Div Grad Curl Solutions

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This is a work in progress. I am using it to learn LaTeX and vector calculus at once.

1 Introduction, Vector Functions, and Electrostatics

Relevant Equations

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{qq_0}{r^2}\hat{\mathbf{u}} \tag{Coulomb's Law}$$

$$\mathbf{E}(\mathbf{r}) = \frac{\mathbf{F}(\mathbf{r})}{q_0} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{u}} \tag{Electric Field}$$

Problems

1. TODO
2. TODO
3. (a) Write a formula for a vector function in two dimensions which is in the positive radial direction and whose magnitude is 1.

   Solution:

   $$\vec{F}(x,y) = \frac{x\mathbf{i} + y\mathbf{j}}{\sqrt{x^2+y^2}}$$

   The length of a vector is given by

   $$|\vec{v}| = \sqrt{x^2+y^2}$$

   For $\vec{F}$ to be positive in $\vec{r}$, we need $\vec{F}(x,y) = x\mathbf{i} + y\mathbf{j}$, and we divide by the length of $\vec{F}$ to get a unit vector in that direction.

   (b) Write a formula for a vector function in two dimensions whose direction makes an angle of 45°with the x-axis and whose magnitude at any point (x,y) is $(x+y)^2$.

   Solution:

   $\vec{F}$ must be the same in the $\vec{x}$ and $\vec{y}$ directions to have a 45°angle with the x-axis

   $$\vec{F}(x,y) = a\mathbf{i} + a\mathbf{j}$$

   To have a magnitude of $(x+y)^2$, we need:

   $$|\vec{F}| = \sqrt{a^2+a^2} = (x+y)^2$$

   $$\sqrt{2}a = (x+y)^2$$

   $$a = \frac{\sqrt{2}(x+y)^2}{2}$$

$$\Rightarrow \vec{F}(x,y) = \frac{\sqrt{2}(x+y)^2}{2}(\mathbf{i}+\mathbf{j})$$

(c) Write a formula for a vector function in two dimensions whose direction is tangential [orthogonal to the radial direction] and whose magnitude at any point (x,y) is equal to its distance from the origin.

Solution:

To get the function orthogonal to radial, we need the **i** to depend on -y, and the **j** to depend on x. So

$$\vec{F}(x,y) = -y\mathbf{i} + x\mathbf{j}$$

The magnitude must be equal to the distance from the origin. The distance is given by $d = \sqrt{x^2+y^2}$, so:

$$|\vec{F}| = \sqrt{a^2+b^2} = \sqrt{x^2+y^2}$$

This works with a=y and b=x, with no other changes to the magnitude of $\vec{F}(x,y)$, so:

$$\vec{F}(x,y) = -y\mathbf{i} + x\mathbf{j}$$

(d) Write a formula for a vector function in three dimensions which is in the positive radial direction and whose magnitude is 1.

Solution:

$$\vec{F}(x,y,z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

To get a unit vector (magnitude = 1), divide this by the length:

$$\vec{F}(x,y,z) = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2+y^2+z^2}}$$

4. An object moves in the xy-plane in such a way that its position vector **r** is given as a function of time t by

$$\mathbf{r} = \mathbf{i}a\cos\omega t + \mathbf{j}b\sin\omega t$$

where a, b, and ω are constants.

(a) How far is the object from the origin at any time t?

Solution:

$$d = \sqrt{x^2+y^2}$$
$$d = \sqrt{a^2\cos^2(\omega t) + b^2\sin^2(\omega t)}$$

The important point is that $\cos^2() + \sin^2() = 1$ cannot be factored out.

(b) Find the object's velocity and acceleration as functions of time.

Solution:

$$\vec{v} = \frac{dx}{dt}$$
$$\vec{v} = \frac{d}{dt}(\mathbf{i}a\cos\omega t + \mathbf{j}b\sin\omega t)$$
$$\vec{v} = -\mathbf{i}a\omega\sin\omega t + \mathbf{j}b\omega\cos\omega t$$

Similarly,

$$\vec{a} = \frac{dv}{dt}$$
$$\vec{a} = \frac{d}{dt}(-\mathbf{i}a\omega\sin\omega t + \mathbf{j}b\omega\cos\omega t)$$
$$\vec{a} = -\mathbf{i}a\omega^2\cos\omega t - \mathbf{j}b\omega^2\sin\omega t$$

Using the definition of $\mathbf{r}$, the last line can also be written:

$$\vec{a} = -\omega^2 \mathbf{r}$$

(c) Show that the object moves on the elliptical path

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

Solution:

The x- and y- coordinates of the object are given by:

$$x = a\cos\omega t \quad y = b\sin\omega t$$

If we plug these into the left hand side of the ellipse equation, we get:

$$\begin{aligned}
\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 &= \left(\frac{\cancel{a^2}\cos^2\omega t}{\cancel{a^2}}\right) + \left(\frac{\cancel{b^2}\sin^2\omega t}{\cancel{b^2}}\right) \\
\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 &= 1
\end{aligned}$$

since $\cos^2() + \sin^2() = 1$.

5. A charge +1 is situated at the point (1, 0, 0) and a charge -1 is situated at the point (-1, 0, 0). Find the electric field of these two charges at an arbitrary point (0, y, 0) on the y-axis.

   Solution:

   An electric field is given by:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{l=1}^{N}\frac{q_l}{|\mathbf{r}-\mathbf{r}_l|^2}\hat{\mathbf{u}}_l$$

   We have two charges at (1,0,0) and (-1,0,0). The y-components will cancel out, and the x-components will reinforce one another. So we need to find the x-component of each charge and add them. For the charge at (1,0,0):

$$\begin{aligned}
E_x &= -\frac{1}{4\pi\epsilon_0}\frac{\cos\theta}{r^2} \\
\cos\theta &= \frac{adj}{hyp} = \frac{x}{r} = \frac{1}{r} \\
r &= \sqrt{1+y^2} \\
\Rightarrow E_x &= -\frac{1}{4\pi\epsilon_0}\frac{1}{(1+y^2)^{\frac{3}{2}}}
\end{aligned}$$

   The field from the charge at (-1,0,0) is the same in the x-direction. Adding them together:

$$E = -\frac{1}{2\pi\epsilon_0}\frac{1}{(1+y^2)^{\frac{3}{2}}}\mathbf{i}$$

6. Instead of using arrows to represent vector functions, we sometimes use families of curves called *field lines.* A curve y = y(x) is a field line of the vector function $\mathbf{F}$(x,y) if at each point (x_0, y_0) on the curve $\mathbf{F}(x_0, y_0)$ is tangent to the curve (see the figure).

   (a) Show that the field lines y = y(x) of a vector function

$$\mathbf{F}(x, y) = \mathbf{i}F_x(x, y) + \mathbf{j}F_y(x, y)$$

are solutions to the differential equation

$$\frac{dy}{dx} = \frac{F_y(x,y)}{F_x(x,y)}$$

Solution:

The field lines are tangent to the vector function. The tangent is the derivative of F, which is given by $\frac{dy}{dx}$. F_y and F_x are the components of F. The slope at any point is given by $\frac{F_y}{F_x}$. The slope is the same as the tangent, so

$$\frac{dy}{dx} = \frac{F_y(x,y)}{F_x(x,y)}$$

(b) Determine the field lines of each of the functions of Problem I-1. Draw the field lines and compare with the arrow diagrams of Problem I-1.

Solution:

We can use the relationship derived in (a) to determine the differential equations we need to solve to get the field lines.

(i) $\mathbf{i}y + \mathbf{j}x$

$$\frac{dy}{dx} = \frac{F_y}{F_x} = \frac{x}{y}$$

This is a separable differential equation.

$$\begin{aligned} y' &= \frac{x}{y} \\ ydy &= xdx \\ \int ydy &= \int xdx \\ y^2 &= x^2 + c \end{aligned}$$

(ii) $(\mathbf{i} + \mathbf{j})/\sqrt{2}$

$$\begin{aligned} \frac{dy}{dx} &= \frac{\sqrt{2}}{\sqrt{2}} = 1 \\ \int dy &= \int dx \\ y &= x + c \end{aligned}$$

(iii) $\mathbf{i}x - \mathbf{j}y$

$$\begin{aligned} \frac{dy}{dx} &= \frac{-y}{x} \\ \int \frac{-dy}{y} &= \int \frac{dx}{x} \\ -\ln y &= \ln x + c \\ c &= \ln x + \ln y = \ln(xy) \\ c &= xy \end{aligned}$$

(iv) $\mathbf{i}$y

$$\begin{aligned} \frac{dy}{dx} &= \frac{0}{y} = 0 \\ \int dy &= 0 \\ y &= c \end{aligned}$$

(v) **j**x

$$\begin{aligned}\frac{dy}{dx} &= \frac{0}{0} = undefined\\ \frac{dx}{dy} &= \frac{0}{x} = 0\\ x &= c\end{aligned}$$

(vi) $(\mathbf{i}y + \mathbf{j}x)/\sqrt{x^2+y^2}, (x, y) \neq (0, 0)$

$$\begin{aligned}\frac{dy}{dx} &= \frac{x}{\sqrt{x^2+y^2}}\frac{\sqrt{x^2+y^2}}{y} = \frac{x}{y}\\ \int ydy &= \int xdx\\ y^2 - x^2 &= c\end{aligned}$$

(vii) **i**y + **j**xy

$$\begin{aligned}\frac{dy}{dx} &= \frac{xy}{y} = x\\ \int dy &= \int xdx\\ y &= \frac{x^2}{2} + c\end{aligned}$$

(viii) **i** + **j**y

$$\begin{aligned}\frac{dy}{dx} &= y\\ \int \frac{dy}{y} &= \int dx\\ \ln y &= x + c\\ y &= e^x + c\end{aligned}$$

2 Surface Integrals and the Divergence

Relevant Equations

$$\hat{\mathbf{n}}(x, y, z) = \frac{\mathbf{u} \times \mathbf{v}}{|\mathbf{u} \times \mathbf{v}|} = \frac{-\mathbf{i}\frac{\partial f}{\partial x} - \mathbf{j}\frac{\partial f}{\partial y} + \mathbf{k}}{\sqrt{1 + (\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2}} \tag{II-4}$$

$$\int\int_S \mathbf{E} \cdot \hat{\mathbf{n}} dS = \frac{q}{\epsilon_0} \tag{Gauss's Law}$$

$$\text{div}\mathbf{E} = \nabla \cdot \mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \frac{\rho}{\epsilon_0} \tag{Gauss's Law Differential Form}$$

Examples

- (Page 25) Solution to double integral

$$\sqrt{3}\int\int_R (1-y)dxdy$$

with $S = z = f(x, y) = 1 - x - y$.

The region R is the triangle in the positive region of the xy-plane delimited by the two axes and the line $y = 1 - x$.

Putting the right limits in the double integral we get:

$$\begin{aligned}
\sqrt{3}\int\int_R (1-y)dxdy &= \sqrt{3}\int_0^1 \int_0^{1-y} (1-y)dxdy \\
&= \sqrt{3}\int_0^1 (1-y)x\Big|_0^{1-y} dy \\
&= \sqrt{3}\int_0^1 (1-y)^2 dy \\
&= \sqrt{3}\int_0^1 (1-2y-y^2)dy \\
&= \sqrt{3}(y - y^2 - \frac{y^3}{3})\Big|_0^1 \\
&= \sqrt{3}(1 - 1 - \frac{1}{3}) \\
&= \frac{1}{\sqrt{3}}
\end{aligned}$$

- (Page 27) Solution to the double integral

$$\int\int_S z^2 dS = \int\int_R \sqrt{1-x^2-y^2}dxdy$$

with $S = x^2 + y^2 + z^2 = 1$, and R is the projection of S in the positive region of the xy-plane.

R is a quarter circle defined by $x^2 + y^2 = 1$. Converting to polar coordinates, R is the region from $r = [0, 1]$ and $\theta = [0, \frac{\pi}{2}]$.

$$\begin{aligned}
\int\int_S z^2 dS &= \int\int_R \sqrt{1-x^2-y^2}dxdy \\
&= \int_0^{\frac{\pi}{2}} \int_0^1 (\sqrt{1 - r^2\cos^2\theta - r^2\sin^2\theta})rdrd\theta \\
&= \int_0^{\frac{\pi}{2}} \int_0^1 r\sqrt{1-r^2}drd\theta
\end{aligned}$$

let $u = \sqrt{1-r^2}$, then

$$\begin{aligned}
du = \frac{1}{2\sqrt{1-r^2}}2rdr &= \frac{rdr}{\sqrt{1-r^2}} = \frac{rdr}{u} \\
\Rightarrow rdr &= udu
\end{aligned}$$

After converting the limits of the definite integral in r to limits in u, we have

$$\begin{aligned}
\int\int_S z^2 dS &= \int_0^{\frac{\pi}{2}} \int_1^0 u^2 dud\theta \\
&= \int_0^{\frac{\pi}{2}} \frac{u^3}{3}\Big|_1^0 d\theta \\
&= \frac{1}{3}\int_0^{\frac{\pi}{2}} d\theta \\
&= \frac{\pi}{6}
\end{aligned}$$

- (Page 29) Solution to the double integral

$$\int\int_S \mathbf{F}\cdot\hat{\mathbf{n}}dS = \int\int_R \left(\frac{3x}{4} - \frac{3y}{2} + \frac{1}{2}\right) dxdy$$

where $S = x + 2y + 2z = 2$, and R is the area in the positve region of the xy-plane with $y = 1 - x/2$.

$$\begin{aligned}
\int\int_S \mathbf{F}\cdot\hat{\mathbf{n}}dS &= \int_0^1 \int_0^{2-2y} \left(\frac{3x}{4} - \frac{3y}{2} + \frac{1}{2}\right) dxdy \\
&= \int_0^1 \left(\frac{3x^2}{8} - \frac{3yx}{2} + \frac{x}{2}\right)\Big|_0^{2-2y} dy \\
&= \int_0^1 \left(\frac{3(2-2y)^2}{8} - \frac{3y(2-2y)}{2} + \frac{2-2y}{2}\right) dy \\
&= \int_0^1 \left(\frac{12-24y+12y^2}{8} + \frac{-6y+6y^2}{2} + \frac{2-2y}{2}\right) dy \\
&= \int_0^1 \left(\frac{12-24y+12y^2}{8} + \frac{2-8y+6y^2}{2}\right) dy \\
&= \int_0^1 \left(\frac{6-12y+6y^2}{4} + \frac{4-16y+12y^2}{4}\right) dy \\
&= \int_0^1 \left(\frac{10-28y+18y^2}{4}\right) dy \\
&= \frac{1}{4}\left(10y - 14y^2 + 6y^3\right)\Big|_0^1 \\
&= \frac{1}{2}
\end{aligned}$$

Problems

1. Find a unit vector $\hat{n}$ normal to each of the following surfaces.

   (a) $z = f(x, y) = 2 - x - y$

   Solution:

   We can use the result derived in equation II-4:

$$\hat{\mathbf{n}}(x, y, z) = \frac{\mathbf{u}\times\mathbf{v}}{|\mathbf{u}\times\mathbf{v}|} = \frac{-\mathbf{i}\frac{\partial f}{\partial x} - \mathbf{j}\frac{\partial f}{\partial y} + \mathbf{k}}{\sqrt{1 + (\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2}}$$

   The partial derivatives are:

$$\frac{\partial f}{\partial x} = -1 \quad \text{and} \quad \frac{\partial f}{\partial y} = -1$$

   Then:

$$\begin{aligned}
\hat{\mathbf{n}} &= \frac{-\mathbf{i}(-1) - \mathbf{j}(-1) + \mathbf{k}}{\sqrt{1 + (-1)^2 + (-1)^2}} \\
&= \frac{\mathbf{i} + \mathbf{j} + \mathbf{k}}{\sqrt{3}}
\end{aligned}$$

   (b) $z = (x^2 + y^2)^{\frac{1}{2}}$

   Solution:

$$\frac{\partial f}{\partial x} = \frac{x}{(x^2 + y^2)^{\frac{1}{2}}} \quad \text{and} \quad \frac{\partial f}{\partial y} = \frac{y}{(x^2 + y^2)^{\frac{1}{2}}}$$

$$
\begin{aligned}
\hat{\mathbf{n}} &= \frac{-\mathbf{i}(\frac{x}{(x^2+y^2)^{\frac{1}{2}}}) - \mathbf{j}(\frac{y}{(x^2+y^2)^{\frac{1}{2}}}) + \mathbf{k}}{\sqrt{1+\left(\frac{x}{(x^2+y^2)^{\frac{1}{2}}}\right)^2 + \left(\frac{y}{(x^2+y^2)^{\frac{1}{2}}}\right)^2}} \\
&= \frac{-\mathbf{i}(\frac{x}{z}) - \mathbf{j}(\frac{y}{z}) + \mathbf{k}}{\sqrt{1+\frac{x^2}{z^2}+\frac{y^2}{z^2}}} \\
&= \frac{-x\mathbf{i} - y\mathbf{j} + z\mathbf{k}}{z\sqrt{1+\frac{x^2}{z^2}+\frac{y^2}{z^2}}} \\
&= \frac{-x\mathbf{i} - y\mathbf{j} + z\mathbf{k}}{\sqrt{z^2+x^2+y^2}} \\
&= \frac{-x\mathbf{i} - y\mathbf{j} + z\mathbf{k}}{\sqrt{z^2+z^2}} \\
&= \frac{-x\mathbf{i} - y\mathbf{j} + z\mathbf{k}}{z\sqrt{2}}
\end{aligned}
$$

(c) $z = (1-x^2)^{\frac{1}{2}}$

Solution:

$$
\frac{\partial f}{\partial x} = \frac{-x}{(1-x^2)^{\frac{1}{2}}} = \frac{-x}{z} \quad \text{and} \quad \frac{\partial f}{\partial y} = 0
$$

$$
\begin{aligned}
\hat{\mathbf{n}} &= \frac{-\mathbf{i}\left(\frac{-x}{z}\right) - \mathbf{j}(0) + \mathbf{k}}{\sqrt{1+\left(\frac{-x}{z}\right)^2 + (0)^2}} \\
&= \frac{x\mathbf{i} + z\mathbf{k}}{z\sqrt{1+\frac{x^2}{z^2}}} \\
&= \frac{x\mathbf{i} + z\mathbf{k}}{\sqrt{z^2+x^2}} \\
&= \frac{x\mathbf{i} + z\mathbf{k}}{\sqrt{((1-x^2)^{\frac{1}{2}})^2+x^2}} \\
&= \frac{x\mathbf{i} + z\mathbf{k}}{\sqrt{1-x^2+x^2}} \\
&= x\mathbf{i} + z\mathbf{k}
\end{aligned}
$$

(d) $z = x^2 + y^2$

Solution:

$$
\frac{\partial f}{\partial x} = 2x \quad \text{and} \quad \frac{\partial f}{\partial y} = 2y
$$

$$
\begin{aligned}
\hat{\mathbf{n}}(x,y,z) &= \frac{-\mathbf{i}2x - \mathbf{j}2y + \mathbf{k}}{\sqrt{1+(2x)^2+(2y)^2}} \\
&= \frac{-\mathbf{i}2x - \mathbf{j}2y + \mathbf{k}}{\sqrt{1+4x^2+4y^2}} \\
&= \frac{-\mathbf{i}2x - \mathbf{j}2y + \mathbf{k}}{\sqrt{1+4z}}
\end{aligned}
$$

(e) $z = \left(1 - \frac{x^2}{a^2} - \frac{y^2}{a^2}\right)^{\frac{1}{2}}$

Solution:

$$\frac{\partial f}{\partial x} = \frac{-x}{a^2 z} \quad \text{and} \quad \frac{\partial f}{\partial y} = \frac{-y}{a^2 z}$$

$$\begin{aligned}
\hat{\mathbf{n}}(x, y, z) &= \frac{-\mathbf{i}\frac{-x}{a^2 z} - \mathbf{j}\frac{-y}{a^2 z} + \mathbf{k}}{\sqrt{1 + (\frac{-x}{a^2 z})^2 + (\frac{-y}{a^2 z})^2}} \\
&= \frac{\mathbf{i}x + \mathbf{j}y + \mathbf{k}a^2 z}{a^2 z\sqrt{1 + \frac{x^2}{a^4 z^2} + \frac{y^2}{a^4 z^2}}} \\
&= \frac{\mathbf{i}x + \mathbf{j}y + \mathbf{k}a^2 z}{a\sqrt{a^2 z^2 + x^2/a^2 + y^2/a^2}} \\
&= \frac{\mathbf{i}x + \mathbf{j}y + \mathbf{k}a^2 z}{a\sqrt{a^2 z^2 + 1 - 1 + x^2/a^2 + y^2/a^2}} \\
&= \frac{\mathbf{i}x + \mathbf{j}y + \mathbf{k}a^2 z}{a\sqrt{a^2 z^2 + 1 - (1 - x^2/a^2 - y^2/a^2)}} \\
&= \frac{\mathbf{i}x + \mathbf{j}y + \mathbf{k}a^2 z}{a\sqrt{a^2 z^2 + 1 - z^2}} \\
&= \frac{\mathbf{i}x + \mathbf{j}y + \mathbf{k}a^2 z}{a\sqrt{1 + (a^2 - 1)z^2}}
\end{aligned}$$

2. (a) Show that the unit vector normal to the plane

$$ax + by + cz = d$$

is given by

$$\hat{\mathbf{n}} = \pm(\mathbf{i}a + \mathbf{b} + \mathbf{c})/(a^2 + b^2 + c^2)^{\frac{1}{2}}$$

Solution:

$$z = f(x, y) = d/c - ax/c - by/c$$

$$\frac{\partial f}{\partial x} = -a/c \quad and \quad \frac{\partial f}{\partial y} = -b/c$$

$$\begin{aligned}
\hat{\mathbf{n}} &= \frac{-\mathbf{i}\,(-a/c) - \mathbf{j}\,(-b/c) + \mathbf{k}}{\sqrt{1 + (-a/c)^2 + (-b/c)^2}} \\
&= \frac{\mathbf{i}(a/c) + \mathbf{j}(b/c) + \mathbf{k}}{\sqrt{1 + a^2/c^2 + b^2/c^2}} \\
&= \frac{\mathbf{i}a + \mathbf{j}b + \mathbf{k}c}{c\sqrt{1 + a^2/c^2 + b^2/c^2}} \\
&= \frac{\mathbf{i}a + \mathbf{j}b + \mathbf{k}c}{\sqrt{c^2 + a^2 + b^2}}
\end{aligned}$$

(b) Explain in geometric terms why this expression for $\hat{\mathbf{n}}$ is independent of the constant d.

Solution:

$\hat{\mathbf{n}}$ being independent of d reflects the fact that there are infinitely many parallel planes that this vector is the normal unit vector to.

3. Derive the expressions for the unit normal vector for surfaces given by $y = g(x, z)$ and by $x = h(y, z)$. Use each to rederive the expression for the normal to the plane given in Problem II-2.

Solution:

For the surface described by $y = g(x, z)$, we need two vectors $\mathbf{u}$ and $\mathbf{v}$ tangent to the surface. To get the first vector, hold z constant, and slice the surface with a plane parallel to the xy-plane. This plane intersects the surface S on a curve. For a point on that curve, the vector $\mathbf{u}$ tangent to the point, the slope $m = \frac{u_y}{u_x}$. The total derivative of the surface at this point is

$$dg = \frac{\partial g}{\partial x} dx + \frac{\partial g}{\partial z} dz$$

On the plane we have used to slice through S, z is fixed, so $dz = 0$:

$$dg = \frac{\partial g}{\partial x} dx \quad \Rightarrow \quad \frac{dg}{dx} = \frac{\partial g}{\partial x}$$

The derivative is the slope of the line tanget to the point, so:

$$\frac{u_y}{u_x} = \frac{\partial g}{\partial x}$$

$$u_y = \left(\frac{\partial g}{\partial x}\right) u_x$$

Thus

$$\mathbf{u} = \mathbf{i} u_x + \mathbf{j} \left(\frac{\partial g}{\partial x}\right) u_x = \left[\mathbf{i} + \mathbf{j}\left(\frac{\partial g}{\partial x}\right)\right] u_x$$

For the other vector $\mathbf{v}$, hold x fixed and allow z to vary so that we slice S with a plane parallel to the yz-plane. Using similar reasoning, the slope of $\mathbf{v}$ tangent to the intersecting curve of S and the slicing plane is:

$$m = \frac{v_y}{v_z} = \frac{\partial g}{\partial z}$$

$$\Rightarrow v_y = \left(\frac{\partial g}{\partial z}\right) v_z$$

Thus

$$\mathbf{v} = \mathbf{j}\left(\frac{\partial g}{\partial z}\right) v_z + \mathbf{k} v_z = \left[\mathbf{j}\left(\frac{\partial g}{\partial z}\right) + \mathbf{k}\right] v_z$$

The unit normal vector is defined as:

$$\hat{\mathbf{n}}(x, y, z) = \frac{\mathbf{u} \times \mathbf{v}}{|\mathbf{u} \times \mathbf{v}|}$$

Then,

$$\begin{aligned}
\mathbf{u} \times \mathbf{v} &= \mathbf{i}(u_z v_y - u_y v_z) + \mathbf{j}(u_x v_z - u_z v_x) + \mathbf{k}(u_y v_x - u_x v_y) \\
\mathbf{u} \times \mathbf{v} &= \mathbf{i}\left((0)\frac{\partial g}{\partial z} v_z - \frac{\partial g}{\partial x} u_x v_z\right) + \mathbf{j}\left(u_x v_z - (0)(0)\right) + \mathbf{k}\left(\frac{\partial g}{\partial x} u_x (0) - u_x \frac{\partial g}{\partial z} v_z\right) \\
\mathbf{u} \times \mathbf{v} &= -\mathbf{i}(\frac{\partial g}{\partial x} u_x v_z) + \mathbf{j}(u_x v_z) - \mathbf{k}(u_x \frac{\partial g}{\partial z} v_z) \\
\mathbf{u} \times \mathbf{v} &= \left[-\mathbf{i}\frac{\partial g}{\partial x} + \mathbf{j} - \mathbf{k}\frac{\partial g}{\partial z}\right] u_x v_z \\
|\mathbf{u} \times \mathbf{v}| &= \sqrt{u_x^2 v_z^2 \left[\left(\frac{\partial g}{\partial x}\right)^2 + 1 + \left(\frac{\partial g}{\partial z}\right)^2\right]}
\end{aligned}$$

$$\Rightarrow \hat{\mathbf{n}} = \frac{-\mathbf{i}\frac{\partial g}{\partial x} + \mathbf{j} - \mathbf{k}\frac{\partial g}{\partial z}}{\sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial z}\right)^2}} \quad \text{for} \quad y = g(x, z)$$

The derivation for $x = h(y, z)$ is similar, and the result is:

$$\hat{\mathbf{n}} = \frac{-\mathbf{i}\frac{\partial h}{\partial y} + \mathbf{j} - \mathbf{k}\frac{\partial h}{\partial z}}{\sqrt{1 + \left(\frac{\partial h}{\partial y}\right)^2 + \left(\frac{\partial h}{\partial z}\right)^2}} \quad \text{for} \quad x = h(y, z)$$

4. In each of the following use Equation II-12 to evaluate the surface integral $\int\int_S G(x, y, z)dS$