# Motion Planning in Topologically Complex Environments via Hybrid Feedback

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***Abstract*—In this article we consider the motion planning problem in an *n*-dimensional Euclidean space, $n \geq 2$, containing finitely many obstacles with boundaries possessing a smooth structure. Obstacle boundaries are assumed to be compact and connected embedded submanifolds of the ambient Euclidean space of codimension 1. The proposed solution guarantees both global asymptotic stability of a target configuration and avoidance of the obstacles. The controller is based on a switching scheme between two modes of operation: 1) a free space mode; and 2) an avoidance mode. A simulation study illustrates and clarifies the theoretical results in a 3-D workspace.**

***Index Terms*—Automatic control, collision avoidance, motion planning, path planning.**

## I. Introduction

MOTION planning in obstacle-cluttered environments is a central problem in the field of robotics. Numerous attempts have appeared in literature in a span of many decades. Despite the intense focus of the robotics and control research communities, there still exists room for the formulation of challenging problems that encapsulate practical engineering scenarios even more effectively.

Several techniques have been employed to tackle motion planning problems. These include discretization of the continuous space and employment of discrete algorithms (e.g., Dijkstra, A*) [1], [2], [3], probabilistic roadmaps [4], sampling-based motion planning [5], and rapidly exploring random trees (RRT) [6], [7], [8], [9]. Feedback-based motion planning [10] is also a commonly applied technique. Compared to algorithms, such as RRT, feedback-based motion planning offers certain advantages. First, the implementation is of significantly lower complexity as the solution is given in terms of a vector field thus eliminating the need for workspace discretization. Algorithms, such as RRT produce highly jagged paths that need to be smoothed in postprocessing thus further increasing complexity of implementation. Moreover, collision avoidance is guaranteed only with a proper step size whose reduction is costly in terms of processing power. Finally, such algorithms are known to have low performance in terms of optimality of path length.

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A feedback-based motion planning methodology, now very well developed, was proposed back in the 90s in a series of works. Artificial potentials pioneered in [11] were employed to tackle motion planning in the so called *sphere worlds*, i.e., Euclidean spaces with the exclusion of a finite number of closed balls.The main idea is to define a vector field on the free space via the negative gradient of a scalar valued function. As Koditschek and Rimon shrewdly pointed out in [12] and [13], artificial potentials lack global convergence properties owing to topological obstructions. To overcome this unfortunate situation, the Koditschek–Rimon navigation function was proposed in [12]. The central idea of a navigation function is similar to that of an artificial potential but does not allow local minima. Further, the requirement of strict navigation, i.e., convergence to the target from all initial conditions, is relaxed to almost global navigation.[1] Constructions of navigation functions were extended to *star worlds* in [14] and [15]. In star worlds, the obstacles are allowed to have boundaries *diffeomorphic* to spheres and constitute a generalization of sphere worlds.

Many researchers have further developed the concept of navigation functions in cases of increasing complexity. In [16], multiagent systems were considered, while a solution to the control aspect of the planning problem is simultaneously addressed. In [17], a convex potential was minimized in a space with convex obstacles utilizing local information.

A slightly different approach on the motion planning problem was introduced in [18], where the workspace is mapped to a punctured Euclidean space via a *navigation transformation*. The critical idea therein, is the construction of a diffeomorphism $\phi : \mathcal{G} \to \mathcal{D}$, where $\mathcal{G}$ is the initial workspace and $\mathcal{D}$ is a punctured disk. The solution for $\mathcal{G}$ will then arise from a solution for $\mathcal{D}$ by employing a pull back. A similar approach is presented in [19], restricted, however, to a 2-D workspace. The authors in [18], constructed the navigation transformation on sphere worlds, which fully incorporate the 2-D workspace of [19]. Further, in [20], a hybrid feedback controller was employed to obtain global convergence on a space with ellipsoidal obstacles.

Unfortunately, a shortcoming of all the aforementioned works is the reliance on the particular topology of sphere worlds, where either the construction of a navigation function is achievable or a navigation transformation exists. For the latter, a diffeomorphism to a punctured space is obtainable exclusively in the case of a sphere/star world, and thus, only allowing obstacles with

[1] In the sense of a zero Lebesgue measure set of initial conditions that do not converge to the target.

boundaries within the diffeomorphism class of spheres. Further, considering more general obstacle boundary typologies—even in a common 3-D scenario—can vastly increase the accuracy of workspace descriptions. For reference, consider the problem of stabilizing the origin in the presence of the torus

$$S : \left(\sqrt{x^2+y^2} - 2^2\right) + z^2 - 1 = 0.$$

A single sphere/ellipsoid containing $S$ would also have to contain the origin, rendering the problem infeasible. Thus, an unnecessary increase in the description's complexity would have to be introduced by trying to cover $S$ with multiple spheres.

Another approach on the problem of addressing state-space constraints along with stabilization is the Barrier Lyapunov function methodology, [21], [22], [23], [24]. In this line of work, a barrier function is introduced to handle state-space constraints along with a control Lyapunov function (CLF) for stabilization. The controller arises from the solution of a quadratic programming optimization problem which has to be solved in each control cycle. However, when examining general constraints, the controller offers only local asymptotic stability [25], while global results can be achieved in some exceptional cases where the constraints are convex [26]. Alternatively, the CBF methodology has to be coupled with RRT type approaches [27] or machine learning [28] to increase convergence performance.

In this work, we present a novel methodology to address the motion planning problem. Our main focus is to present a *unified*[2] approach for obstacles whose boundaries possess a smooth structure. The solution proposed can address workspaces cluttered by finitely many obstacles having as boundary an *arbitrary hypersurface*, thus extending sphere/star worlds. In contrast with various methods proposed in literature, the solution is obtained by studying each obstacle in isolation rather than seeking a solution in terms of a vector field defined on the "free" space, i.e., the workspace minus the obstacles. The proposed solution is global, i.e., the system converges to the desired configuration for all allowed initial conditions. Our approach, which relies on the use of a switching scheme, consists of employing two distinct modes of operation: 1) an *avoidance mode*; and 2) a *free-space mode*. While in free-space mode, the system follows a linear path to the desired configuration until an obstacle is encountered (in the sense of a suitable criterion). While in avoidance mode, motion is dictated by a vector field defined on a suitable *tubular* neighborhood of the encountered obstacle.

It is noteworthy that there exist approaches that also utilize tangent structures to address feasible/safe sets in nonsmooth and convex analysis, such as complementarity systems/projected systems [29]. In the mentioned work, convex feasible sets are considered even allowing nonsmooth boundaries. Further, the use of hybrid system techniques for safe reaching (along with several other objectives) can be seen in [30]. The convex constraints addressed by the former are bound to be topologically simple while the latter deals with constraints in the sense of construction of basins of attraction via CLFs.

[2]In the sense that the obsatacles' topological/geometric properties do not alter the design procedure.

The rest of the article is organized as follows. In Section II, a mathematical formulation of the problem addressed is stated. Section III is devoted to the presentation of the control scheme proposed, while in Section IV the main result of this work is stated and proved. Section V contains a simulation study which demonstrates the effectiveness of the proposed controller. Finally, Section VI concludes this article.

### A. Notations

We will denote as $\mathbb{R}, \mathbb{N}, \mathbb{N}_{\geq k}$ the sets of real and natural numbers and $\mathbb{N} \setminus \{0, \ldots, k-1\}$, respectively. The sets of nonnegative and positive real numbers are denoted as $\mathbb{R}_{\geq 0}, \mathbb{R}_{>0}$, respectively. Further, let $\mathbb{S}^n$ denote the standard $n$-sphere, i.e., $\mathbb{S}^n := \{x \in \mathbb{R}^{n+1} | |x| = 1\}$ and $\mathring{I}$ denote the open-unit interval. The Euclidean norm of the vector $v \in \mathbb{R}^n$ will be denoted as $|v|$. The distance of a point $x \in \mathbb{R}^n$ from a subset $S \subseteq \mathbb{R}^n$, i.e., the quantity $\inf\{|x-y| \, | y \in S\}$ will be denoted as $d(x, S)$. The gradient of a smooth mapping $f : \mathbb{R}^n \to \mathbb{R}$ at $x \in \mathbb{R}^n$ will be denoted as $\nabla f(x) \in \mathbb{R}^n$. The convex hull of a set $S \subseteq \mathbb{R}^n$ (the smallest convex set containing $S$), will be denoted as $\mathrm{conv(S)}$. For a Cartesian product $X = \prod_{j=1}^n X_j$, the mapping $\pi_j^X : X \to X_j, x = (x_1, \ldots, x_j, \ldots x_n) \mapsto x_j$ denotes projection onto the $j$th coordinate. For a family of sets $(X_j)_{j \in J}$, we will denote their union as $\coprod_{j \in J} X_j$ when these sets are pairwise disjoint, i.e., $X_i \cap X_j = \emptyset$ for all $(i, j) \in J^2$ such that $i \neq j$, for emphasis. For a mapping $f : X \to Y$ and a subset $S \subseteq Y$, we use the standard notation $f^{-1}(S) := \{x \in X | f(x) \in S\}$ for the preimage of $S$. In the special case of a real function $f : X \to \mathbb{R}$ and $a \in \mathbb{R}$, we let $[f \leq a]$ denote the set $\{x \in X | f(x) \leq a\}$. The sets $[f < a], [f > a], [f \geq a]$ are defined in an analogous manner. For a pair of sets $X, Y$, we will use the standard notation $X \setminus Y$ for the set $\{x \in X | x \notin Y\}$. In the case of a topological space $X$ and a subset $Y \subseteq X$, the closure (i.e., the smallest closed subset of $X$ containing $Y$) and the interior (the largest open subset of $X$ contained in $Y$) of $Y$ in $X$ will be denoted as $\bar{Y}$ and $\mathring{Y}$, respectively. The *closed* disk (or ball) with center $x_0 \in \mathbb{R}^n, n \in \mathbb{N}_{\geq 1}$ and radius $r > 0$ will be denoted as $D(x_0, R) := \{x \in \mathbb{R}^n | |x - x^*| \leq R\}$. In the case of a smooth manifold $M$, we will denote the tangent space at a point $p \in M$ as $T_pM$ and the tangent bundle of $M$ will be denoted as $TM$. The tangent map at $p \in M$ (also sometimes called pushforward or differential) of a smooth map $f : M \to N$ between smooth manifolds will be denoted as $T_pf : T_pM \to T_pN$. A vector field on a smooth manifold $M$ is to be understood as a mapping $X : M \to TM$ such that $X(p) \in T_pM$ for all $p \in M$. If $M$ is a manifold with boundary, then $\partial M$ will denote said boundary. When $\psi$ is a hybrid arc, we shall denote as $\mathrm{dom}\psi$ its associated hybrid time domain. Given a hybrid time domain $E$, we shall denote as $\sup_\nu E$ the number $\sup_\nu E := \sup\{\nu \in \mathbb{N} | \exists t \in \mathbb{R} : (t, \nu) \in E\}$.

## II. Problem Formulation

We consider a Euclidean space having excluded finitely many pairwise disjoint subsets, modeling obstacles occupying a workspace. The obstacles possess a smooth structure and are compact, connected subsets of the ambient Euclidean space. The following definitions follow closely [13, Definition 1].

*Definition 1:* A "smooth cluttered space" of dimension $n+1$, is a triple $\mathcal{S} := (\mathbb{R}^{n+1}, \{M_j\}_{j\in J}, \{\phi_j\}_{j\in J})$, where $n, N \in \mathbb{N}_{\geq 1}$, $J = \{1,\ldots,N\}$ and $\{M_j\}_{j\in J}$ is a family of pairwise disjoint subsets of $\mathbb{R}^{n+1}$ such that the following conditions hold.

1) For each $j \in J$ there exists a real smooth function $\phi_j : \mathbb{R}^{n+1} \to \mathbb{R}$, such that $M_j = \phi_j^{-1}(\{0\})$ and 0 is a regular value[3] of $\phi_j$.
2) $M_j$ is compact and connected for all $j \in J$.

If $\mathcal{S}$ is a smooth cluttered space as above, $\mathcal{S}_a$ denotes the set $\mathcal{S}_a := \mathbb{R}^{n+1} \setminus \coprod_{j\in J} M_j$.

*Remark 1:* Definition 1 and Theorem 2 provided in the Appendix, imply that each $M_j$ is a connected, compact embedded submanifold of $\mathbb{R}^{n+1}$ of dimension $n \in \mathbb{N}_{\geq 1}$. Conversely, any codimension 1 embedded submanifold of $\mathbb{R}^{n+1}$ can be obtained as the zero locus of a single smooth real function. Further, it is natural to require each $M_j$ be connected. If that was not the case, we would simply obtain a larger set $J$ by decomposing $M_j$ into its connected components, which are finite owing to compactness. Finally, notice that the case of a noncompact $M_j$ is not really of interest in any practical scenario.

*Remark 2:* In a smooth cluttered space of dimension 2, i.e., a planar space, Definition 1 allows—up to diffeomorphism—the $M_j$ to be circles. In dimension 3, we might encounter an orientable surface of any genus $g$, including spheres, tori, or the connected sum of $g$ tori. This list exhausts orientable surfaces in view of the classification of closed surfaces [31]. In dimension 4 cluttered spaces, the situation can be quite complex. Examples would include $\mathbb{S}^3$, $\mathbb{T}^3 := \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$, $\mathbb{S}^2 \times \mathbb{S}^1$ and many others.[4] Generally, increasing the dimension yields increasingly complex manifolds.

We let $\mathcal{S}$ be a smooth cluttered space. With the notation of Definition 1 we make the following observations. For each $j \in J$, the set $\mathbb{R}^{n+1} \setminus M_j$ has two connected components. Namely, $[\phi_j < 0], [\phi_j > 0]$ with exactly one of them bounded. We will always assume, without any loss of generality, that $[\phi_j < 0]$ is the bounded component for all $j \in J$. Further, note that for each $(i,j) \in J^2$ one of two cases holds true:

1) $[\phi_j \leq 0] \cap [\phi_i \leq 0] = \emptyset$;
2) $[\phi_j \leq 0] \subset [\phi_i < 0]$ or $[\phi_i \leq 0] \subset [\phi_j < 0]$.

The above observations justify the following definition.

*Definition 2:* Let $\mathcal{S}$ be a smooth cluttered space. With notation as in Definition 1, we say that $\mathcal{S}$ is "nonredundant" if the following condition holds.

1) For all $(i,j) \in J^2$, $[\phi_j \leq 0] \cap [\phi_i \leq 0] = \emptyset$.

*Definition 3:* Let $\mathcal{S}$ be a nonredundant smooth cluttered space. The sets

$$\mathcal{O}_j := [\phi_j \leq 0] \tag{1}$$

are called the "obstacles" of $\mathcal{S}$. The set

$$\mathcal{F} := \mathbb{R}^{n+1} \setminus \coprod_{j\in J} \mathcal{O}_j \tag{2}$$

is called the "free space" of $\mathcal{S}$.

In what follows, we formulate the problem addressed in this work.

*Problem 1 (Motion planning problem):* Consider the single integrator system

$$\dot{x} = u \tag{3}$$

where $x \in \mathbb{R}^{n+1}$ is the state and $u \in \mathbb{R}^{n+1}$ is the control input. Given a nonredundant smooth cluttered space $\mathcal{S} = (\mathbb{R}^{n+1}, \{M_j\}_{j\in J}, \{\phi_j\}_{j\in J})$, $J = \{1,\ldots,N\}$ and a point $x^* \in \mathcal{F}$ (serving as a desired configuration), design a control law for (3), such that for all initial conditions $x_0 \in \mathbb{R}^{n+1}$ of (3) satisfying $x_0 \in \mathcal{F}$ the following hold true.

1) The system's (3) trajectory is forward complete and satisfies $\lim_{t\to\infty} |x(t) - x^*| = 0$.
2) $x(t) \in \mathcal{F}$ for all $t \in \mathbb{R}_{>0}$.

*Remark 3:* There is also the possibility of a *bounded* nonredundant smooth cluttered space, where the following condition holds.

1) There exists $j_0 \in J$ such that for all $j \in J \setminus \{j_0\}$, $[\phi_j \leq 0] \subset [\phi_{j_0} < 0]$ and for all $(i,j) \in (J \setminus \{j_0\})^2$, $[\phi_j \leq 0] \cap [\phi_i \leq 0] = \emptyset$.

In this case, there is a single surface that encloses all other obstacles and the workspace would be taken to be its interior. In what follows we shall not address bounded spaces for simplicity of presentation. If neither the above nor the condition of Definition 2 hold then some obstacles will lie in the interior of other obstacles or lie outside the workspace, thus constituting a redundant description.

*Remark 4:* In Definition 1, we deliberately excluded a dimension 1 cluttered space. The problem of motion planning in dimension 1 is trivial. A zero-dimensional connected manifold is just a point and a smooth cluttered space of dimension 1 is the real line $\mathbb{R}$, with the exclusion of a discrete subset of the form $\{x_1,\ldots,x_N\} \subseteq \mathbb{R}$. As it becomes evident, the problem degenerates to that of stabilizing a single integrator within the confines of an interval.

*Remark 5:* Single integrator dynamics (3) are commonly utilized with the hope of decomposing the problem of motion planning in two components: 1) a reference generation problem; and 2) a subsequent tracking problem. In this work, the focus is on the former. However, given a tracking controller that can achieve prescribed tracking error accuracy, systems with more complicated dynamics can be incorporated. Viable solutions to the former have been provided with the help of prescribed performance control methodology (see indicatively [33], [34], [35], and [36]).

## III. Closed-Loop System

This section is devoted to the presentation of the controller proposed to address Problem 1. The design is split into two parts. The first addresses auxiliary problems that have to be solved

[3] See Definition 4 in the Appendix.

[4] As it stands, the question of which 3-manifolds embed in $\mathbb{R}^4$ is open. In [32], it is shown that at least 41 3-manifolds embed in $\mathbb{R}^4$ and there are 37 unresolved cases within the Burton/Martelli/Matveev/Petronio census of triangulated prime closed 3-manifolds built from 11 or less tetrahedra.

offline and require as input the data defining the considered smooth nonredundant cluttered space. The second part presents the actual control law, comprised of a switching scheme. Striving for clarity of presentation, the switching scheme is presented in a nonformal fashion with four rules. Subsequently an example is provided to clarify this set of rules. In what follows, we assume that $\mathcal{S} = (\mathbb{R}^{n+1}, \{M_j\}_{j\in J}, \{\phi_j\}_{j\in J}), J = \{1,\ldots,N\}$ is a smooth, nonredundant cluttered space, and that $x^* \in \mathcal{F}$ is a specified desired output configuration (i.e., the data of Problem 1), where $\mathcal{F}$ is the free space.

*Part 1 (Offline preparation)*:

*(AP 1)* Set,

$$r_j := \min\{|x - x^*|\,|x \in M_j\}, \quad j \in J \tag{4}$$

and specify a point $p_j^* \in M_j$ satisfying

$$|x^* - p_j^*| = r_j. \tag{5}$$

*(AP 2)* For each $j \in J$ construct a Filippov system $\mathfrak{F}_j = (G_{j,\kappa}, (X_{j,\kappa})_{\kappa\in\mathcal{K}_j})$ on $M_j$ that flows arbitrarily close to $p_j^*$ in the sense of Definition 9 (see Appendix). The differential inclusion defined by the vector fields $(X_{j,\kappa})_{\kappa\in\mathcal{K}_j}$ via (43) and (44), will be denoted as $X_j$.

*(AP 3)* For any $j \in J$, choose $\epsilon_j \in \mathbb{R}_{>0}$, such that the set

$$N_j := \{x \in \mathbb{R}^{n+1}\,|d(x, M_j) < \epsilon_j\} \tag{6}$$

is a *tubular* neighborhood of $M_j$ (see Definition 5 and Theorem 3 in the Appendix), such that $N_j \cap \coprod_{i\in J\setminus j} M_j = \emptyset$. In addition, specify positive constants $\tau_j$ such that

$$0 < \tau_j < \min\{\epsilon_j, r_j\}. \tag{7}$$

For any $j \in J$, let $\theta_j : N_j \to M_j$ be the projection of Definition 5. For any $j \in J$, we can define a Filippov system $\theta^\star\mathfrak{F}_j$ on $N_j$. Indeed, assume that $\mathfrak{F}_j = (G_{j,\kappa}, X_{j,\kappa})_{\kappa\in\mathcal{K}_j}$. Then, it is easily verified that $\theta_j^{-1}(G_{j,\kappa}), \kappa \in \mathcal{K}_j$ forms a collection of open subsets of $N_j$ that satisfy the conditions of Definition 8 (see Appendix). Furthermore, for any vector field $X_{j,\kappa}$, a new vector field can be defined on $N_j$ via

$$\theta^* X_{j,\kappa}(p) := X_{j,\kappa}(\theta_j p). \tag{8}$$

Thus, $\theta^\star\mathfrak{F}_j := (\theta_j^{-1}(G_{j,\kappa}), \theta^\star X_{j,\kappa})_{\kappa\in\mathcal{K}_j}$ is a Filippov system on $N_j$. For any $p \in N_j$ let $\Theta_p := \mathrm{sgn}(\phi_j(p))|\theta_j p - p|$. We denote as $\theta^\star X_j$ the differential inclusion defined by the vector fields $(\theta^\star X_{j,\kappa})_{\kappa\in\mathcal{K}_j}$. Note that for any $p \in N_j$ and any solution $\chi^{N_j}$ of $\theta^\star X_j$ initializing from $p$ there is a corresponding solution $\chi^{M_j}$ of $X_j$ initializing from $\theta_j p$ such that,

$$\chi^{N_j}(t) = \chi^{M_j}(t) + \Theta_p \frac{\nabla\phi_j\left(\chi^{M_j}(t)\right)}{|\nabla\phi_j\left(\chi^{M_j}(t)\right)|}. \tag{9}$$

*Remark 6:* A point as in (5) always exists. Indeed, the function $d(-, x^*) : \mathbb{R}^{n+1} \to \mathbb{R}, x \mapsto |x - x^*|$ is continuous and each $M_j$ is a compact subset of $\mathbb{R}^{n+1}$. The conclusion follows from the extreme value theorem. From a computational standpoint, the resulting optimization problems, i.e., minimizing $d = |x - x^*|^2$ subject to the constraint $\phi_j(x) = 0, j \in J$ can be solved in many cases of interest. For example, for any polynomial $\phi_j$ one can use Lagrange multipliers to obtain a system of polynomial equations, which can be solved numerically. The global minimum is then to be found within the (finite) set of critical points of the Lagrangian

$$\mathcal{L}_j(x, \lambda) := |x - x^*|^2 + \lambda\phi_j(x).$$

*Remark 7:* The stated condition of (AP2) makes sure that for any initial point, any Filippov solution of $X_j$ has points arbitrarily close to $p_j^*$. Ideally, we would like to have $p_j^*$ as a globally asymptotically stable equilibrium of a smooth vector field defined on $M_j$ but that is too strong a condition for an arbitrary manifold owing to topological obstructions. To overcome such difficulties, one could consider time-dependent smooth vector fields as done for (11) in the Appendix or consider discontinuous vector fields via Filippov systems. The latter could be useful in case there is a convenient tessellation of $M_j$ and navigation in each tile is simple. In fact, this procedure is demonstrated in the Appendix for the 2-D surfaces of genus $g$, also entirely settling the problem for 3-D spaces.

*Remark 8:* Notice that in the setting of a Euclidean space the condition of Definition 9 in the Appendix is equivalent to the following: For any $p \in M$,

$$\inf\left\{\left|\chi^{M_j}(t) - p_j^*\right|\,|t \in \mathbb{R}_{\geq 0}\right\} = 0 \tag{10}$$

for any Filippov solution, $\chi$, of $\dot{\chi} \in X_j$ with $\chi(0) = p$.

*Remark 9:* Construction of vector fields that satisfy (AP2) on the family of manifolds of the form

$$M := \mathbb{T}^{n_0} \times \mathbb{S}^{n_1} \times \cdots \times \mathbb{S}^{n_k} \tag{11}$$

where $k \in \mathbb{N}_{\geq 1}, (n_1, \ldots, n_k) \in \mathbb{N}_{\geq 2}^k$ and $n_0 \in \mathbb{N}$ is straightforward and provided in the Appendix. This family can be used to construct very intricate cluttered spaces as knots and links can be formed. To the best of our knowledge, such a family has not been addressed in literature. Furthermore, note that once the problem of construction of a Filippov system that flows arbitrarily close to an arbitrary point of a manifold $M$ has been solved, it is immediately solved for any manifold diffeomorphic to $M$. For if $f : M \to N$ is a diffeomorphism and $(G_\kappa, X_\kappa)_{\kappa\in\mathcal{K}}$ is a Filippov system for $M$ that flows arbitrarily close to $p$, it is easily verified that $(f(G_\kappa), f_\star X_\kappa)$—where $f_\star X$ is the pushforward of $X$—is a Filippov system for $N$ that flows arbitrarily close to $f(p)$.

*Remark 10:* Unfortunately, there is no general procedure to obtain the constants $\epsilon_j$ of (6). Nevertheless, in a practical situation, one would use tuning to obtain them. It is also worth noting that the requirement $N_j \bigcap \coprod_{i\in J, i\neq j} M_i = \emptyset$ can always be met by choosing $\epsilon_j$ to satisfy

$$\epsilon_j < \min\{d(M_j, M_i)\,|i \in J, i \neq j\} \tag{12}$$

where the right-hand side above is well defined since $M_i$ is compact for all $i \in J$. Furthermore, a nonpoint mass system, e.g., bounded by a sphere of some radius $r > 0$, can be handled without any modification if for all $j \in J$ the constants $\epsilon_j, j \in J$ can be chosen to satisfy $\epsilon_j > r$.

In what follows, it is assumed that (by introducing a permutation on $J$ if necessary)

$$0 < r_1 \leq \ldots \leq r_N. \tag{13}$$

That is, we linearly order the obstacles according to their distance from the target configuration.

*Part 2 (Control Scheme):* We distinguish two modes of operation and we assign the binary index $q \in \{0,1\}$ to indicate which mode is currently active. In particular, when $q = 0$ system (3) operates under the free-space mode, while when $q = 1$ the obstacle avoidance mode is activated. We further define the index $j \in (J \cup \{0\})$ (henceforth called obstacle indicator index) to determine the distance of the systems configuration to the target in relation to (13). In particular, the value $j$ implies that the state $x$ lies in the exterior of the disk $D(x_0, r_j)$ [$r_j$ as in (4)] but lies within the disk $D(x_0, r_{j+1})$, where the extreme cases $j = 0, N$ imply that $x$ lies within $D(x, r_1)$ and outside $D(x, r_N)$, respectively.

(i) *Initialization:* For any initial condition $x_0 \in \mathbb{R}^{n+1}$ of (3) we set $(q_0, j_0) = (0, N)$.

(ii) *Free-Space mode:* While $q = 0$ we consider two kinds of events. If the systems state enters the disk $D(x^*, r_j)$, the index $j$ is decreased by 1. If the systems state satisfies

$$\phi_i\left(x - \tau_i \frac{\nabla\phi_i(x)}{|\nabla\phi_i(x)|}\right) = 0 \tag{14}$$

for some $i \in \{1, \ldots, j\}$, we set $q = 1, j = i$. While $q = 0$, the right-hand side of (3) becomes

$$u = \zeta(x^* - x) \tag{15}$$

where $\zeta > 0$ is a design constant. In the case where (14) is satisfied for multiple $i \in \{1, \ldots, j\}$, we set $j = i$ for the smallest such $i$.

(iii) *$j$th Avoidance Mode:* While in avoidance mode, the index $j$ indicates that the $j$th obstacle is being encountered. The system remains in avoidance mode until the event $|x - x^*| \leq r_j$. Then, $q$ is set to 0 and $j$ is reduced by 1. While in avoidance mode,

$$u = \theta^* X_j(x) \tag{16}$$

where $\theta^\star X_j$ is the Filippov system defined in Part 1, (AP3).

(iv) When $j = 0$ the systems state $x$ lies within the disk $D(x^*, r_1)$. As no obstacle can be encountered any more, $q$ is no longer altered and the systems trajectory will reach the target configuration $x^*$ following a straight line.

*Remark 11:* Rules (ii) and (iii) above rely on the constructions of (AP1)—(AP3). Indeed, rule (ii) utilizes the constants $\tau_j, j \in J$ chosen in (AP3). Rule (iii) utilizes the Fillipov systems defined below (9) which in turn rest on the constructions of (AP2). Further, the switching policy of rule (iii) relies on the constants $r_j, j \in J$ defined in (AP1).

*Remark 12:* The purpose of (14) is to provide a criterion whether the system is sufficiently close to an obstacles surface. Suppose that $j \in J$ and the systems state satisfies $\phi_j(x - \tau_j \frac{\nabla\phi_j(x)}{|\nabla\phi_j(x)|}) = 0$. It will be proved (Lemmas 1 and 6 of the Appendix) that in this case $x \in N_j$ but also $x \notin M_j$. For our purposes, it is important that the $jth$ avoidance mode is activated only when we are within $N_j$. In Fig. 2 , the criterion in the simple case where $M_j$ is a circle is demonstrated.

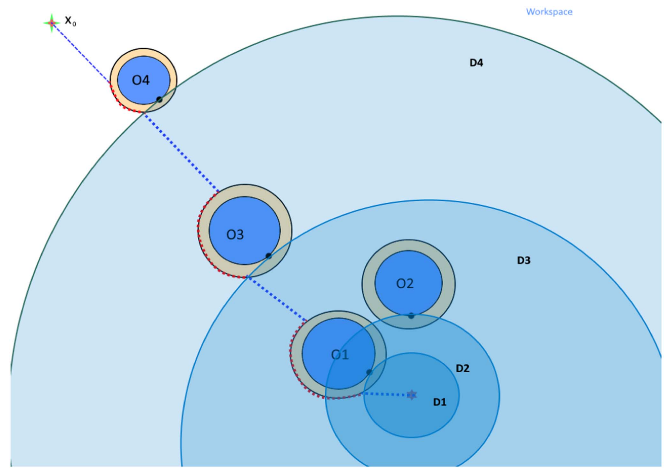


Fig. 1. Planar cluttered space. Dotted path: System trajectory. $O_1 - O_4$: Obstacles with their boundaries $M_1 - M_4$ surrounded by their respective tubular neighborhoods (in tan color), notice that the neighborhoods are depicted only in the exterior of each obstacle. Four point star: Initial position. Six point star: Desired position, $x^*$. $D_1 - D_4$: Disks centered at $x^*$ with radii $d(x^*, Mj)$, $j = 1, 2, 3, 4$. Black dots: Points in $D_j \cap M_j$, $j = 1, 2, 3, 4$.

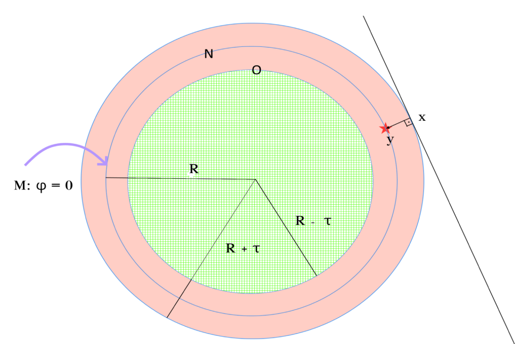


Fig. 2. Further elaboration on the example of Fig. 1. The obstacle $O$ is depicted with its boundary $M$ (a circle of radius $R$). $M$ is surrounded by its tubular neighborhood $N$ [see (6)], i.e., an annulus of inner radius $R - \tau$ and outer radius $R + \tau$. Any point of $p \in N$ projects uniquely onto a point $\theta p \in M$. Note that part of $N$ is contained in the interior of $O$. On the right, we can see a point $x$ on the boundary of $N$ and the point $y = x - \tau \frac{\nabla\phi(x)}{|\nabla\phi(x)|}$ which lies on $M$ and satisfies $\phi(y) = 0$ (depicted with a star). In this simple case, $\theta x = y$ holds.

*Remark 13:* The definition of the Filippov systems $\theta^\star X_j, j \in J$ in (16) is not constructive. Nonetheless, it is straightforward to implement such a system on the basis of (9). Indeed, let $p \in N_j$ be the position of the system at the time instant of a switch to $j$th avoidance mode. Then, the projection of $p$ onto $M_j$ is to be found. Subsequently, the solution $\chi_j^M$ to the corresponding Filippov system is obtained via numerical integration. Finally, $\chi^{M_j}$ is lifted to $N_j$ as (9) prescribes to obtain the curve $t \mapsto c_p(t) := \chi^{M_j}(t) + \lambda \frac{\nabla\phi_j(\chi^{M_j}(t))}{|\nabla\phi_j(\chi^{M_j}(t))|}$.

*Remark 14:* It can be easily verified that as long as the state $x$ remains bounded the control input cannot grow unbounded. This is always the case because, as a consequence of Theorem 1, the state converges to the desired configuration, and thus, always remains bounded. As such no issue of unbounded control input can arise.

To increase clarity, the operation of the proposed control scheme is illustrated in Fig. 1, where a simplified planar cluttered space is visualized. $N = 4$ obstacles $O_j, j = 1, 2, 3, 4$ lie in the workspace, whose boundaries $M_j, j = 1, 2, 3, 4$ are assumed to be circles for simplicity. Mappings $\phi_j, j = 1, 2, 3, 4$ of Definition 1 would then be given by

$$\phi_j(x) = (x - x_{0,j})^2 + (y - y_{0,j})^2 - R_j$$

where $(x_{0,j}, y_{0,j}), j = 1, 2, 3, 4$ are the circles centers and $R_j, j = 1, 2, 3, 4$ their radii. Following the rules *Part 2*(i)–(iv), the system's trajectory evolves as follows. Initially the system is at rest and will follow, under the guidance of (15), a linear path to the desired position $x^*$, depicted with a six-point star in Fig. 1. However, this linear path eventually forces the quantity $\phi_4(x - \tau_4 \frac{\nabla\phi_4}{|\nabla\phi_4|})$ to zero (see Fig. 2) and a switch to $4-$avoidance mode will occur, as *Part 2*(ii) prescribes. While in 4-avoidance mode, the system is governed by a vector field as (16) in *Part 2*(iii) prescribes. As long as the system flows with (16), it is intended that the distance to $M_4$ is kept constant, disallowing any collision with $M_4$. Furthermore, as the trajectory is kept within $N_4$, which does not contain any point of $M_i$ for any $i \neq 4$, a collision with any of the rest of the obstacles is avoided. Finally, the construction of the right-hand side of (16) will force the system's trajectory to eventually reach $D_4$, owing to (9) and (10). Whence, a switch back to free-space mode occurs, again according to rule *Part 2*(iii). In free-space mode, the system will revert to following a linear path to the desired position. It is important to note that no point of $M_4$ can lie on this path. The system will then go on to switch to 3-avoidance mode. The process repeats until, eventually, $D_1$ is reached. As $D_1$ is free of obstacles, the system cannot update its discrete state variables any more and the trajectory will converge (on a linear path) to the desired position. In the following section, it is precisely this informal discussion that will be converted to a rigorous proof.

## IV. Hybrid System Analysis

### A. Hybrid System Formulation

In what follows, we assume that the data of Problem 1 are given as in the beginning of Section III. Further, we assume $r_j, p_j^*, j \in J$ are as in (4) and (5), respectively, and that (13) is satisfied. Set $J' = J \cup \{0\}$. In view of the discussion of Section III, the following hybrid system is constructed:

$$\begin{aligned} \dot{\psi} &\in F(\psi), \quad \psi \in C \\ \psi^+ &= G(\psi), \quad \psi \in D \end{aligned} \tag{17}$$

where the state $\psi = (q, j, x)$ takes values in the state space

$$\Omega := \{0\} \times J' \times \mathbb{R}^{n+1} \bigcup \coprod_{j\in J} \{(1, j)\} \times N_j \tag{18}$$

where $q$ takes values in $\{0, 1\}$ to indicate which mode is active, $j$ is the obstacle indicator index described in Section III-Part 2 taking values in $J \cup \{0\}$ and $x$ is the system's position. Note that in view of *Part 2*(iv) we disallow the pair (1,0) for the variables $(q, j)$. Furthermore,

$$\begin{aligned} C &:= \{0\} \times C_0 \coprod \{1\} \times C_1, \qquad F(\xi) := (0, 0, u_{q,j}(x)) \\ D &:= \{0\} \times D_0 \coprod \{1\} \times D_1, \quad G(\xi) := (g_1(\xi), g_2(\xi), x) \end{aligned} \tag{19}$$

with,

$$\begin{aligned} C_0 &:= \coprod_{j\in J'} \{j\} \times C_{0,j}, D_0 := \coprod_{j\in J'} \{j\} \times D_{0,j} \\ C_1 &:= \coprod_{j\in J} \{j\} \times C_{1,j}, D_1 := \coprod_{j\in J} \{j\} \times D_{1,j}. \end{aligned} \tag{20}$$

To define the sets $C_{0,j}, D_{0,j}, C_{1,j}, D_{1,j}$ for $j \in J$, we set

$$K_i = \mathbb{R}^{n+1} \setminus \bar{L}_i \tag{21}$$

$$L_i := \left\{ x \in \mathbb{R}^{n+1} \middle| \nabla\phi_i(x) \neq 0, \phi_i\left(x - \tau_i \frac{\nabla\phi_i(x)}{|\nabla\phi_i(x)|}\right) \leq 0 \right\} \tag{22}$$

for all $i \in J$. Then,

$$\begin{aligned} C_{0,j} &:= \coprod_{i=1}^{j} \bar{K}_i \bigcap \left(\mathbb{R}^{n+1} \setminus \mathring{D}(x^*, r_j)\right) \\ D_{0,j} &:= \coprod_{i=1}^{j} \bar{L}_i \bigcup D(x^*, r_j) \\ C_{1,j} &:= \left(\mathbb{R}^{n+1} \setminus \mathring{D}(x^*, r_j)\right) \cap N_j \\ D_{1,j} &:= D(x^*, r_j) \cap N_j. \end{aligned} \tag{23}$$

Finally, in view of *Part 2*(iv),

$$C_{0,0} = \mathbb{R}^{n+1}, D_{0,0} = \emptyset. \tag{24}$$

In view of the switching protocol of *Part 2*(ii) and (iii), the maps $g_1 : \Omega \to \{0, 1\}, g_2 : \Omega \to J'$ are defined as follows:

$$\begin{aligned} g_1(1, j, x) &= 0, \quad j \in J \\ g_1(0, j, x) &= \begin{cases} 0, & \text{if } |x - x^*| \leq r_j \\ 1 & \text{otherwise} \end{cases} \end{aligned} \tag{25}$$

$$\begin{aligned} g_2(0, j, x) &= \begin{cases} j - 1, & \text{if } |x - x^*| \leq r_j \\ j^+ & \text{otherwise} \end{cases} \\ g_2(1, j, x) &= j - 1 \end{aligned} \tag{26}$$

where $j^+ := \min\{i \in J | \phi_i(x - \tau_i \frac{\nabla\phi_i(x)}{|\nabla\phi_i(x)|}) \leq 0\}$. For $q = 0$, the mappings $u_{(0,j)} : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ of (19) coincide for all $j \in J$. In view of (15), we set

$$u_{0,j}(x) = u_0(x) := \zeta(x^* - x) \tag{27}$$

where $\zeta \in \mathbb{R}_{>0}$ is a design constant. Finally, in view of (16), we set

$$u_{1,j}(x) := \theta^\star X_j(x) \tag{28}$$

for all $j \in J$. For the remainder, we define

$$T := \{0\} \times \{N\} \times \mathcal{F}. \tag{29}$$

### B. Statement of Main Results

*Proposition 1:* Let $\mathcal{S}$ be a smooth nonredundant cluttered space. For hybrid system (17), defined via (20)–(28), the following holds true. For any $\psi_0 \in T$, a hybrid solution $\psi$ of (17) with $\psi(0,0) = \psi_0$ exists. Furthermore, any maximal solution to (17) with $\psi(0,0) = \psi_0$ is complete.

*Proof:* See Section A2. □

On the basis of Proposition 1, the main result of this work is stated.

*Theorem 1:* Let $\mathcal{S}$ be a smooth nonredundant cluttered space. Any maximal solution to the hybrid system (17) with $\psi(0,0) \in T$ is eventually continuous and satisfies the following.

i) $\lim_{t\to\infty} \pi_3^\Omega \psi(t, \mathfrak{n}) = x^*$, where $\mathfrak{n} = \sup_\nu \operatorname{dom}\psi$.

ii) $\pi_3^\Omega \psi(t,\nu) \in \mathcal{F}$ for all $(t,\nu) \in \operatorname{dom}\psi$, where $\mathcal{F}$ is the free space of $\mathcal{S}$.

In the following subsection, a proof of Theorem 1 is presented as a result of a sequence of four lemmas. The first lemma establishes a general result that will be used throughout the proof. The next lemma establishes item ii), i.e., that no obstacle is entered as the solution evolves. The last two lemmas are concerned with establishing that the number of switches is finite (consequently also disallowing Zeno solutions) and that item i) of Theorem 1 holds true, i.e., that the system's (3) trajectory converges to the desired configuration, effectively establishing the desired results.

### C. Proof of Theorem 1

The subsequent result is essentially a gradient descent argument and is used in the proof of Lemma 2.

*Lemma 1:* Let $j \in J$ and $p \in M_j$. Set $p_{\tau_j} := p - \tau_j \frac{\nabla\phi_j(p)}{|\nabla\phi_j(p)|}$. Then, the following hold true.

(i) $\phi_j(p_{\tau_j}) < 0$.

(ii) $\phi_j(p + \tau_j \frac{\nabla\phi_j(p)}{|\nabla\phi_j(p)|}) > 0$.

*Proof:* It suffices to prove (i) as the proof of (ii) is obtained in a similar manner. Notice that for all $\gamma \in (0, \tau_j]$ it holds that $\phi_j(p_\gamma) \neq 0$, where $p_\gamma := p - \gamma\frac{\nabla\phi_j(p)}{|\nabla\phi_j(p)|}$. Indeed, in view of Corollary 1 (see Appendix) it holds that $d(p_\gamma, M_j) = \gamma > 0$ for all $\gamma \in (0,\tau_j]$. Furthermore, for a sufficiently small $\gamma > 0$, it holds that $\phi_j(p_\gamma) \le \phi_j(p) = 0$. Since, $\phi_j(p_\gamma)$ cannot be equal to zero we must have $\phi_j(p_\gamma) < 0$. If $\phi_j(p_{\tau_j}) > 0$ we would obtain a $\gamma' \in [\gamma, \tau_j]$ such that $\phi_j(p_{\gamma'}) = 0$, a contradiction. □

In the subsequent lemma, we establish that the system's motion remains confined within $\mathcal{F}$.

*Lemma 2:* Let $\psi : \operatorname{dom}\psi \to \Omega$ be a maximal (hybrid) solution to (17) with $\psi(0,0) \in T$. Let $\nu \in \mathbb{N}$ and $I_\nu := \{t \in \mathbb{R} | (t,\nu) \in \operatorname{dom}\psi\}$. Then, for all $t \in I_\nu$ it holds that $\psi(t,\nu) \in \mathcal{F}$, where $\mathcal{F}$ is the free space of $\mathcal{S}$.

*Proof:* We shall make use of an inductive argument. Initially, the conclusion for $\nu = 0$ is established. Indeed, note that $\pi_3^\Omega\psi(0,0) \in \mathcal{F}$ since we always initialize within $T$. If $I_0$ has empty interior the conclusion is obvious. Assume $I_0$ has nonempty interior and let $0 < T \in I_0, i \in J$ be such that $\pi_3^\Omega\psi(T,0) \in \mathcal{O}_i$. Then, the intermediate value theorem implies the existence of $T \ge T' > 0$ such that $x := \pi_3^\Omega\psi(T',0) \in M_i$. Then, in view of Lemma 1 it holds that $\phi_i(y) < 0$, where $y = x - \tau_1\frac{\nabla\phi_i(x)}{|\nabla\phi_i(x)|}$. Hence, in view of (21) and (24), $x \notin C_{0,N}$, a contradiction. Let $\nu \in \mathbb{N}$ and assume that the conclusion holds true for any $k < \nu$. Let $\mathfrak{t} := \inf I_\nu$, $j_\nu := \pi_2^\Omega\psi(\mathfrak{t},\nu)$ and $x_\nu := \pi_3^\Omega\psi(\mathfrak{t},\nu)$. Owing to the inductive hypothesis,

$$\phi_i(x_\nu) > 0 \quad \forall i \in J. \tag{30}$$

To proceed, it is claimed that

$$\pi_3^\Omega\psi(t,\nu)) \in \mathcal{S}_a \quad \forall i \in J, t \in I_\nu \tag{31}$$

with $\mathcal{S}_a$ as in Definition 1. Two cases are distinguished as follows.

*Case I:* $\pi_1^\Omega\psi(t,\nu) = 1$ for all $t \in I_\nu$. Then, in view of (9) and Corollary 1 it is obtained,

$$\begin{aligned} d(\pi_3^\Omega\psi(t,\nu), M_{j_\nu}) &= \left|\theta_{j_\nu}\left(\pi_3^\Omega\psi(t,\nu)\right)\right) - \pi_3^\Omega\psi(t,\nu))\right| \\ &= |\theta_{j_\nu}x_\nu - x_\nu| > 0 \end{aligned} \tag{32}$$

for all $t \in I_\nu$. Furthermore, $\pi_3^\Omega\psi(t,\nu) \in N_{j_\nu} \subseteq \mathbb{R}^{n+1} \setminus \coprod_{i\in J, i\neq j_\nu} M_i$. Hence,

$$\pi_3^\Omega\psi(t,\nu) \in \mathcal{S}_a \tag{33}$$

for all $t \in I_\nu$, $i \in J$.

*Case II:* $\pi_1^\Omega\psi(t,\nu) = 0$ for all $t \in I_\nu$. For all indexes $i \in J, i \le j_\nu$ the claim is established as in the case $\nu = 0$ above. For the rest, we proceed as follows. In view of (24), either $x_\nu \in D(x^*, r_{j_\nu+1})$ holds true. It also holds that,

$$\pi_3^\Omega\psi(t,\nu) = x_\nu e^{-\zeta\tau} + x^*\left(1 - e^{-\zeta\tau}\right).$$

If $x_\nu \in D(x^*, r_{j_\nu+1})$, then in view of (13) and the above result,

$$\left|\pi_3^\Omega\psi(t,\nu)) - x^*\right| < |x_\nu - x^*| \le r_{j_\nu+1} \le r_i$$

for all $t \in I_\nu, t > \mathfrak{t}, i \in \{j_\nu+1,\ldots,N\}$. This further implies that, $\pi_3^\Omega\psi(t,\nu)) \notin M_i$ for all $i \in \{j_\nu+1,\ldots,N\}$. Indeed, if that was not the case and $\pi_3^\Omega\psi(T,\nu)) \in M_i$ for some $i$ as above and $T > \mathfrak{t}$, then $r_i = d(x^*, M_i) \le |\pi_3^\Omega\psi(T,\nu)) - x^*| < r_i$, a contradiction.

Evidently, (31) is established in both cases. Finally, in view of (30) and (31) and the intermediate value theorem, the proof is concluded. □

The next two results address the issue of convergence.

*Lemma 3:* Let $\psi : \operatorname{dom}\psi \to \Omega$ be a maximal (hybrid) solution to (17) with $\psi(0,0) \in T$. Let $\nu \in \mathbb{N}$ be such that $\pi_1^\Omega\psi(t,\nu) = 1$ for all $t \in I_\nu := \{t \in \mathbb{R} | (t,\nu) \in \operatorname{dom}\psi\}$. Then, $I_\nu$ is bounded.

*Proof:* If $I_\nu$ has empty interior the conclusion is obvious. Suppose that $I_\nu$ is unbounded and let $\mathfrak{t} := \inf I_\nu, j_\nu := \pi_2^\Omega(\psi(\mathfrak{t},\nu)), x_0 := \pi_3^\Omega\psi(\mathfrak{t},\nu) \in N_j$ (see Lemma 6 in the Appendix) , $\lambda_0 = |\theta_{j_\nu}x_0 - x_0| < \epsilon_{j_\nu}$. Let $\chi^{N_{j_\nu}}$ denote any solution to the differential inclusion

$$\dot{x} \in \theta^\star X_{j_\nu}$$

initializing from $x_0$. In view of (28) and (9) and the fact that $\phi_{j_\nu}(x_0) > 0$, it is obtained

$$\pi_3^\Omega \psi(t,\nu) = \chi^{M_{j_\nu}}(\tau) + \lambda_0 \frac{\nabla\phi_{j_\nu}\left(\chi^{M_{j_\nu}}(\tau)\right)}{|\nabla\phi_{j_\nu}\left(\chi^{M_{j_\nu}}(\tau)\right)|} \tag{34}$$

where $\tau = t - \mathfrak{t}$ and $\chi^{M_{j_\nu}} = \theta_{j_\nu} \circ \chi^{N_{j_\nu}}$. Let $y^* := p^*_{j_\nu} + \lambda_0 \frac{\nabla\phi_{j_\nu}(p^*_{j_\nu})}{|\nabla\phi_{j_\nu}(p^*_{j_\nu})|}$. In view of (34) and (10), it is obtained

$$\inf_{t\geq 0} \left|\pi_3^\Omega \psi(t,\nu) - y^*\right| = 0. \tag{35}$$

It is claimed that the points $p^*_{j_\nu}, y^*, x^*$ are collinear. Indeed, Proposition 2 stated in the Appendix, implies that both $y^* - p^*_{j_\nu}, x^* - p^*_{j_\nu}$ are normal to $M_{j_\nu}$ at $p^*_{j_\nu}$. But the normal space to $T_{p^*_{j_\nu}} M_{j_\nu}$ is just a 1-D subspace of $\mathbb{R}^{n+1}$ (as the dimension of the tangent space is $n$), i.e., the linear span of a single vector. Hence, the claim. In view of (7), it must be the case that $y^*$ lies between $p^*_{j_\nu}, x^*$. Therefore,

$$|x^* - p^*_{j_\nu}| = |x^* - y^*| + |p^*_{j_\nu} - y^*| > 0.$$

Furthermore, (35) implies the existence of a $T > 0$ such that $0 < |\pi_3^\Omega\psi(T,\nu) - y^*| < |x^* - p^*_{j_\nu}| - |y^* - x^*| = r_{j_\nu} - |y^* - x^*|$. Employing the triangle inequality and the latter, it is obtained,

$$\left|\pi_3^\Omega\psi(T,\nu) - x^*\right| < r_{j_\nu}.$$

The latter implies that the solution lies in the interior of the disk $D(x^*, r_{j_\nu})$, and thus, $\pi_3^\Omega\psi(T,\nu) \notin C_{1,j_\nu}$, a contradiction. $\square$

*Lemma 4:* Let $\psi : \mathrm{dom}\psi \to \Omega$ be a maximal (hybrid) solution to (17) with $\psi(0,0) \in T$. Then,

1) let $\mathfrak{n} := \sup_\nu \mathrm{dom}\psi$. Then, $\mathfrak{n} < \infty$;
2) $\lim_{t\to\infty} \pi_3^\Omega\psi(t,\mathfrak{n}) = x^*$.

*Proof:*

1) In what follows, for each $(t_\nu,\nu) \in \mathrm{dom}\psi$ such that $(t_\nu,\nu+1) \in \mathrm{dom}\psi$, we let $j_\nu = \pi_2^\Omega\psi(t_\nu,\nu), x_\nu = \pi_3^\Omega\psi(t_\nu,\nu)$. Let $(t_\nu,\nu) \in \mathrm{dom}\psi$ such that $\mathfrak{n} \geq \nu + 2$. It is claimed that $j_{\nu+2} \leq j_\nu - 1$. Indeed, in view of (25) and (26), we either have $j_{\nu+1} = g_2(0, j_\nu, x_\nu) \leq j_\nu$ or $j_{\nu+1} = g_2(1, j_\nu, x_\nu) = j_\nu - 1$. In the latter case, the claim is proved. Assume the former case. If $x_\nu \in D(x^*, r_{j_\nu})$, then $j_{\nu+1} = j_\nu - 1$ and the claim is proved. If $|x_\nu - x^*| > r_{j_\nu}$, then $\pi_1^\Omega\psi(t_\nu, j_{\nu+1}) = 1$. But in this case $j_{\nu+2} = g_2(1, j_{\nu+1}, x_{\nu+1}) = j_{\nu+1} - 1$ and the claim is proved. Hence, we have covered all possible cases and the claim holds true. Let $(t_\nu,\nu) \in \mathrm{dom}\psi$ and $m \in \mathbb{N}$ such that $\nu + m \leq \mathfrak{n}$, then utilizing the claim, it is obtained
$$j_{\nu+2m} \leq j_\nu - m.$$
Let $j_0 = \pi_2^\Omega\psi(0,1) = N$ and assume that $\mathfrak{n} \geq 2N$. Then, $j_{2N} \leq 0$. In view of (24), it must be the case that $j_{2N} = 0$ and $\mathfrak{n} = 2N$ as no more switches are allowed.
2) It is claimed that $\pi_1^\Omega\psi(t,\mathfrak{n}) = 0$ for all $t \in I_\mathfrak{n} = \{t \in \mathbb{R}_{\geq 0} | (t,\mathfrak{n}) \in \mathrm{dom}\psi\}$. Indeed, this is a direct application of Lemma 3 and the fact that $I_\mathfrak{n}$ is unbounded as our solution is complete. Furthermore, in view of (27), the following is obtained:
$$\pi_3^\Omega\psi(t,\mathfrak{n}) = x_0 e^{-\zeta\tau} + x^*(1 - e^{-\zeta\tau}) \tag{36}$$
where $x_0 = \pi_3^\Omega\psi(\inf I_\mathfrak{n}, \mathfrak{n}), \tau = t - \inf I_\mathfrak{n}$. Then, $\lim_{t\to\infty}\pi_3^\Omega\psi(t,\mathfrak{n}) = x^*$ and the proof is concluded. $\square$

Theorem 1 has henceforth been established on the basis of Lemmas 2–4.

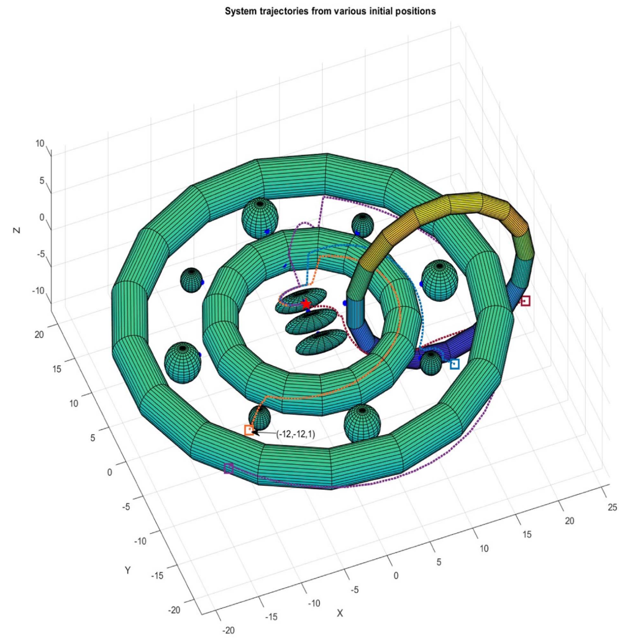


Fig. 3. System trajectories from various initial positions. Pentagram: Target position. Squares: Initial positions. Dotted lines: System trajectories. Dots: Minima of $|x - x^*|$ on the obstacles' surfaces.

## V. Simulation Study

To validate the theoretical findings, we conducted simulations on the single integrator system of (3), within the nonredundant smooth cluttered space $\mathcal{S} = (\mathbb{R}^3, \{M_j\}_{j\in J}, \{\phi_j\}_{j\in J}), J = \{1, 2, \ldots, 14\}$ depcited in Fig 3. It can be observed that the workspace contains 14 obstacles.

Three of them are bounded by toroidal surfaces which are the zero locus of the smooth real functions

$$\phi_1(x,y,z) = \left(\sqrt{x^2+y^2} - 20\right)^2 + z^2 - 2^2$$

$$\phi_2(x,y,z) = \left(\sqrt{x^2+y^2} - 10\right)^2 + z^2 - 2^2$$

$$\phi_3(x,y,z) = \left(\sqrt{(x-15)^2+z^2} - 10\right)^2 + y^2 - 1.$$

Eight of them are bounded by spheres centered at $(0, \pm 15, 0)$, $(\pm 15, 0, 0)$ with radius 2 and at $(\pm 10, \pm 10, 0)$ with radius 1.2. The last three obstacles are bounded by ellipsoids which are the zero locus of the mappings

$$\phi_{12}(x,y,z) = \frac{x^2}{3} + y^2 + z^2 - 1$$

$$\phi_{13}(x,y,z) = \frac{x^2}{3} + (y-3)^2 + z^2 - 1$$

$$\phi_{14}(x,y,z) = \frac{x^2}{3} + (y+3)^2 + z^2 - 1.$$

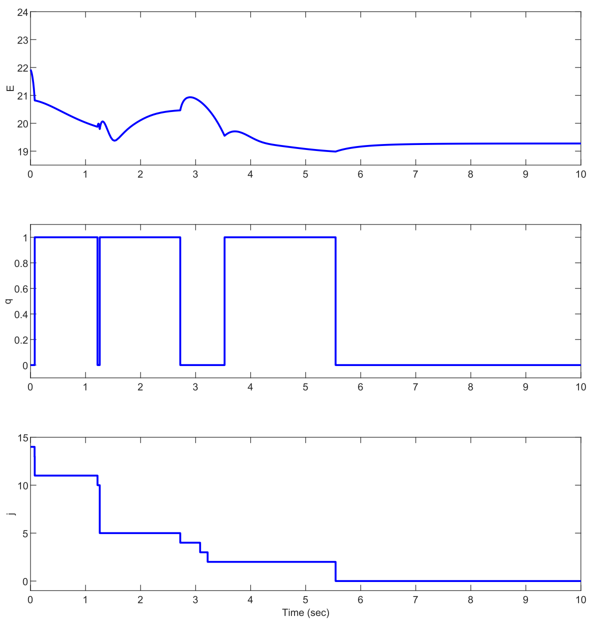


Fig. 4. System data evolution from $(-12, -12, 1)$. Upper subplot - evolution of $E$ (37) over time. Middle subplot: Evolution of mode indicator state $q$ over time. Lower subplot - evolution of the obstacle indicator index.

The desired position (depicted with a pentagram in Fig. 3) is the point $x^* = (0, 1.5, 1)$. To implement the controller, (AP1)–(AP3) were required. For (AP1), we are required to find the minimum of $d = |x - x^*|^2$ satisfying a single equality constraint, i.e., $\phi_j = 0$ for each $j \in J$. In this case, we acquired the points depicted with a blue dot on each surface in Fig. 3. For (AP3), $\tau_j$ [see (7)] was set to 0.4 for all $j \in J$. In Fig. 3, trajectories of the system for various initial positions are depicted. It can be observed that the target position is reached while motion strictly within the free space is maintained. This is further elaborated in the first subplot of Fig. 4, where the evolution of the quantity

$$E := \log\left(1 + \prod_{j=1}^{14} \log\left(1 + \phi_j(x(t))\right)\right) \tag{37}$$

is monitored, corresponding to the trajectory originating from $(-12, -12, 1)$. It is observed that $E$ remains strictly above zero for all time, thus establishing that the system remains within the free space. In the second subplot, the mode indicator state $q$ is depicted. Finally, the third subplot depicts the evolution of the obstacle indicator index $j$ over time. The plots depicted are justified as follows. The system initializes from $(-12, -12, 1)$ with $j = 14$ and $q = 0$ and remains in free-space mode until $t_1 = 0.074$ when the disk $D(r_{14}, x^*)$ is entered and $j$ is set to 13. At $t_2 = 0.0766$ (14) is satisfied for $i = 12$ (note the corresponding sphere in Fig. 3), so $j$ is set to 12 and $q$ to 1. The system evolves in 12-avoidance mode until $t_3 = 1.03977$ when the system reverts to free-space mode. An instantaneous jump occurs as the system is already within $D(r_{11}, x^*)$ and $j$ is set to 10. At $t_4 = 1.09087$ (14) is satisfied for $i = 5$ so $j$ is set to 5 and the 5-avoidance mode begins and lasts until $t_5 = 2.67275$. Note the corresponding part of the system trajectory in Fig. 3, evolving above the torus encircling the ellipsoids. At $t_5$ the system reverts to free-space mode and $j$ is set to 4. Subsequently, the system enters the disks $D(r_4, x^*)$ at $t_6 = 3.0339$ and at $t_7 = 3.17036$ (14) is satisfied for $i = 2$ and $j$ is set to 2. The 2-avoidance mode lasts until $t_8 = 5.46985$. Again, an instantaneous switch occurs as the system is already within $D(r_1, x^*)$. For the remaining time the system is in free-space mode and converges to the target.

## VI. Conclusion

In this work, a hybrid control architecture is proposed to address the motion planning problem in topologically complex spaces. The hybrid scheme consists of two modes of operation: 1) a free-space mode where the solution takes the simplest form, i.e., that of a proportional controller; 2) an avoidance mode where the system's motion is dictated by a possibly discontinuous vector field, defined on a tubular neighborhood of the obstacle encountered. We recognize several possibilities for generalization of the present results. In the direction of examining more complex spaces, one could allow immersed submanifolds instead of embedded ones. The possibility of submanifolds with boundary is also of interest. However, in our immediate research interests is the introduction of more complex dynamics instead of (3) to the motion planning problem.

## Appendix

### A. Preliminaries

Throughout this work we assume familiarity with the topological notions of connectedness (and related concepts, such as connected components) and compactness. We also assume familiarity with the notion of a differentiable manifold (involving charts and a maximal atlas), tangent structures, tangent maps, vector fields, and the notion of embedded submanifolds. For an exposition, the reader can see [37, ch. I and II] or [38, ch. 1–3]. We also assume familiarity with the notion of a hybrid system (hybrid arcs, hybrid solutions, and their properties, etc.), as exposed in [39] and [40].

*Definition 4 (Regular Values [38]):* Let $f : M \to N$ be a smooth map between smooth manifolds and $p \in M$. Then, $p$ is called a regular point if the tangent map $T_p f : T_p M \to T_p N$ is a surjection. A point $q \in N$ is called a regular value if every point in $f^{-1}(q)$ is a regular point. In the case of a map $f : \mathbb{R}^n \to \mathbb{R}$ and $y \in \mathbb{R}$, this is equivalent to $\nabla f(x) \neq 0$ for any $x \in f^{-1}(y)$. Furthermore, the "rank" of $f$ is defined as the rank of the tangent map, $T_p f : T_p M \to T_p N$.

*Theorem 2 (Level Submanifold Theorem [38]):* Let $f : M \to N$ be a smooth map and consider the level set $f^{-1}(q)$ for a point $q \in N$. If $f$ has constant rank $k$ on an open neighborhood of each $p \in f^{-1}(q)$, then $S = f^{-1}(q)$ is a closed in $M$, embedded submanifold of codimension $k$. In particular, if $N = \mathbb{R}$ and $q$ is a regular value, then $S$ is an embedded submanifold of codimension 1.

An important concept is that of a *tubular neighborhood*. The tubular neighborhood theorem presented below is just the

Euclidean version (sometimes also called $\epsilon$-neighborhood theorem) of a much more general theorem. Nonetheless, this weaker version will suffice for our needs in this work.

*Definition 5:* Let $n \in \mathbb{N}$ and $M \subset \mathbb{R}^n$ be an embedded compact smooth submanifold. A "neighborhood" of $M$ in $\mathbb{R}^n$ is an open subset of $\mathbb{R}^n$ with $M \subset U$. $U$ is said to have the "unique nearest point property" if for every $x \in U$ there exists a unique point $\theta(x) \in M$, such that $d(x, M) = |x - \theta(x)|$. The map $\theta : U \to M, x \mapsto \theta(x)$ is called the "projection" onto $M$.

*Definition 6:* Let $M \subset \mathbb{R}^n$ be an embedded compact smooth submanifold and $p \in M$. Under the identification of $T_pM$ to a subspace of $\mathbb{R}^n$, the "normal space" to $T_pM$ at $p$ is defined as

$$N_pM := \{v \in \mathbb{R}^n \,|(v', v) = 0 \quad \forall v \in T_pM\} \tag{38}$$

where $(\cdot, \cdot) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ denotes the standard inner product. A vector $v \in \mathbb{R}^n$ is said to be "orthogonal" to $M$ at $p \in M$ (and we write $v \perp T_pM$) if $v \in N_pM$. The "normal bundle" of $M$ in $\mathbb{R}^n$ is

$$NM := \{(x, v) \in M \times \mathbb{R}^n \,|v \perp T_xM\}. \tag{39}$$

The natural projection $\pi : NM \to M$ is the mapping $(x, v) \mapsto x$.

*Theorem 3 (Tubular neighborhood theorem [37]):* Let $n \in \mathbb{N}$, $M \subset \mathbb{R}^n$ be an embedded compact smooth submanifold and let $\mathfrak{p} : NM \to \mathbb{R}^n$ be the mapping

$$(x, v) \mapsto x + v. \tag{40}$$

Then, there exists an $\epsilon < 0$ such that $\mathfrak{p} : NM(\epsilon) \to M^\epsilon$ is a diffeomorphism, where

$$\begin{aligned} M^\epsilon &:= \{x \in \mathbb{R}^n \,|d(x, M) < \epsilon\} \\ NM(\epsilon) &:= \{(x, v) \in NM \,||v| < \epsilon\}. \end{aligned} \tag{41}$$

*Definition 7:* Let $M \subset \mathbb{R}^n$ satisfy the conditions of Theorem 3 and $\epsilon > 0$. $M^\epsilon$ (see 41) is called a "tubular neighborhood" of $M$ in $\mathbb{R}^n$ if the mapping $\mathfrak{p} : NM(\epsilon) \to M^\epsilon$ of (40) is a diffeomorphism.

We shall also require the following well-known fact about geodesics [41].

*Proposition 2:* Let $M$ be an embedded smooth submanifold of $\mathbb{R}^n$ and $p \in \mathbb{R}^n \setminus M$. Assume also that $M$ is a closed subset of $\mathbb{R}^n$. Then, there is a point $q \in M$ such that $d(p, M) = |p - q|$. Furthermore, $(p - q) \perp T_qM$.

Combining Theorem 3 and Proposition 2 we get the following result, which is most convenient for our needs in this work.

*Corollary 1:* Let $M \subset \mathbb{R}^n$ satisfy the conditions of Theorem 3 and $\epsilon > 0$. Suppose that $M^\epsilon$ is a tubular neighborhood of $M$ in $\mathbb{R}^n$ and consider the mapping,

$$\theta := \pi \circ \mathfrak{p}^{-1} : M^\epsilon \to M$$

where $\mathfrak{p}$ is as in (40). Then,

1) for any $(x, v) \in NM(\epsilon)$ it holds that $\theta(x + v) = x$;
2) $M^\epsilon$ has the unique nearest point property with respect to $M$ and $\theta$ is the projection onto $M$.

A straightforward example of $M^\epsilon$ in the case where $M$ is a circle in $\mathbb{R}^2$ is displayed in Fig. 2.

We shall also require some definitions and facts regarding piecewise smooth vector fields on manifolds. We shall follow the exposition of [42].

*Definition 8:* Let $M$ be a smooth manifold and assume $M$ compact and connected. A Filippov system on $M$ consists of (A) a finite collection of open sets $G_i, i \in I$ such that

1) $\bigcup_{i\in I} \bar{G}_i = M$;
2) $G_i \cap G_j = \emptyset$ for all $(i, j) \in I$ such that $i \neq j$;
3) $\Sigma := \bigcup_{i\in I} \partial G_i$ is of measure zero;
4) for each $i \in I$, there exists an open subset $U$ of $M$ such that $\bar{G}_i \subset U \subseteq M$ and a smooth vector field $X_i : U \to TU$;

and (B) the differential inclusion

$$\dot{x} \in X(x) \tag{42}$$

where $X$ is the set valued mapping such that

$$X(x) := \{X_i(x)\} \tag{43}$$

for all $x \in G_i$ and

$$X(x) = \operatorname{conv}\left(\left\{X_i(x) \,\middle|i \in I \text{ such that } x \in \bar{G}_i\right\}\right). \tag{44}$$

Naturally, we interpret solutions to a Filippov system [i.e., solutions to the differential inclusion (42)] as Filippov solutions [43]. A Filippov system consisting of the open sets and vector fields $G_i, X_i, i \in I$, respectively, will be denoted as $\mathfrak{F} := (G_i, X_i)_{i\in I}$.

*Definition 9:* Let $M$ be as in Definition 8 and assume $p \in M$. Assume that $\mathfrak{F}$ is a Filippov system on $M$. We shall say that $\mathfrak{F}$ flows arbitrarily close to $p$ if for any solution $x$ of (42) and any open set $U$ containing $p$, there exists $T > 0$ such that $x(T) \in U$.

Finally, we introduce basic notions on hybrid systems following the exposition of [39], [40], and [44].

*Definition 10:* A set $E \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is called a "hybrid time domain" if for any $(T, J) \in E$, the set $E \cap ([0, T] \times \{0, 1, \dots, J\})$ is a "compact hybrid time domain", i.e., it has the form $\bigcup_{j=0}^{J-1}([t_j, t_{j+1}], j)$ for some finite sequence $0 \leq t_1 \leq \dots \leq t_J$. Given a hybrid time domain $E$, we can define

$$\begin{aligned} \sup_t E &:= \sup\{t \in \mathbb{R}_{\geq 0} | \exists j \in \mathbb{N} :\in E\} \\ \sup_j E &:= \sup\{j \in \mathbb{N} | \exists t \in \mathbb{R}_{\geq 0} : (t, j) \in E\}. \end{aligned} \tag{45}$$

*Definition 11:* Consider a hybrid time domain $E$. A mapping $\phi : E \to \mathbb{R}^n$ is called a "hybrid arc" if for any $j \in \mathbb{N}$, the mapping $t \mapsto \phi(t, j)$ is absolutely continuous on $I_j := \{t \in \mathbb{R}_{\geq 0} : (t, j) \in E\}$. A hybrid arc $\phi : E \to \mathbb{R}^n$ is called:

(a) complete when $E$ is unbounded;
(b) "Zeno" if it is complete and $\sup_t E < \infty$;
(c) "eventually continuous" if $J = \sup_j E < \infty$ and $E \cap (\mathbb{R}_{\geq 0} \times \{J\})$ contains at least two points.

*Definition 12:* Let $\mathcal{H} = (C, F, D, G)$ be a hybrid system on the state space $\mathcal{O} \subseteq \mathbb{R}^n$. A hybrid arc $\phi : \mathrm{dom}\phi \to \mathbb{R}^n$ such that $\phi(0, 0) \in C \cup D$ is called a "hybrid solution" of $\mathcal{H}$ if

*H1* for any $j \in \mathbb{N}$ such that $I_j := \{t \in \mathbb{R}_{\geq 0} : (t,j) \in \mathrm{dom}\phi\}$ has nonempty interior

$$\phi(t,j) \in C \qquad \text{for all } \; t \in [\min I_j, \sup I_j)$$
$$\dot{\phi}(t,j) \in F(\phi(t,j)) \qquad \text{for almost all } \; t \in I_j \tag{46}$$

*H2* for any $(t,j) \in \mathrm{dom}\phi$ such that $(t,j+1) \in \mathrm{dom}\phi$,

$$\phi(t,j) \in D \;\text{ and }\; \phi(t,j+1) \in G(\phi(t,j)). \tag{47}$$

A hybrid solution $\phi : \mathrm{dom}\phi \to \mathbb{R}^n$ is called "maximal" if there does not exist a hybrid solution $\phi' : \mathrm{dom}\phi' \to \mathbb{R}^n$, such that $\mathrm{dom}\phi \subsetneq \mathrm{dom}\phi'$ and $\phi'_{|_{\mathrm{dom}\phi}} = \phi$, i.e., $\phi$ does not admit an extension on a larger hybrid time domain.

## B. Proof of Proposition 1

The purpose of the following results is to establish Proposition 1. Initially, we establish that hybrid solutions exist for any initial state which lies on $T$ [see (29)].

*Lemma 5:* Consider hybrid system (17). Then, a nontrivial solution $\psi$ to (17) with $\psi(0,0) = \psi_0$, exists for any $\psi_0 \in C \cup D$.

*Proof:* We will initially establish that $C, D$ of (19) are closed (within $\Omega$). To this end, it suffices to show that for all $j \in J'$ the sets $C_{0,j}, D_{0,j}, C_{1,j}, D_{1,j}$ defined in (23) and (24) are closed in $\mathbb{R}^{n+1}$. Let $j \in J'$, if $j = 0$ the conclusion is obvious by inspection of (24). Suppose $j \in J$. Then, $\bar{L}_j$ and $\bar{K}_j$ are closed by construction for all $j \in J$. Therefore, the desired result is obtained by mere inspection of (23). Furthermore, note that for all $j \in J$, $C_{1,j} \setminus D_{1,j}, C_{0,j} \setminus D_{0,j}$ are open in $\mathbb{R}^{n+1}$. Indeed, for the former note that

$$C_{1,j} \setminus D_{1,j} = \left(\mathbb{R}^{n+1} \setminus D(x^*, r_j)\right)$$

which is an open set. To prove the latter, it suffices to show that $\bar{K}_j \setminus \bar{L}_j$ is open for all $j \in J$ but that is just $\mathbb{R}^{n+1} \setminus \bar{L}_j$. The first conclusion is now a direct consequence of [39, Proposition 6.11]. Note that the condition (VC) therein is trivially satisfied owing to the discussion above. □

To obtain a proof of Proposition 1 we aim to exclude cases (b) and (c) [39, Proposition 6.11]. The following shall be required.

*Lemma 6:* Consider the set $L_j$ defined in (22). Then, $\bar{L}_j \subseteq N_j$, where $N_j$ was defined in (6).

*Proof:* We will initially consider the case where $x \in L_j$. Then, the point $y = x - \tau_j \frac{\nabla \phi_j(x)}{|\nabla \phi_j(x)|}$ lies on $\mathcal{O}_j$. Let $z$ be the point on the intersection of the line defined by $x, y$ and $M_j$. Then, $|x - z| \leq |x - y| = \tau_j$ and $d(x, M) \leq |x - z| \leq \tau_j < \epsilon_j$, i.e., $x \in N_j$. Now, let $x \in \bar{L}_j$. Then, there is a sequence $\{x_n\}_{n \in \mathbb{N}}, x_n \in L_j \; \forall n \in \mathbb{N}$, such that $\lim_{n \to \infty} x_n = x$. Then,

$$d(x, M) \leq d(x_n, M) + d(x, x_n) \tag{48}$$

and passing to the limit, it is obtained

$$d(x, M) \leq \tau_j < \epsilon_j.$$

The proof is concluded. □

The following lemma excludes case (c) as discussed above.

*Lemma 7:* Let $\psi : \mathrm{dom}\psi \to \Omega$ be a maximal (hybrid) solution to (17). Then, for any $(t, \nu) \in \mathrm{dom}\psi$ such that $(t, \nu+1) \in \mathrm{dom}\psi$ it holds that $\psi(t, \nu+1) \subseteq C \cup D$.

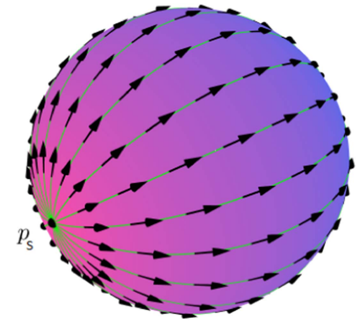


Fig. 5 Vector field $X_2$ on $\mathbb{S}^2$. The integral curves are great circles connecting $p_S$ (source) and $p_N$ (sink).

*Proof:* Let $x := \pi_3^{\Omega}\psi(t, \nu+1), j := \pi_2^{\Omega}(t, \nu+1), q := \pi_1^{\Omega}(t, \nu+1)$. It is noted that $g_1(D) \subseteq \{0,1\}$ and $g_2(D) \subseteq J'$. Thus, the only real issue is to establish that $x \in C_{q,j} \cup D_{q,j}$. If $q = 0$, this is trivial as $C_{0,j} \cup D_{0,j} = \mathbb{R}^{n+1}$ for any $j \in J'$. If $q = 1$, then there exists $j \in J$ such that $x = \pi_3^{\Omega}\psi(t,\nu) \in \bar{L}_j \subseteq N_j$, with the last inclusion owing to Lemma 6. To obtain the result just note that $C_{1,j} \cup D_{1,j} = N_j$. □

The following lemma concludes the proof.

*Lemma 8:* Let $\psi : \mathrm{dom}\psi \to \Omega$ be a maximal (hybrid) solution to (17). Then, $\psi$ is complete.

*Proof:* Seeking for a contradiction, suppose this is not the case. In view of Lemma 7, case (b) of [39, Proposition 6.11] must hold true. Let $\mathfrak{n} := \sup_{\nu} \mathrm{dom}\psi, I_{\mathfrak{n}} := \{t \in \mathbb{R} | (t, \mathfrak{n}) \in \mathrm{dom}\psi\}, \mathfrak{t} := \inf I_{\nu}$. If $\pi_1^{\Omega}\psi(\mathfrak{t}, \mathfrak{n}) = 0$, $\pi_3^{\Omega}\psi$ will be contained within a finite line segment—see also (36), obtained by the usual solution to $\dot{x} = \zeta(x^* - x)$—a compact subset of $\mathbb{R}^{n+1}$. In the case $\pi_1^{\Omega}\psi(\mathfrak{t}, \mathfrak{n}) = 1$, $\pi_3^{\Omega}\psi$ is contained within $\mathcal{W} := \{x \in \mathbb{R}^{n+1} | d(x, M_{j_\nu})) = \lambda_0\}$, where $j_\nu := \pi_2^{\Omega}\psi(\mathfrak{t}, \mathfrak{n})$, $\lambda_0 = |\theta_{j_\nu} x_0 - x_0|$ [see also (32)]. Further, $\mathcal{W}$ is closed as the preimage of a singleton under a continuous mapping and it is also bounded as $M_{j_\nu}$ is compact and thus itself bounded. Hence, in either case the solution remains within a compact subset of $\Omega$ for all $t \in I_{\mathfrak{n}}$, a contradiction. □

## C. Constructions for (11)

The purpose of this section is the construction of vector fields with the properties specified in (AP2) in the case of manifolds that belong to the family defined in (11). In this direction, let $n \in \mathbb{N}_{\geq 1}$ and suppose we want to construct a vector field that flows arbitrarily close to $p_{N_n} = (0, \ldots, 1) \in \mathbb{S}^n$, i.e., the *north pole*. Consider the vector field

$$X_n(x) := p_{N_n} - x_{n+1} x \tag{49}$$

where $x = (x_1, \ldots, x_{n+1})$. It is easily verified that

$$(p_{N_n} - x_{n+1} x, x) = 0 \tag{50}$$

for any $x \in \mathbb{S}^n$, and thus, $X_n$ is well defined. For any $x \in \mathbb{S}^n$, $X_n(x)$ is just the tangential component of the constant field $\tilde{X}_n(x) := v$. It is easily verified that any integral curve of $X_n$ is a great circle connecting $p_{N_n} = (0, \ldots, 1)$ and $p_{S_n} = (0, \ldots, -1)$, as in Fig. 5. Hence, for any $p \in \mathbb{S}^n \setminus \{p_S\}$ it holds that $\phi_t^{X_n}(p) \to p_{N_n}$ for $t \to \infty$. Notice that the choice of the north pole is not restrictive as we can always find a rotation that takes $p_{N_n}$ to any other point $p$ that we wish to flow arbitrarily close to.

*Remark 15:* Unfortunately, $X_n$ of (49) has an unwanted fixed point, i.e., $p_{S_n}$. Further, in odd-dimensional spheres this is bound to be the case because of the Hopf–Poincare theorem. This is undesirable, but such an isolated fixed point can be easily eliminated by adding time dependence. Any small, time-dependent vanishing disturbance will take the system to the stable manifold of $p_{N_n}$ which is of full measure. In what follows, we will assume that $X_n$ is such a time-dependent vector field and for "any" point $p \in \mathbb{S}^n$ it holds that

$$\phi_t^{X_n}(p) \to p_{N_n}, t \to \infty. \tag{51}$$

*Lemma 9:* Let $\mathbb{T}^n := \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ be the n-torus and consider the point $p := (p_{N_1}, \ldots, p_{N_1})$, where $p_{N_1} = (0, 1) \in \mathbb{S}^1$. Consider the vector field

$$X_{\mathbb{T}^n} := X_1 \times \cdots \times X_1 \tag{52}$$

where $X_1$ is defined in (49) with $n = 1$. Then, for all $x \in \mathbb{T}^n$ it holds that

$$\lim_{t\to\infty} \phi_t^{X_{\mathbb{T}^n}}(x) = p \tag{53}$$

where $\phi^{X_{\mathbb{T}^n}}$ is the flow corresponding to $X_{\mathbb{T}^n}$.

*Proof:* Similar to the proof of Lemma 2. □

*Corollary 2:* Let $M$ be a smooth manifold as in (11). For each $j \in \{1, \ldots, k\}$ let $X_{n_j}$ be the vector field defined in (49) with $n = n_j$. Consider the vector field

$$X_M := X_{\mathbb{T}_{n_0}} \times X_{n_1} \times \ldots \times X_{n_k} \tag{54}$$

such that

$$M \ni x := (p_0, \ldots, p_k) \mapsto X_M(x) := (X_{\mathbb{T}^{n_0}}(p_0), \ldots, X_k(p_k)) \tag{55}$$

where $X_{\mathbb{T}^{n_0}}$ is as in (52) with $n = n_0$. Then, for all $x := (x_0, \ldots, x_k) \in M$ it holds that

$$\lim_{t\to\infty} \phi_t^{X_M}(x) = p^* := \left(p, \ldots, p_{N_{n_k}}\right) \tag{56}$$

where $p = (p_{N_1}, \ldots, p_{N_1}) \in \mathbb{T}^{n_0}$.

*Proof:* Let $U = U_0 \times U_1 \times \cdots \times U_k$ be an open neighborhood of $p^*$. Then, owing to (51) and Lemma 9, there exist $T_0, T_1, \ldots, T_k$ such that

$$\phi_t^{\mathbb{T}^{n_0}}(x_0) \in U_0 \text{ for all } t > T_0 \tag{57}$$

and for all $j \in \{1, \ldots, k\}$,

$$\phi_t^{X_{n_j}}(x_j) \in U_j \text{ for all } t > T_j. \tag{58}$$

Further, the flow $\phi^{X_M}$ corresponding to $X_M$ is related to $\phi_{\mathbb{T}^{n_0}}, \phi^{X_{n_j}}, j \in \{1, \ldots, k\}$ via

$$\phi_t^{X_M}(x) = \left(\phi_t^{X_{\mathbb{T}^{n_0}}}(x_0), \phi_t^{X_{n_1}}(x_1), \ldots, \phi_t^{X_{n_k}}(x_k)\right) \tag{59}$$

where $x = (x_0, x_1, \ldots, x_k)$. The desired conclusion is now a direct consequence of (57)–(59). □

### D. Constructions for Orientable Surfaces of Genus $g$

*Definition 13:* Let $g \in \mathbb{N}_{\geq 2}$. A manifold of the form

$$\Sigma_g := \underbrace{\mathbb{T}^2 \# \cdots \# \mathbb{T}^2}_{g\text{-times}} \tag{60}$$

where $\mathbb{T}^2$ is the 2−torus and $M \# N$ denotes the "connected sum" [31] of $M, N$ is called "orientable surface of genus g."

In what follows, we shall outline the construction of Filippov systems for (60) to be utilized in (AP2). In this direction, suppose $p^*$ is a point in $\Sigma_2$ for which we wish to construct a Filippov system. We can view $\Sigma_2 := \mathbb{T}^2 \# \mathbb{T}^2$ as formed by gluing together $M_1 := \mathbb{T}^2 \setminus D_1$ and $M_2 := \mathbb{T}^2 \setminus D_2$ by a diffeomorphism $f : \partial D_1 \to \partial D_2$, where $D_1$ and $D_2$ are open discs (see, e.g., [31, Ch. 9]). Then, there are smooth embeddings $i_1 : M_1 \to \Sigma_2, i_2 : M_2 \to \Sigma_2$ such that

1) $U \cap V = \emptyset$, where $U := i_1(\mathring{M}_1), V := i_2(\mathring{M}_2)$;
2) $i_1(M_1) = \bar{U}, i_2(M_2) = \bar{V}$ and $\Sigma_2 = \bar{U} \cup \bar{V}$;
3) $\bar{U} \cap \bar{V} = \partial\bar{U} = \partial\bar{V}$.

Without loss of generality, it is assumed that $p^* \in \bar{V}$. Let us pick an arbitrary point $p_0 \in D_1$. Owing to Lemma 9, we can always find a time-varying vector field $X_1$ such that all its integral curves converge to $p_0$. Subsequently, the pushforward via $i_1$ on $\bar{U}$ can be defined, i.e., $X_U = i_{1_*}X_1$. Let $X_2$ be a vector field on $\mathbb{T}^2$ such that all its integral curves converge to $i_2^{-1}(p^*)$ and obtain the pushforward $X_V := i_{2_*}X_2$. Evidently, the collection $\mathfrak{S}_2 := \{(U, X_U), (V, X_V)\}$ is a Filippov system on $\Sigma_2$. Notice that when initializing from $U$, all Filippov solutions run up to the common boundary $\bar{U} \cap \bar{V}$ in finite time owing to the definition of $X_U$. Subsequently $X_V$ applies to take the solution to $p^*$. Thus, it is easily deduced that all Filippov solutions of $\mathfrak{S}_2$ converge to $p^*$. For an arbitrary genus we make use of an inductive argument. It is noted that $\Sigma_g := \Sigma_{g-1} \# \mathbb{T}^2$. If a Filippov system $\Sigma_{g-1}$ has been defined on $\Sigma_{g-1}$ we can repeat the process of $\Sigma_2$ with $\Sigma_{g-1}$ assuming the role of one of the two tori and $\mathfrak{S}_{g-1}$ assuming the role of one of the two vector fields $X_1, X_2$.

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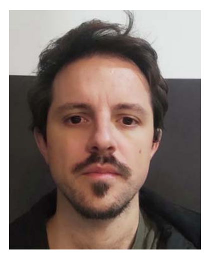

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