

8.3 Constructing tight examples for our conjectures

We begin by generalizing Halin trees. A tree is an r -tree if all of its non-leaves have degree at least r , and it is a *regular r -tree* if all of its non-leaves have degree exactly r . As per this, Halin trees are precisely the 3-trees, whereas cubic Halin trees are precisely the regular 3-trees. Next, we generalize the isomorphic leaf matching operation.

Definition 8.18. [ISOMORPHIC k -LEAF MATCHING]

Let T denote a nontrivial tree, and L its set of leaves. For a positive integer k , let H denote the (bipartite) graph obtained from the disjoint union of k copies of T , say T_1, T_2, \dots, T_k , by identifying, for each member of L , all of its k copies into a single vertex; we let $L(H) := L$. Now, let H' denote a copy of H . The (bipartite) graph G obtained from $H \cup H'$ by adding a matching, each of whose edges joins a member of $L(H)$ with the corresponding member of $L(H')$ as per some fixed isomorphism between H and H' , is said to be obtained from T by isomorphic k -leaf matching; furthermore, we let $L(G) := L(H) \cup L(H')$.

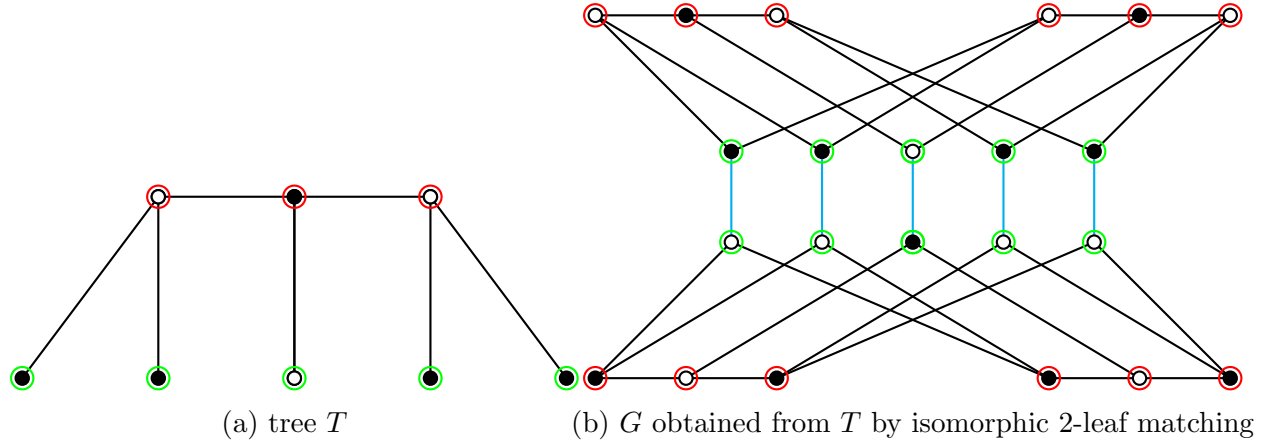


Figure 17: an example of isomorphic k -leaf matching operation

For instance, the graph G shown in Figure 17b is obtained from the tree T shown in Figure 17a by isomorphic 2-leaf matching; the green coloured vertices represent $L(G)$ and $L(T)$, respectively. Figures 18 and 19 also depict examples of the isomorphic k -leaf matching operation. The reader may verify that the isomorphic leaf matching operation defined in Subsection 1.2 is precisely the isomorphic 1-leaf matching operation. We now invite the reader to observe the following that will be used later, together with Corollary 8.17, to prove minimality of our examples.

Proposition 8.19. *Let G be a graph obtained from a tree T by isomorphic k -leaf matching where $k \geq 1$; adopt notation from Definition 8.18. For an edge e of $H \cup H'$, let e_i and e'_i denote the copies of e in T_i and T'_i , respectively, for each $i \in \{1, 2, \dots, k\}$. Then, the set $\{e_1, e_2, \dots, e_k, e'_1, e'_2, \dots, e'_k\}$ is a $2k$ -cut of G containing the edge e . \square*

We will prove a generalization (namely, Proposition 8.24) of Proposition 1.7 in terms of $(k + 1)$ -trees and the isomorphic k -leaf matching operation. In its proof, we induct on the number of non-leaves of the $(k + 1)$ -tree, say T , to prove that the constructed graphs are indeed k -extendable. The base case is when the number of non-leaves is at most two. If T has precisely one non-leaf then T is a star. On the other hand, if T has precisely two non-leaves, we call it a *double star* and denote it as $D_{p,q}$ where p and q are the degrees of the non-leaves. The induction step of the proof of Proposition 8.24 will be handled by Lemma 8.23. However, before that, we prove a couple of lemmas to address the base case.

We define $J_{p,r}$, where $r \geq p \geq 1$, as the graph obtained from $K_{1,r}$ by isomorphic p -leaf matching; see Figure 18. The graph $J_{0,r}$, where $r \geq 0$, is defined to be the disjoint union of r copies of K_2 . The reader may easily observe that $J_{p,r}$ is matchable for all $r \geq p \geq 0$. For the graph $J_{p,r}[A, B]$ where $r \geq p \geq 1$, we adopt notation from Definition 8.18, and we let $A_r := L(J_{p,r}) \cap A$ and $B_r := L(J_{p,r}) \cap B$; on the other hand, for $J_{0,r}[A, B]$, we let $A_r := A$ and $B_r := B$. Note that $|A_r| = |B_r| = r$. We now prove the following stronger property.

Lemma 8.20. *The bipartite graph $J_{k,r}[A, B]$, where $r \geq k \geq 0$, is $\min\{k, r - k\}$ -extendable.*

Proof. Let S be a nonempty subset of A . Note that $|A - A_r| = k$. Observe that if S contains a vertex from each of $A - A_r$ and A_r , then $N(S) = B$. Otherwise, either $S \subseteq A - A_r$ or $S \subseteq A_r$. In the former case, $|N(S)| = r \geq |S| + r - k$ and in the latter case, $|N(S)| = |S| + k$. Thus, by Theorem 8.3, $J_{k,r}$ is $\min\{k, r - k\}$ -extendable. \square

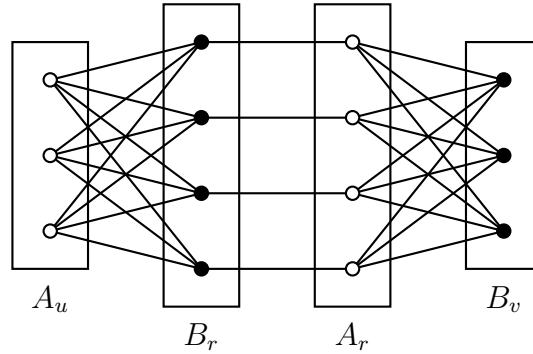


Figure 18: $J_{3,4}$ obtained from $K_{1,4}$ by isomorphic 3-leaf matching

Now, let G be the graph obtained from a double star $D_{p,q}$, where $p, q \geq 2$, by isomorphic k -leaf matching. We provide an alternative viewpoint for constructing G . Let M_1, M_2, M_3 and M_4 be the graphs obtained from the disjoint union of $p - 1, k, q - 1$ and k copies of K_2 , respectively, and let A_i and B_i denote fixed color classes of M_i for each $i \in \{1, 2, 3, 4\}$. Now, G may be obtained from $M_1 \cup M_2 \cup M_3 \cup M_4$ by joining, for each $i \in \{1, 2, 3, 4\}$, each vertex of A_i with every vertex of B_{i+1} , where arithmetic is modulo four; see Figure 19. The following lemma addresses the double star base case.

Lemma 8.21. *For any positive integer k , the bipartite graph $G[A, B]$ obtained from a double star $D_{p,q}$, where $p, q \geq k + 1$, by isomorphic k -leaf matching is k -extendable.*

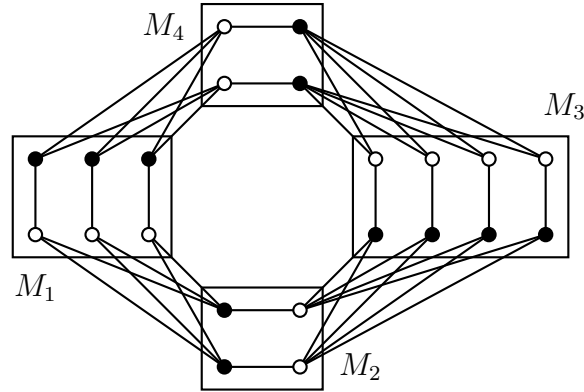


Figure 19: The graph obtained from $D_{4,5}$ by isomorphic 2-leaf matching

Proof. We use the alternative viewpoint that was described in the paragraph preceding the lemma statement, and the notation defined therein. Let $A := A_1 \cup A_2 \cup A_3 \cup A_4$, and likewise for B . Let S be any nonempty subset of A . If S meets each A_i , where $i \in \{1, 2, 3, 4\}$, then $N(S) = B$. Otherwise, there exists an $i \in \{1, 2, 3, 4\}$ such that $S \cap A_i \neq \emptyset$ but $S \cap A_{i+1} = \emptyset$. Now, note that the graph $H := G - V(M_{i+1})$ is matchable. As a result, $|N_H(S)| \geq |S|$. Furthermore, since $S \cap A_i$ is nonempty, $B_{i+1} \subseteq N(S)$. Thus, $|N_G(S)| = |N_H(S)| + |B_{i+1}| \geq |S| + k$. Consequently, by Theorem 8.3, G is k -extendable. \square

Before proving Lemma 8.23, we state and prove an easy consequence of Theorem 8.3.

Corollary 8.22. *If $G[A, B]$ is a k -extendable bipartite graph, where $k \geq 1$, then $G - e - e'$ is $(k - 1)$ -extendable for any two nonadjacent edges e and e' such that an end of e is adjacent with an end of e' .*

Proof. Let $e := ab$ and $e' := a'b'$, where $a, a' \in A$ and $b, b' \in B$, so that $ab' \in E(G)$. We let $H := G - e - e'$. For any $S \subseteq A$, observe that $|N_H(S)| \geq |N_G(S)| - 2$ and equality holds only if $a, a' \in S$ and $b, b' \notin N_H(S)$. However, since $ab' \in E(H)$, equality does not hold and $|N_H(S)| \geq |N_G(S)| - 1$. We intend to show that H satisfies statement (ii) of Theorem 8.3 with $k - 1$ playing the role of k .

Now, by applying Theorem 8.3 to G , either $|N_G(S)| \geq |S| + k$ or $N_G(S) = B$. If $|N_G(S)| \geq |S| + k$ then $|N_H(S)| \geq |S| + k - 1$, and we are done. Now, suppose that $N_G(S) = B$ and $|N_G(S)| \leq |S| + k - 1$; these imply that $|S| \geq |A| - k + 1$. Since $d_H(b) = d_G(b) - 1 \geq k$, we infer that $b \in N_H(S)$; likewise, $b' \in N_H(S)$. Consequently, $N_H(S) = N_G(S) = B$. Thus, by Theorem 8.3, H is $(k - 1)$ -extendable. \square

Next, we describe an operation that appears in the induction step of the proof of Proposition 8.24. Let G be a simple bipartite graph and uv be an edge such that $d(u) = d(v) = p + 1$. Recall the definition of $J_{p,r}$ that appears in the paragraph preceding Lemma 8.20, and the notation therein; let $A_u := A - A_r$ and $B_v := B - B_r$. Now, the (bipartite) graph $G' —$

constructed from the disjoint union of $H := G - u - v$ and $J_{p,r}$ by adding two matchings, each of size p , one between $N_G(u) - v$ and A_u , and another between $N_G(v) - u$ and B_v — is said to be obtained *from G by replacing the edge uv with $J_{p,r}$* . See Figure 20 for an illustration. We now show that this operation preserves k -extendability.

Lemma 8.23. *The bipartite graph G' obtained from a k -extendable bipartite graph G by replacing an edge uv , where $d(u) = d(v) = p + 1$, with $J_{p,r}$ is also k -extendable.*

Proof. We adopt notation from the paragraph preceding the lemma statement, and we let y_i, u_i, v_i and x_i , for each $i \in \{1, 2, \dots, p\}$, denote the vertices in the sets $N_G(u) - v, A_u, B_v$ and $N_G(v) - u$, respectively. We let a_i and b_i , for each $i \in \{1, 2, \dots, r\}$, denote the vertices in A_r and B_r , respectively; see Figure 20. We proceed by induction on k ; the following observation proves the statement for $k = 0$, and will also come in handy later.

8.23.1. *For each perfect matching M of G , the restriction of M to $G - u - v$ extends to a perfect matching of G' .*

Proof. First, suppose that $uv \in M$. Then, let M' be any perfect matching of $J_{p,r}$; see Lemma 8.20. Then, $M - uv + M'$ is the desired perfect matching of G' .

Otherwise, $p \geq 1$. Adjust notation so that $uy_1, vx_1 \in M$. Now, let M' be any perfect matching of $J_{p,r} - u_1 - v_1 \cong J_{p-1,r}$; see Lemma 8.20. Then, $M - uy_1 - vx_1 + M' + u_1y_1 + v_1x_1$ is the desired perfect matching of G' . \square

We use J to denote the subgraph of G' induced by the vertices of $J_{p,r}$. For disjoint sets of vertices of G' , say X and Y , we let $\partial(X, Y)$ denote the set of those edges whose one end is in X and the other end is in Y . We let M_u, M_r and M_v denote the matchings $\partial(N_G(u) - v, A_u), \partial(A_r, B_r)$ and $\partial(B_v, N_G(v) - u)$, respectively, and adjust notation so that both ends of each edge in $M_u \cup M_r \cup M_v$ have the same subscript.

Now, let $k \geq 1$ and M be a matching of size k in G' . We consider two cases depending on whether M is a subset of $E(G' - J)$ or not. In each case, we argue that M extends to a perfect matching of G' .

Firstly, suppose that $M \subseteq E(G' - J)$. As G is k -extendable, there is a perfect matching M' of G containing M . Note that the restriction of M' to $G - u - v$ also contains M ; by 8.23.1, this extends to a perfect matching of G' , and we are done.

Secondly, suppose that $M \not\subseteq E(G' - J)$. In other words, M contains at least one edge, say e , from $E(J) \cup \partial(J)$. Observe that $(M_u, M_v, M_r, \partial(A_u, B_r), \partial(B_v, A_r))$ is a partition of $E(J) \cup \partial(J)$. We now consider two subcases depending on whether $e \in M_u \cup M_r \cup M_v$ or not.

First, suppose that $e \in M_u \cup M_r \cup M_v$. Note that $r \geq p \geq k$; consequently, M_u, M_r and M_v are disjoint induced matchings of size at least k . Using this fact, we choose one edge from each of these sets as follows. For each $F \in \{M_u, M_r, M_v\}$, we pick e if $e \in F$; otherwise, we pick any edge e' whose both ends are M -exposed; let u_iy_i, v_jx_j and $a_\ell b_\ell$ denote these three edges. By Corollary 8.4, the graph $H := G - y_i - x_j$ is $(k - 1)$ -extendable. Also, observe

that the graph $H' := G' - u_i - y_i - v_j - x_j - a_\ell - b_\ell$ is obtained from H by replacing uv by $J_{p-1, r-1}$. Thus, by the induction hypothesis, H' is $(k-1)$ -extendable. Consequently, there is a perfect matching M' of H' containing $M - e$. Thus, $M' + u_i y_i + v_j x_j + a_\ell b_\ell$ is a perfect matching of G' containing M .

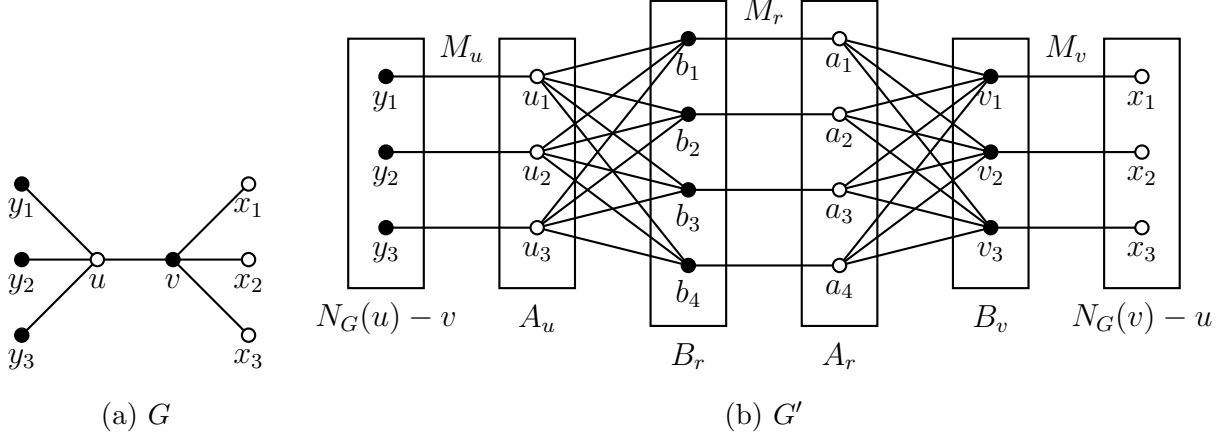


Figure 20: an illustration of replacing uv by $J_{3,4}$

Now, suppose that $e \notin M_u \cup M_r \cup M_v$; consequently, $e \in \partial(A_u, B_r) \cup \partial(B_v, A_r)$. Adjust notation so that $e := u_i b_j$. If a_j is matched in M then let $v_\ell \in B_v$ denote its matched neighbour; otherwise, since $p \geq k$, let v_ℓ denote any unmatched vertex in B_v ; we shall define a perfect matching M'' of G' that contains M in both cases. By Corollary 8.22, the graph $H := G - u y_i - v x_\ell$ is $(k-1)$ -extendable. Also, observe that $H' := G' - u_i - b_j - a_j - v_\ell$ is obtained from H by replacing uv by $J_{p-1, r-1}$. Thus, by the induction hypothesis, H' is $(k-1)$ -extendable. Consequently, $M - e - a_j v_\ell$ extends to a perfect matching M' of H' . Observe that $M'' := M' + u_i b_j + a_j v_\ell$ is the desired perfect matching of G' .

This completes the proof of Lemma 8.23. \square

We are now ready to prove the following generalization of Proposition 1.7.

Proposition 8.24. *For a positive integer k , any graph G — obtained from a $(k+1)$ -tree T , that is neither K_2 nor a star $K_{1,p}$ where $p < 2k$, by isomorphic k -leaf matching — is a minimal k -extendable bipartite graph.*

Proof. We will induct on the number of non-leaves of T . Firstly, if T has exactly one non-leaf, then it is a star and we are done by Lemma 8.20 as $p - k \geq k$. Secondly, if T has exactly two non-leaves, then it is a double star with $p, q \geq k+1$ and we are done by Lemma 8.21.

Now, suppose that T has at least three non-leaves. Let u be a leaf of the tree obtained from T by deleting all of its leaves. Observe that, in T , each neighbour of u , except one, is a leaf. Let T' be the tree obtained from T by deleting those leaves that are neighbours of u . Let us relate the leaves and non-leaves of T and T' with each other.

Note that u is a leaf in T' . Furthermore, for each vertex $w \in V(T') - u$, we have $N_{T'}(w) = N_T(w)$. Thus, the non-leaves of T' are precisely the non-leaves of T minus u . Also, each leaf of T' , except u , is a leaf of T . Ergo, T' is also a $(k+1)$ -tree with precisely one non-leaf fewer than T ; in particular, T' has at least two non-leaves. Hence, by the induction hypothesis, the graph G' obtained from T' by isomorphic k -leaf matching is k -extendable.

As u is a leaf in T' , let u and v be the two vertices corresponding to u in G' . Now, observe that G may be obtained from G' by replacing the edge uv by $J_{k,r}$, where $r := d_T(u) - 1 \geq k$. Since G' is k -extendable, by Lemma 8.23, G is also k -extendable.

Next, we prove minimality of G . By Corollary 8.17 (i), we only need to inspect those edges of G each of whose ends has degree at least $k+2$; let e denote such an edge. We adopt notation from Definition 8.18, and observe that $e \in H \cup H'$. By Proposition 8.19, the set of edges $\{e_1, e_2, \dots, e_k, e'_1, e'_2, \dots, e'_k\}$ is a $2k$ -cut (of G) that contains e . Thus, by Corollary 8.17 (ii), e is not superfluous. We thus infer that G is minimal. \square

Finally, we provide our constructions of the promised tight examples, and use the above proposition to validate them. The reader may compare the statement to Theorem 1.8.

Theorem 8.25. *For a positive integer k , any graph G obtained from a $(k+2)$ -tree T , that is neither K_2 nor a star $K_{1,p}$ where $p < 2k$, by isomorphic k -leaf matching is a minimal k -extendable bipartite graph that satisfies the bound in Conjecture 8.9 with equality. Furthermore:*

- (i) *if T is a regular $(k+2)$ -tree then G satisfies the bound in Conjecture 8.10 with equality, whereas*
- (ii) *if T is a star then G satisfies the bound in Conjecture 8.11 with equality.*

Proof. By Proposition 8.24, G is a minimal k -extendable bipartite graph. We adopt all of the notation present in Definition 8.18. We first argue that G satisfies the bound in Conjecture 8.9 with equality.

Since T is a $(k+2)$ -tree, $V_{k+1}(G) = L(G)$. Note that $|E_{k+1}| = \frac{1}{2}|L(G)| = \frac{1}{2}|V_{k+1}|$ and $|\partial(V_{k+1})| = k|V_{k+1}|$. Consequently, $|E_{k+2}| = m - |E_{k+1}| - |\partial(V_{k+1})| = m - \frac{2k+1}{2}|V_{k+1}|$.

On the other hand, observe that $G[V_{k+2}] = G - V_{k+1}$ is a forest with $2k$ components. So, $|E_{k+2}| = |V_{k+2}| - 2k = n - |V_{k+1}| - 2k$.

It follows from the preceding two paragraphs that $m - \frac{2k+1}{2}|V_{k+1}| = n - |V_{k+1}| - 2k$. By rearranging, $\frac{2k-1}{2}|V_{k+1}| = m - n + 2k$ which is precisely the bound in Conjecture 8.9. It remains to prove statements (i) and (ii).

First, suppose that T is a regular $(k+2)$ -tree. By double counting, $\sum_{v \in V_{k+2}} d(v) = 2|E_{k+2}| + |\partial(V_{k+2})|$. As noted earlier, $|E_{k+2}| = |V_{k+2}| - 2k$ and $|\partial(V_{k+2})| = k|V_{k+1}|$. By substituting, $(k+2)|V_{k+2}| = 2(|V_{k+2}| - 2k) + k|V_{k+1}|$. By rearranging and dividing by k , $|V_{k+2}| = |V_{k+1}| - 4$. Now, by plugging $|V_{k+2}| = n - |V_{k+1}|$ and rearranging, $|V_{k+1}| = \frac{n}{2} + 2$ which is precisely the bound in Conjecture 8.10.

Finally, suppose that T is a star $K_{1,p}$, where $p \geq 2k$. Note that $n = 2p + 2k$. Observe that $m = (2k + 1)p = \frac{(2k+1)(n-2k)}{2}$ which is precisely the bound in Conjecture 8.11. This completes the proof of Theorem 8.25. \square

We are unable to construct any examples, apart from the ones described in the above theorem statement, that satisfy the bounds in Conjectures 8.9, 8.10 or 8.11 with equality, and are thus tempted to further conjecture that these are the only such examples.

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