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How Should an Autonomous Vehicle Overtake a Slower Moving Vehicle: Design and Analysis of an Optimal Trajectory

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***Abstract*—Over the past few years, there has been much research on various aspects of control of autonomous vehicles. However, it seems that the problem of overtaking a slower moving vehicle has been somewhat neglected. This note deals with the three-phase overtaking maneuver and with designing a smooth and ergonomic optimal lane-change trajectory to be used under normal conditions. It is shown that the absolute shape, size, and time of the first-phase trajectory do not depend on the velocity of the leading, slower moving vehicle. Only the absolute point for initiating the diversion is affected. The relatively simple mathematical model for each lane-change trajectory is based on minimizing the total kinetic energy during the maneuver, superimposed on a "minimum-jerk trajectory." For high enough initial velocities, explicit formulas are obtained for the optimal distance and the optimal time of the maneuver. It is also shown that the total time is bounded from above and below, regardless of the velocity. By using the results of the suggested model, an autonomous vehicle, equipped with appropriate sensors, can estimate the best time and place to begin and end the overtaking and its total time and distance. This may help to make a decision whether to overtake or not.**

***Index Terms*—Autonomous vehicles, nonlinear optimization, obstacle avoidance.**

I. Introduction

The purpose of this note is to design an optimal trajectory for one vehicle under normal conditions, in order to overtake a single, slower-moving vehicle on a predetermined road. The following questions arise: At what distance from the other vehicle should the diversion from the lane begin? How long will each lane-change maneuver take? What is an optimal trajectory during each lane-change? Where and when can the overtaking vehicle return to its lane? How long will the complete overtaking maneuver take? The answers to these questions may aid the system to decide whether to overtake or not. In particular, if an emergency arises such as an oncoming vehicle, this decision could be crucial.

We first analyze the general three-phase overtaking maneuver and show that the absolute shape, size and time of the lane-change trajectory do not depend on the velocity of the obstacle. Then we design a smooth, optimal lane-change trajectory that is also ergonomic and comfortable for the passenger. The solution of the optimization problem determines the time and distance of the trajectory. It relies on formulating a nonlinear constrained optimization problem, and using optimization software to find an approximate closed-form solution. For the sake of simplicity and generality, the model does not explicitly take into account the dynamics of the vehicle or vehicle model. Therefore it can be applied to any kind of vehicle, including possible future concepts and technologies. All the forces acting upon the vehicle are embedded into one parameter—the maximal acceleration during the maneuver.

There is a vast amount of work on collision avoidance and trajectory design for autonomous vehicles. The specific problem of lane changing maneuvers is treated in [2], [4]–[6], and [8]–[10] using geometric reasoning, control theory or other methods. Some of them impose specific constraints on the dynamic variables of the vehicle or on parameters like acceleration, curvature, jerk, etc. and some specifically minimize parameters like time, distance, acceleration, curvature and such.

In [10] and [2], results are obtained for the distance to begin the diversion and the total time the lane-change maneuver takes, considering the vehicle dynamics. In [10], the objective is to minimize the clearing distance for emergency maneuvers in such a way that the diversion is still safe and feasible. Although there is some similarity between the results of [10] and [2] and the ones obtained in this note for the lane-change trajectory, the trajectories they generate are not necessarily smooth and they do not obtain closed-form formulas. Furthermore, they only consider lane-change maneuvers and not overtaking a moving vehicle.

II. General Overtaking Maneuver

Suppose an autonomous vehicle P (passing) is driving at a velocity V, say in the x direction. In front of it another car, O (obstacle) is driving at a constant velocity $0 < V_1 < V$ in the same direction. Vehicle P intends to pass it.

An overtaking maneuver consists of three phases: 1) diverting from the original lane, 2) driving straight in the adjacent lane, and 3) returning to the lane.

We shall consider planar translation, i.e., $x(t)$ and $y(t)$, where x is the original direction of motion and y is the orthogonal direction of diversion. The lane-changing trajectories could be any one of those suggested in the literature, as presented in the introduction. However, a different option is suggested in the following sections.

A. Phase 1): Diverting From the Lane

Denote the total x-direction distance traveled during the lane-change maneuver by D, the total y-direction distance by W (the width of the lane or of the diversion) and the total time duration by T. These values are determined by the specific lane-change trajectory. We set the origin of the (x, y, t) system at the point on the x-axis next to the peak of the diversion, where the front of vehicle P is located in the adjacent lane next to the rear of vehicle O, and set the time at that point to zero. In

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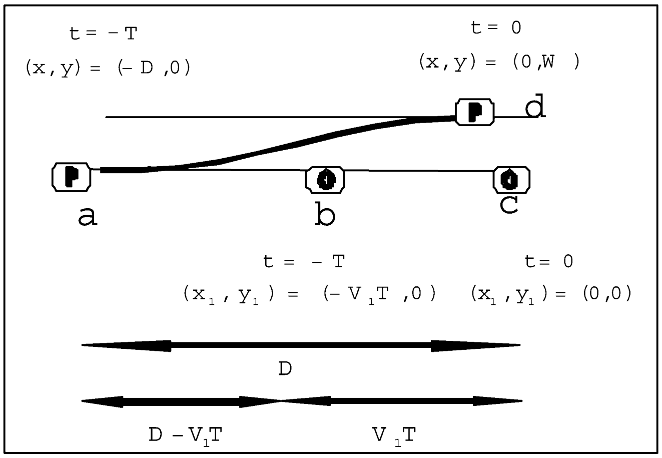

Fig. 1. Overtaking a moving vehicle. At time $\boldsymbol{t = -T}$, vehicle $\boldsymbol{P}$ is located at the point **a**, and vehicle $\boldsymbol{O}$ is at the point **b**. After $\boldsymbol{T}$ seconds, at time $\boldsymbol{t = 0}$, vehicle $\boldsymbol{O}$ is located at **c** and $\boldsymbol{P}$ is at **d**. Vehicle **P** needs to travel from point **a** to point **d** during the same time interval that $\boldsymbol{O}$ travels from **b** to **c**. The time interval $\boldsymbol{T}$ and horizontal distance $\boldsymbol{D}$ depend only on the lane-change trajectory of vehicle $\boldsymbol{P}$ and not on $\boldsymbol{O}$. The only role of vehicle $\boldsymbol{O}$ is to determine the absolute location of the point **c**, hence that of the point **a**.

TABLE I
SOME EXAMPLES OF THE OPTIMAL VALUES AND RELATIVE DISTANCES

V (m/s)	*W (meters)*	*A (m/s^2)*	*D*(meters)*	*T* (seconds)*	*V_1 (m/s)*	*D_rel (meters)*
15	3	3	36	2.47	12	6.36
25	3	4	52	2.1	15	20.38
25	4	2	84.96	3.43	20	16.38
35	3.5	4	78.67	2.26	20	33.35

other words, the rear of vehicle O is located at (0,0,0) at the same time ($t = 0$) when the front of vehicle P is at $(0, W, 0)$; see Fig. 1.[1]

It turns out that the absolute shape, size and time of the trajectory of vehicle P do not depend on the velocity of vehicle O. This has a simple explanation.[2] If the velocity of vehicle O is $0 < V_1 = \text{const}$, then its position at each instant $t \in [-T, 0]$, is:$x_1(t) = -V_1 t$, $y_1(t) \equiv 0$. Vehicle P begins its diversion at time $t = -T$ and point $(-D, 0)$, where T is the time it will take both vehicles to reach the points where $x = 0$. At this time, vehicle O is located at a distance of $V_1 T$ from the origin, and vehicle P at a distance D, so the distance between the two vehicles is $D_{\text{rel}} = D - V_1 T$. Vehicle P needs to move from point $(-D, 0)$ to point $(0, W)$ during a time interval T regardless of the other vehicle. So the only role of vehicle O is to determine the position of the origin in the (x, y, t)-space. Fig. 1 explains this observation.

To determine the point for beginning the diversion, vehicle P calculates *its own* T and D, based on the parameters of its own lane-changing trajectory and the width W. It estimates the velocity V_1 of vehicle O, and calculates the relative distance $D_{\text{rel}} = D - V_1 T$. At this distance from O, it begins the diversion. Some examples are presented in Table I.

[1]The lane-change trajectory shown in the figure was generated by the model presented in this note.

[2]Due to the nonlinear nature of the problem, it is impossible to use relative velocities.

B. Phase 2): Driving Straight in the Adjacent Lane

For simplicity, we assume that while vehicle P is moving in parallel to vehicle O, both vehicles maintain their original velocities V and V_1, respectively. If the length of vehicle P is L and that of O is L_1, then the passing vehicle must travel a *relative* distance of at least $L + L_1$ at a relative velocity of $V - V_1$ before it can begin returning to its lane. This would take an amount of time $T_{(b)} = (L + L_1)/(V - V_1)$. Therefore, the *absolute* distance that P must travel is at least $D_{(b)} = ((L + L_1)/(V - V_1))V$. For example, if $L = 5$ m and $L_1 = 6$ m, $V = 25\ \mathrm{m/s}$ and $V_1 = 20\ \mathrm{m/s}$, then vehicle P must travel at least $D_{(b)} = ((5 + 6)/(25 - 20)) \cdot 25 = 55$ m. This would take $T_{(b)} = (5 + 6)/(25 - 20) = 2.2$ s. Of course, P must keep a security distance before returning, since its velocity may decrease slightly during lane-change.

C. Phase 3): Returning to the Lane

Vehicle P can now use the symmetric lane-change trajectory to return. We simply use symmetry and time reversal. The origin of the new space-time coordinate system is located in front of vehicle O, next to the rear of vehicle P that is traveling in the adjacent lane. In the new lane-change trajectory we simply substitute $x_{\text{new}}(t) = -x(-t)$, $y_{\text{new}}(t) = y(-t)$.

To conclude, if both vehicles maintain their parameters and velocities, then the total overtaking maneuver would take *at least* $2T + T_{(b)}$ seconds and the total x-direction distance is at least $2D + D_{(b)}$ m. If P changes its velocity while in the adjacent lane (say, if an emergency situation arose), then the previous formulas for $T_{(b)}$ and $D_{(b)}$ will pertain to its average velocity. If it also changes its dynamic variables when returning to the lane, then its trajectory is determined accordingly, hence the T and D values for returning will also be the new ones. In any case, it is possible to estimate the total time and distance of the complete overtaking maneuver.

III. DESIGN OF AN OPTIMAL LANE-CHANGE TRAJECTORY

We now suggest a design of an optimal lane-change trajectory for vehicle P. We regard vehicle P as a point mass.[3] The maximal resultant force acting on the vehicle, which is proportional to the norm of the maximal acceleration vector, depends on various external and internal conditions such as inclination, friction and especially the amount of power that the (human or automatic) driver chooses to exert.[4] In an emergency situation, the system may use higher acceleration, which is of course bounded by the vehicle's capabilities and the external conditions. These factors vary with different conditions, but are taken as constant during the short time interval of the maneuver. Thus the assumption is that the acceleration has a constant bound. If the trajectory is designed for comfort, then the system should choose the acceleration bound so that the lateral acceleration[5] does not exceed the recommended bound for passenger comfort, which is 3–4 $\mathrm{m/s^2}$ (for example, see [7]). However, it is stated in [11] that human drivers, under normal conditions, usually use an acceleration of about 1 $\mathrm{m/s^2}$ during overtakes.

A. Formulating the Equations of Motion

The suggested design of a lane-change trajectory is based on underlying polynomial equations, superimposed with minimization of the

[3]Although we consider vehicle **P** as a point mass located at its CG, all distances relate to its front for phase 1) and 2) maneuvers and to its rear for phase 3) maneuvers.

[4]For a conventional human-driven vehicle this may be the pressure on the gas pedal or brakes, the gear being used, etc.

[5]By analyzing acceleration profiles, it can be shown that the lateral component of the acceleration vector is the dominant one.

total kinetic energy during the maneuver. It is convenient to consider the maneuver for phase 3), returning to the lane.

To determine the trajectory of vehicle P, we fit a polynomial expression for $x(t)$ and $y(t)$, satisfying appropriate boundary conditions. For simplicity, we assume the accelerations at the initial and final points of the lane-changing maneuver are both zero, and the initial and final velocities are equal. These assumptions may not be realistic, but they are a simplifying approximation. During the maneuver itself, the velocity and acceleration are not assumed to be constant.

Let D and T be as in Section II. These values are as yet unknown, but are determined by the optimization. The known parameters of the optimization problem are: V = the initial and final velocity of P; W = the width of the lane or of the diversion; A = the magnitude of the maximal resultant acceleration of P. All the known and unknown parameters are positive. The boundary conditions are

$$\begin{aligned}&x(0)=0,\ x(T)=D,\ \dot{x}(0)=\dot{x}(T)=V,\ \ddot{x}(0)=\ddot{x}(T)=0\\&y(0)=W,\ y(T)=0,\ \dot{y}(0)=\dot{y}(T)=0,\ \ddot{y}(0)=\ddot{y}(T)=0.\end{aligned} \tag{3.1}$$

By writing down a general fifth-deree polynomial and applying the boundary conditions (3.1), we obtain the following equations:

$$x(t)=Vt+(VT-D)\left(-10\left(\frac{t}{T}\right)^3+15\left(\frac{t}{T}\right)^4-6\left(\frac{t}{T}\right)^5\right) \tag{3.3}$$

$$y(t)=W+W\left(-10\left(\frac{t}{T}\right)^3+15\left(\frac{t}{T}\right)^4-6\left(\frac{t}{T}\right)^5\right). \tag{3.4}$$

The form of these equations is well known and is often used to model biological motion [3]. In that context it is called a "minimal jerk" trajectory. Similar equations for the trajectory of an autonomous vehicle were suggested in [2].

IV. Formulating the Optimization Model

The coefficient $(VT-D)$ in (3.3) represents an upper bound on the *additional* distance that the vehicle traveled due to the diversion. Denote this difference by $S=VT-D$. It can be shown that this difference is positive.[6] The variable S is just a substitution that makes the expressions simpler. It is more convenient to use the variables (T,S) instead of (T,D), so we shall use them henceforward. However, since we are actually seeking the optimal value of D, then we find the optimal S^* and T^* and substitute

$$D^*=VT^*-S^*. \tag{4.1}$$

The constraints of the optimization problem are as follows.

1) **The velocity constraint**: We want to insure that the motion in the x-direction is always forward, or that $\dot{x}(t)\geq 0$ for all $0\leq t\leq T$. From (3.3) it can be shown that $\dot{x}_{\min}=V-(15S/8T)$. Therefore, we must satisfy the *velocity constraint*

$$8VT\geq 15S. \tag{4.2}$$

2) **The acceleration constraint**: The maximal acceleration or deceleration from (3.3) and (3.4) are: $\max\sqrt{\ddot{x}^2+\ddot{y}^2} = 10(\sqrt{3}/3)(\sqrt{S^2+W^2}/T^2)$. This should be equal to A, the norm of the maximal acceleration vector of vehicle P, chosen to use during the maneuver. Thus, the following *acceleration constraint* must be satisfied:

$$g(T,S)=\frac{S^2+W^2}{T^4}=\frac{3}{100}A^2. \tag{4.3}$$

[6]Because of the diversion and the slight reduction of velocity, $\boldsymbol{T}$ is a bit larger than the original time-to-collision.

The objective function to be minimized is the total kinetic energy[7]. In terms of x and y from (3.3), (3.4) and omitting the constant mass coefficient, it is

$$TKE=f(T,S)=\int_0^T\left(\dot{x}^2+\dot{y}^2\right)dt=\frac{10}{7T}\left(S^2+W^2\right)-2VS+V^2T. \tag{4.4}$$

The optimization problem we wish to solve for vehicle P is

$$\begin{cases}\text{Minimize} & f(T,S)=\frac{10}{7T}\left(S^2+W^2\right)-2VS+V^2T\\ \text{Subject} & \text{to}\\ 1) & 8VT\geq 15S\\ 2) & g(T,S)=\left(\frac{W^2+S^2}{T^4}\right)=\frac{3}{100}A^2.\end{cases} \tag{4.5}$$

Claim 1 in the Appendix proves that (4.5) attains a unique global solution.[8]

V. Approximating the Optimal D^* and T^*

Finding an analytic solution for problem (4.5) using methods like Karush–Kuhn–Tucker or Lagrange is extremely difficult. Therefore a numerical method was used, together with the optimization software Lingo. The results were obtained by running the program with various values of the parameters and seeking the optimal values of the unknown variables. Each parameter was isolated and varied while the others were kept constant.

The resulting values for D^* and S^* were plotted against each varying parameter separately. For high enough velocities, (say $V\geq\sim 5$ m/s), it was obvious that the optimal value of D^* is approximately proportional to V, to the square root of W, and inversely proportional to the square root of A. These proportions were also obtained in [2]. The proportionality constant was found to be approximately[9] $\alpha=2.4$. The nature and justification of this coefficient is given in Section VI and Claim 2 in the Appendix . It was also quite obvious for high enough velocities, that S^* is approximately proportional to $W^{3/2}$, to $\sqrt{A}$, and inversely proportional to V. The proportionality constant was found to be approximately[10] $\beta=1.73(\approx\sqrt{3})$. Therefore, it can be concluded for high enough values of V, that

$$D^*\approx 2.4V\sqrt{\frac{W}{A}} \tag{5.1}$$

$$S^*\approx\sqrt{3}\frac{W^{3/2}\sqrt{A}}{V}. \tag{5.2}$$

By substituting $T^*=(S^*+D^*)/V=(S^*/V)+(D^*/V)$ from (4.1), we obtain

$$T^*\approx\sqrt{3}\frac{W^{3/2}\sqrt{A}}{V^2}+2.4\frac{\sqrt{W}}{\sqrt{A}}. \tag{5.3}$$

These formulas were validated against the results obtained from the optimization software with many test runs, and gave a highly accurate approximation. There were relatively significant deviations only for cases of very low velocities.[11]

Some examples of the optimization are presented in Table I, as well as the relative distances for beginning the diversion in phase 1).

[7]Minimizing the TKE, bounds both the time and arc-length.

[8]By checking the Hessian of $\boldsymbol{f(T,S)}$, this function is strictly convex for $\boldsymbol{T>0}$.

[9]$\mathbf{n=50,ave=2.4041,SD=0.0008}$

[10]$\mathbf{n=50,ave=1.7268,SD=0.027}$

[11]The fact that (5.1)–(5.3) are not as accurate for low velocities as they are for higher ones can be explained, but this case is not in the scope of the present note, which deals with relatively high velocities.

VI. Bounds on T^*

According to the model (4.5), the optimal value of T is bounded, independent of the velocity. The acceleration constraint in (4.3) implies the following lower bound: $T^4 > (100W^2)/(3A^2)$ or (since $T > 0$):

$$T^* > \frac{\sqrt{10}}{\sqrt[4]{3}}\sqrt{\frac{W}{A}} \approx 2.4028\sqrt{\frac{W}{A}}. \tag{6.1}$$

The coefficient here is very close to the constant α that was found independently when deriving the formula (5.1). It is also shown in Claim 2 of the Appendix that $T^{*2} \leq 10\sqrt{5}(W/A)$, or $T^* \leq \sqrt{10}\sqrt[4]{5}\sqrt{W/A} \approx 4.7287\sqrt{W/A}$. Therefore

$$2.4028\sqrt{\frac{W}{A}} \leq T^* < 4.7287\sqrt{\frac{W}{A}}. \tag{6.2}$$

In *all* test runs, these bounds were satisfied.

It is also proven in Claim 2 that the optimal values of T^* actually tend to the lower bound as V gets larger, and those of S^* tend to zero.[12] As a result, for high enough velocities, we can roughly "substitute" $S^* \approx 0$, $T^* \approx 2.4028\sqrt{W/A}$ into (4.1) and partially support the approximation (5.1).

VII. Conclusion

This note deals with overtaking a slower-moving vehicle. The first part analyzes the three-phase maneuver and shows that the velocity of the obstacle does not affect the absolute shape or time of the lane-change trajectory. The second part designs a smooth and comfortable optimal lane-change trajectory. The optimization objective is to minimize the total kinetic energy that is exerted during the lane-changing maneuver, superimposed on a "minimal-jerk" trajectory. Two constraints are imposed, one to guarantee that the motion of the overtaking vehicle is always forward, and one to comply with the maximal acceleration chosen to use during the maneuver.

The solution of the optimization problem determines the optimal time and distance of the maneuver. These values in turn, determine the trajectory itself. For high enough initial velocities, explicit closed-form formulas were developed, which approximate the optimal values of these parameters. The formulas were compared with results obtained from the optimization software and showed high accuracy. It is also shown that the optimal time is bounded, independent of the velocity.

An autonomous vehicle equipped with appropriate sensors and programmed with these formulas, can calculate its trajectory and the best place to begin and end the maneuver.

Appendix

The first purpose of this section is to prove that the optimization problem (4.5) attains a unique solution. The reader is referred to nonlinear programming and convexity theory (for example, [1]).

Claim 1: **The optimization problem (4.5) attains a unique solution for $T > 0$, $S > 0$.**

Proof: Constraint 2 implies $S^2 + W^2 = (3/100)A^2T^4$. Isolating $S = (1/10)\sqrt{3A^2T^4 - 100W^2}$ and substituting this in the objective function $f(T, S)$, yields the following function of T only:

$$F(T) = \frac{3}{70}A^2T^3 - \frac{V}{5}\sqrt{3A^2T^4 - 100W^2} + V^2T. \tag{A.1}$$

The square-root expression in (A.1) is equal to $10S$. For simplicity, we denote it by $B(T)$.

$$B(T) = \sqrt{3A^2T^4 - 100W^2} = 10S. \tag{A.2}$$

[12]Although this can be seen from the form of (5.2) and (5.3), they were obtained numerically and not analytically.

By twice differentiating $F(T)$ and repeated use of (4.3), it can be shown that it is strictly convex. Since constraint 1 is linear therefore convex, then by [1], the problem attains a unique solution. □

Now, we wish to prove, as stated in Section VI that as V becomes larger, the optimal S^* tends to zero and the optimal T^* tends to its lower bound given in (6.1). If the derivative of $F(T)$ from (A.1) with respect to T is zero, then

$$\frac{dF}{dT} = \frac{9}{70}A^2T^2 + V^2 - \frac{6}{5}\frac{VA^2T^3}{B(T)} = 0. \tag{A.3}$$

Although (A.3) is very difficult to solve explicitly for T^*, it can help in the proof.

Claim 2: **If we let the values of V become higher** (and assuming constant values of A and W), **then i) The optimal values of S^* tend to zero. ii) The optimal values of T^* tend to the lower bound of (6.1).**

Proof: Assuming $dF/dT = 0$ as in (A.3) for T^* and isolating $B(T^*)$, yields

$$B(T^*) = \frac{cVA^2T^{*2}}{9A^2T^{*2} + 70V^2} \tag{A.4}$$

where $c = 84/5$ is a numerical constant. We would like to keep all the other variables constant and increase V alone, but T^* is not independent of V. However, finding an upper bound on T^* will help. If we regard (A.3) as a quadratic equation in V and solve it for V using the regular formula for quadratic equations, then the discriminant must be nonnegative since V is a real-valued parameter. After some manipulation, this implies that $T^{*2} \leq 10\sqrt{5}(W/A)$, as mentioned in (6.2). Plugging this back in (A.4) yields: $0 < B(T^*) \leq (cVA^2/(9A^2 + 7V^2(A/\sqrt{5}W)))$. Letting V increase, we get $\lim_{V\to\infty} B(T^*) = 0$. Since $B(T^*) = 10S^*$, this implies that $\lim_{V\to\infty} S^* = 0$. By (A.2), this in turn implies that $\lim_{V\to\infty} T^* = (\sqrt{10}/\sqrt[4]{3})\sqrt{(W/A)}$. □

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