

# Mathematics for Machine Learning

## Multivariate Calculus

### *Formula sheet*

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#### Definition of a derivative

$$f'(x) = \frac{df(x)}{dx} = \lim_{\Delta x \rightarrow 0} \left( \frac{f(x + \Delta x) - f(x)}{\Delta x} \right)$$

#### Time saving rules

- Sum Rule:

$$\frac{d}{dx} (f(x) + g(x)) = \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x))$$

- Power Rule:

$$\begin{aligned} f(x) &= ax^b \\ f'(x) &= abx^{(b-1)} \end{aligned}$$

- Product Rule:

$$\begin{aligned} A(x) &= f(x)g(x) \\ A'(x) &= f'(x)g(x) + f(x)g'(x) \end{aligned}$$

- Chain Rule:

$$\begin{aligned} \text{If } h &= h(p) \quad \text{and} \quad p = p(m) \\ \text{then } \frac{dh}{dm} &= \frac{dh}{dp} \times \frac{dp}{dm} \end{aligned}$$

- Total derivative:

For the function  $f(x, y, z, \dots)$ , where each variable is a function of parameter  $t$ , the total derivative is

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial z} \frac{dz}{dt} + \dots$$

#### Derivatives of named functions

$$\frac{\partial}{\partial x} \frac{1}{x} = -\frac{1}{x^2} \tag{1}$$

$$\frac{\partial}{\partial x} \sin x = \cos x \tag{2}$$

$$\frac{\partial}{\partial x} \cos x = -\sin x \tag{3}$$

$$\frac{\partial}{\partial x} \exp x = \exp x \tag{4}$$

#### Derivative structures

$$f = f(x, y, z)$$

- Jacobian:

$$\mathbf{J}_f = \left[ \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right]$$

- Hessian:

$$\mathbf{H}_f = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial x \partial z} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial y \partial z} \\ \frac{\partial^2 f}{\partial z \partial x} & \frac{\partial^2 f}{\partial z \partial y} & \frac{\partial^2 f}{\partial z^2} \end{bmatrix}$$

## Taylor Series

- Univariate:

$$\begin{aligned} f(x) &= f(c) + f'(c)(x - c) + \frac{1}{2}f''(c)(x - c)^2 + \dots \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x - c)^n \end{aligned}$$

- Multivariate:

$$\begin{aligned} f(\mathbf{x}) &= f(\mathbf{c}) + \mathbf{J}_f(\mathbf{c})(\mathbf{x} - \mathbf{c}) + \dots \\ &\quad \frac{1}{2}(\mathbf{x} - \mathbf{c})^t \mathbf{H}_f(\mathbf{c})(\mathbf{x} - \mathbf{c}) + \dots \end{aligned}$$

## Optimization and Vector Calculus

- Newton-Raphson:

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

- Grad:

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial z} \\ \frac{\partial f}{\partial z} \end{bmatrix}$$

- Directional Gradient:

$$\nabla f \cdot \hat{r}$$

- Gradient Descent:

$$s_{n+1} = s_n - \gamma \nabla f$$

- Lagrange Multipliers  $\lambda$ :

$$\nabla f = \lambda \nabla g$$

- Least Squares -  $\chi^2$  minimization:

$$\chi^2 = \sum_i^n \frac{(y_i - y(x_i; a_k))^2}{\sigma_i}$$

$$\text{criterion: } \nabla \chi^2 = 0$$

$$a_{\text{next}} = a_{\text{cur}} - \gamma \nabla \chi^2$$

$$= a_{\text{cur}} + \gamma \sum_i^n \frac{(y_i - y(x_i; a_k))}{\sigma_i} \frac{\partial y}{\partial a_k}$$