Nonlinear Adaptive PID Control for Nonlinear Systems

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***Abstract*—This article introduces a nonlinear adaptive proportional-integral-derivative (PID) tracking control methodology suitable for a class of multi-in-multi-out nonlinear systems. By incorporating three nonlinear modulating (trimming) functions to "recalibrate" the generalized tracking error, a nonlinear PID control framework is constructed that possesses several notable features. First, the user-assignable parameters in the modulating functions allow the resultant control strategy to analytically bridge fixed gain linear PID control with nonlinear PID control, covering both traditional fixed-gain and time-varying gain PID control as special cases. Second, the inherently nonlinear nature of the proposed control methodology effectively counteracts system nonlinearities, thereby enhancing the overall performance of the control scheme. Third, unlike conventional PID control that often involves integration saturation, the proposed control method obliterates such "windup" phenomenon by utilizing the integral trimming function that remains bounded or self-vanishes. Fourth, eliminating the necessity for a "trial-and-error" process for PID gain determination, the proposed control not only ensures asymptotic tracking but also guarantees a bounded comprehensive performance index in the presence of modeling uncertainties and external disturbances. The effectiveness and benefits of the proposed method are also confirmed by numerical simulations.**

***Index Terms*—Asymptotic tracking control, multi-in-multi-out (MIMO) nonlinear systems, nonlinear adaptive proportional-integral-derivative (PID) control.**

I. Introduction

For any dynamic control system, maintaining specific performance metrics while ensuring stable operation has always been a fundamental goal pursued by all control systems. The most typical methods for performance preservation include those based on funnel functions [1], prescribed performance functions [2] and barrier functions [3]. In this work, motivated by the structural simplicity and conceptual intuitiveness of the proportional-integral-derivative (PID) control, we explore a PID-like control method to ensure asymptotic tracking with bounded comprehensive performance characteristics for a class of nonlinear systems.

PID control, renowned for its inexpensive computation and ease of implementation, has garnered widespread acceptance in both academic and industrial spheres, with numerous successful applications in various industrial systems [4], [5], [6]. In recent years, significant progress has been made in the field, including advances for linear systems [7], [8], [9], [10], [11], [12], and several studies addressing input delay [7], [9], [10], [11], [12]. More recently, there have been substantial results for nonlinear systems. Most of these studies [13], [14], [15], [16], [17], [18], [19], [20], [21] are based on first or second-order nonlinear systems, except for [17] and [18], which achieve bounded tracking for higher-order nonlinear systems.

One common thread running through these methods is the nature of constant PID gain, which literally posits PID control as a fixed-gain and fixed linear structure control mode. This control approach, with its fixed parameters embedded in a fixed linear structure, normally encounters difficulties when dealing with systems characterized by nonlinearities and uncertainties, or those operating under continuously changing conditions. As a result, control performance can often be unpredictable. This "one size fits all" control approach literally has certain limitations. In light of this, efforts (e.g., [17], [18], etc.) have been made in exploiting a linear PID control structure with time-varying gains, thereby ensuring bounded stable tracking control in high-order nonlinear systems. In order to improve the performance of conventional PID, some scholars have ventured into the domain of constant-gain based nonlinear PID control methods, such as [16], [19], [20], [21], which developed a kind of nonlinear PID controller with the use of nonlinear characteristics (i.e., sign functions), achieving promising results, although the results are based on linear systems or low-order nonlinear systems and lack rigorous theoretical backing.

These developments lead to an intriguing proposition. The possibility of integrating variable gains with nonlinear structures to establish a new nonlinear PID control method with self-adjusting gains. This could form a "respond change with adaptation" control mechanism, which logically should have its effectiveness. This article makes a preliminary attempt in this direction, focusing on how to adjust PID parameters, what form of nonlinear structure to use, and more importantly, how to effectively combine the parameter adjustment mechanism with nonlinear structure to form an analytical and interpretable solution.

The main contributions and features of this work can be summarized as follows: 1) We propose an adaptive single-parameter tuning algorithm that avoids the tedious manual trial-and-error process of determining PID gains; 2) We introduce the nonlinear PID trimmers with user-adjustable structure parameters, allowing for the integral saturation (windup) and the over-penalization phenomena inherent in existing PID controls to be eliminated elegantly. By appropriately selecting design parameters within the trimming functions, it is possible to seamlessly connect and naturally switch between adaptive linear PID, adaptive partially nonlinear PID, and adaptive fully nonlinear PID control modes; 3) We integrate the self-tuning gain and variable nonlinear structure to form a nonlinear adaptive PID control framework that is able to cope with system nonlinearities and uncertainties, achieving asymptotically stable tracking and ensuring that the comprehensive performance index (cumulative generalized error) remains bounded. The developed control algorithms are both robust and adaptive, capable of ensuring global full-state tracking for high-order multi-in-multi-out

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(MIMO) normal form-like uncertain nonlinear systems. Theoretical analysis and simulation tests confirm the advantages and effectiveness of the proposed method.

Notations: Throughout this article, $\mathbb{R}$, $\mathbb{R}_{\geq 0}$, $\mathbb{R}_+$, and $\mathbb{R}^n$ indicate the spaces of real numbers, nonnegative real numbers, positive numbers, and real n-vectors, respectively. $\max\{\cdot\}$ and $\min\{\cdot\}$ stand for the maximum and minimum operators. $\|\cdot\|$ denotes the Euclidean norm of $\{\cdot\}$. L_∞ denotes the space of bounded signals and L_2 denotes the space of square integrable functions.

II. Problem Setting

Consider the following nth order nonsquare MIMO nonlinear feedforward-like systems

$$\begin{aligned} \dot{x}_i &= x_{i+1}, \quad i = 1, 2, \ldots, n-1 \\ \dot{x}_n &= G(x, \theta, u)u + F(x, \theta, u) \\ y &= x_1 \end{aligned} \tag{1}$$

where $x_i \in \mathbb{R}^m$ $(i = 1, \ldots, n)$ are the substate vectors of the system, $x = [x_1^T, \ldots, x_n^T]^T \in \mathbb{R}^{mn}$ represents the full state vector of the system, $u \in \mathbb{R}^\nu$ is the system control input, $G(x, \theta, u) \in \mathbb{R}^{m\times\nu}$ is the time-varying yet uncertain control gain matrix, $F(x, \theta, u) \in \mathbb{R}^m$ denotes the unknown nonlinear function vector, and $\theta \in \mathbb{R}^r$ represents an unknown and possibly time-varying parameter vector. Note that n, m, and ν are positive integers, sharing the following three possible relationships: $\nu = m$ (square system); $\nu > m$ (nonsquare and over-actuated systems); $\nu < m$ (nonsquare and under-actuated systems), here in this work, we focus on the first two cases. Define $E = x_1 - y^* \in \mathbb{R}^m$ as the output tracking error with $y^* \in \mathbb{R}^m$ being the desired trajectory vector. It is assumed that y^* and its derivatives up to the nth order are known and bounded. Hereafter, in order to make the presentation simpler, we sometimes drop the dependence of the function on its variable(s).

Our objective in this work is to develop a PID-like control methodology featuring adaptive gains and a user-adjustable nonlinear structure to achieve global full-state asymptotic tracking with a bounded performance index in the presence of uncertainties in $G(x, \theta, u)$ and $F(x, \theta, u)$, that is

$$\lim_{t\to\infty} E^{(k)}(t) = \mathbf{0}_m \quad (k = 0, \ldots, n-1) \quad \forall x(0) \in \mathbb{R}^{mn} \tag{2}$$

$$J_E = \int_0^\infty \|S(E(\tau)\|^2\, d\tau < \infty \tag{3}$$

where S represents a generalized or filtered error to be defined later and J_E denotes the accumulated generalized tracking error performance index.

III. Preliminary Results

In this section, we introduce three modulating (trimming) functions and establish two lemmas to facilitate the development of our nonlinear PID control.

A. Adjustable PID Trimming Functions

Proportional modulating function (or proportional trimmer): A continuously differentiable function $\mathcal{N}_P(Z, \boldsymbol{a}) = [\mathcal{N}_{Pi}(Z_i, a_i)] \in \mathbb{R}^m$ with $\mathcal{N}_{Pi} : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ is referred to as proportional modulating function (or proportional trimmer) if, for any $a_i > 0$, the following properties hold:

1) $\mathcal{N}_{Pi}(0, a_i) = 0$;
2) $\mathcal{N}_{Pi}(Z_i, a_i)$ is monotonically increasing with respect to Z_i;
3) there exist some positive constants μ_1 and μ_2 with $\mu_2 > \mu_1 > 0$ such that $\mu_1 Z_i^2 \leq Z_i \mathcal{N}_{Pi}(Z_i, a_i) \leq \mu_2 Z_i^2$ for all $Z_i \in (-\infty, +\infty)$;
4) $\lim_{a_i\to 0} \mathcal{N}_{Pi}(Z_i, a_i) = Z_i$.

Some examples of such $\mathcal{N}_{Pi}(Z_i, a_i)$ include (but are not limited to)

$$\begin{cases} \mathcal{N}_{1_{Pi}}(Z_i, a_i) = \beta_1 \frac{2}{a_i} \frac{1-\mathrm{e}^{-a_i Z_i}}{1+\mathrm{e}^{-a_i Z_i}} + \beta_2 Z_i & \forall a_i > 0 \\ \mathcal{N}_{2_{Pi}}(Z_i, a_i) = \beta_1 \frac{Z_i}{\sqrt{(a_i Z_i)^2+1}} + \beta_2 Z_i & \forall a_i > 0 \\ \mathcal{N}_{3_{Pi}}(Z_i, a_i) = \frac{Z_i^3}{a_i \arctan^2(Z_i) + Z_i^2} & \forall a_i > 0 \end{cases} \tag{4}$$

where $\beta_1+\beta_2=1$ and $\beta_1, \beta_2 > 0$.

Remark 1: The selection of various design parameters "a_i" in $\mathcal{N}_{Pi}(Z_i, a_i)$ results in different trimming/modulating capabilities, highlighting the design's versatility and adaptability. Interestingly, when "a_i" becomes sufficiently small, $\mathcal{N}_{Pi}(Z_i, a_i)$ tends to align with "Z_i," implying that $\mathcal{N}_{Pi}(Z_i, a_i)$ becomes a linear function. In other words, depending on the design parameter "a_i", the proportional trimmer processes/utilizes the feedback signal "Z_i" in different ways—in general, "Z_i" is manipulated/processed via the trimming function $\mathcal{N}_{Pi}(Z_i, a_i)$, such that "$Z_i$" is either expanded or compressed, and by setting "a_i" sufficiently small, $\mathcal{N}_{Pi}(Z_i, a_i)$ gets closer to "Z_i," i.e., the nonlinear trimming function becomes linear one naturally and seamlessly, and in this case the signal "Z_i" moves through the trimmer directly with no compression or expansion.

Integral modulating function (or integral trimmer): A continuously differentiable function $\mathcal{N}_I(\int_0^t Z(\tau)d\tau, Z, \boldsymbol{b}) = [\mathcal{N}_{Ii}(\int_0^t Z_i(\tau)d\tau, Z_i, b_i)] \in \mathbb{R}^m$ with $\mathcal{N}_{Ii} : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ is referred to as integral modulating function (or integral trimmer) if, for any $b_i > 0$, the following conditions are satisfied:

1) $\mathcal{N}_{Ii}(\int_0^t Z_i(\tau)d\tau, Z_i, b_i)$ is bounded for any $\int_0^t Z_i(\tau)d\tau$;
2) $Z_i\mathcal{N}_{Ii}(\int_0^t Z_i(\tau)d\tau, Z_i, b_i) \geq 0$ for $Z_i \in (-\infty, +\infty)$ and $Z_i = 0 \Rightarrow \mathcal{N}_{Ii}(\int_0^t Z_i(\tau)d\tau, Z_i, b_i) = 0$;
3) $\lim_{b_i\to 0} \mathcal{N}_{Ii}(\int_0^t Z_i(\tau)d\tau, Z_i, b_i) = \int_0^t Z_i(\tau)d\tau$.

There are many functions with the above properties, such as

Case 1—Bounded integration $\mathcal{N}_{Ii}(\int_0^t Z_i(\tau)d\tau, Z_i, b_i)$:

$$\begin{cases} \mathcal{N}_{1_{Ii}}(\int_0^t Z_i(\tau)d\tau, Z_i, b_i) = \frac{2}{b_i} \frac{1-\mathrm{e}^{-b_i\int_0^t Z_i(\tau)d\tau}}{1+\mathrm{e}^{-b_i\int_0^t Z_i(\tau)d\tau}} \Gamma_i \\ \mathcal{N}_{2_{Ii}}(\int_0^t Z_i(\tau)d\tau, Z_i, b_i) = \frac{\int_0^t Z_i(\tau)d\tau}{\sqrt{\left(b_i\int_0^t Z_i(\tau)d\tau\right)^2+1}} \Gamma_i \\ \mathcal{N}_{3_{Ii}}(\int_0^t Z_i(\tau)d\tau, Z_i, b_i) = \frac{\int_0^t Z_i(\tau)d\tau}{b_i\left(\int_0^t Z_i(\tau)d\tau\right)^2+1} \Gamma_i \end{cases} \tag{5}$$

where $b_i > 0$ and $\Gamma_i = (\tanh(\int_0^t Z_i(\tau)d\tau)\tanh(Z_i))^{\mu_i}$ with $\mu_i = \frac{q_i}{p_i} > 0$ being q_i and p_i are some positive odd integers.

Case 2—Bounded and eventually vanishing integration $\mathcal{N}_{Ii}(\int_0^t Z_i(\tau)d\tau, Z_i, b_i)$:

$$\begin{cases} \mathcal{N}_{1_{Ii},t}(\int_0^t Z_i(\tau)d\tau, Z_i, b_i, t) = \frac{2\mathcal{N}_{1Ii}}{t^{b_i}+1} \\ \mathcal{N}_{2_{Ii},t}(\int_0^t Z_i(\tau)d\tau, Z_i, b_i, t) = \frac{2\mathcal{N}_{2Ii}}{t^{b_i}+1} \\ \mathcal{N}_{3_{Ii},t}(\int_0^t Z_i(\tau)d\tau, Z_i, b_i, t) = \frac{2\mathcal{N}_{3Ii}}{t^{b_i}+1} \end{cases} \tag{6}$$

with $b_i > 0$.

Remark 2: According to the original definition of integration, "$\int_0^t Z_i(\tau)d\tau$" represents the accumulation of the signal "Z_i" over time interval $[0, t]$. Incorporating this in control design can significantly improve the steady-state accuracy. However, it is worth emphasizing that, in the absence of suitable constraints, this integration can readily lead to saturation—the long-standing "windup" phenomenon in most PID/PI control schemes. While various "anti-windup" methods are reported in the literature [25], [26], most of them lack a systematic or analytical rigor, casting doubts on their effectiveness in practical applications. In this study, we propose an alternative approach to process and utilize the feedback signal "$\int_0^t Z_i(\tau)d\tau$" through the integral trimmer $\mathcal{N}_{Ii}(\int_0^t Z_i(\tau)d\tau, Z_i, b_i)$, which is designed to be uniformly bounded or ultimately diminishing. This unique characteristic enables it to effectively circumvent the "windup phenomenon" triggered by the integration term.

Derivative modulating function (or derivative trimmer): A continuously differentiable function $\mathcal{N}_D(\dot{Z}, \mathbf{c}) = [\mathcal{N}_{Di}(\dot{Z}_i, c_i)] \in \mathbb{R}^m$ with $\mathcal{N}_{Di} : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ is referred to as derivative modulating function (or derivative trimmer) if, for any $c_i > 0$, the following conditions are satisfied:

1) $\mathcal{N}_{Di}(0, c_i) = 0$;
2) $\frac{\mathcal{N}_{Di}(\dot{Z}_i, c_i)}{\partial \dot{Z}_i}$ is bounded and positive for any $\dot{Z}_i$;
3) $\lim_{c_i \to 0} \mathcal{N}_{Di}(\dot{Z}_i, c_i) = \dot{Z}_i$.

The following $\mathcal{N}_{Di}(\dot{Z}_i, c_i)$ are some examples:

$$\begin{cases} \mathcal{N}_{1Di}(\dot{Z}_i, c_i) = r_1 \frac{2}{c_i} \frac{1-\mathbf{e}^{-c_i \dot{Z}_i}}{1+\mathbf{e}^{-c_i \dot{Z}_i}} + r_2 \dot{Z}_i \\ \mathcal{N}_{2Di}(\dot{Z}_i, c_i) = r_1 \frac{\dot{Z}_i}{\sqrt{(c_i \dot{Z}_i)^2 + 1}} + r_2 \dot{Z}_i \end{cases}$$

$$\left\{ \mathcal{N}_{3Di}(\dot{Z}_i, c_i) = \frac{\dot{Z}_i^3}{c_i \arctan^2(\dot{Z}_i) + \dot{Z}_i^2} \right. \tag{7}$$

with $c_i > 0$, $r_1, r_2 > 0$, and $r_1 + r_2 = 1$.

Remark 3: In accordance with the definition of the derivative, it is evident that this operation captures the inherent fluctuation trend of the signal, thus embodying a valuable "predictive" capability. To strategically harness this predictive capacity in alignment with the designer's intention, we deliberately impose a constraint on this predictive action by directing "$\dot{Z}_i$" into $\mathcal{N}_{Di}(\dot{Z}_i, c_i)$. It is important to note that $\mathcal{N}_{Di}(\dot{Z}_i, c_i)$ exhibits different slopes and curvatures with different parameter values of "c_i."

In our subsequent control development, these three trimming functions will be integrated into the control scheme, culminating in a nonlinear adaptive PID control strategy, with each function serving the following role.

1) The proportional trimming function—providing an overall control action proportional to the error signal through the all-pass gain factor. By feeding Z_i into $\mathcal{N}_{Pi}(Z_i, a_i)$ and properly choosing the parameter "a_i," we can control the incremental change of $\mathcal{N}_{Pi}(Z_i, a_i)$ and further modulate the control action. Such action is actually based on the penalty/punishment mechanism in that if the error (mistake) Z_i is larger, then the "punishment" is more severe, and if there is no error (mistake) (i.e., $Z_i = 0$), then there is no punishment. The function $\mathcal{N}_{Pi}(Z_i, a_i)$ plays a role of constraining the penalty.
2) The integral trimming function—reducing steady-state errors through low-frequency compensation by an integrator. By inputting $\int_0^t Z_i(\tau)d\tau$ into $\mathcal{N}_{Ii}(\int_0^t Z_i(\tau)d\tau, Z_i, b_i)$ with b_i chosen properly, we can impose certain constraints on $\int_0^t Z_i(\tau)d\tau$ to effectively prevent the occurrence of integral saturation.
3) The derivative trimming function—improving transient response through high-frequency compensation by a differentiator. By feeding $\dot{Z}_i$ into $\mathcal{N}_{Di}(\dot{Z}_i, c_i)$, and choosing c_i properly, we can adjust the impact of the derivative (predictive) term.

It is interesting to note that our constructed $\mathcal{N}_{Pi}(Z_i, a_i)$, $\mathcal{N}_{Ii}(\int_0^t Z_i(\tau)d\tau, Z_i, b_i)$, and $\mathcal{N}_{Di}(\dot{Z}_i, c_i)$ achieve a fusion of linear and nonlinear components, allowing for a smooth and natural transition from nonlinear to linear behavior. The elegance of this approach is quite remarkable. In fact, note that the parameters a_i, b_i, and c_i in these trimmers play the role of dictating the nonlinearity (range, magnitude, and curvature) of the corresponding function. Interestingly, as $r_1 + r_2 = 1$, $\beta_1 + \beta_2 = 1$, and $\mu_i \to 0$, it is straightforward to show that $\lim_{a_i \to 0} \mathcal{N}_{Pi}(Z_i, a_i) = Z_i$, $\lim_{b_i \to 0} \mathcal{N}_{Ii}(\int_0^t Z_i(\tau)d\tau, Z_i, b_i) = \int_0^t Z_i(\tau)d\tau$, and $\lim_{c_i \to 0} \mathcal{N}_{Di}(\dot{Z}_i, c_i) = \dot{Z}_i$, implying that the three trimmers explicitly include the standard PID components commonly employed in many existing PID control methods as special cases.

B. Two Useful Lemmas

Lemma 1: Define $Z \in R^m$ as

$$Z = \tilde{E}^T \tilde{\iota} \tag{8}$$

where $\tilde{E} = [E, \ldots, E^{(n-2)}]^T \in \mathbb{R}^{(n-1) \times m}$ with $E = x_1 - y^*$ being the output tracking error in (1) and $\tilde{\iota} = [\iota_0, \ldots, \iota_{n-3}, 1]^T \in \mathbb{R}^{(n-1)}$ with ι_0 to ι_{n-3} being positive constants chosen such that the roots of the polynomial $\theta^{(n-2)} + \iota_{n-3}\theta^{(n-3)} + \cdots + \iota_0 = 0$ have negative real parts. If $Z(t) \to \mathbf{0}_m$ and $\dot{Z}(t) \to \mathbf{0}_m$ as $t \to \infty$, then $E^{(k)}(t) \to \mathbf{0}_m$ as $t \to \infty$ $(k = 0, \ldots, n-1)$.

Proof: See [28]. ■

Lemma 2: Consider the continuous function $\eta : \mathbb{R}_{\geq 0} \to \mathbb{R}$, if a continuously differentiable function $\mathbf{V} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ satisfies

$$\dot{\mathbf{V}}(t) \leq -\alpha \mathbf{V}(t) + \sqrt{2\mathbf{V}(t)}\eta(t) \tag{9}$$

for positive constant α, and $\eta(t)$ tends to zero as $t \to \infty$, then $\mathbf{V}(t)$ converges to zero as $t \to \infty$.

Proof: Obviously, if $\mathbf{V} = 0$, the result holds naturally. So, for $\mathbf{V} > 0$, dividing by $\sqrt{\mathbf{V}}$ and multiplying by $\mathbf{e}^{\frac{1}{2}\alpha t}$on both sides of (9), yields

$$\frac{d(2\mathbf{e}^{\frac{1}{2}\alpha t}\sqrt{\mathbf{V}})}{dt} \leq \mathbf{e}^{\frac{1}{2}\alpha t}\sqrt{2}\eta(t). \tag{10}$$

Integrating both sides of (10) over the interval $[0, t)$ and carrying out certain calculations, yields

$$\sqrt{\mathbf{V}} \leq \mathbf{e}^{-\frac{1}{2}\alpha t}\sqrt{\mathbf{V}(0)} + \frac{\sqrt{2}}{2}\mathbf{e}^{-\frac{1}{2}\alpha t}\int_0^t \mathbf{e}^{\frac{1}{2}\alpha\tau}\eta(\tau)d\tau. \tag{11}$$

From (11), it is clear that if $\int_0^t \mathbf{e}^{\frac{1}{2}\alpha\tau}\eta(\tau)d\tau$ is bounded, then, $\lim_{t \to \infty}\sqrt{\mathbf{V}(t)} = 0$, and even if $\int_0^t \mathbf{e}^{\frac{1}{2}\alpha\tau}\eta(\tau)d\tau$ is unbounded, by using L'Hôpital's rule on $\frac{\sqrt{2}}{2}\mathbf{e}^{-\frac{1}{2}\alpha t}\int_0^t \mathbf{e}^{\frac{1}{2}\alpha\tau}\eta(\tau)d\tau$, it follows that:

$$\begin{aligned} \lim_{t\to\infty}\frac{\sqrt{2}}{2}\mathbf{e}^{-\frac{1}{2}\alpha t}\int_0^t \mathbf{e}^{\frac{1}{2}\alpha\tau}\eta(\tau)d\tau &= \frac{\sqrt{2}}{2}\lim_{t\to\infty}\frac{\int_0^t \mathbf{e}^{\frac{1}{2}\alpha\tau}\eta(\tau)d\tau}{\mathbf{e}^{\frac{1}{2}\alpha t}} \\ &= \lim_{t\to\infty}\frac{\sqrt{2}}{\alpha}\eta(t) \end{aligned} \tag{12}$$

indicating that the second term in (11) goes to zero if $\eta(t)$ tends to zero as time goes to infinity. Then from (11), it is concluded that $\sqrt{\mathbf{V}}$ (i.e., $\mathbf{V}$) tends to zero as $\eta(t)$ goes to zero.

Remark 4: Note that in (9), even if $\eta(t) = \eta(\mathbf{V}(t))$, the result still holds as long as $\eta(\mathbf{V}(t))$ approaches zero as t tends to infinity, which can be easily demonstrated by following a similar reasoning to that used in the proof of Lemma 2. This lemma is crucial for deriving the main results presented in the subsequent sections.

IV. Main Results

Utilizing the results presented in Section III, we now establish a nonlinear adaptive PID control scheme for square and nonsquare MIMO nonlinear systems.

For later technical development, we introduce the filtered (generalized) error vector as given in (8). Thus, (1) can be expressed in terms of Z as

$$\ddot{Z} = Gu + B \tag{13}$$

where $B = F - y^{*(n)} + \iota_{n-3}E^{(n-1)} + \cdots + \iota_0\ddot{E}$ represents lumped uncertainty.

Using Z and $\dot{Z}$ (both are available for feedback), we construct the following nonlinear filter to generate a new signal with nonlinearly modulated PID ingredients:

$$S = \omega_1\mathcal{N}_P(Z,\boldsymbol{a}) + \omega_2\mathcal{N}_I\left(\int_0^t Z(\tau)d\tau, Z, \boldsymbol{b}\right) + \omega_3\mathcal{N}_D(\dot{Z},\boldsymbol{c}) \tag{14}$$

where $\omega_j > 0$ $(j = 1,2,3)$ are user-chosen positive constants, $\mathcal{N}_P(Z,\boldsymbol{a}) = [\mathcal{N}_{Pi}(Z_i,a_i)] \in \mathbb{R}^m$, $\mathcal{N}_I(\int_0^t Z(\tau)d\tau, Z, \boldsymbol{b}) = [\mathcal{N}_{Ii}(\int_0^t Z_i(\tau)d\tau, Z_i, b_i)] \in \mathbb{R}^m$ and $\mathcal{N}_D(\dot{Z},\boldsymbol{c}) = [\mathcal{N}_{Di}(\dot{Z}_i,c_i)] \in \mathbb{R}^m$ $(i = 1,\ldots,m)$ with $\mathcal{N}_{Pi}(Z_i,a_i) \in \mathbb{R}$, $\mathcal{N}_{Ii}(\int_0^t Z_i(\tau)d\tau, Z_i, b_i) \in \mathbb{R}$, and $\mathcal{N}_{Di}(\dot{Z}_i,c_i) \in \mathbb{R}$ being the PID trimmers satisfying the conditions as described in Section III and a, b, and \$c\$ being user-design parameter vectors with appropriate dimensions.

The design philosophy is to make $S(t)$ in (14) converge to zero asymptotically, then by the following lemma it is established that $Z(t)$ and $\dot{Z}(t)$ (thus $E^{(k)}(t)$ by Lemma 1) converge to zero asymptotically.

Lemma 3: Let $S(t)$ be generated by (14), if $S(t) \to \mathbf{0}_m$ as $t \to \infty$, then $Z(t) \to \mathbf{0}_m$ and $\dot{Z}(t) \to \mathbf{0}_m$ as $t \to \infty$.

Proof: By using the mean value theorem [22], [24], there exist points $\xi_i \in (\min\{0,\dot{Z}_i\}, \max\{0,\dot{Z}_i\})$, such that the function $\mathcal{N}_D(\dot{Z},\boldsymbol{c})$ can be expressed as

$$\mathcal{N}_D(\dot{Z},\boldsymbol{c}) = \mathcal{N}_D(\mathbf{0},\boldsymbol{c}) + \frac{\partial\mathcal{N}_D(\xi,\boldsymbol{c})}{\partial\xi}\dot{Z} = \frac{\partial\mathcal{N}_D(\xi,\boldsymbol{c})}{\partial\xi}\dot{Z} \tag{15}$$

where $\mathcal{N}_D(\mathbf{0},\boldsymbol{c}) = \mathbf{0}_m$ and

$$\frac{\partial\mathcal{N}_D(\xi,\boldsymbol{c})}{\partial\xi} = \operatorname{diag}\left\{\frac{\partial\mathcal{N}_{D1}(\xi_1,c_1)}{\partial\xi_1},\ldots,\frac{\partial\mathcal{N}_{Dm}(\xi_m,c_m)}{\partial\xi_m}\right\} \tag{16}$$

with $0 < \underline{\gamma}_i \leq \frac{\partial\mathcal{N}_{Di}(\xi_i,c_i)}{\partial\xi_i} \leq \bar{\gamma}_i < \infty$. Based on (14) and (15), we have

$$\dot{Z} = -\frac{\omega_1}{\omega_3}\left(\frac{\partial\mathcal{N}_D(\xi,\boldsymbol{c})}{\partial\xi}\right)^{-1}\mathcal{N}_P(Z,\boldsymbol{a}) + \frac{1}{\omega_3}\left(\frac{\partial\mathcal{N}_D(\xi,\boldsymbol{c})}{\partial\xi}\right)^{-1} \times S - \frac{\omega_2}{\omega_3}\left(\frac{\partial\mathcal{N}_D(\xi,\boldsymbol{c})}{\partial\xi}\right)^{-1}\mathcal{N}_I\left(\int_0^t Z(\tau)d\tau, Z, \boldsymbol{b}\right). \tag{17}$$

Construct the following Lyapunov function:

$$\mathbf{V}_0 = \frac{1}{2}Z^TZ. \tag{18}$$

Taking time derivative of (18) along (17) to get

$$\begin{aligned}\dot{\mathbf{V}}_0 = &-\frac{\omega_1}{\omega_3}Z^T\left(\frac{\partial\mathcal{N}_D(\xi,\boldsymbol{c})}{\partial\xi}\right)^{-1}\mathcal{N}_P(Z,\boldsymbol{a})\\ &+\frac{1}{\omega_3}Z^T\left(\frac{\partial\mathcal{N}_D(\xi,\boldsymbol{c})}{\partial\xi}\right)^{-1}S\\ &-\frac{\omega_2}{\omega_3}Z^T\left(\frac{\partial\mathcal{N}_D(\xi,\boldsymbol{c})}{\partial\xi}\right)^{-1}\mathcal{N}_I\left(\int_0^t Z(\tau)d\tau, Z, \boldsymbol{b}\right).\end{aligned} \tag{19}$$

Based on property 2) of integral trimmer $(Z_i\mathcal{N}_{Ii}(\int_0^t Z_i(\tau)d\tau, Z_i, b_i) \geq 0)$, we have

$$\begin{aligned}&Z^T\left(\frac{\partial\mathcal{N}_D(\xi,\boldsymbol{c})}{\partial\xi}\right)^{-1}\mathcal{N}_I\left(\int_0^t Z(\tau)d\tau, Z, \boldsymbol{b}\right)\\ &= \sum_{i=1}^m \frac{1}{\left(\frac{\partial\mathcal{N}_{Di}(\xi_i,c_i)}{\partial\xi_i}\right)}Z_i\mathcal{N}_{Ii}\left(\int_0^t Z_i(\tau)d\tau, Z_i, b_i\right) \geq 0.\end{aligned} \tag{20}$$

Then, we have

$$\begin{aligned}\dot{\mathbf{V}}_0 \leq &-\frac{\omega_1}{\omega_3}Z^T\left(\frac{\partial\mathcal{N}_D(\xi,\boldsymbol{c})}{\partial\xi}\right)^{-1}\mathcal{N}_P(Z,\boldsymbol{a})\\ &+\frac{1}{\omega_3}Z^T\left(\frac{\partial\mathcal{N}_D(\xi,\boldsymbol{c})}{\partial\xi}\right)^{-1}S.\end{aligned} \tag{21}$$

Using the fact that $Z_i\mathcal{N}_{Pi}(Z_i,a_i) \geq \mu_1 Z_i^2$ with $\mu_1 > 0$, we obtain

$$\begin{aligned}&-Z^T\left(\frac{\partial\mathcal{N}_D(\xi,\boldsymbol{c})}{\partial\xi}\right)^{-1}\mathcal{N}_P(Z,\boldsymbol{a})\\ &\leq -\min\left\{\frac{\mu_1}{\frac{\partial\mathcal{N}_{Di}(\xi_i,c_i)}{\partial\xi_i}}\right\}Z^TZ\end{aligned} \tag{22}$$

it follows from (21) that:

$$\dot{\mathbf{V}}_0 \leq -\beta\mathbf{V}_0 + \sqrt{2\mathbf{V}_0}\eta_0(t) \tag{23}$$

where $\beta = 2\frac{\omega_1}{\omega_3}\min\{\frac{\mu_1}{\frac{\partial\mathcal{N}_{Di}(\xi_i,c_i)}{\partial\xi_i}}\}$ and $\eta_0(t) = \frac{1}{\omega_3}\|(\frac{\partial\mathcal{N}_D(\xi,\mathbf{c})}{\partial\xi})^{-1}\|\|S\|$.

Using Lemma 2, we have that $\sqrt{\mathbf{V}_0}$ (thus $\mathbf{V}_0$) tends to zero, thus $Z(t) \to \mathbf{0}_m$. Now with $Z(t) \to \mathbf{0}_m$ (thus $\mathcal{N}_P(\mathbf{0}_m,\boldsymbol{c}) = \mathbf{0}_m$ and $\mathcal{N}_I(\int_0^t Z(\tau)d\tau, \mathbf{0}_m, \boldsymbol{c}) = \mathbf{0}_m$) and $S(t) \to \mathbf{0}_m$, it follows that $\lim_{t\to\infty}\dot{Z}(t) = \mathbf{0}_m$ from (17). Hence, it is established that $\lim_{t\to\infty}Z(t) = \mathbf{0}_m$ and $\lim_{t\to\infty}\dot{Z}(t) = \mathbf{0}_m$ as $\lim_{t\to\infty}S(t) = \mathbf{0}_m$. ■

A. Nonlinear PID Control for Nonlinear Square MIMO Systems

We first explore full state asymptotic tracking control of the nth order square MIMO systems (1) with $\nu = m$.

Assumption 1: The matrix $G \in \mathbb{R}^{m\times m}$ is unknown and time-varying, not necessarily symmetric, and belongs to

$$\mathbb{G}_\Omega := \left\{G \in \mathbb{R}^{m\times m}\ \middle|\ \frac{G+G^T}{2}\text{is positive definite}\quad \forall x,\theta,u\right\}. \tag{24}$$

Assumption 2: Certain crude structural information on F is available to allow unknown constant b^* and known scalar-valued smooth function $\psi^*(x)$ to be extracted, such that F belongs to

$$\mathbb{F}_{b^*,\psi^*(x)} := \left\{F \in \mathbb{R}^m\ \middle|\ \|F\| \leq b^*\psi^*(x)\quad \forall x,\theta,u\right\} \tag{25}$$

where $\psi^*(x)$ is bounded, if x is bounded. As $\psi^*(0)$ might be nonzero, $\psi^*(x)$ is nonvanishing. ■

Remark 5: For an MIMO system, Assumption 1 is a necessary condition to ensure its controllability, and most practical systems [22], [23] (e.g., Euler–Lagrange systems) satisfy such condition, which implies that there exists some unknown constant λ_1, such that $0 < \lambda_1 \leq \lambda_{\min}(\frac{G+G^T}{2}) < +\infty$ with $\lambda_{\min}(\cdot)$ being minimal eigenvalue of $(\cdot)$, and $\|G\| \leq \bar{G} < \infty$ with $\bar{G}$ being a unknown constant. As for Assumption 2, only certain crude structure information on nonlinear functions is needed to extract the core function of the system.

By making use of the three modulating (trimming) functions $\mathcal{N}_P$, $\mathcal{N}_I$, and $\mathcal{N}_D$, we generate the signal S driven by the nonlinear PID components as in (14), upon which we build the following nonlinear adaptive PID-like control:

$$u = u_r + u_a \tag{26}$$

with

$$u_r = -k_d \frac{\partial \mathcal{N}_D(\dot{Z}, \boldsymbol{c})}{\partial \dot{Z}} S \tag{27}$$

$$u_a = -\frac{\hat{b}_r \psi_r^2}{\psi_r \|S\| + \varepsilon_r(t)} \frac{\partial \mathcal{N}_D(\dot{Z}, \boldsymbol{c})}{\partial \dot{Z}} S \tag{28}$$

where $k_d > 0$ and the adaptive parameter $\hat{b}_r$ needs to be updated by the algorithm defined by the following set $\Omega_{\dot{\hat{b}}_r}$:

$$\Omega_{\dot{\hat{b}}_r} := \left\{ \dot{\hat{b}}_r \in \mathbb{R}_{\geq 0} \,\middle|\, \dot{\hat{b}}_r = \tfrac{l_r \psi_r^2 \|S\|^2}{\psi_r \|S\| + \varepsilon_r(t)},\ \hat{b}_r(0) \geq 0 \right\} \tag{29}$$

where $\hat{b}_r$ is the estimate of b_r (a virtual parameter to be defined later), $l_r > 0$ a user designed constant, ψ_r is a nonlinear function as defined by (35), and $\varepsilon_r(t)$ is a positive scalar function chosen to satisfy $\int_0^t \varepsilon_r(\tau) d\tau < \bar{\varepsilon}_r < \infty$.

Theorem 1: Consider the square MIMO system (1), satisfying Assumptions 1 and 2, let the control scheme (26) with adaptive law (29) be applied, where the weighting factors $\omega_i > 0$ $(i = 1, 2, 3)$ are freely user-chosen constants. Then, full-state asymptotic tracking with a bounded comprehensive performance index, as characterized in (2) and (3), is achieved with bounded control action.

Proof: Using (13) and (14), we have

$$\begin{aligned} \dot{S} &= \omega_1 \dot{\mathcal{N}}_P(Z, \boldsymbol{a}) + \omega_2 \dot{\mathcal{N}}_I \left(\textstyle\int_0^t Z(\tau) d\tau, Z, \boldsymbol{b} \right) \\ &\quad + \omega_3 \frac{\partial \mathcal{N}_D(\dot{Z}, \boldsymbol{c})}{\partial \dot{Z}} \ddot{Z} \\ &= \omega_3 \frac{\partial \mathcal{N}_D(\dot{Z}, \boldsymbol{c})}{\partial \dot{Z}} G u + R \end{aligned} \tag{30}$$

where $R = \omega_1 \dot{\mathcal{N}}_P + \omega_2 \dot{\mathcal{N}}_I + \omega_3 \frac{\partial \mathcal{N}_D}{\partial \dot{Z}} B$.

As the resultant control gain matrix $\frac{\partial \mathcal{N}_D}{\partial \dot{Z}} G$ in (30) is unknown and time-varying yet not even symmetric, in order to carry out Lyapunov based stability analysis, we introduce the matrix

$$\begin{cases} G_D = \dfrac{\frac{\partial \mathcal{N}_D}{\partial \dot{Z}} (G + G^T) \frac{\partial \mathcal{N}_D}{\partial \dot{Z}}}{2} \\ \tilde{G}_D = \dfrac{\frac{\partial \mathcal{N}_D}{\partial \dot{Z}} (G - G^T) \frac{\partial \mathcal{N}_D}{\partial \dot{Z}}}{2} \end{cases} \tag{31}$$

such that

$$\frac{\partial \mathcal{N}_D}{\partial \dot{Z}} G \frac{\partial \mathcal{N}_D}{\partial \dot{Z}} = \underset{\text{positive definite}}{G_D} + \underset{\text{skew symmetric}}{\tilde{G}_D}. \tag{32}$$

Note that with Assumption 1, G_D is positive definite and $\tilde{G}_D$ is skew symmetric, thus there exists some unknown constant λ_r such that $0 < \lambda_r \leq \lambda_{\min}(G_D)$ and $S^T \tilde{G}_D S = 0$.

Choosing the Lyapunov function candidate as

$$V_r = \frac{1}{2} S^T S + \frac{1}{2\omega_3 l_r \lambda_r} \tilde{b}_r^2 \tag{33}$$

where the estimate error is defined as $\tilde{b}_r = b_r - \omega_3 \lambda_r \hat{b}_r$ with b_r being a virtual parameter defined below. Taking the time derivative of (33) along (30) gives

$$\dot{V}_r = \omega_3 S^T \frac{\partial \mathcal{N}_D(\dot{Z}, \boldsymbol{c})}{\partial \dot{Z}} G u + S^T R - \frac{1}{l_r} \tilde{b}_r \dot{\hat{b}}_r \tag{34}$$

similarly, using Assumption 2 for R, we have $\|R\| \leq b_r \psi_r$ where $b_r = \max\{1, b^*\} > 0$ and

$$\begin{aligned} \psi_r &= \|\omega_1 \dot{\mathcal{N}}_P + \omega_2 \dot{\mathcal{N}}_I\| + \omega_3 \left\| \frac{\partial \mathcal{N}_D}{\partial \dot{Z}} \right\| \Big(\psi^* + \| - y^{*(n)} \\ &\quad + \iota_{n-3} E^{(n-1)} + \cdots + \iota_0 \ddot{E} \| \Big). \end{aligned} \tag{35}$$

Therefore, it follows that:

$$S^T R \leq b_r \varepsilon_r(t) + \frac{b_r \psi_r^2 \|S\|^2}{\psi_r \|S\| + \varepsilon_r(t)}. \tag{36}$$

By (36), using control scheme (26) and updating law $\dot{\hat{b}}_r$ (29), it is not difficult to show from (34) that

$$\dot{V}_r \leq -\bar{\sigma} \|S\|^2 + b_r \varepsilon_r(t) \tag{37}$$

where $\bar{\sigma} = \omega_3 \lambda_r k_d > 0$. By integrating (37) over \$[0, t]\$, we get

$$V_r \leq V_r(0) - \bar{\sigma} \int_0^t \|S\|^2 d\tau + b_r \bar{\varepsilon}_r. \tag{38}$$

From (38), it is obvious that $V_r(t)$ and $\int_0^t \|S\|^2 d\tau$ are less than $V_r(0) + b_r \bar{\varepsilon}_r$, thus it can be concluded that $V_r(t) \in L_\infty$ and $\int_0^t \|S\|^2 d\tau \in L_\infty$, and therefore $\|S\|$ belongs to L_2. Due to the boundedness of $V_r(t)$, it is not difficult to prove that $\|S\|$ and $\|\dot{S}\|$ are also bounded, in the light of $\|S\| \in L_\infty \cap L_2$ and $\|\dot{S}\| \in L_\infty$, based on Barbalat Lemma [28], it is established that $\lim_{t \to \infty} S(t) = \mathbf{0}_m$, then by Lemma 3, it is obtained that $\lim_{t \to \infty} Z(t) = \mathbf{0}_m$ and $\lim_{t \to \infty} \dot{Z}(t) = \mathbf{0}_m$, therefore, it follows by Lemma 1 that $\lim_{t \to \infty} E^{(k)}(t) = \mathbf{0}_m$.

Furthermore, the tracking performance index $J_E = \int_0^\infty \|S(\tau)\|^2 d\tau$ is bounded. Because, from (38), it holds that

$$J_E = \lim_{t \to \infty} \int_0^t \|S(\tau)\|^2 \, d\tau \leq \frac{1}{\bar{\sigma}} (V_r(0) + b_r \bar{\varepsilon}_r) < \infty. \tag{39}$$

In addition, during the entire process of system operation, the proposed control action u remains bounded. In fact, from (26), (27), and (28), we have

$$\begin{aligned} \|u\| &= \left\| -k_d \frac{\partial \mathcal{N}_D(\dot{Z}, \boldsymbol{c})}{\partial \dot{Z}} S - \frac{\hat{b}_r \psi_r^2}{\psi_r \|S\| + \varepsilon_r(t)} \frac{\partial \mathcal{N}_D(\dot{Z}, \boldsymbol{c})}{\partial \dot{Z}} S \right\| \\ &\leq k_d \left\| \frac{\partial \mathcal{N}_D(\dot{Z}, \boldsymbol{c})}{\partial \dot{Z}} \right\| \|S\| + \hat{b}_r \psi_r \left\| \frac{\partial \mathcal{N}_D(\dot{Z}, \boldsymbol{c})}{\partial \dot{Z}} \right\| \\ &\quad \times \frac{\psi_r \|S\|}{\psi_r \|S\| + \varepsilon_r(t)} \\ &\leq k_d \left\| \frac{\partial \mathcal{N}_D(\dot{Z}, \boldsymbol{c})}{\partial \dot{Z}} \right\| \|S\| + \hat{b}_r \psi_r \left\| \frac{\partial \mathcal{N}_D(\dot{Z}, \boldsymbol{c})}{\partial \dot{Z}} \right\|. \end{aligned}$$

The boundedness of $\hat{b}_r$, ψ_r, and $\|S\|$ can be ensured by the boundedness of the Lyapunov function V_r. The boundedness of $\|\frac{\partial \mathcal{N}D(\dot{Z}, c)}{\partial \dot{Z}}\|$ is derived from the properties of the derivative trimming function (or derivative trimmer). Thus, our controller remains bounded throughout the entire time range, maintaining stability and effectiveness throughout the operation. ■

B. Nonlinear PID Control for Nonlinear Nonsquare MIMO Systems

Let us consider a more generic class of nonsquare uncertain systems (1) with the following assumption being imposed.

Assumption 3: The matrix $G \in \mathbb{R}^{m\times\nu}$ can be decomposed as $G = AN$, where $A \in \mathbb{R}^{m\times\nu}$ is known and bounded full row rank with elements consisting of 0s and 1s, $N \in \mathbb{R}^{\nu\times\nu}$ is an unknown square matrix factored from G. There exists an unknown diagonal positive definite matrix $M \in \mathbb{R}^{m\times m}$ such that the matrix $G_M := MGA^T + AG^TM$ is uniformly positive definite or negative definite (without losing generality the former is assumed here). In addition, $\|M\| \le \bar{M}$ and $\|\dot{M}\| \le \bar{\dot{M}}$ for some positive constants $\bar{M}$ and $\bar{\dot{M}}$.

Remark 6: It is worth mentioning that for nonsquare systems, the unknown control gain G poses significant challenges in control design and stability analysis. When utilizing the decomposition of the gain matrix $G = AN$ as in [17], [18], and [27], it is typically required that the matrix $N + N^T$ be positive definite, which is quite restrictive. In contrast, our proposed decomposition method employs a matrix A that consists solely of 0s and 1s with simple constituent elements. Simultaneously, we introduce an unknown diagonal positive definite matrix $M \in \mathbb{R}^{m\times m}$. This approach allows us to gracefully relax the previous conditions and broaden the applicability of the proposed method to nonsquare systems. To illustrate this more clearly, let $G = \begin{bmatrix} 0.72 + 7.2\sin(x_1) & 36 & 0 \\ 0 & 72 + 7.2\sin(x_2) & 36 \end{bmatrix}$, and by selecting $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$, we then obtain $N = \begin{bmatrix} 0.72 + 7.2\sin(x_1) & -36 - 7.2\sin(x_2) & 0 \\ 0 & 72 + 7.2\sin(x_2) & 0 \\ 0 & 0 & 36 \end{bmatrix}$. It can be observed that $N + N^T$ is not positive definite, while the condition specified in Assumption 3 remains valid because there exists a diagonal positive definite matrix $M = \begin{bmatrix} 2 + 0.1\sin(x_1) & 0 \\ 0 & 1 \end{bmatrix}$ such that the matrix $G_M := MGA^T + AG^TM$ is uniformly positive definite, showing that the controllability condition here is less demanding as compared with [17], [18], [27].

For nonlinear system (1) in nonsquare form, we present the following nonlinear adaptive PID control scheme

$$u = u_r + u_a \tag{40}$$

with

$$u_r = -k_d \frac{A^T}{\|A\|} \frac{\partial \mathcal{N}_D(\dot{Z}, \boldsymbol{c})}{\partial \dot{Z}} S \tag{41}$$

$$u_a = -\frac{\hat{b}_M \psi_M^2}{\psi_M \|S\| + \varepsilon_M(t)} \frac{A^T}{\|A\|} \frac{\partial \mathcal{N}_D(\dot{Z}, \boldsymbol{c})}{\partial \dot{Z}} S \tag{42}$$

where $k_d > 0$ and the adaptive parameter $\hat{b}_M$ needs to be updated by the algorithm defined by the following set $\Omega_{\dot{\hat{b}}_M}$:

$$\Omega_{\dot{\hat{b}}_M} := \left\{ \dot{\hat{b}}_M \in \mathbb{R}_{\ge 0} \left| \dot{\hat{b}}_M = \frac{l_M \psi_M^2 \|S\|^2}{\psi_M \|S\| + \varepsilon_M(t)},\ \hat{b}_M(0) \ge 0 \right. \right\} \tag{43}$$

where $\hat{b}_M$ is the estimate of b_M (a virtual parameter to be defined later), $l_M > 0$ a user designed constant, $\psi_M = \psi_r + \|S\|$ is a nonlinear function, and $\varepsilon_M(t)$ is a positive scalar function chosen to satisfy $\int_0^t \varepsilon_M(\tau)d\tau < \bar{\varepsilon}_M < \infty$.

Corollary 1: Consider the nonsquare MIMO system described in (1), which satisfies Assumptions 2 and 3. Let the control scheme outlined in (40) be applied with the adaptive law given in (43), where the weighting factors $\omega_i > 0$ $(i = 1, 2, 3)$ are freely chosen by the user. Then, full-state asymptotic tracking with a bounded tracking performance index, as characterized in (2) and (3), is ensured with bounded control action.

Proof: Choosing the Lyapunov function candidate as

$$V_M = \frac{1}{2} S^T M S + \frac{1}{2\omega_3 \lambda_M l_M} \tilde{b}_M^2 \tag{44}$$

where the estimate error is defined as $\tilde{b}_M = b_M - \omega_3 \lambda_M \hat{b}_M$ with $b_M = \max\{\bar{M} b_r, \frac{1}{2}\bar{\dot{M}}\} > 0$ being a virtual parameter.

The proof process is similar to Theorem 1, it is omitted here. Finally, using control scheme (40) and updating law $\dot{\hat{b}}_M$ (43), it is not difficult to show that

$$\dot{V}_M \le -\bar{\sigma}_M \|S\|^2 + b_M \varepsilon_M(t) \tag{45}$$

where $\bar{\sigma}_M = \omega_3 \lambda_M k_d > 0$ with $0 < \lambda_M \le \frac{\lambda_{\min}(G_M)}{\|A\|}$ and $G_M = \frac{\frac{\partial \mathcal{N}_D}{\partial \dot{Z}} (MGA^T + AG^TM) \frac{\partial \mathcal{N}_D}{\partial \dot{Z}}}{2}$. By using the argument similar to that in the proof of Theorem 1, we get that $\lim_{t\to\infty} S(t) = \mathbf{0}_m$, then by Lemma 3, we have $\lim_{t\to\infty} Z(t) = \mathbf{0}_m$ and $\lim_{t\to\infty} \dot{Z}(t) = \mathbf{0}_m$. The analysis on the boundedness of the tracking performance index and u is analogous to that presented in Theorem 1. ■

Remark 7: A notable feature of the proposed nonlinear adaptive PID control method is its ability to encompass existing linear PID controls as special forms. Our approach, which involves integrating nonlinear trimming functions and user-adjustable parameters, enables the creation of a variety of nonlinear PID controllers. These controllers can seamlessly transition between linear and nonlinear behaviors, a transition made possible through appropriate parameter selection. Furthermore, the method offers a wide range of options for nonlinear trimming functions, with those presented in this article representing just a subset, highlighting significant design flexibility and diversity. By transitioning from the traditional "linear combination" framework to a "nonlinear recalibration" strategy, our efforts result in a novel and compelling control solution.

V. Simulation Verification

We now examine the applicability of the proposed nonlinear PID design approach for the well-known robotic systems. Consider a two joint rigid-link robotic manipulator with the following joint space dynamics [23]:

$$\mathcal{M}(q, \ell)\ddot{q} + \mathcal{W}_c(q, \dot{q}, \ell)\dot{q} + \mathcal{G}_g(q, \ell) + \tau_d = u \tag{46}$$

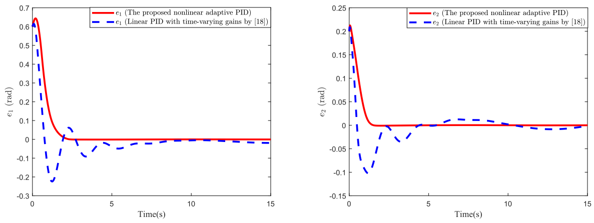
where $\mathcal{M}(q, \ell) \in \mathbb{R}^{2\times 2}$ is the generalized inertia matrix, $\mathcal{W}_c(q, \dot{q}, \ell) \in \mathbb{R}^{2\times 2}$ represents coriolis and centrifugal forces, $\mathcal{G}_g(q, \ell) \in \mathbb{R}^2$ denotes the gravitation vector, $\tau_d \in \mathbb{R}^2$ models the unknown external disturbance, $q(t), \dot{q}(t), \ddot{q}(t) \in \mathbb{R}^2$ denote the link position, velocity, and acceleration vectors, respectively. $u \in \mathbb{R}^2$ is the control input and $\ell \in \mathbb{R}^6$ represents the unknown parameter vector. Specifically, the values are assigned to these variables, see [23] for details.

For the purpose of simulation, we define the tracking output error as $Z = e = q - y^*$, where the desired signal $y^* = [0.3\cos(t), 0.4\cos(t)]^T(\text{rad})q(0) =, \ [0.9, 0.6]^T(\text{rad}),\ \dot{q}(0) = [0, 0]^T(\text{rad/s})$, and $\hat{b}_r(0) = 0$. The nonlinear adaptive PID tracking control for robotic systems is given as (26), where $S = \omega_1 \mathcal{N}_P(e, \boldsymbol{a}) + w_2 \mathcal{N}_I(\int_0^t e(\tau)d\tau, e, \boldsymbol{b}) + \omega_3 \mathcal{N}_D(\dot{e}, \boldsymbol{c})$. The PID trimming functions are selected as

$$\mathcal{N}_P(e, \boldsymbol{a}) = \left[\frac{1}{a_1} \frac{1 - \mathbf{e}^{-a_1 e_1}}{1 + \mathbf{e}^{-a_1 e_1}} + \frac{1}{2} e_1, \frac{1}{a_2} \frac{1 - \mathbf{e}^{-a_2 e_2}}{1 + \mathbf{e}^{-a_2 e_2}} + \frac{1}{2} e_2 \right]^T$$

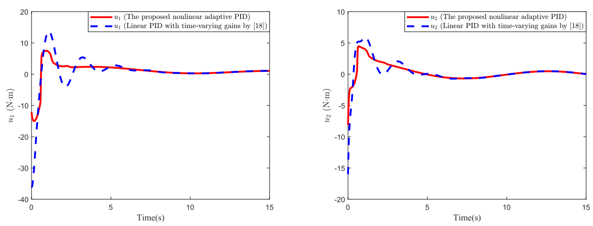
$$\mathcal{N}_I\left(\int_0^t e(\tau)d\tau, \boldsymbol{b} \right) = \left[\frac{2}{b_1} \frac{1 - \mathbf{e}^{-b_1 \int_0^t e_1(\tau)d\tau}}{1 + \mathbf{e}^{-b_1 \int_0^t e_1(\tau)d\tau}} \Gamma_1, \frac{2}{b_2} \frac{1 - \mathbf{e}^{-b_2 \int_0^t e_2(\tau)d\tau}}{1 + \mathbf{e}^{-b_2 \int_0^t e_2(\tau)d\tau}} \Gamma_2 \right]^T$$

$$\mathcal{N}_D(\dot{e}, \boldsymbol{c}) = \left[\frac{\dot{e}_1}{2\sqrt{(c_1 \dot{e}_1)^2 + 1}} + \frac{1}{2}\dot{e}_1, \frac{\dot{e}_2}{2(\sqrt{(c_2 \dot{e}_2)^2 + 1})} + \frac{1}{2}\dot{e}_2 \right]^T$$



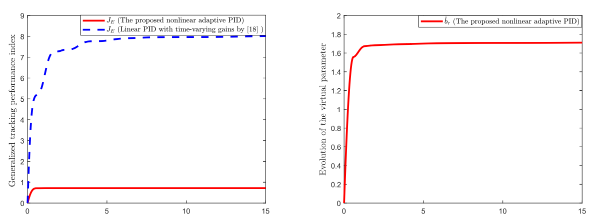
(a) (b)

Fig. 1. Comparison of the tracking process e_i $(i = 1, 2)$. (a) and (b) exhibit the comparisons of the tracking processes e_1 and e_2, respectively.



(a) (b)

Fig. 2. Comparison of the control action u_i $(i = 1, 2)$. (a) and (b) show the comparisons of the control actions u_1 and u_2, respectively.



(a) (b)

Fig. 3. Generalized tracking performance index J_E comparison and the evolution of the virtual parameter $\hat{b}_r$. (a) and (b) illustrate the comparison of the generalized tracking performance index J_E and the evolution of the virtual parameter $\hat{b}_r$, respectively.

where $a_j = c_j = b_j = 0.2$, $\Gamma_j = \tanh(\int_0^t e_j(\tau)d\tau)\tanh(e_j)$ $(j = 1, 2)$. Based on Assumptions 1 and 2, the core function is $\psi_r = \|\dot{\mathcal{N}}_P\| + \|\dot{\mathcal{N}}_I\| + \|\frac{\partial \mathcal{N}_D}{\partial \dot{e}}\|(\|\dot{q}\|^2 + 3)$. Also, the relevant parameters selection are: $k_d = 8$, $\omega_1 = 7$, $\omega_2 = 12$, $\omega_3 = 1$, and $l_r = 0.5$, and the function $\varepsilon_r(t) = \frac{1}{(1+t)^2}$.

To visually demonstrate the effectiveness of our proposed control scheme in handling nonlinear model with mutual coupling, we conducted a comparison between our proposed nonlinear adaptive PID control, linear fixed-gain PID, and linear time-varying gain PID. It is important to note that linear PID with fixed gains faces challenges in meeting the control requirements of such systems at a theoretical level, let alone in simulation. Therefore, we compared the linear PID with time-varying gains with our proposed control algorithm. The simulation results are illustrated in Figs. 1– 3, where it is evident from Fig. 1 that the proposed nonlinear adaptive PID scheme significantly outperforms the linear PID with time-varying gains in terms of transient tracking behavior and steady-state performance. Fig. 2 illustrates the magnitude and variability of the control input, revealing that our proposed nonlinear PID exhibits well-behaved control action. This is evidenced by the reduced amplitude and smoother variation of the control input when compared to the linear PID with time-varying gains. Furthermore, Fig. 3 demonstrates that the generalized tracking performance index and the evolution of the virtual parameter remain bounded throughout the process. While the simulation verification is not exhaustive, it clearly indicates that our method achieves superior performance relative to the existing linear PID control with time-varying gains [17]. Specifically, our approach enables full state asymptotic tracking with reduced and smoother control effort, along with an improved overall performance index (i.e., smaller accumulated tracking error).

VI. Conclusion

In this work, a nonlinear adaptive PID tracking control method is introduced for a class of MIMO nonlinear systems. This method incorporates three trimming (modulating) functions that reshape or nonlinearly transform the generalized tracking errors, establishing an analytical connection between fixed-gain linear PID control and nonlinear PID control. It encompasses both traditional fixed-gain PID control and time-varying gain PID control as special cases. The developed control algorithms ensure full-state asymptotic tracking while maintaining a bounded performance index, elegantly circumventing the long-standing "windup" issue associated with unrestricted integration in traditional PID control methods. Future research could focus on extending this method to more general nonlinear systems, including nonaffine and strict-feedback systems, as well as systems with delayed input, representing intriguing avenues for further investigation.

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