Div Grad Curl Solutions

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This is a work in progress. I am using it to learn LaTeX and vector calculus at once.

1 Introduction, Vector Functions, and Electrostatics

Relevant Equations

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{qq_0}{r^2}\hat{\mathbf{u}} \tag{Coulomb's Law}$$

$$\mathbf{E}(\mathbf{r}) = \frac{\mathbf{F}(\mathbf{r})}{q_0} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{u}} \tag{Electric Field}$$

Problems

1. TODO
2. TODO
3. (a) Write a formula for a vector function in two dimensions which is in the positive radial direction and whose magnitude is 1.

   Solution:

   $$\vec{F}(x,y) = \frac{x\mathbf{i} + y\mathbf{j}}{\sqrt{x^2+y^2}}$$

   The length of a vector is given by

   $$|\vec{v}| = \sqrt{x^2+y^2}$$

   For $\vec{F}$ to be positive in $\vec{r}$, we need $\vec{F}(x,y) = x\mathbf{i} + y\mathbf{j}$, and we divide by the length of $\vec{F}$ to get a unit vector in that direction.

   (b) Write a formula for a vector function in two dimensions whose direction makes an angle of 45°with the x-axis and whose magnitude at any point (x,y) is $(x+y)^2$.

   Solution:

   $\vec{F}$ must be the same in the $\vec{x}$ and $\vec{y}$ directions to have a 45°angle with the x-axis

   $$\vec{F}(x,y) = a\mathbf{i} + a\mathbf{j}$$

   To have a magnitude of $(x+y)^2$, we need:

   $$|\vec{F}| = \sqrt{a^2+a^2} = (x+y)^2$$

   $$\sqrt{2}a = (x+y)^2$$

   $$a = \frac{\sqrt{2}(x+y)^2}{2}$$

$$\Rightarrow \vec{F}(x,y) = \frac{\sqrt{2}(x+y)^2}{2}(\mathbf{i}+\mathbf{j})$$

(c) Write a formula for a vector function in two dimensions whose direction is tangential [orthogonal to the radial direction] and whose magnitude at any point (x,y) is equal to its distance from the origin.

Solution:

To get the function orthogonal to radial, we need the **i** to depend on -y, and the **j** to depend on x. So

$$\vec{F}(x,y) = -y\mathbf{i} + x\mathbf{j}$$

The magnitude must be equal to the distance from the origin. The distance is given by $d = \sqrt{x^2+y^2}$, so:

$$|\vec{F}| = \sqrt{a^2+b^2} = \sqrt{x^2+y^2}$$

This works with a=y and b=x, with no other changes to the magnitude of $\vec{F}(x,y)$, so:

$$\vec{F}(x,y) = -y\mathbf{i} + x\mathbf{j}$$

(d) Write a formula for a vector function in three dimensions which is in the positive radial direction and whose magnitude is 1.

Solution:

$$\vec{F}(x,y,z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

To get a unit vector (magnitude = 1), divide this by the length:

$$\vec{F}(x,y,z) = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2+y^2+z^2}}$$

4. An object moves in the xy-plane in such a way that its position vector **r** is given as a function of time t by

$$\mathbf{r} = \mathbf{i}a\cos\omega t + \mathbf{j}b\sin\omega t$$

where a, b, and ω are constants.

(a) How far is the object from the origin at any time t?

Solution:

$$d = \sqrt{x^2+y^2}$$
$$d = \sqrt{a^2\cos^2(\omega t) + b^2\sin^2(\omega t)}$$

The important point is that $\cos^2() + \sin^2() = 1$ cannot be factored out.

(b) Find the object's velocity and acceleration as functions of time.

Solution:

$$\vec{v} = \frac{dx}{dt}$$
$$\vec{v} = \frac{d}{dt}(\mathbf{i}a\cos\omega t + \mathbf{j}b\sin\omega t)$$
$$\vec{v} = -\mathbf{i}a\omega\sin\omega t + \mathbf{j}b\omega\cos\omega t$$

Similarly,

$$\vec{a} = \frac{dv}{dt}$$
$$\vec{a} = \frac{d}{dt}(-\mathbf{i}a\omega\sin\omega t + \mathbf{j}b\omega\cos\omega t)$$
$$\vec{a} = -\mathbf{i}a\omega^2\cos\omega t - \mathbf{j}b\omega^2\sin\omega t$$

Using the definition of $\mathbf{r}$, the last line can also be written:

$$\vec{a} = -\omega^2 \mathbf{r}$$

(c) Show that the object moves on the elliptical path

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

Solution:

The x- and y- coordinates of the object are given by:

$$x = a\cos\omega t \quad y = b\sin\omega t$$

If we plug these into the left hand side of the ellipse equation, we get:

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = \left(\frac{\cancel{a^2}\cos^2\omega t}{\cancel{a^2}}\right) + \left(\frac{\cancel{b^2}\sin^2\omega t}{\cancel{b^2}}\right)$$
$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

since $\cos^2() + \sin^2() = 1$.

5. A charge +1 is situated at the point (1, 0, 0) and a charge -1 is situated at the point (-1, 0, 0). Find the electric field of these two charges at an arbitrary point (0, y, 0) on the y-axis.

   Solution:

   An electric field is given by:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{l=1}^{N}\frac{q_l}{|\mathbf{r}-\mathbf{r}_l|^2}\hat{\mathbf{u}}_l$$

   We have two charges at (1,0,0) and (-1,0,0). The y-components will cancel out, and the x-components will reinforce one another. So we need to find the x-component of each charge and add them. For the charge at (1,0,0):

$$E_x = -\frac{1}{4\pi\epsilon_0}\frac{\cos\theta}{r^2}$$
$$\cos\theta = \frac{adj}{hyp} = \frac{x}{r} = \frac{1}{r}$$
$$r = \sqrt{1+y^2}$$
$$\Rightarrow E_x = -\frac{1}{4\pi\epsilon_0}\frac{1}{(1+y^2)^{\frac{3}{2}}}$$

   The field from the charge at (-1,0,0) is the same in the x-direction. Adding them together:

$$E = -\frac{1}{2\pi\epsilon_0}\frac{1}{(1+y^2)^{\frac{3}{2}}}\mathbf{i}$$

6. Instead of using arrows to represent vector functions, we sometimes use families of curves called *field lines.* A curve y = y(x) is a field line of the vector function $\mathbf{F}$(x,y) if at each point (x_0, y_0) on the curve $\mathbf{F}(x_0, y_0)$ is tangent to the curve (see the figure).

   (a) Show that the field lines y = y(x) of a vector function

$$\mathbf{F}(x,y) = \mathbf{i}F_x(x,y) + \mathbf{j}F_y(x,y)$$

are solutions to the differential equation

$$\frac{dy}{dx} = \frac{F_y(x,y)}{F_x(x,y)}$$

Solution:

The field lines are tangent to the vector function. The tangent is the derivative of F, which is given by $\frac{dy}{dx}$. F_y and F_x are the components of F. The slope at any point is given by $\frac{F_y}{F_x}$. The slope is the same as the tangent, so

$$\frac{dy}{dx} = \frac{F_y(x,y)}{F_x(x,y)}$$

(b) Determine the field lines of each of the functions of Problem I-1. Draw the field lines and compare with the arrow diagrams of Problem I-1.

Solution:

We can use the relationship derived in (a) to determine the differential equations we need to solve to get the field lines.

(i) $\mathbf{i}y + \mathbf{j}x$

$$\frac{dy}{dx} = \frac{F_y}{F_x} = \frac{x}{y}$$

This is a separable differential equation.

$$\begin{aligned} y' &= \frac{x}{y} \\ ydy &= xdx \\ \int ydy &= \int xdx \\ y^2 &= x^2 + c \end{aligned}$$

(ii) $(\mathbf{i} + \mathbf{j})/\sqrt{2}$

$$\begin{aligned} \frac{dy}{dx} &= \frac{\sqrt{2}}{\sqrt{2}} = 1 \\ \int dy &= \int dx \\ y &= x + c \end{aligned}$$

(iii) $\mathbf{i}x - \mathbf{j}y$

$$\begin{aligned} \frac{dy}{dx} &= \frac{-y}{x} \\ \int \frac{-dy}{y} &= \int \frac{dx}{x} \\ -\ln y &= \ln x + c \\ c &= \ln x + \ln y = \ln(xy) \\ c &= xy \end{aligned}$$

(iv) $\mathbf{i}$y

$$\begin{aligned} \frac{dy}{dx} &= \frac{0}{y} = 0 \\ \int dy &= 0 \\ y &= c \end{aligned}$$

(v) **j**x

$$\frac{dy}{dx} = \frac{0}{0} = undefined$$
$$\frac{dx}{dy} = \frac{0}{x} = 0$$
$$x = c$$

(vi) $(\mathbf{i}y + \mathbf{j}x)/\sqrt{x^2+y^2}, (x, y) \neq (0, 0)$

$$\frac{dy}{dx} = \frac{x}{\sqrt{x^2+y^2}} \frac{\sqrt{x^2+y^2}}{y} = \frac{x}{y}$$
$$\int ydy = \int xdx$$
$$y^2 - x^2 = c$$

(vii) **i**y + **j**xy

$$\frac{dy}{dx} = \frac{xy}{y} = x$$
$$\int dy = \int xdx$$
$$y = \frac{x^2}{2} + c$$

(viii) **i** + **j**y

$$\frac{dy}{dx} = y$$
$$\int \frac{dy}{y} = \int dx$$
$$\ln y = x + c$$
$$y = e^x + c$$

2 Surface Integrals and the Divergence

Relevant Equations

$$\hat{\mathbf{n}}(x, y, z) = \frac{\mathbf{u} \times \mathbf{v}}{|\mathbf{u} \times \mathbf{v}|} = \frac{-\mathbf{i}\frac{\partial f}{\partial x} - \mathbf{j}\frac{\partial f}{\partial y} + \mathbf{k}}{\sqrt{1 + (\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2}} \tag{II-4}$$

$$\iint_S G(x, y, z)dS = \iint_R G(x, y, f(x, y)) \cdot \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} dxdy \tag{II-12}$$

$$\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}dS = \iint_R \left\{-F_x[x, y, f(x, y)]\frac{\partial f}{\partial x} - F_y[x, y, f(x, y)]\frac{\partial f}{\partial y} + F_z[x, y, f(x, y)]\right\} dxdy \tag{II-13}$$

$$\iint_S \mathbf{E} \cdot \hat{\mathbf{n}}dS = \frac{q}{\epsilon_0} \tag{Gauss's Law}$$

$$\text{div}\mathbf{E} = \nabla \cdot \mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \frac{\rho}{\epsilon_0} \tag{Gauss's Law Differential Form}$$

$$\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}dS = \iiint_V \nabla \cdot \mathbf{F}dV \tag{Divergence Theorem}$$

Examples

- (Page 25) Solution to double integral

$$\sqrt{3}\int\int_R (1-y)dxdy$$

with $S = z = f(x,y) = 1 - x - y$.

The region R is the triangle in the positive region of the xy-plane delimited by the two axes and the line $y = 1 - x$.

Putting the right limits in the double integral we get:

$$\begin{aligned}
\sqrt{3}\int\int_R (1-y)dxdy &= \sqrt{3}\int_0^1\int_0^{1-y}(1-y)dxdy \\
&= \sqrt{3}\int_0^1 (1-y)x\Big|_0^{1-y}dy \\
&= \sqrt{3}\int_0^1 (1-y)^2 dy \\
&= \sqrt{3}\int_0^1 (1-2y-y^2)dy \\
&= \sqrt{3}(y - y^2 - \frac{y^3}{3})\Big|_0^1 \\
&= \sqrt{3}(1 - 1 - \frac{1}{3}) \\
&= \frac{1}{\sqrt{3}}
\end{aligned}$$

- (Page 27) Solution to the double integral

$$\int\int_S z^2 dS = \int\int_R \sqrt{1-x^2-y^2}dxdy$$

with $S = x^2 + y^2 + z^2 = 1$, and R is the projection of S in the positive region of the xy-plane.

R is a quarter circle defined by $x^2 + y^2 = 1$. Converting to polar coordinates, R is the region from $r = [0,1]$ and $\theta = [0, \frac{\pi}{2}]$.

$$\begin{aligned}
\int\int_S z^2 dS &= \int\int_R \sqrt{1-x^2-y^2}dxdy \\
&= \int_0^{\frac{\pi}{2}}\int_0^1 (\sqrt{1-r^2\cos^2\theta - r^2\sin^2\theta})rdrd\theta \\
&= \int_0^{\frac{\pi}{2}}\int_0^1 r\sqrt{1-r^2}drd\theta
\end{aligned}$$

let $u = \sqrt{1-r^2}$, then

$$du = \frac{1}{2\sqrt{1-r^2}}2rdr = \frac{rdr}{\sqrt{1-r^2}} = \frac{rdr}{u}$$
$$\Rightarrow rdr = udu$$

After converting the limits of the definite integral in r to limits in u, we have

$$\begin{aligned}
\int\int_S z^2 dS &= \int_0^{\frac{\pi}{2}} \int_1^0 u^2 du d\theta \\
&= \int_0^{\frac{\pi}{2}} \frac{u^3}{3}\bigg|_1^0 d\theta \\
&= \frac{1}{3}\int_0^{\frac{\pi}{2}} d\theta \\
&= \frac{\pi}{6}
\end{aligned}$$

- (Page 29) Solution to the double integral

$$\int\int_S \mathbf{F}\cdot\hat{\mathbf{n}} dS = \int\int_R \left(\frac{3x}{4} - \frac{3y}{2} + \frac{1}{2}\right) dxdy$$

where $S = x + 2y + 2z = 2$, and R is the area in the positve region of the xy-plane with $y = 1 - x/2$.

$$\begin{aligned}
\int\int_S \mathbf{F}\cdot\hat{\mathbf{n}} dS &= \int_0^1 \int_0^{2-2y} \left(\frac{3x}{4} - \frac{3y}{2} + \frac{1}{2}\right) dxdy \\
&= \int_0^1 \left(\frac{3x^2}{8} - \frac{3yx}{2} + \frac{x}{2}\right)\bigg|_0^{2-2y} dy \\
&= \int_0^1 \left(\frac{3(2-2y)^2}{8} - \frac{3y(2-2y)}{2} + \frac{2-2y}{2}\right) dy \\
&= \int_0^1 \left(\frac{12-24y+12y^2}{8} + \frac{-6y+6y^2}{2} + \frac{2-2y}{2}\right) dy \\
&= \int_0^1 \left(\frac{12-24y+12y^2}{8} + \frac{2-8y+6y^2}{2}\right) dy \\
&= \int_0^1 \left(\frac{6-12y+6y^2}{4} + \frac{4-16y+12y^2}{4}\right) dy \\
&= \int_0^1 \left(\frac{10-28y+18y^2}{4}\right) dy \\
&= \frac{1}{4}\left(10y - 14y^2 + 6y^3\right)\big|_0^1 \\
&= \frac{1}{2}
\end{aligned}$$

Problems

1. Find a unit vector $\hat{n}$ normal to each of the following surfaces.

   (a) $z = f(x, y) = 2 - x - y$

   Solution:

   We can use the result derived in equation II-4:

$$\hat{\mathbf{n}}(x,y,z) = \frac{\mathbf{u}\times\mathbf{v}}{|\mathbf{u}\times\mathbf{v}|} = \frac{-\mathbf{i}\frac{\partial f}{\partial x} - \mathbf{j}\frac{\partial f}{\partial y} + \mathbf{k}}{\sqrt{1 + (\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2}}$$

The partial derivatives are:

$$\frac{\partial f}{\partial x} = -1 \quad \text{and} \quad \frac{\partial f}{\partial y} = -1$$

Then:

$$\begin{aligned}
\hat{\mathbf{n}} &= \frac{-\mathbf{i}(-1) - \mathbf{j}(-1) + \mathbf{k}}{\sqrt{1 + (-1)^2 + (-1)^2}} \\
&= \frac{\mathbf{i} + \mathbf{j} + \mathbf{k}}{\sqrt{3}}
\end{aligned}$$

(b) $z = (x^2 + y^2)^{\frac{1}{2}}$

Solution:

$$\frac{\partial f}{\partial x} = \frac{x}{(x^2 + y^2)^{\frac{1}{2}}} \quad \text{and} \quad \frac{\partial f}{\partial y} = \frac{y}{(x^2 + y^2)^{\frac{1}{2}}}$$

$$\begin{aligned}
\hat{\mathbf{n}} &= \frac{-\mathbf{i}(\frac{x}{(x^2+y^2)^{\frac{1}{2}}}) - \mathbf{j}(\frac{y}{(x^2+y^2)^{\frac{1}{2}}}) + \mathbf{k}}{\sqrt{1 + \left(\frac{x}{(x^2+y^2)^{\frac{1}{2}}}\right)^2 + \left(\frac{y}{(x^2+y^2)^{\frac{1}{2}}}\right)^2}} \\
&= \frac{-\mathbf{i}(\frac{x}{z}) - \mathbf{j}(\frac{y}{z}) + \mathbf{k}}{\sqrt{1 + \frac{x^2}{z^2} + \frac{y^2}{z^2}}} \\
&= \frac{-x\mathbf{i} - y\mathbf{j} + z\mathbf{k}}{z\sqrt{1 + \frac{x^2}{z^2} + \frac{y^2}{z^2}}} \\
&= \frac{-x\mathbf{i} - y\mathbf{j} + z\mathbf{k}}{\sqrt{z^2 + x^2 + y^2}} \\
&= \frac{-x\mathbf{i} - y\mathbf{j} + z\mathbf{k}}{\sqrt{z^2 + z^2}} \\
&= \frac{-x\mathbf{i} - y\mathbf{j} + z\mathbf{k}}{z\sqrt{2}}
\end{aligned}$$

(c) $z = (1 - x^2)^{\frac{1}{2}}$

Solution:

$$\frac{\partial f}{\partial x} = \frac{-x}{(1 - x^2)^{\frac{1}{2}}} = \frac{-x}{z} \quad \text{and} \quad \frac{\partial f}{\partial y} = 0$$

$$\begin{aligned}
\hat{\mathbf{n}} &= \frac{-\mathbf{i}\left(\frac{-x}{z}\right) - \mathbf{j}(0) + \mathbf{k}}{\sqrt{1 + \left(\frac{-x}{z}\right)^2 + (0)^2}} \\
&= \frac{x\mathbf{i} + z\mathbf{k}}{z\sqrt{1 + \frac{x^2}{z^2}}} \\
&= \frac{x\mathbf{i} + z\mathbf{k}}{\sqrt{z^2 + x^2}} \\
&= \frac{x\mathbf{i} + z\mathbf{k}}{\sqrt{((1 - x^2)^{\frac{1}{2}})^2 + x^2}} \\
&= \frac{x\mathbf{i} + z\mathbf{k}}{\sqrt{1 - x^2 + x^2}} \\
&= x\mathbf{i} + z\mathbf{k}
\end{aligned}$$

(d) $z = x^2 + y^2$

Solution:

$$\frac{\partial f}{\partial x} = 2x \quad \text{and} \quad \frac{\partial f}{\partial y} = 2y$$

$$\begin{aligned}
\hat{\mathbf{n}}(x,y,z) &= \frac{-\mathbf{i}2x - \mathbf{j}2y + \mathbf{k}}{\sqrt{1 + (2x)^2 + (2y)^2}} \\
&= \frac{-\mathbf{i}2x - \mathbf{j}2y + \mathbf{k}}{\sqrt{1 + 4x^2 + 4y^2}} \\
&= \frac{-\mathbf{i}2x - \mathbf{j}2y + \mathbf{k}}{\sqrt{1 + 4z}}
\end{aligned}$$

(e) $z = \left(1 - \frac{x^2}{a^2} - \frac{y^2}{a^2}\right)^{\frac{1}{2}}$

Solution:

$$\frac{\partial f}{\partial x} = \frac{-x}{a^2 z} \quad \text{and} \quad \frac{\partial f}{\partial y} = \frac{-y}{a^2 z}$$

$$\begin{aligned}
\hat{\mathbf{n}}(x,y,z) &= \frac{-\mathbf{i}\frac{-x}{a^2 z} - \mathbf{j}\frac{-y}{a^2 z} + \mathbf{k}}{\sqrt{1 + (\frac{-x}{a^2 z})^2 + (\frac{-y}{a^2 z})^2}} \\
&= \frac{\mathbf{i}x + \mathbf{j}y + \mathbf{k}a^2 z}{a^2 z\sqrt{1 + \frac{x^2}{a^4 z^2} + \frac{y^2}{a^4 z^2}}} \\
&= \frac{\mathbf{i}x + \mathbf{j}y + \mathbf{k}a^2 z}{a\sqrt{a^2 z^2 + x^2/a^2 + y^2/a^2}} \\
&= \frac{\mathbf{i}x + \mathbf{j}y + \mathbf{k}a^2 z}{a\sqrt{a^2 z^2 + 1 - 1 + x^2/a^2 + y^2/a^2}} \\
&= \frac{\mathbf{i}x + \mathbf{j}y + \mathbf{k}a^2 z}{a\sqrt{a^2 z^2 + 1 - (1 - x^2/a^2 - y^2/a^2)}} \\
&= \frac{\mathbf{i}x + \mathbf{j}y + \mathbf{k}a^2 z}{a\sqrt{a^2 z^2 + 1 - z^2}} \\
&= \frac{\mathbf{i}x + \mathbf{j}y + \mathbf{k}a^2 z}{a\sqrt{1 + (a^2 - 1)z^2}}
\end{aligned}$$

2. (a) Show that the unit vector normal to the plane

$$ax + by + cz = d$$

is given by

$$\hat{\mathbf{n}} = \pm(\mathbf{i}a + \mathbf{b} + \mathbf{c})/(a^2 + b^2 + c^2)^{\frac{1}{2}}$$

Solution:

$$z = f(x,y) = d/c - ax/c - by/c$$

$$\frac{\partial f}{\partial x} = -a/c \quad \textit{and} \quad \frac{\partial f}{\partial y} = -b/c$$

$$\begin{aligned}
\hat{\mathbf{n}} &= \frac{-\mathbf{i}\,(-a/c) - \mathbf{j}\,(-b/c) + \mathbf{k}}{\sqrt{1 + (-a/c)^2 + (-b/c)^2}} \\
&= \frac{\mathbf{i}(a/c) + \mathbf{j}(b/c) + \mathbf{k}}{\sqrt{1 + a^2/c^2 + b^2/c^2}} \\
&= \frac{\mathbf{i}a + \mathbf{j}b + \mathbf{k}c}{c\sqrt{1 + a^2/c^2 + b^2/c^2}} \\
&= \frac{\mathbf{i}a + \mathbf{j}b + \mathbf{k}c}{\sqrt{c^2 + a^2 + b^2}}
\end{aligned}$$

(b) Explain in geometric terms why this expression for $\hat{\mathbf{n}}$ is independent of the constant d.

Solution:

$\hat{\mathbf{n}}$ being independent of d reflects the fact that there are infinitely many parallel planes that this vector is the normal unit vector to.

3. Derive the expressions for the unit normal vector for surfaces given by $y = g(x, z)$ and by $x = h(y, z)$. Use each to rederive the expression for the normal to the plane given in Problem II-2.

   Solution:

   For the surface described by $y = g(x, z)$, we need two vectors $\mathbf{u}$ and $\mathbf{v}$ tangent to the surface. To get the first vector, hold z constant, and slice the surface with a plane parallel to the xy-plane. This plane intersects the surface S on a curve. For a point on that curve, the vector $\mathbf{u}$ tangent to the point, the slope $m = \frac{u_y}{u_x}$. The total derivative of the surface at this point is

$$dg = \frac{\partial g}{\partial x} dx + \frac{\partial g}{\partial z} dz$$

   On the plane we have used to slice through S, z is fixed, so $dz = 0$:

$$dg = \frac{\partial g}{\partial x} dx \quad \Rightarrow \quad \frac{dg}{dx} = \frac{\partial g}{\partial x}$$

   The derivative is the slope of the line tanget to the point, so:

$$\frac{u_y}{u_x} = \frac{\partial g}{\partial x}$$

$$u_y = \left(\frac{\partial g}{\partial x}\right) u_x$$

   Thus

$$\mathbf{u} = \mathbf{i}u_x + \mathbf{j}\left(\frac{\partial g}{\partial x}\right) u_x = \left[\mathbf{i} + \mathbf{j}\left(\frac{\partial g}{\partial x}\right)\right] u_x$$

   For the other vector $\mathbf{v}$, hold x fixed and allow z to vary so that we slice S with a plane parallel to the yz-plane. Using similar reasoning, the slope of $\mathbf{v}$ tangent to the intersecting curve of S and the slicing plane is:

$$m = \frac{v_y}{v_z} = \frac{\partial g}{\partial z}$$

$$\Rightarrow v_y = \left(\frac{\partial g}{\partial z}\right) v_z$$

   Thus

$$\mathbf{v} = \mathbf{j}\left(\frac{\partial g}{\partial z}\right) v_z + \mathbf{k}v_z = \left[\mathbf{j}\left(\frac{\partial g}{\partial z}\right) + \mathbf{k}\right] v_z$$

The unit normal vector is defined as:

$$\hat{\mathbf{n}}(x, y, z) = \frac{\mathbf{u} \times \mathbf{v}}{|\mathbf{u} \times \mathbf{v}|}$$

Then,

$$\begin{aligned}
\mathbf{u} \times \mathbf{v} &= \mathbf{i}(u_z v_y - u_y v_z) + \mathbf{j}(u_x v_z - u_z v_x) + \mathbf{k}(u_y v_x - u_x v_y) \\
\mathbf{u} \times \mathbf{v} &= \mathbf{i}\left((0)\frac{\partial g}{\partial z} v_z - \frac{\partial g}{\partial x} u_x v_z\right) + \mathbf{j}\left(u_x v_z - (0)(0)\right) + \mathbf{k}\left(\frac{\partial g}{\partial x} u_x (0) - u_x \frac{\partial g}{\partial z} v_z\right) \\
\mathbf{u} \times \mathbf{v} &= -\mathbf{i}(\frac{\partial g}{\partial x} u_x v_z) + \mathbf{j}(u_x v_z) - \mathbf{k}(u_x \frac{\partial g}{\partial z} v_z) \\
\mathbf{u} \times \mathbf{v} &= \left[-\mathbf{i}\frac{\partial g}{\partial x} + \mathbf{j} - \mathbf{k}\frac{\partial g}{\partial z}\right] u_x v_z \\
|\mathbf{u} \times \mathbf{v}| &= \sqrt{u_x^2 v_z^2 \left[\left(\frac{\partial g}{\partial x}\right)^2 + 1 + \left(\frac{\partial g}{\partial z}\right)^2\right]}
\end{aligned}$$

$$\Rightarrow \hat{\mathbf{n}} = \frac{-\mathbf{i}\frac{\partial g}{\partial x} + \mathbf{j} - \mathbf{k}\frac{\partial g}{\partial z}}{\sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial z}\right)^2}} \quad \text{for} \quad y = g(x, z)$$

The derivation for $x = h(y, z)$ is similar, and the result is:

$$\hat{\mathbf{n}} = \frac{-\mathbf{i}\frac{\partial h}{\partial y} + \mathbf{j} - \mathbf{k}\frac{\partial h}{\partial z}}{\sqrt{1 + \left(\frac{\partial h}{\partial y}\right)^2 + \left(\frac{\partial h}{\partial z}\right)^2}} \quad \text{for} \quad x = h(y, z)$$

4. In each of the following use Equation II-12 to evaluate the surface integral $\int\int_S G(x, y, z) dS$

(a) $G(x, y, z) = z$, where S is the portion of the plane $x + y + z = 1$ in the first octant.

Solution:

$$z = f(x, y) = 1 - x - y$$

$$\frac{\partial f}{\partial x} = -1 \quad \mathit{and} \quad \frac{\partial f}{\partial y} = -1$$

$$\int\int_S G(x, y, f(x, y)) \cdot \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} dx dy = \int\int_R z \cdot \sqrt{1 + (-1)^2 + (-1)^2} dx dy$$

$$= \int\int_R (1 - x - y)\sqrt{3} dx dy$$

Where the integral is still expressed merely over R. To get the limits of integration, we must find $f(x, y)$ in the first octant, which is when $x \geq 0, y \geq 0, z \geq 0$. R is the projection of S onto the xy-plane, so z = 0.

$$0 \leq 1 - x - y$$

$$x \leq 1 - y$$

So the limits of integration for x are $x \in (0, 1 - y)$. For y, the lower limit is 0 since y must be positive. The maximum for y is when x = 0:

$$0 = 1 - 0 - y$$

$$y = 1$$

Thus $y \in (0,1)$. The integral then is:

$$
\begin{aligned}
&= \int_0^1 \int_0^{1-y} (1-x-y)\sqrt{3}dxdy \\
&= \sqrt{3}\int_0^1 (x - x^2/2 - yx)|_0^{1-y} dy \\
&= \sqrt{3}\int_0^1 (1-y-(1-y)^2/2 - y(1-y))|_0^{1-y} dy \\
&= \frac{\sqrt{3}}{2}\int_0^1 (2-2y-(1-2y+y^2)-2y+2y^2)dy \\
&= \frac{\sqrt{3}}{2}\int_0^1 (1-2y+y^2)dy \\
&= \frac{\sqrt{3}}{2}(y-y^2+y^3/3)|_0^1 \\
&= \frac{\sqrt{3}}{2}(1-(1)^2+(1)^3/3) \\
&= \frac{\sqrt{3}}{6}
\end{aligned}
$$

(b) $G(x,y,z) = \frac{1}{1+4(x^2+y^2)}$ where S is the portion of the paraboloid $z = x^2+y^2$ between z=0 and z=1.

$$\int\int_S G(x,y,f(x,y))\cdot\sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}dxdy = \int\int_R \frac{1}{1+4(x^2+y^2)}\cdot\sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}dxdy$$

$$
\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}(x^2+y^2) = 2x \\
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(x^2+y^2) = 2y
\end{aligned}
$$

The region R is the projection of S onto the xy-plane. Since the surface is a paraboloid around the z-axis, R is a circle centered at the origin. Using z=1,

$$1 = x^2 + y^2$$

This is a circle of radius 1. We can convert to polar coordinates, with $r \in (0,1)$ and $\theta \in (0, 2\pi)$.

$dxdy = rdrd\theta$.

$$
\begin{aligned}
&= \int\int_R \frac{1}{1+4(x^2+y^2)} \cdot \sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} dxdy \\
&= \int\int_R \frac{1}{1+4(x^2+y^2)} \cdot \sqrt{1+(2x)^2+(2y)^2} dxdy \\
&= \int\int_R \frac{1}{1+4(x^2+y^2)} \cdot \sqrt{1+4(x^2+y^2)} dxdy \\
&= \int\int_R \frac{1}{\sqrt{1+4(x^2+y^2)}} dxdy \\
&= \int_0^{2\pi}\int_0^1 \frac{1}{\sqrt{1+4r^2}} rdrd\theta \\
&\quad \text{let } u = 1+4r^2 \\
&\quad du = 8rdr \\
&\quad r = 0 \to u = 1 \\
&\quad r = 1 \to u = 5 \\
&= \frac{1}{8}\int_0^{2\pi}\int_1^5 \frac{1}{\sqrt{u}} dud\theta \\
&= \frac{1}{8}\int_0^{2\pi}\int_1^5 \frac{1}{\sqrt{u}} dud\theta \\
&= \frac{1}{8}\int_0^{2\pi} 2u^{1/2}\Big|_1^5 d\theta \\
&= \frac{1}{8}\int_0^{2\pi} 2\sqrt{5}-2d\theta \\
&= \frac{1}{8}2\pi\left(2\sqrt{5}-2\right) \\
&= \frac{\pi}{2}\left(\sqrt{5}-1\right)
\end{aligned}
$$

(c) $G(x,y,z) = (1-x^2-y^2)^{\frac{3}{2}}$, where S is the hemisphere $z = (1-x^2-y^2)^{\frac{1}{2}}$.

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(1-x^2-y^2)^{\frac{1}{2}} = \frac{-2x}{2\sqrt{1-x^2-y^2}} = \frac{-x}{z}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(1-x^2-y^2)^{\frac{1}{2}} = \frac{-2y}{2\sqrt{1-x^2-y^2}} = \frac{-y}{z}$$

$$\int\int_S G(x,y,f(x,y)) \cdot \sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} dxdy = \int\int_R (1-x^2-y^2)^{\frac{3}{2}} \cdot \sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} dxdy$$

$$
\begin{aligned}
&= \int\int_R (z^2)^{\frac{3}{2}} \cdot \sqrt{1+(-x/z)^2+(-y/z)^2}dxdy \\
&= \int\int_R z^3\sqrt{1+x^2/z^2+y^2/z^2}dxdy \\
&= \int\int_R \frac{z^3}{z}\sqrt{z^2+x^2+y^2}dxdy \\
&= \int\int_R z^2\sqrt{z^2+x^2+y^2}dxdy \\
&= \int\int_R z^2\sqrt{(1-x^2-y^2)+x^2+y^2}dxdy \\
&= \int\int_R z^2\sqrt{1}dxdy \\
&= \int\int_R 1-x^2-y^2dxdy
\end{aligned}
$$

The region R is the projection of the hemisphere onto the xy-plane, which is a circle centered at the origin with radius 1. Converting to polar coordinates:

$$
\begin{aligned}
&= \int_0^{2\pi}\int_0^1 (1-r^2)rdrd\theta \\
&\text{let } u = 1-r^2 \\
&du = -2rdr \\
&r = 0 \rightarrow u = 1 \\
&r = 1 \rightarrow u = 0 \\
&= -\frac{1}{2}\int_0^{2\pi}\int_1^0 (u)dud\theta \\
&= -\frac{1}{4}\int_0^{2\pi} u^2|_1^0 d\theta \\
&= \frac{1}{4}\int_0^{2\pi} d\theta \\
&= \frac{2\pi}{4} \\
&= \frac{\pi}{2}
\end{aligned}
$$

5. In each of the following use Equation II-13 to evaluate the surface integral $\int\int_S \mathbf{F}\cdot\mathbf{n}dS$

(a) $\mathbf{F}(x,y,z) = \mathbf{i}x - \mathbf{k}z$, where S is the portion of the plane x + y + 2z = 2 in the first octant.

Solution:

Equation II-13 is:

$$\int\int_S \mathbf{F}\cdot\hat{\mathbf{n}}dS = \int\int_R \left\{-F_x[x,y,f(x,y)]\frac{\partial f}{\partial x} - F_y[x,y,f(x,y)]\frac{\partial f}{\partial y} + F_z[x,y,f(x,y)]\right\} dxdy$$

Thus:

$$
\begin{aligned}
&F_x = x \quad F_y = 0 \quad F_z = -z \\
&z = f(x,y) = 1 - x/2 - y/2 \\
&\frac{\partial f}{\partial x} = -1/2 \quad \frac{\partial f}{\partial y} = -1/2
\end{aligned}
$$

R is the projection of S onto the xy-plane, where z=0, so R is bounded by

$$x + y + 2(0) = 2$$

$$x = 2 - y$$

$$\Rightarrow x \in (0, 2-y) \quad y \in (0, 2)$$

$$\begin{aligned}
\int\int_S \mathbf{F}\cdot\hat{\mathbf{n}}dS &= \int_0^2\int_0^{2-y}\{-(x)(-1/2) - (0) + (-z)\}\,dxdy \\
&= \int_0^2\int_0^{2-y}(x/2 - 1 + x/2 + y/2)dxdy \\
&= \int_0^2\int_0^{2-y}(-1 + x + y/2)dxdy \\
&= 2\int_0^2\int_0^{2-y}(-2 + 2x + y)dxdy \\
&= 2\int_0^2(-2x + x^2 + yx)\big|_0^{2-y}dy \\
&= 2\int_0^2(-2(2-y) + (2-y)^2 + y(2-y))dy \\
&= 2\int_0^2(-4 + 2y + 4 - 4y + y^2 + 2y - y^2)dy \\
&= 2\int_0^2(0)dy \\
&= 0
\end{aligned}$$

(b) $\mathbf{F}(x,y,z) = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$, where S is the hemisphere $z = \sqrt{a^2 - x^2 - y^2}$.
Solution:

$$F_x = x \quad F_y = y \quad F_z = z$$

$$\frac{\partial f}{\partial x} = \frac{-2x}{2\sqrt{a^2 - x^2 - y^2}} = \frac{-x}{z} \quad \frac{\partial f}{\partial y} = \frac{-y}{z}$$

R is:

$$0 = \sqrt{a^2 - x^2 - y^2}$$

$$x^2 + y^2 = a^2$$

This is a circle in the xy-plane with radius a.

$$\begin{aligned}
\int\int_S \mathbf{F}\cdot\hat{\mathbf{n}}dS &= \int\int_R\left\{-F_x[x,y,f(x,y)]\frac{\partial f}{\partial x} - F_y[x,y,f(x,y)]\frac{\partial f}{\partial y} + F_z[x,y,f(x,y)]\right\}dxdy \\
&= \int\int_R\{-(x)(-x/z) - (y)(-y/z) + z\}\,dxdy \\
&= \int\int_R\left\{x^2/z + y^2/z + z\right\}dxdy \\
&= \int\int_R(1/z)\left(x^2 + y^2 + z^2\right)dxdy \\
&= \int\int_R\frac{a^2}{\sqrt{a^2 - (x^2 + y^2)}}dxdy
\end{aligned}$$

Converting to polar coordinates:

$$
\begin{aligned}
&= \int_0^{2\pi} \int_0^a \frac{a^2}{\sqrt{a^2 - r^2}} r dr d\theta \\
&u = a^2 - r^2 \\
&du = -2rdr \\
&r = 0 \rightarrow u = a^2 \\
&r = a \rightarrow u = 0 \\
&= \frac{-a^2}{2} \int_0^{2\pi} \int_{a^2}^0 \frac{1}{\sqrt{u}} du d\theta \\
&= \frac{-a^2}{2} \int_0^{2\pi} 2u^{1/2}\Big|_{a^2}^0 d\theta \\
&= -a^2 \int_0^{2\pi} -(a^2)^{1/2} d\theta \\
&= a^3 \int_0^{2\pi} d\theta \\
&= 2\pi a^3
\end{aligned}
$$

(c) $\mathbf{F}(x, y, z) = \mathbf{j}y + \mathbf{k}$, where S is the portion of the paraboloid $z = 1 - x^2 - y^2$ above the xy-plane.
Solution:

$$
\begin{aligned}
&F_x = 0 \quad F_y = y \quad F_z = 1 \\
&\frac{\partial f}{\partial x} = -2x \quad \frac{\partial f}{\partial y} = -2y
\end{aligned}
$$

R is:

$$
\begin{aligned}
0 &= 1 - x^2 - y^2 \\
x^2 + y^2 &= 1
\end{aligned}
$$

R is a circle on the xy-plane with radius 1.

$$
\begin{aligned}
\int\int_S \mathbf{F} \cdot \hat{\mathbf{n}} dS &= \int\int_R \left\{ -F_x[x, y, f(x, y)] \frac{\partial f}{\partial x} - F_y[x, y, f(x, y)] \frac{\partial f}{\partial y} + F_z[x, y, f(x, y)] \right\} dxdy \\
&= \int\int_R \{-(0)(-2x) - (y)(-2y) + 1\} dxdy \\
&= \int\int_R \left(2y^2 + 1\right) dxdy \\
&= \int\int_R 1 dxdy + \int\int_R 2y^2 dxdy
\end{aligned}
$$

In polar coordinates, $y = r\sin\theta$:

$$
\begin{aligned}
&= \int_0^{2\pi}\int_0^1 r dr d\theta + \int_0^{2\pi}\int_0^1 2r^2\sin^2\theta r dr d\theta \\
&= \pi + 2\int_0^{2\pi}\int_0^1 r^3\sin^2\theta dr d\theta \\
&= \pi + 2\int_0^{2\pi} \frac{r^4}{4}\Big|_0^1 \sin^2\theta d\theta \\
&= \pi + (1/2)\int_0^{2\pi} \sin^2\theta d\theta \\
&= \pi + (1/2)\int_0^{2\pi} \{(1/2) - (1/2)\cos 2\theta\}\, d\theta \\
&= \pi + (1/2)\int_0^{2\pi} (1/2) d\theta - (1/2)\int_0^{2\pi} (1/2)\cos 2\theta d\theta \\
&= \pi + (1/2)(2\pi/2) - (1/2)(1/4)\sin 2\theta\Big|_0^{2\pi} \\
&= \pi + (\pi/2) - (1/8)(\sin 4\pi - \sin 0) \\
&= 3\pi/2
\end{aligned}
$$

6. The distribution of mass on the hemispherical shell $z = (R^2 - x^2 - y^2)^{1/2}$ is given by $\sigma(x,y,z) = (\sigma_0/R^2)(x^2+y^2)$ where σ_0 is a constant. Find an expression in terms of σ_0 and R for the total mass of the shell.

   Solution:

   To get the total mass of a shell from the mass density, integrate the density over the surface of the shell:

$$M = \iint_S \sigma(x,y,z) dS$$

   The means of solving this in the spirit of the book is to evaluate the integral of σ over the projection of S onto the xy-plane. This result is derived in the book, and is:

$$M = \iint_S \sigma(x,y,z) dS = \iint_R \sigma(x,y,z)\cdot\sqrt{1 + \frac{\partial f}{\partial x}^2 + \frac{\partial f}{\partial y}^2}\, dxdy \tag{II-12}$$

$$f(x,y) = z(x,y,z) \qquad \frac{\partial f}{\partial x} = \frac{-2x}{2\sqrt{R^2 - x^2 - y^2}} = \frac{-x}{z} \qquad \frac{\partial f}{\partial y} = \frac{-y}{z}$$

$$
\begin{aligned}
M &= \iint_S \sigma(x,y,z) dS = \iint_R \frac{\sigma_0}{R^2}(x^2+y^2)\cdot\sqrt{1 + \left(\frac{-x}{z}\right)^2 + \left(\frac{-y}{z}\right)^2}\, dxdy \\
&= \frac{\sigma_0}{R^2}\iint_R (x^2+y^2)\cdot\frac{1}{z}\sqrt{z^2 + x^2 + y^2} dxdy \\
&= \frac{\sigma_0}{R^2}\iint_R (x^2+y^2)\cdot\frac{1}{(R^2 - x^2 - y^2)^{1/2}}\sqrt{((R^2 - x^2 - y^2)^{1/2})^2 + x^2 + y^2} dxdy \\
&= \frac{\sigma_0}{R^2}\iint_R (x^2+y^2)\cdot\frac{R}{(R^2 - x^2 - y^2)^{1/2}} dxdy
\end{aligned}
$$

   At this point we can convert to polar coordinates, which will also help define the region R (different from the constant R).

$$r^2 = x^2 + y^2 \qquad dxdy = r dr d\theta$$

$$R : r \in (0, R), \theta \in (0, 2\pi)$$
$$M = \frac{\sigma_0}{R} \int_0^{2\pi} \int_0^R \frac{r^3}{\sqrt{R^2 - r^2}} dr d\theta$$

We can solve this through integration by parts:

$$\int u dv = uv - \int v du$$

$$u = r^2 \quad du = 2rdr$$
$$dv = \frac{-2rdr}{(R^2 - r^2)^{1/2}} \quad v = (R^2 - r^2)^{1/2}$$

(admittedly, this takes some jiggery pokery to figure out.)

$$\begin{aligned}
M &= \frac{2\pi\sigma_0}{R} \int_0^R \frac{r^3}{\sqrt{R^2 - r^2}} dr \\
&= \frac{2\pi\sigma_0}{R} \left((r^2)(R^2 - r^2)^{1/2} \Big|_0^R - \int_0^R 2r\sqrt{R^2 - r^2} dr \right) \\
&= \frac{2\pi\sigma_0}{R} \left((R^2)(R^2 - R^2)^{1/2} - (0)(R^2) - \int_0^R 2r\sqrt{R^2 - r^2} dr \right) \\
&= -\frac{2\pi\sigma_0}{R} \int_0^R 2r\sqrt{R^2 - r^2} dr \\
& let \quad u = (R^2 - r^2) \quad du = -2rdr \\
&= \frac{2\pi\sigma_0}{R} \int_{R^2}^0 \sqrt{u} du \\
&= \frac{2\pi\sigma_0}{R} \frac{2u^{3/2}}{3} \Big|_{R^2}^0 \\
&= \frac{4\pi\sigma_0}{3R} (R^2)^{3/2} \\
&= \frac{4\pi\sigma_0 R^2}{3}
\end{aligned}$$

7. Find the moment of inertia about the z-axis of the hemispherical shell of Problem II-6.

   Solution:

   In general:

$$I = \int_0^M r^2 dm$$

   In this case, the mass distribution function is over a surface:

$$I = \iint_S \sigma_0(x, y, z) r^2 dS$$

$$I = \iint_S \sigma_0(x, y, z)(x^2 + y^2) dS$$

$$I = \iint_S (\sigma_0/R^2)(x^2 + y^2)(x^2 + y^2) dS$$

   This is solved similarly to the integral in II-6. The surface S is the same, so the partial derivatives and R are the same as well:

$$I = \frac{\sigma_0}{R^2} \iint_R (x^2+y^2)^2 \cdot \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} dxdy$$

$$= \frac{\sigma_0}{R^2} \iint_R (x^2+y^2)^2 \cdot \sqrt{1 + \left(\frac{-x}{z}\right)^2 + \left(\frac{-y}{z}\right)^2} dxdy$$

$$= \frac{\sigma_0}{R^2} \iint_R (x^2+y^2)^2 \cdot \frac{R}{(R^2-x^2-y^2)^{1/2}} dxdy$$

$$= \frac{\sigma_0}{R^2} \int_0^{2\pi} \int_0^R r^4 \cdot \frac{R}{(R^2-r^2)^{1/2}} rdrd\theta$$

$$= \frac{\sigma_0}{R} \int_0^{2\pi} \int_0^R \frac{r^5}{(R^2-r^2)^{1/2}} drd\theta$$

$$\text{let} \quad u = r^4 \quad du = 4r^3dr$$

$$dv = \frac{-2rdr}{2(R^2-r^2)^{1/2}} = \frac{-rdr}{(R^2-r^2)^{1/2}} \quad v = (R^2-r^2)^{1/2}$$

$$= \frac{-\sigma_0}{R} \int_0^{2\pi} \int_0^R r^4 \frac{-r}{(R^2-r^2)^{1/2}} drd\theta$$

$$= \frac{-\sigma_0}{R} \left[\int_0^{2\pi} (r^4)(R^2-r^2)^{1/2} \Big|_0^R d\theta - \int_0^{2\pi} \int_0^R 4r^3(R^2-r^2)^{1/2} drd\theta \right]$$

$$= \frac{\sigma_0}{R} \int_0^{2\pi} \int_0^R 4r^3(R^2-r^2)^{1/2} drd\theta$$

$$\text{let} \quad u = 4r^2 \quad du = 8rdr$$

$$dv = (-3r)(R^2-r^2)^{1/2} \quad v = (R^2-r^2)^{3/2}$$

$$= \frac{\sigma_0}{R} \int_0^{2\pi} \int_0^R 4r^2 \left(\frac{-3r}{-3}\right) (R^2-r^2)^{1/2} drd\theta$$

$$= \frac{-\sigma_0}{3R} \left[\int_0^{2\pi} (4r^2)(R^2-r^2)^{3/2} \Big|_0^R d\theta - \int_0^{2\pi} \int_0^R 8r(R^2-r^2)^{3/2} drd\theta \right]$$

$$= \frac{8\sigma_0}{3R} \int_0^{2\pi} \int_0^R r(R^2-r^2)^{3/2} drd\theta$$

$$\text{let} \quad u = R^2 - r^2 \quad du = -2rdr$$

$$r = 0 \to u = R^2 \quad r = R \to u = 0$$

$$= \frac{8\sigma_0}{3R} \int_0^{2\pi} \int_0^R \frac{-2r}{-2} (R^2-r^2)^{3/2} drd\theta$$

$$= \frac{-4\sigma_0}{3R} \int_0^{2\pi} \int_{R^2}^0 (u)^{3/2} dud\theta$$

$$= \frac{-4\sigma_0}{3R} \int_0^{2\pi} (2/5)(u)^{5/2} \Big|_{R^2}^0 d\theta$$

$$= \frac{-8\sigma_0}{15R} \int_0^{2\pi} \left[(0)^{5/2} - (R^2)^{5/2} \right] d\theta$$

$$= \frac{8\sigma_0}{15R} \int_0^{2\pi} R^5 d\theta$$

$$= \frac{8\sigma_0 R^4}{15} 2\pi$$

$$= \frac{16\pi\sigma_0 R^4}{15}$$

8. An electrostatic field is given by $\mathbf{E} = \lambda(\mathbf{i}yz + \mathbf{j}xz + \mathbf{k}xy)$, where λ is a constant. Use Gauss' law to find the total charge enclosed by the surface shown in the figure consisting of S_1, the hemisphere $z = (R^2 - x^2 - y^2)^{1/2}$, and S_2, its circular base in the xy-plane.

Solution:

Gauss's Law is

$$\iint_S \mathbf{E} \cdot \hat{\mathbf{n}} dS = \frac{q}{\epsilon_0}$$

We have to evaluate Gauss's Law over the two surfaces separately. For the base $\hat{\mathbf{n}} = \hat{\mathbf{k}}$ since the normal vector will point up (or down). Thus

$$\mathbf{E} \cdot \hat{\mathbf{n}} = E_z = \lambda xy$$

$$\frac{q}{\epsilon_0} = \lambda \iint_R (xy) dS$$

Converting to polar coordinates

$$x = r\cos\theta \quad y = r\sin\theta$$

$$\theta \in (0, 2\pi) \quad r \in (0, R)$$

$$\frac{q}{\epsilon_0} = \lambda \int_0^{2\pi} \int_0^R (r^2 \cos\theta \sin\theta) r dr d\theta$$

The integral $\int_0^{2\pi} \sin\theta \cos\theta d\theta$ is well known to be zero, since it is an odd function being integrated over a symmetric area. In an odd function, $f(-x) = -f(x)$, so integrating over a symmetric area results in zero. Thus for the base,

$$\frac{q}{\epsilon_0} = 0 \implies q = 0$$

For the second part of the surface, the hemisphere:

$$f = z(x, y) = (R^2 - x^2 - y^2)^{1/2}$$

$$\frac{\partial f}{\partial x} = (-x/z) \quad \frac{\partial f}{\partial y} = (-y/z)$$

We can use II-13 to evaluate this since $\mathbf{E}$ is not always directed radially:

$$\iint_S \mathbf{E} \cdot \hat{\mathbf{n}} dS = \iint_R \left\{ -E_x[x, y, f(x, y)] \frac{\partial f}{\partial x} - E_y[x, y, f(x, y)] \frac{\partial f}{\partial y} + E_z[x, y, f(x, y)] \right\} dxdy$$

$$\frac{q}{\epsilon_0} = \lambda \iint_R \{-(yz)(-x/z) - (xz)(-y/z) + xy\} dxdy$$

$$= \lambda \iint_R (yx + xy + xy) dxdy$$

$$= \lambda \iint_R (3xy) dxdy$$

Converting to polar coordinates

$$x = r\cos\theta \quad y = r\sin\theta$$

$$\theta \in (0, 2\pi) \quad r \in (0, R)$$

$$= 3\lambda \int_0^{2\pi} \int_0^R (r^2 \sin\theta \cos\theta) r dr d\theta$$

This is the same integral as we had for the base, so the answer is zero. Thus in total:

$$q = 0$$

9. An electrostatic field is given by $\mathbf{E} = \lambda(\mathbf{i}x + \mathbf{j}y + \mathbf{k}0)$, where λ is a constant. Use Gauss' Law to find the total charge enclosed by the surface shown in the figure consisting of S_1, the curved portion of the half-cylinder $z = (r^2 - y^2)^{1/2}$ of length h; S_2 and S_3, the two semicircular plane end pieces; and S_4, the rectangular portion of the xy-plane. Express your results in terms of λ, r, and h.

   Solution:

   We have to apply Gauss' Law to each of the four surfaces independently. However, by inspection, the two ends of the half-cylinder will yield the same result. Gauss's Law is:

$$\iint_S \mathbf{E} \cdot \hat{\mathbf{n}} dS = \frac{q}{\epsilon_0}$$

   We will take S_4, the long base on the xy-plane, first. The normal vector to this surface is $\hat{\mathbf{n}} = \hat{\mathbf{k}}$. Thus:

$$\mathbf{E} \cdot \hat{\mathbf{n}} = \mathbf{E} \cdot \hat{\mathbf{k}} = E_z = 0$$

$$\Longrightarrow \iint_{S_4} \mathbf{E} \cdot \hat{\mathbf{n}} dS = \frac{q}{\epsilon_0} = 0$$

   Next we take S_2, the half-circle end to the cylinder, and will double the result to account for S_3. The normal vector is $\hat{\mathbf{n}} = \hat{\mathbf{i}}$. Thus:

$$\mathbf{E} \cdot \hat{\mathbf{n}} = \mathbf{E} \cdot \hat{\mathbf{i}} = E_x = \lambda h/2$$

$$\Longrightarrow \iint_{S_2} \mathbf{E} \cdot \hat{\mathbf{n}} dS = \frac{q}{\epsilon_0} = \lambda \iint_{S_2} \frac{h}{2} dS$$

   Converting to polar coordinates:

$$\theta \in (0, \pi) \quad \rho \in (0, r)$$

$$dS = \rho d\rho d\theta$$

$$\frac{q}{\epsilon_0} = \lambda \frac{h}{2} \iint_{S_2} dS = \lambda \frac{h}{2} \int_0^\pi \int_0^r \rho d\rho d\theta$$

   where we have use ρ as the polar radial coordinate because r in this case is a constant.

$$\frac{q}{\epsilon_0} = \lambda \frac{h}{2} \frac{r^2 \pi}{2}$$

$$\Longrightarrow q = \frac{\lambda h r^2 \pi \epsilon_0}{4}$$

   for both S_2 and S_3.

   Lastly, S_1, the half-cylindrical shell.

$$f = z(x, y) = (r^2 - y^2)^{1/2}$$

$$\frac{\partial f}{\partial x} = (0) \quad \frac{\partial f}{\partial y} = (-y/z)$$

   Applying II-13 to Gauss's Law:

$$\iint_S \mathbf{E} \cdot \hat{\mathbf{n}} dS = \iint_R \left\{ -E_x[x, y, f(x, y)] \frac{\partial f}{\partial x} - E_y[x, y, f(x, y)] \frac{\partial f}{\partial y} + E_z[x, y, f(x, y)] \right\} dx dy$$

$$\frac{q}{\epsilon_0} = \lambda \iint_R \{-x(0) - y(-y/z) + (0)\} dx dy$$

$$= \lambda \int_{-h/2}^{h/2} \int_{-r}^{r} \frac{y^2}{(r^2 - y^2)^{1/2}} dx dy$$

$$= \lambda \left[x\Big|_{-h/2}^{h/2} \right] \int_{-r}^{r} \frac{y^2}{(r^2 - y^2)^{1/2}} dy$$

$$= h\lambda \int_{-r}^{r} \frac{y^2}{(r^2 - y^2)^{1/2}} dy$$

TODO - work out this integral through trig substitution

$$\frac{q}{\epsilon_0} = h\lambda \frac{\pi r^2}{2}$$

$$\implies q = h\lambda \frac{\pi r^2}{2} \epsilon_0$$

Adding together the contributions from each surface:

$$q = q_{S_1} + q_{S_2} + q_{S_3} + q_{S_4}$$

$$q = 0 + \frac{\lambda h r^2 \pi \epsilon_0}{4} + \frac{\lambda h r^2 \pi \epsilon_0}{4} + h\lambda \frac{\pi r^2}{2} \epsilon_0$$

$$q = \lambda h r^2 \pi \epsilon_0$$

10. It sometimes happens that surface integrals can be evaluated without using the long-winded procedures outlined in the text. Try evaluating $\iint_S \mathbf{F} \cdot \hat{\mathbf{n}} dS$ for each of the following; think a bit and avoid a lot of work!

(a) $\mathbf{F} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$. S, the three squares each of side b as shown in the figure (in the xz-plane, xy-plane, and yz-plane).

Solution:

On the square on the xy-plane, $z = 0$, and $\hat{\mathbf{n}} = -\hat{\mathbf{k}}$. Thus:

$$\mathbf{F} \cdot \mathbf{n} = F_z = z = 0$$

It is a similar result for each of the other faces. In the xz-plane:

$$y = 0 \quad \hat{\mathbf{n}} = -\hat{\mathbf{y}}$$

$$\mathbf{F} \cdot \mathbf{n} = F_y = y = 0$$

In the yz-plane:

$$x = 0 \quad \hat{\mathbf{n}} = -\hat{\mathbf{x}}$$

$$\mathbf{F} \cdot \mathbf{n} = F_x = x = 0$$

Thus, $\iint_S \mathbf{F} \cdot \hat{\mathbf{n}} dS = 0$

(b) $\mathbf{F} = (\mathbf{i}x + \mathbf{j}y + \mathbf{k}0) \ln(x^2 + y^2)$. S, the cylinder (including the top and bottom) of radius R and height h shown in the figure (centered on z-axis).

Solution:

Treat the cylinder as three surfaces, the top circle, bottom circle, and ring in the middle. The top and bottom surfaces will be the same. In each case, $\hat{\mathbf{n}} = \pm\hat{\mathbf{k}}$. Thus for the top and bottom circle:

$$\mathbf{F} \cdot \mathbf{n} = \mathbf{F} \cdot \mathbf{k} = E_z = 0$$

$$\implies \iint_S \mathbf{F} \cdot \hat{\mathbf{n}} dS = 0$$

For the ring, we should use cylindrical coordinates where:

$$z \in (0, h), \quad r = R, \quad \phi \in (0, 2\pi)$$

$$r^2 = x^2 + y^2$$
$$x = r\cos\phi \quad y = r\sin\phi$$

The difficulty is finding the unit normal vector in a useful form. In cylindrical coordinates, it is $\hat{\mathbf{n}} = \hat{\mathbf{r}}$, but we need this in Cartesian coordinates so that we can perform the dot product. The vector $\vec{r}$ is the sum of the vectors $\vec{x}$ and $\vec{y}$:

$$\vec{r} = \vec{x} + \vec{y} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$$
$$\implies \hat{\mathbf{n}} = \hat{\mathbf{r}} = \frac{\vec{r}}{|r|} = \frac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}}}{R}$$

We can now perform the dot product:

$$\mathbf{F}\cdot\hat{\mathbf{n}} = \ln(x^2+y^2)\frac{(F_x)(x) + (F_y)(y)}{R} = \ln(R^2)\frac{x^2+y^2}{R} = (R^2/R)2\ln R = 2R\ln R$$

This is a constant since the surface is a shell and r=R, so it can be taken out of the integral, which otherwise gives the surface area of a cylinder less the two ends:

$$\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}dS = \int_0^{2\pi}\int_0^h (2R\ln R)rdzd\phi = 4\pi R^2 h\ln R$$

(c) $\mathbf{F} = (\mathbf{i}x + \mathbf{j}y + \mathbf{k}z)e^{-(x^2+y^2+z^2)}$. S, the surface of the sphere of radius R centered at the origin as shown in the figure.

Solution:

Similar to the previous problem but in spherical coordinates, the normal vector to the sphere's surface is:

$$\hat{\mathbf{n}} = \hat{\mathbf{r}} = \frac{(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})}{r}$$
$$\mathbf{F}\cdot\hat{\mathbf{n}} = \frac{x^2+y^2+z^2}{r}e^{-(x^2+y^2+z^2)} = re^{-(r^2)}$$

For this spherical shell:

$$r = R, \quad \theta \in (0, 2\pi), \quad \phi \in (0, \pi)$$

The dot product will be a constant again, and the rest of the integral gives the surface area of a sphere of radius R:

$$\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}dS = \int_0^{2\pi}\int_0^{\pi} re^{-(r^2)}r^2\sin\theta d\phi d\theta = R^3e^{-(R^2)}\int_0^{2\pi}\int_0^{\pi}\sin\theta d\phi d\theta$$
$$= 4\pi R^3 e^{-(R^2)}$$

(d) $\mathbf{F} = \mathbf{i}E(x)$ where E(x) is an arbitrary scalar function of x. S, the surface of the cube of side b shown in the figure (bottom corner at the origin, and the entire cube in the first octant).

Solution:

Faces with a unit normal vector of $\hat{\mathbf{j}}$ or $\hat{\mathbf{k}}$ will have yield a dot product with $\mathbf{F}$ of zero. The nonzero faces are the two parallel to the yz-plane, one of which has a +**i** normal vector and the other has a -**i**. Thus the surface integral is broken into two smaller surface integrals:

$$\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}dS = \iint_S E(x)dydz\bigg|_{x=b} + \iint_S -E(x)dydz\bigg|_{x=0}$$
$$= \iint_S E(b)dydz + \iint_S -E(0)dydz$$

The double integral is the area of the face of the cube, which is b^2.

$$= E(b)b^2 - E(0)b^2$$
$$= (E(b) - E(0))b^2$$

11. (a) Use Gauss' law and symmetry to find the electrostatic field as a function of position for an infinite uniform plane of charge. Let the charge lie in the yz-plane and denote the charge per unit area by σ.

Solution:

$$\iint_S \mathbf{E} \cdot \mathbf{n} dS = \frac{q}{\epsilon_0}$$

The components of the field in every direction except the $\pm\hat{\mathbf{x}}$ direction will cancel out. We will use a cylinder of radius $r = \sqrt{y^2 + z^2}$ and height x in either direction. The charge q in terms of σ is

$$q = \sigma \pi r^2$$

There are two surfaces we need to consider, the ends of the cylinder. The body of the cylinder has a dot product with the field of zero since the field is parallel to the edge.

$$\mathbf{E}(x) = \mathbf{i}E(x)\ \{x > 0\} = -\mathbf{i}E(x)\ \{x < 0\}$$

$$E(x) = E(-x)$$

$$\iint_S \mathbf{E(x)} dS = \iint_S E(x) dS + \iint_S E(-x) dS = \frac{\sigma \pi r^2}{\epsilon_0}$$

$$2\iint_S E(x) dS = \frac{\sigma \pi r^2}{\epsilon_0}$$

$$2E(x)\pi r^2 = \frac{\sigma \pi r^2}{\epsilon_0}$$

$$E(x) = \frac{\sigma}{2\epsilon_0}$$

Thus, for the positive and negative x-directions:

$$E(x) = \hat{\mathbf{i}}\frac{\sigma}{2\epsilon_0},\ x > 0$$

$$E(x) = -\hat{\mathbf{i}}\frac{\sigma}{2\epsilon_0},\ x < 0$$

(b) Repeat part (a) for an infinite slab of charge parallel to the yz-plane, whose density is given by

$$\rho(x) = \begin{cases} \rho_0, -b < x < b, \\ 0, |x| \geq b, \end{cases}$$

where ρ_0 and b are constants.

Solution:

Similar to part a, the components in every direction except $\pm\hat{\mathbf{x}}$ will cancel out. Thus:

$$\iint_S \mathbf{E(x)} dS = 2E(x)A$$

where we have used A as the area of the end of whatever shape we use. In part (a) we used a cylinder, but we could use any shape, and the area will cancel out, so we will leave it as a general area. We have to separately consider the region within the slab and the region without.

Within the slab $|x| < b$. The trick is determining the correct expression for the enclosed charge.

$$q = \rho_0 \cdot volume = \rho_0 A(2x)$$

$$\therefore 2E(x)A = \frac{2x\rho_0 A}{\epsilon_0}$$

$$E(x) = \frac{x\rho_0}{\epsilon_0},\ -b < x < b$$

Outside the slab, $|x| > b$. The enclosed charge is:

$$q = \rho_0 \cdot volume = \rho_0 A(2b)$$

$$\therefore 2E(x)A = \frac{2b\rho_0 A}{\epsilon_0}$$

$$E(x) = \frac{b\rho_0}{\epsilon_0}$$

Thus:

$$E(x) = \hat{\mathbf{i}}\frac{x\rho_0}{\epsilon_0},\ -b < x < b$$

$$E(x) = \hat{\mathbf{i}}\frac{b\rho_0}{\epsilon_0},\ x > b$$

$$E(x) = -\hat{\mathbf{i}}\frac{b\rho_0}{\epsilon_0},\ x < -b$$

(c) Repeat part (b) with $\rho(x) = \rho_0 e^{-|x/b|}$.

Solution:

The charged area is the same slab, so all the same symmetry considerations abide. The enclosed charge in this case is:

$$q = \iiint \rho(x)dV = \iiint \rho_0 e^{-|x/b|}dV$$

$$q = A\rho_0 \int_x^x e^{-|x'/b|}dx'$$

$$q = A\rho_0 2\int_0^x e^{-x'/b}dx'$$

$$q = 2A\rho_0 \left[(-b)e^{-x'/b}\right]\Big|_0^x$$

$$q = 2A\rho_0[(-b)e^{-x/b} - (-b)e^{-0/b}]$$

$$q = 2A\rho_0[(-b)e^{-x/b} + b]$$

$$q = 2Ab\rho_0[1 - e^{-x/b}]$$

$$\therefore 2E(x)A = \frac{2Ab\rho_0[1 - e^{-x/b}]}{\epsilon_0}$$

$$E(x) = \frac{b\rho_0[1 - e^{-x/b}]}{\epsilon_0}$$

$$\implies E(x) = \hat{\mathbf{i}}\frac{b\rho_0[1 - e^{-x/b}]}{\epsilon_0},\ |x| < b$$

$$\implies E(x) = -\hat{\mathbf{i}}\frac{b\rho_0[1 - e^{x/b}]}{\epsilon_0},\ |x| > b$$

12. (a) Use Gauss' Law and symmetry to find the electrostatic field as a function of position for an infinite line of charge. Let the charge lie along the z-axis and denote the charge per unit length by λ.

Solution:

By symmetry, the field will only point radially outward from the line of charge. The field in every other direction will cancel out. We'll use a cylinder around the z-axis for the surrounding surface. The ends of the cylinder do not contribute to the integral since the dot product with the field is zero. The only surface that matters is the outside of the cylinder, which we can treat as a rectangle of length L. Gauss' Law is:

$$\iint_S \mathbf{E(r)}dS = \frac{q}{\epsilon_0}$$

and the enclosed charge is:

$$q = \lambda L$$

$$\iint_S \mathbf{E(r)}dS = \frac{\lambda L}{\epsilon_0}$$

$$E(r)(L)(2\pi r) = \frac{\lambda L}{\epsilon_0}$$

$$E(r) = \frac{\lambda}{2\pi r\epsilon_0}$$

$$E(r) = \frac{\lambda}{2\pi\epsilon_0} \cdot \frac{\hat{\mathbf{e}}_r}{r}$$

(b) Repeat part (a) for an infinite cylinder of charge whose axis coincides with the z-axis and whose density is given in cylindrical coordinates by

$$\rho(r) = \begin{cases} \rho_0, r < x, \\ 0, r \geq b, \end{cases}$$

where ρ_0 and b are constants.

Solution:

All the same symmetry considerations abide as in part (a). There are two regions that matter, inside and outside the cylinder of charge. For $r < b$, the enclosed charge is:

$$q = \rho_0 * volume = \rho_0 \pi r^2 L$$

$$\iint_S \mathbf{E(r)}dS = \frac{\rho_0 \pi r^2 L}{\epsilon_0}$$

$$E(r)2\pi r L = \frac{\rho_0 \pi r^2 L}{\epsilon_0}$$

$$E(r) = \frac{\rho_0 r}{2\epsilon_0}$$

$$E(r) = \frac{\rho_0 r}{2\epsilon_0}\hat{\mathbf{e}}_r, \; r < b$$

For $r > b$, the enclosed charge is:

$$q = \rho_0 \pi b^2 L$$

$$\iint_S \mathbf{E(r)}dS = \frac{\rho_0 \pi b^2 L}{\epsilon_0}$$

$$E(r)2\pi r L = \frac{\rho_0 \pi b^2 L}{\epsilon_0}$$

$$E(r) = \frac{\rho_0 b^2}{2r\epsilon_0}$$

$$E(r) = \frac{\rho_0 b^2}{2r\epsilon_0}\hat{\mathbf{e}}_r, \; r > b$$

(c) Repeat part (b) with $\rho(r) = \rho_0 e^{-r/b}$.

Solution:

$$\iint_S \mathbf{E(r)} dS = \frac{q}{\epsilon_0}$$

$$q = \iiint_V \rho(r) dV = \iiint_V \rho_0 e^{(-r/b)} dV$$

The volume in this case is a cylinder, so we can use cylindrical coordinates:

$$q = \rho_0 \int_0^L \int_0^{2\pi} \int_0^r e^{(-x/b)} x dx d\phi dz$$

$$q = \rho_0 2\pi L \int_0^r e^{(-x/b)} x dx$$

TODO evaluate through integration by parts

$$q = \rho_0 2\pi L b \left[b - e^{-r/b}(b+r)\right]$$

$$q = \rho_0 2\pi L b^2 \left[1 - e^{-r/b}(1 + r/b)\right]$$

$$E(r) \iint_S dS = \frac{\rho_0 2\pi L b^2 \left[1 - (1 + r/b)e^{-r/b}\right]}{\epsilon_0}$$

$$E(r)(2\pi r L) = \frac{\rho_0 2\pi L b^2 \left[1 - (1 + r/b)e^{-r/b}\right]}{\epsilon_0}$$

$$E(r) = \frac{\rho_0}{\epsilon_0}\frac{b^2}{r}\left[1 - (1 + r/b)e^{-r/b}\right]\hat{\mathbf{e}}_r$$

13. (a) Use Gauss' law and symmetry to find the electrostatic field as a function of position for the spherically symmetric charge distribution whose density is given in spherical coordinates by

$$\rho(r) = \begin{cases} \rho_0, r < b, \\ 0, r \geq b, \end{cases}$$

where ρ_0 and b are constants.

Solution:

For the region inside the sphere, $r < b$:

$$\iint_S \mathbf{E(r)} dS = \frac{q}{\epsilon_0}$$

$$q = \iiint_V \rho dV$$

$$q = \int_0^{2\pi} \int_0^{\pi} \int_0^r \rho_0 r^2 \sin\theta dr d\theta d\phi$$

$$q = \frac{4}{3}\pi r^3 \rho_0$$

$$E(r)(4\pi r^2) = \frac{4}{3\epsilon_0}\pi r^3 \rho_0$$

$$E(r) = \frac{1}{3\epsilon_0} r\rho_0 \hat{\mathbf{e}}_r \quad (r < b)$$

For the region outside the sphere, $r \geq b$:

$$q = \int_0^{2\pi}\int_0^{\pi}\int_0^{b} \rho_0 r^2 \sin\theta dr d\theta d\phi$$

$$q = \frac{4}{3}\pi b^3$$

$$E(r)(4\pi r^2) = \frac{4}{3\epsilon_0}\pi b^3 \rho_0$$

$$E(r) = \frac{\rho_0}{3\epsilon_0}\frac{b^3}{r^2}\hat{\mathbf{e}}_r \quad (r \geq b)$$

(b) Repeat part (a) for $\rho(r) = \rho_0 e^{-r/b}$.

Solution:

This distribution is not separated into regions.

$$q = \int_0^{2\pi}\int_0^{\pi}\int_0^{r} \rho_0 e^{-r'/b} r'^2 \sin\theta dr' d\theta d\phi$$

$$q = 4\pi\rho_0 \int_0^{r} e^{-r'/b} r'^2 dr'$$

TODO - solve this through integration by parts

$$q = 4\pi\rho_0 \left[2b^3 - be^{-r/b}(2b^2 + 2br + r^2)\right]$$

$$q = 4\pi b^3\rho_0 \left[2 - be^{-r/b}(2/b + 2r/b^2 + r^2/b^3)\right]$$

$$E(4\pi r^2) = 4\pi b^3\rho_0/\epsilon_0 \left[2 - e^{-r/b}(2 + 2r/b + r^2/b^2)\right]$$

$$E = \frac{b^3\rho_0}{r^2\epsilon_0}\left[2 - e^{-r/b}(2 + 2r/b + r^2/b^2)\right]\hat{\mathbf{e}}_r$$

(c) Repeat part (a) for

$$\rho(r) = \begin{cases} \rho_0, r < b, \\ \rho_1, b \leq r < 2b, \\ 0, r \geq 2b \end{cases}$$

How must ρ_0 and ρ_1 be related so that the field will be zero for $r > 2b$? What is the total charge of this distribution under these circumstances?

Solution:

For the region $r < b$:

$$q = \int_0^{\pi}\int_0^{2\pi}\int_0^{r} \rho_0 r^2 \sin\theta dr d\phi d\theta$$

$$q = 4\pi r^3\rho_0/3$$

$$E(4\pi r^2) = \frac{4\pi r^3\rho_0}{3\epsilon_0}$$

$$E = \frac{r\rho_0}{3\epsilon_0}\hat{\mathbf{e}}_r$$

For the region $b < r \leq 2b$:

$$q = \int_0^{\pi}\int_0^{2\pi}\int_0^{b} \rho_0 r^2 \sin\theta dr d\phi d\theta + \int_0^{\pi}\int_0^{2\pi}\int_b^{r} \rho_1 r'^2 \sin\theta dr' d\phi d\theta$$

$$q = \frac{4\pi b^3 \rho_0}{3} + \frac{4\pi\rho_1}{3}(r^3 - b^3)$$

$$E(4\pi r^2) = \frac{4\pi}{3\epsilon_0}(\rho_0 b^3 + \rho_1(r^3 - b^3))$$

$$E = \frac{1}{3\epsilon_0 r^2}(\rho_0 b^3 + \rho_1(r^3 - b^3))\hat{\mathbf{e}}_r$$

For the region $r > 2b$:

$$q = \int_0^{\pi}\int_0^{2\pi}\int_0^{b} \rho_0 r^2 \sin\theta dr d\phi d\theta + \int_0^{\pi}\int_0^{2\pi}\int_b^{2b} \rho_1 r'^2 \sin\theta dr' d\phi d\theta$$

$$q = \frac{4\pi b^3 \rho_0}{3} + \frac{4\pi\rho_1}{3}(8b^3 - b^3)$$

$$q = \frac{4\pi b^3}{3}(\rho_0 + 7\rho_1)$$

$$E(4\pi r^2) = \frac{4\pi b^3}{3\epsilon_0}(\rho_0 + 7\rho_1)$$

$$E = \frac{b^3}{3r^2\epsilon_0}(\rho_0 + 7\rho_1)\hat{\mathbf{e}}_r$$

For the total field outside $r > 2b$ to be zero, the total charge inside the sphere must be zero.

$$q = \int_0^{\pi}\int_0^{2\pi}\int_0^{b} \rho_0 r^2 \sin\theta dr d\phi d\theta + \int_0^{\pi}\int_0^{2\pi}\int_b^{2b} \rho_1 r^2 \sin\theta dr d\phi d\theta = 0$$

$$4\pi\rho_0 b^3/3 = -4\pi\rho_1 r^3/3\big|_b^{2b}$$

$$\rho_0 b^3 = -\rho_1(8b^3 - b^3)$$

$$\rho_0 = -7\rho_1$$

14. Calculate the divergence of each of the following fucntinos using Equation (II-22):

(a) $\mathbf{i}x^2 + \mathbf{j}y^2 + \mathbf{k}z^2$

Solution:

$$\text{div}\mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

$$F_x = x^2 \quad F_y = y^2 \quad F_z = z^2$$

$$\nabla \cdot \mathbf{F} = 2x + 2y + 2z$$

(b) $\mathbf{i}yz + \mathbf{j}xz + \mathbf{k}xy$

Solution:

$$\nabla \cdot \mathbf{F} = 0 + 0 + 0$$

(c) $\mathbf{i}e^{-x} + \mathbf{j}e^{-y} + \mathbf{k}e^{-z}$

Solution:

$$\nabla \cdot \mathbf{F} = -e^{-x} - e^{-y} - e^{-z}$$

(d) $\mathbf{i}1 + \mathbf{j}{-3} + \mathbf{k}z^2$

Solution:

$$\nabla \cdot \mathbf{F} = 0 + 0 + 2z = 2z$$

(e) $(\mathbf{i}(-xy) + \mathbf{j}x^2 + \mathbf{k}0)/(x^2 + y^2)$

Solution:

TODO - show these derivatives

$$\nabla \cdot \mathbf{F} = \frac{y(x^2 - y^2)}{(x^2 + y^2)^2} - \frac{2x^2 y}{(x^2 + y^2)^2}$$

$$\nabla \cdot \mathbf{F} = \frac{yx^2 - y^3 - 2x^2 y}{(x^2 + y^2)^2}$$

$$\nabla \cdot \mathbf{F} = \frac{y(-x^2 - y^2)}{(x^2 + y^2)^2}$$

$$\nabla \cdot \mathbf{F} = \frac{-y}{(x^2 + y^2)}$$

(f) $\mathbf{k}\sqrt{x^2 + y^2}$

Solution:

$$\nabla \cdot \mathbf{F} = 0 + 0 + \frac{\partial}{\partial z}\sqrt{x^2 + y^2} = 0$$

(g) $\mathbf{i}x + \mathbf{j}y + \mathbf{k}z$

Solution:

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}x + \frac{\partial}{\partial y}y + \frac{\partial}{\partial z}z$$

$$\nabla \cdot \mathbf{F} = 1 + 1 + 1 = 3$$

(h) $\mathbf{i}(-y) + \mathbf{j}x + \mathbf{k}0/\sqrt{x^2 + y^2}$

Solution:

$$\nabla \cdot \mathbf{F} = -y(-1/2)(x^2 + y^2)^{-3/2}(2x) + x(-1/2)(x^2 + y^2)^{-3/2}(2y)$$

$$\nabla \cdot \mathbf{F} = \frac{xy}{(x^2 + y^2)^{-3/2}} + \frac{-xy}{(x^2 + y^2)^{-3/2}}$$

$$\nabla \cdot \mathbf{F} = 0$$

15. (a) Calculate $\iint_S \mathbf{F} \cdot \hat{\mathbf{n}} dS$ for the function in Problem II-14(a) over the surface of a cube of side s whose center is at (x_0, y_0, z_0) and whose faces are parallel to the coordinate planes.

Solution:

From the previous problem, we have:

$$\nabla \cdot \mathbf{F} = 2x + 2y + 2z$$

$$\mathbf{F} = x^2 + y^2 + z^2$$

The surface is the sides of the cube. There are six sides, such that:

$$\mathbf{F} \cdot \hat{\mathbf{i}} = F_x = (x_0 + s/2)^2$$

$$\mathbf{F}\cdot -\hat{\mathbf{i}} = -F_x = -(x_0 - s/2)^2$$
$$\mathbf{F}\cdot \hat{\mathbf{j}} = F_y = (y_0 + s/2)^2$$
$$\mathbf{F}\cdot -\hat{\mathbf{j}} = -F_y = -(y_0 - s/2)^2$$
$$\mathbf{F}\cdot \hat{\mathbf{k}} = F_z = (z_0 + s/2)^2$$
$$\mathbf{F}\cdot -\hat{\mathbf{k}} = -F_z = -(z_0 - s/2)^2$$

and the area of each surface is s^2.

$$\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}dS = s^2[(x_0 + s/2)^2 - (x_0 - s/2)^2] + (\text{similar terms in y and z})$$
$$\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}dS = s^2[(x_0^2 + sx_0 + s^2/4) - (x_0^2 - sx_0 + s^2/4)] + (\text{similar terms in y and z})$$
$$\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}dS = s^2(2sx_0) + (\text{similar terms in y and z})$$
$$\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}dS = 2s^3x_0 + (\text{similar terms in y and z})$$
$$\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}dS = 2s^3(x_0 + y_0 + z_0)$$

(b) Divide the above result by the volume of the cube and calculate the limit of the quotient as $s \to 0$. Compare your result with the divergence found in Problem II-14(a).

Solution:

The volume is $V = s^3$. Thus:

$$\lim_{s\to 0}\frac{1}{\Delta V}\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}dS = \frac{1}{s^3}2s^3(x_0 + y_0 + z_0)$$
$$= 2(x_0 + y_0 + z_0)$$

The divergence from II-14(a) is:

$$\nabla\cdot\mathbf{F} = 2x + 2y + 2z = 2(x_0 + y_0 + z_0)$$

(c) Repeat parts (a) and (b) for the function of Problem II-14(b) and (c).

Solution:

from II-14(b)

$$\mathbf{F} = \mathbf{i}yz + \mathbf{j}xz + \mathbf{k}xy$$
$$\mathbf{F}\cdot\hat{\mathbf{i}} = F_x = y_0z_0$$
$$\mathbf{F}\cdot -\hat{\mathbf{i}} = -F_x = -y_0z_0$$

Similar terms for F_y and F_z

$$\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}dS = s^2[\mathbf{F}\cdot\hat{\mathbf{i}} + \mathbf{F}\cdot -\hat{\mathbf{i}} + \mathbf{F}\cdot\hat{\mathbf{j}} + \mathbf{F}\cdot -\hat{\mathbf{j}} + \mathbf{F}\cdot\hat{\mathbf{k}} + \mathbf{F}\cdot -\hat{\mathbf{k}}]$$
$$= s^2[y_0z_0 - y_0z_0 + x_0z_0 - x_0z_0 + x_0y_0 - x_0y_0]$$
$$= 0$$
$$\implies \frac{1}{\Delta V}\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}dS = 0$$

from II-14(c)

$$\mathbf{F} = \mathbf{i}e^{-x} + \mathbf{j}e^{-y} + \mathbf{k}e^{-z}$$

$$\mathbf{F} \cdot \hat{\mathbf{i}} = F_x = e^{-(x_0+s/2)} = e^{-x_0}e^{-s/2}$$

$$\mathbf{F} \cdot -\hat{\mathbf{i}} = -F_x = -e^{-(x_0-s/2)} = -e^{-x_0}e^{s/2}$$

$$\iint_S \mathbf{F} \cdot \hat{\mathbf{n}} dS = s^2[(e^{-s/2} - e^{s/2})(e^{-x_0} + e^{-y_0} + e^{-z_0})]$$

$$\lim_{s \to 0} \frac{1}{\Delta V} \iint_S \mathbf{F} \cdot \hat{\mathbf{n}} dS = \frac{1}{s^3} s^2[(e^{-s/2} - e^{s/2})(e^{-x_0} + e^{-y_0} + e^{-z_0})]$$

Use L'Hopital's rule

$$\lim_{s \to 0} \frac{(e^{-s/2} - e^{s/2})}{s} = \lim_{s \to 0} \frac{(-1/2)e^{-s/2} - (1/2)e^{s/2}}{1}$$

$$\lim_{s \to 0} \frac{(e^{-s/2} - e^{s/2})}{s} = (1/2)[-(1) - (1)] = -1$$

$$\implies \lim_{s \to 0} \frac{1}{\Delta V} \iint_S \mathbf{F} \cdot \hat{\mathbf{n}} dS = (-1)(e^{-x_0} + e^{-y_0} + e^{-z_0})$$

$$= e^{-x_0} - e^{-y_0} - e^{-z_0}$$

16. (a) Calculate the divergence of the function

$$\mathbf{F}(x, y, z) = \mathbf{i}f(x) + \mathbf{j}f(y) + \mathbf{k}f(-2z)$$

and show that it is zero at the point $(c, c, -c/2)$.

Solution:

The divergence is:

$$\nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

$$\nabla \cdot \mathbf{F} = f'(x) + f'(y) + -2f'(-2z)$$

$$= f'(x) + f'(y) + -2f'(-2z)$$

$$= f'(c) + f'(c) + -2f'(-2(-c/2))$$

$$= f'(c) + f'(c) + -2f'(c) = 0$$

(b) Calculate the divergence of

$$\mathbf{G}(x, y, z) = \mathbf{i}f(y, z) + \mathbf{j}g(x, z) + \mathbf{k}h(x, y)$$

Solution:

$$\nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

$$\frac{\partial F_x}{\partial x} = \frac{\partial f(y, z)}{\partial x} = 0$$

$$\frac{\partial F_y}{\partial y} = \frac{\partial g(x, z)}{\partial y} = 0$$

$$\frac{\partial F_z}{\partial z} = \frac{\partial h(x, y)}{\partial z} = 0$$

$$\implies \nabla \cdot \mathbf{F} = 0$$

17. In the text we obtained the result

$$\nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

by integrating the surface of a small rectangular parallelepiped. As an example of the fact that this result is independent of the usrface, rederive it using the wedge-shaped surface shown in the figure.

Solution:

We need to evaluate the surface integral of F:

$$\iint_s \mathbf{F} \cdot \hat{\mathbf{n}} dS$$

on each of the five surfaces of the wedge. The two surfaces parallel to the xy-plane and the xz-plane are identical to derivations in the text:

$$\iint_{S1} F_y \cdot -\hat{\mathbf{j}} dS = F_y(x, y - \frac{\Delta y}{2}, z)\Delta x \Delta z$$

$$\iint_{S2} F_z \cdot -\hat{\mathbf{k}} dS = F_z(x, y, z - \frac{\Delta z}{2})\Delta x \Delta y$$

For the two opposite triangular sides:

$$\iint_{S3+S4} F_x \cdot \hat{\mathbf{n}} dS = \iint_{S3} F_x \cdot \hat{\mathbf{i}} dS + \iint_{S4} F_x \cdot -\hat{\mathbf{i}} dS$$

$$= F_x(x + \frac{\Delta x}{2}, y, z)\frac{\Delta y \Delta z}{2} - F_x(x - \frac{\Delta x}{2}, y, z)\frac{\Delta y \Delta z}{2}$$

The trick with the slanted side is to get the normal unit vector correct:

$$\hat{\mathbf{n}} = \frac{\hat{\mathbf{j}}\Delta z}{\sqrt{\Delta y^2 + \Delta z^2}} + \frac{\hat{\mathbf{k}}\Delta y}{\sqrt{\Delta y^2 + \Delta z^2}}$$

For this side, we evaluate the function F at the point (x,y,z) exactly without adding any Δs.

$$\iint_{S5} F(x, y, z) \cdot \hat{\mathbf{n}} dS = \iint_{S5} \left(F_y \frac{\hat{\mathbf{j}}\Delta z}{\sqrt{\Delta y^2 + \Delta z^2}} + F_z \frac{\hat{\mathbf{k}}\Delta y}{\sqrt{\Delta y^2 + \Delta z^2}} \right) dS$$

$$= \frac{\Delta z}{\sqrt{\Delta y^2 + \Delta z^2}} \iint_{S5} F_y \hat{\mathbf{j}} dS + \frac{\Delta y}{\sqrt{\Delta y^2 + \Delta z^2}} \iint_{S5} F_z \hat{\mathbf{k}} dS$$

$$= \frac{\Delta z}{\sqrt{\Delta y^2 + \Delta z^2}} F_y(x, y, z)\Delta x \sqrt{\Delta z^2 + \Delta y^2} + \frac{\Delta y}{\sqrt{\Delta y^2 + \Delta z^2}} F_z(x, y, z)\Delta x \sqrt{\Delta z^2 + \Delta y^2}$$

The volume of the wedge is

$$\Delta V = \frac{\Delta x \Delta y \Delta z}{2}$$

Adding the surface integrals of all the sides together, and dividing by ΔV:

$$\frac{1}{\Delta V} \iint_S \mathbf{F} \cdot \hat{\mathbf{n}} dS = \frac{1}{\frac{1}{2}\Delta x \Delta y \Delta z} \Bigg[F_x(x + \frac{\Delta x}{2}, y, z)\frac{\Delta y \Delta z}{2} - F_x(x - \frac{\Delta x}{2}, y, z)\frac{\Delta y \Delta z}{2} +$$

$$F_y(x, y - \frac{\Delta y}{2}, z)\Delta x \Delta z + F_z(x, y, z - \frac{\Delta z}{2})\Delta x \Delta y +$$

$$\frac{\Delta z}{\sqrt{\Delta y^2 + \Delta z^2}} F_y(x, y, z)\Delta x \sqrt{\Delta z^2 + \Delta y^2} + \frac{\Delta y}{\sqrt{\Delta y^2 + \Delta z^2}} F_z(x, y, z)\Delta x \sqrt{\Delta z^2 + \Delta y^2} \Bigg]$$

$$\frac{1}{\Delta V}\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}dS = \frac{1}{\frac{1}{2}\Delta x\Delta y\Delta z}\left[\left(F_x(x+\frac{\Delta x}{2},y,z)-F_x(x-\frac{\Delta x}{2},y,z)\right)\frac{\Delta y\Delta z}{2}+\right.$$

$$\left(F_y(x,y-\frac{\Delta y}{2},z)+F_y(x,y,z)\right)\Delta x\Delta z+$$

$$\left.\left(F_z(x,y,z-\frac{\Delta z}{2})+F_z(x,y,z)\right)\Delta x\Delta y\right]$$

$$\frac{1}{\Delta V}\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}dS = \frac{\left(F_x(x+\frac{\Delta x}{2},y,z)-F_x(x-\frac{\Delta x}{2},y,z)\right)}{\Delta x}+$$

$$\frac{\left(F_y(x,y-\frac{\Delta y}{2},z)+F_y(x,y,z)\right)}{\frac{\Delta y}{2}}+\frac{\left(F_z(x,y,z-\frac{\Delta z}{2})+F_z(x,y,z)\right)}{\frac{\Delta z}{2}}$$

$$\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}dS = \frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}+\frac{\partial F_z}{\partial z}$$