Div Grad Curl Solutions

Dan Henriksen

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This is a work in progress. I am using it to learn LaTeX and vector calculus at once.

1 Introduction, Vector Functions, and Electrostatics

Relevant Equations

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{qq_0}{r^2}\hat{\mathbf{u}} \tag{Coulomb's Law}$$

$$\mathbf{E}(\mathbf{r}) = \frac{\mathbf{F}(\mathbf{r})}{q_0} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{u}} \tag{Electric Field}$$

Problems

1. TODO
2. TODO
3. (a) Write a formula for a vector function in two dimensions which is in the positive radial direction and whose magnitude is 1.

   Solution:
   $$\vec{F}(x,y) = \frac{x\mathbf{i} + y\mathbf{j}}{\sqrt{x^2+y^2}}$$
   The length of a vector is given by
   $$|\vec{v}| = \sqrt{x^2+y^2}$$
   For $\vec{F}$ to be positive in $\vec{r}$, we need $\vec{F}(x,y) = x\mathbf{i} + y\mathbf{j}$, and we divide by the length of $\vec{F}$ to get a unit vector in that direction.

   (b) Write a formula for a vector function in two dimensions whose direction makes an angle of 45°with the x-axis and whose magnitude at any point (x,y) is $(x+y)^2$.

   Solution:

   $\vec{F}$ must be the same in the $\vec{x}$ and $\vec{y}$ directions to have a 45°angle with the x-axis
   $$\vec{F}(x,y) = a\mathbf{i} + a\mathbf{j}$$
   To have a magnitude of $(x+y)^2$, we need:
   $$|\vec{F}| = \sqrt{a^2+a^2} = (x+y)^2$$
   $$\sqrt{2}a = (x+y)^2$$
   $$a = \frac{\sqrt{2}(x+y)^2}{2}$$

$$\Rightarrow \vec{F}(x,y) = \frac{\sqrt{2}(x+y)^2}{2}(\mathbf{i}+\mathbf{j})$$

(c) Write a formula for a vector function in two dimensions whose direction is tangential [orthogonal to the radial direction] and whose magnitude at any point (x,y) is equal to its distance from the origin.

Solution:

To get the function orthogonal to radial, we need the **i** to depend on -y, and the **j** to depend on x. So

$$\vec{F}(x,y) = -y\mathbf{i} + x\mathbf{j}$$

The magnitude must be equal to the distance from the origin. The distance is given by $d = \sqrt{x^2+y^2}$, so:

$$|\vec{F}| = \sqrt{a^2+b^2} = \sqrt{x^2+y^2}$$

This works with a=y and b=x, with no other changes to the magnitude of $\vec{F}(x,y)$, so:

$$\vec{F}(x,y) = -y\mathbf{i} + x\mathbf{j}$$

(d) Write a formula for a vector function in three dimensions which is in the positive radial direction and whose magnitude is 1.

Solution:

$$\vec{F}(x,y,z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

To get a unit vector (magnitude = 1), divide this by the length:

$$\vec{F}(x,y,z) = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2+y^2+z^2}}$$

4. An object moves in the xy-plane in such a way that its position vector **r** is given as a function of time t by

$$\mathbf{r} = \mathbf{i}a\cos\omega t + \mathbf{j}b\sin\omega t$$

where a, b, and ω are constants.

(a) How far is the object from the origin at any time t?

Solution:

$$d = \sqrt{x^2+y^2}$$
$$d = \sqrt{a^2\cos^2(\omega t) + b^2\sin^2(\omega t)}$$

The important point is that $\cos^2() + \sin^2() = 1$ cannot be factored out.

(b) Find the object's velocity and acceleration as functions of time.

Solution:

$$\vec{v} = \frac{dx}{dt}$$
$$\vec{v} = \frac{d}{dt}(\mathbf{i}a\cos\omega t + \mathbf{j}b\sin\omega t)$$
$$\vec{v} = -\mathbf{i}a\omega\sin\omega t + \mathbf{j}b\omega\cos\omega t$$

Similarly,

$$\vec{a} = \frac{dv}{dt}$$
$$\vec{a} = \frac{d}{dt}(-\mathbf{i}a\omega\sin\omega t + \mathbf{j}b\omega\cos\omega t)$$
$$\vec{a} = -\mathbf{i}a\omega^2\cos\omega t - \mathbf{j}b\omega^2\sin\omega t$$

Using the definition of $\mathbf{r}$, the last line can also be written:

$$\vec{a} = -\omega^2 \mathbf{r}$$

(c) Show that the object moves on the elliptical path

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

Solution:

The x- and y- coordinates of the object are given by:

$$x = a\cos\omega t \quad y = b\sin\omega t$$

If we plug these into the left hand side of the ellipse equation, we get:

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = \left(\frac{a^2\cos^2\omega t}{a^2}\right) + \left(\frac{b^2\sin^2\omega t}{b^2}\right)$$
$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

since the constants a and b cancel, and using $\cos^2() + \sin^2() = 1$.

5. A charge +1 is situated at the point (1, 0, 0) and a charge -1 is situated at the point (-1, 0, 0). Find the electric field of these two charges at an arbitrary point (0, y, 0) on the y-axis.

   Solution:

   An electric field is given by:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{l=1}^{N}\frac{q_l}{|\mathbf{r}-\mathbf{r}_l|^2}\hat{\mathbf{u}}_l$$

   We have two charges at (1,0,0) and (-1,0,0). The y-components will cancel out, and the x-components will reinforce one another. So we need to find the x-component of each charge and add them. For the charge at (1,0,0):

$$E_x = -\frac{1}{4\pi\epsilon_0}\frac{\cos\theta}{r^2}$$
$$\cos\theta = \frac{adj}{hyp} = \frac{x}{r} = \frac{1}{r}$$
$$r = \sqrt{1+y^2}$$
$$\Rightarrow E_x = -\frac{1}{4\pi\epsilon_0}\frac{1}{(1+y^2)^{\frac{3}{2}}}$$

   The field from the charge at (-1,0,0) is the same in the x-direction. Adding them together:

$$E = -\frac{1}{2\pi\epsilon_0}\frac{1}{(1+y^2)^{\frac{3}{2}}}\mathbf{i}$$

6. Instead of using arrows to represent vector functions, we sometimes use families of curves called *field lines*. A curve y = y(x) is a field line of the vector function **F**(x,y) if at each point (x_0, y_0) on the curve $\mathbf{F}(x_0, y_0)$ is tangent to the curve (see the figure).

   (a) Show that the field lines y = y(x) of a vector function

$$\mathbf{F}(x,y) = \mathbf{i}F_x(x,y) + \mathbf{j}F_y(x,y)$$

are solutions to the differential equation

$$\frac{dy}{dx} = \frac{F_y(x,y)}{F_x(x,y)}$$

Solution:

The field lines are tangent to the vector function. The tangent is the derivative of F, which is given by $\frac{dy}{dx}$. F_y and F_x are the components of F. The slope at any point is given by $\frac{F_y}{F_x}$. The slope is the same as the tangent, so

$$\frac{dy}{dx} = \frac{F_y(x,y)}{F_x(x,y)}$$

(b) Determine the field lines of each of the functions of Problem I-1. Draw the field lines and compare with the arrow diagrams of Problem I-1.

Solution:

We can use the relationship derived in (a) to determine the differential equations we need to solve to get the field lines.

(i) $\mathbf{i}y + \mathbf{j}x$

$$\frac{dy}{dx} = \frac{F_y}{F_x} = \frac{x}{y}$$

This is a separable differential equation.

$$\begin{aligned} y' &= \frac{x}{y} \\ ydy &= xdx \\ \int ydy &= \int xdx \\ y^2 &= x^2 + c \end{aligned}$$

(ii) $(\mathbf{i} + \mathbf{j})/\sqrt{2}$

$$\begin{aligned} \frac{dy}{dx} &= \frac{\sqrt{2}}{\sqrt{2}} = 1 \\ \int dy &= \int dx \\ y &= x + c \end{aligned}$$

(iii) $\mathbf{i}x - \mathbf{j}y$

$$\begin{aligned} \frac{dy}{dx} &= \frac{-y}{x} \\ \int \frac{-dy}{y} &= \int \frac{dx}{x} \\ -\ln y &= \ln x + c \\ c &= \ln x + \ln y = \ln(xy) \\ c &= xy \end{aligned}$$

(iv) $\mathbf{i}$y

$$\begin{aligned} \frac{dy}{dx} &= \frac{0}{y} = 0 \\ \int dy &= 0 \\ y &= c \end{aligned}$$

(v) **j**x

$$\begin{aligned}
\frac{dy}{dx} &= \frac{0}{0} = undefined \\
\frac{dx}{dy} &= \frac{0}{x} = 0 \\
x &= c
\end{aligned}$$

(vi) $(\mathbf{i}y + \mathbf{j}x)/\sqrt{x^2+y^2}, (x, y) \neq (0, 0)$

$$\begin{aligned}
\frac{dy}{dx} &= \frac{x}{\sqrt{x^2+y^2}} \frac{\sqrt{x^2+y^2}}{y} = \frac{x}{y} \\
\int ydy &= \int xdx \\
y^2 - x^2 &= c
\end{aligned}$$

(vii) **i**y + **j**xy

$$\begin{aligned}
\frac{dy}{dx} &= \frac{xy}{y} = x \\
\int dy &= \int xdx \\
y &= \frac{x^2}{2} + c
\end{aligned}$$

(viii) **i** + **j**y

$$\begin{aligned}
\frac{dy}{dx} &= y \\
\int \frac{dy}{y} &= \int dx \\
\ln y &= x + c \\
y &= e^x + c
\end{aligned}$$

2 Surface Integrals and the Divergence

Relevant Equations

$$\int \int_S \mathbf{E} \cdot \hat{\mathbf{n}} dS = \frac{q}{\epsilon_0} \qquad \text{(Gauss's Law)}$$

$$\text{div}\mathbf{E} = \nabla \cdot \mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \frac{\rho}{\epsilon_0} \qquad \text{(Gauss's Law Differential Form)}$$

Examples

- (Page 25) Solution to double integral

$$\sqrt{3} \int \int_R (1-y) dxdy$$

with $S = z = f(x, y) = 1 - x - y$.

The region R is the triangle in the positive region of the xy-plane delimited by the two axes and the line $y = 1 - x$.

Putting the right limits in the double integral we get:

$$\begin{aligned}
\sqrt{3}\int\int_R (1-y)dxdy &= \sqrt{3}\int_0^1\int_0^{1-y}(1-y)dxdy \\
&= \sqrt{3}\int_0^1 (1-y)x\Big|_0^{1-y}dy \\
&= \sqrt{3}\int_0^1 (1-y)^2 dy \\
&= \sqrt{3}\int_0^1 (1-2y-y^2)dy \\
&= \sqrt{3}(y-y^2-\frac{y^3}{3})\Big|_0^1 \\
&= \sqrt{3}(1-1-\frac{1}{3}) \\
&= \frac{1}{\sqrt{3}}
\end{aligned}$$

- (Page 27) Solution to the double integral

$$\int\int_S z^2 dS = \int\int_R \sqrt{1-x^2-y^2}dxdy$$

with $S = x^2+y^2+z^2 = 1$, and R is the projection of S in the positive region of the xy-plane.
R is a quarter circle defined by $x^2+y^2=1$. Converting to polar coordinates, R is the region from $r=[0,1]$ and $\theta=[0,\frac{\pi}{2}]$.

$$\begin{aligned}
\int\int_S z^2 dS &= \int\int_R \sqrt{1-x^2-y^2}dxdy \\
&= \int_0^{\frac{\pi}{2}}\int_0^1 (\sqrt{1-r^2\cos^2\theta-r^2\sin^2\theta})rdrd\theta \\
&= \int_0^{\frac{\pi}{2}}\int_0^1 r\sqrt{1-r^2}drd\theta
\end{aligned}$$

let $u=\sqrt{1-r^2}$, then

$$du = \frac{1}{2\sqrt{1-r^2}}2rdr = \frac{rdr}{\sqrt{1-r^2}} = \frac{rdr}{u}$$

$$\Rightarrow rdr = udu$$

After converting the limits of the definite integral in r to limits in u, we have

$$\begin{aligned}
\int\int_S z^2 dS &= \int_0^{\frac{\pi}{2}}\int_1^0 u^2 dud\theta \\
&= \int_0^{\frac{\pi}{2}} \frac{u^3}{3}\Big|_1^0 d\theta \\
&= \frac{1}{3}\int_0^{\frac{\pi}{2}} d\theta \\
&= \frac{\pi}{6}
\end{aligned}$$

- (Page 29) Solution to the double integral

$$\int\int_S \mathbf{F}\cdot\hat{\mathbf{n}}dS = \int\int_R \left(\frac{3x}{4} - \frac{3y}{2} + \frac{1}{2}\right) dxdy$$

where $S = x + 2y + 2z = 2$, and R is the area in the positve region of the xy-plane with $y = 1 - x/2$.

$$\begin{aligned}
\int\int_S \mathbf{F}\cdot\hat{\mathbf{n}}dS &= \int_0^1 \int_0^{2-2y} \left(\frac{3x}{4} - \frac{3y}{2} + \frac{1}{2}\right) dxdy \\
&= \int_0^1 \left(\frac{3x^2}{8} - \frac{3yx}{2} + \frac{x}{2}\right)\Big|_0^{2-2y} dy \\
&= \int_0^1 \left(\frac{3(2-2y)^2}{8} - \frac{3y(2-2y)}{2} + \frac{2-2y}{2}\right) dy \\
&= \int_0^1 \left(\frac{12-24y+12y^2}{8} + \frac{-6y+6y^2}{2} + \frac{2-2y}{2}\right) dy \\
&= \int_0^1 \left(\frac{12-24y+12y^2}{8} + \frac{2-8y+6y^2}{2}\right) dy \\
&= \int_0^1 \left(\frac{6-12y+6y^2}{4} + \frac{4-16y+12y^2}{4}\right) dy \\
&= \int_0^1 \left(\frac{10-28y+18y^2}{4}\right) dy \\
&= \frac{1}{4}\left(10y - 14y^2 + 6y^3\right)\Big|_0^1 \\
&= \frac{1}{2}
\end{aligned}$$

Problems

1. Find a unit vector $\hat{n}$ normal to each of the following surfaces.

   (a) $z = 2 - x - y$

   Solution:

   We can use the result derived in equation II-4:

$$\hat{\mathbf{n}}(x,y,z) = \frac{\mathbf{u}\times\mathbf{v}}{|\mathbf{u}\times\mathbf{v}|} = \frac{-\mathbf{i}\frac{\partial f}{\partial x} - \mathbf{j}\frac{\partial f}{\partial y} + \mathbf{k}}{\sqrt{1 + (\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2}}$$

   The partial derivatives are:

$$\frac{\partial f}{\partial x} = -1 \quad \text{and} \quad \frac{\partial f}{\partial y} = -1$$

   Then:

$$\begin{aligned}
\hat{\mathbf{n}} &= \frac{-\mathbf{i}(-1) - \mathbf{j}(-1) + \mathbf{k}}{\sqrt{1 + (-1)^2 + (-1)^2}} \\
&= \frac{\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{3}}
\end{aligned}$$

   (b) $z = (x^2+y^2)^{\frac{1}{2}}$

   Solution:

$$\frac{\partial f}{\partial x} = \frac{x}{(x^2+y^2)^{\frac{1}{2}}} \quad \text{and} \quad \frac{\partial f}{\partial y} = \frac{y}{(x^2+y^2)^{\frac{1}{2}}}$$

$$
\begin{aligned}
\hat{\mathbf{n}} &= \frac{-\mathbf{i}(\frac{x}{(x^2+y^2)^{\frac{1}{2}}}) - \mathbf{j}(\frac{y}{(x^2+y^2)^{\frac{1}{2}}}) + \mathbf{k}}{\sqrt{1 + \left(\frac{x}{(x^2+y^2)^{\frac{1}{2}}}\right)^2 + \left(\frac{y}{(x^2+y^2)^{\frac{1}{2}}}\right)^2}} \\
&= \frac{-\mathbf{i}(\frac{x}{z}) - \mathbf{j}(\frac{y}{z}) + \mathbf{k}}{\sqrt{1 + \frac{x^2}{z^2} + \frac{y^2}{z^2}}} \\
&= \frac{-x\mathbf{i} - y\mathbf{j} + z\mathbf{k}}{z\sqrt{1 + \frac{x^2}{z^2} + \frac{y^2}{z^2}}} \\
&= \frac{-x\mathbf{i} - y\mathbf{j} + z\mathbf{k}}{\sqrt{z^2 + x^2 + y^2}} \\
&= \frac{-x\mathbf{i} - y\mathbf{j} + z\mathbf{k}}{\sqrt{z^2 + z^2}} \\
&= \frac{-x\mathbf{i} - y\mathbf{j} + z\mathbf{k}}{z\sqrt{2}}
\end{aligned}
$$

(c) $z = (1 - x^2)^{\frac{1}{2}}$

Solution:

$$
\frac{\partial f}{\partial x} = \frac{-x}{(1-x^2)^{\frac{1}{2}}} = \frac{-x}{z} \quad \text{and} \quad \frac{\partial f}{\partial y} = 0
$$

$$
\begin{aligned}
\hat{\mathbf{n}} &= \frac{-\mathbf{i}\left(\frac{-x}{z}\right) - \mathbf{j}(0) + \mathbf{k}}{\sqrt{1 + \left(\frac{-x}{z}\right)^2 + (0)^2}} \\
&= \frac{x\mathbf{i} + z\mathbf{k}}{z\sqrt{1 + \frac{x^2}{z^2}}} \\
&= \frac{x\mathbf{i} + z\mathbf{k}}{\sqrt{z^2 + x^2}} \\
&= \frac{x\mathbf{i} + z\mathbf{k}}{\sqrt{((1-x^2)^{\frac{1}{2}})^2 + x^2}} \\
&= \frac{x\mathbf{i} + z\mathbf{k}}{\sqrt{1 - x^2 + x^2}} \\
&= x\mathbf{i} + z\mathbf{k}
\end{aligned}
$$

(d) $z = x^2 + y^2$

Solution:

$$
\frac{\partial f}{\partial x} = 2x \quad \text{and} \quad \frac{\partial f}{\partial y} = 2y
$$

$$
\begin{aligned}
\hat{\mathbf{n}}(x,y,z) &= \frac{-\mathbf{i}2x - \mathbf{j}2y + \mathbf{k}}{\sqrt{1 + (2x)^2 + (2y)^2}} \\
&= \frac{-\mathbf{i}2x - \mathbf{j}2y + \mathbf{k}}{\sqrt{1 + 4x^2 + 4y^2}} \\
&= \frac{-\mathbf{i}2x - \mathbf{j}2y + \mathbf{k}}{\sqrt{1 + 4z}}
\end{aligned}
$$

(e) $z = \left(1 - \frac{x^2}{a^2} - \frac{y^2}{a^2}\right)^{\frac{1}{2}}$

Solution:

$$\frac{\partial f}{\partial x} = \frac{-x}{a^2 z} \quad \text{and} \quad \frac{\partial f}{\partial y} = \frac{-y}{a^2 z}$$

$$\begin{aligned}
\hat{\mathbf{n}}(x,y,z) &= \frac{-\mathbf{i}\frac{-x}{a^2 z} - \mathbf{j}\frac{-y}{a^2 z} + \mathbf{k}}{\sqrt{1 + (\frac{-x}{a^2 z})^2 + (\frac{-y}{a^2 z})^2}} \\
&= \frac{\mathbf{i}x + \mathbf{j}y + \mathbf{k}a^2 z}{a^2 z\sqrt{1 + \frac{x^2}{a^4 z^2} + \frac{y^2}{a^4 z^2}}} \\
&= \frac{\mathbf{i}x + \mathbf{j}y + \mathbf{k}a^2 z}{a\sqrt{a^2 z^2 + x^2/a^2 + y^2/a^2}} \\
&= \frac{\mathbf{i}x + \mathbf{j}y + \mathbf{k}a^2 z}{a\sqrt{a^2 z^2 + 1 - 1 + x^2/a^2 + y^2/a^2}} \\
&= \frac{\mathbf{i}x + \mathbf{j}y + \mathbf{k}a^2 z}{a\sqrt{a^2 z^2 + 1 - (1 - x^2/a^2 - y^2/a^2)}} \\
&= \frac{\mathbf{i}x + \mathbf{j}y + \mathbf{k}a^2 z}{a\sqrt{a^2 z^2 + 1 - z^2}} \\
&= \frac{\mathbf{i}x + \mathbf{j}y + \mathbf{k}a^2 z}{a\sqrt{1 + (a^2 - 1)z^2}}
\end{aligned}$$