Technische Universität Berlin
Faculty II - Mathematics and Natural Sciences
Institute of Mathematics

Project in Numerical Analysis SS 2019

FEM modeling for elastic deformation of Bernoulli beam

Name	Course	Matric.	E-mail
Juan Sebastian Diaz Boada	COSSE	405385	diazboada@tu-berlin.de
Sigurdur Ag. Sigurdsson	COSSE	405725	sig.ag.sig@gmail.com
Traianos Karageorgiou	COSSE	406070	karageorgiou@campus.tu-berlin.de

Lecturer Michael Karow

Berlin, July 20, 2019

Contents

Contents **I**

List of Figures **II**

List of Symbols **III**

1 Introduction **1**
- 1.1 Bending equation 1
- 1.2 FEM 1
- 1.3 Newmark Method 2

2 1D Beam **3**
- 2.1 Results 3
  - 2.1.1 Starting from horizontal position 3
  - 2.1.2 Starting from position of stationary solution 5

3 2D Prism **6**

4 2D Beam **7**
- 4.1 Method 7
- 4.2 Results 7
  - 4.2.1 Without holes 8
  - 4.2.2 With one hole 9
  - 4.2.3 With two holes 9

5 Conclusion **11**

List of Figures

1 Results of starting the simulation from horizontal position for different shear forces. 4
2 Results of starting the simulation from horizontal position for different shear forces. 4
3 Results of starting the simulation from the stationary state after applying a force and letting it go at $t = 5$. The movement after its release is clearly harmonic. . . 5
4 Comparison of the formulation of reference prism and the results of the simulation. 6
5 Long beam in time $t = 100$ after 500 iterations. The red dotted lines represent the coordinate axes. 8
6 Displacement in the x direction of every element showed with color gradient. The displaced elements almost reach the fixed end of the beam. 8
7 Displacement in the x direction of every element showed with color gradient. The beam seems more deformed than in the previous case but the displaced elements seem fewer. 9
8 Displacement in the x direction of every element showed with color gradient.The beam looks less deformed than in the previous case, but more elements are displaced. Still fewer than in the first case with no hole. 10

List of Symbols

The next list describes several symbols that will be later used within the body of the document

δt	time step
ϵ	strain tensor
λ	Lamé's first parameter
μ	Lamé's second parameter: shear modulus
ρ	density
σ	stress
τ	boundary force density in both x,y directions in 2D
τ^N	Neumann boundary condition applied in the middle of each edge
$\tilde{f}$	external force f multiplied by Vol(B)
C	sparse matrix which defines in which edge the Dirichlet B.C. apply
E	Young's modulus
f	external force vector applying at the whole structure in both x,y directions at 2D
h	thickness of the beam at 2D case only
I	are moment of inertia
i	the i-th node
M	global mass matrix with dimensions 2nx2n for 2D
m	local mass matrix with dimensions 6x6 for 2D
M_e	global extended mass matrix
$M_x(w)$	bending moment at position x
n	number of nodes
p_i	position of a node described by x,y coordinates in 2D
$q(x)$	load (force density) at position x
$Q_x(w)$	shear force at x
S	global stiffness matrix with dimensions 2nx2n for 2D

S_e global extended stiffness matrix

T matrix multiplied with boundary force densities of all edges

t time

V volume of all prisms

$vol(B)$ volume of a prism

w displacement of a node from the reference position p in both x,y directions in 2D

w^D Dirichlet boundary condition at both x,y coordinates at 2D

x horizontal coordinate of position

y vertical coordinate of position

1 Introduction

The bending equation for the deformation of an elastic Bernoulli beam via Finite Element method (FEM) is analyzed in this report.The analysis was performed for 1D and 2D case and a simulation of this system run using MATLAB. In order to keep things simple, the following assumptions are considered:

- Piecewise differentiable functions are used instead of Sobolev spaces.
- The mass density is considered as constant.
- There is no damping effect on the beam oscillation.
- The traction densities τ at the common edge of two different triangles in 2D case are equal in absolute values from Newton's third law.
- For the model to hold we need a small angle of deflection that is $w'(x) << 1$

1.1 Bending equation

Static bending equation

$$\left(EIw''\right)'' x = q(x) \tag{1}$$

In order to specify a particular solution at the static bending equation, we need two Dirichlet and two Neumann boundary conditions:

$$w(0) = a, w'(0) = b, Q^L(w) = Q_L, M^L(w) = M_L \tag{2}$$

Dynamic bending equation

$$\rho\ddot{w} + \left(EIw''\right)'' x = q(x) \tag{3}$$

For solving the dynamic bending equation is necessary to use a method of computation for structural dynamics such as Euler, Runge-Kutta or Newmark method.

1.2 FEM

For a deformable body, the total external complementary work is equal to the total internal complementary work for every system of virtual forces and stresses that satisfy the equations of equilibrium.The principle of virtual work is the starting point for the formulation of the finite element method for solids and structures and is used to calculate the displacements of the beam.

Weak formulation in the static case

$$Sw = q \tag{4}$$

Weak formulation in the dynamic case

$$M\ddot{w} + Sw = q \tag{5}$$

As mentioned before, we need the necessary boundary conditions and a direct integration method for solving the ordinary differential equation of motion of elastic beam.

1.3 Newmark Method

The Newmark-beta method is a method of numerical integration used to solve differential equations. It is widely used in numerical evaluation of the dynamic response of structures and solids such as in finite element analysis to model dynamic systems. The following equations define the Newmark method:

$$w_{j+1} = w_j + \dot{w}_j \tau + \left(\left(\frac{1}{2} - \beta \right) \ddot{w}_j + \beta \ddot{w}_{j+1} \right) \tau^2 \tag{6}$$

$$\dot{w}_{j+1} = \dot{w}_j + \left((1 - \gamma) \ddot{w}_j + \gamma \ddot{w}_{j+1} \right) \tau \tag{7}$$

The optimal parameter are $\beta = 1/4$ and $\gamma = 1/2$. With these parameters, the method is stable for any step size τ.
For calculations using the Newmark method intermediary steps are needed to calculate $\ddot{w_{j+1}}$. This is done using the following steps

$$\hat{w}_j = w_j + \dot{w}_j \tau_j + \left(\frac{1}{2} - \beta \right) \ddot{w}_j \tau^2 \tag{8}$$

$$\widehat{\dot{w}}_j = \dot{w}_j + (1 - \gamma) \ddot{w}_j \tau \tag{9}$$

Then using these steps you solve for $\ddot{w}_{j+1}$

$$\ddot{w}_{j+1} = (M_e + \beta \tau^2 S_e)^{-1} (f - S_e \hat{w}) \tag{10}$$

2 1D Beam

Firstly, the 1D analysis of the deformation of an elastic Bernoulli beam is performed. Piecewise differentiable functions are used. The static bending equation (1) describes the relation between displacement w in the direction only to x (recall that the beam is modeled as a one-dimensional object) and the applied load q, in other words a force per unit length (analogous to pressure being a force per area). E is the elastic modulus and I is the second moment of area of the beam's cross-section.

2.1 Results

In 1D case, there are two main configurations which depend on the starting positions of the beam:

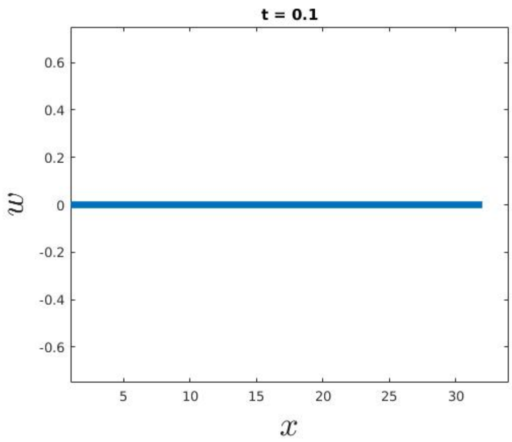
- The initial position of the beam is the horizontal position.
- The initial position of the beam is the position of the stationary solution.

It is considered that there is undamped oscillation of the beam and that the beam density is constant. In both configurations, we chose the following values:

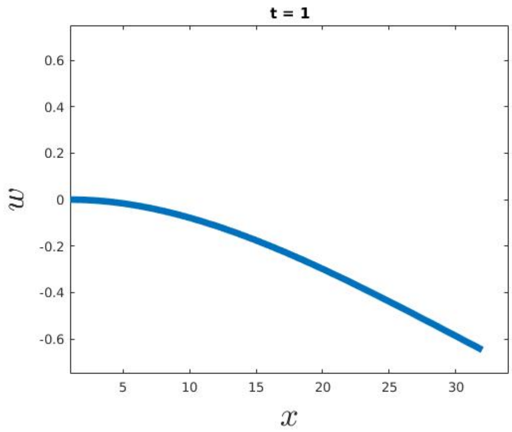
- Young modulus $E = 1$ with beam density $\rho = 1$ and moment of inertia $I = 1$
- Length $L = 1$ with nodes n=32 with 31 elements
- the beam is fixed at the left so w(0,t)=0 and w'(0,t)=0
- Moment in position L $M_L = 0$ and load function $q = 0$
- Time evolution from $t = 0$ to $t = 20$ with 200 time steps of length $\delta t = 0.1$

2.1.1 Starting from horizontal position

In the first configuration, a force $Q_L = -1$ is acting at the upper right corner at y-direction (since we are in 1D case) and the beam starts at horizontal position.The beam then is bending up to a lowest limit where velocity becomes zero and the displacement from the reference position is (at absolute value) maximum then it moves upward until it reaches the horizontal position again. The movement is without damping so it will oscillate forever between the horizontal position ($w_x = 0$) and the lowest position (w_x =maximum at absolute value). The position of the beam at time t=0.1 and t=1.0 can be seen at figures 1(a) and 1(b) .



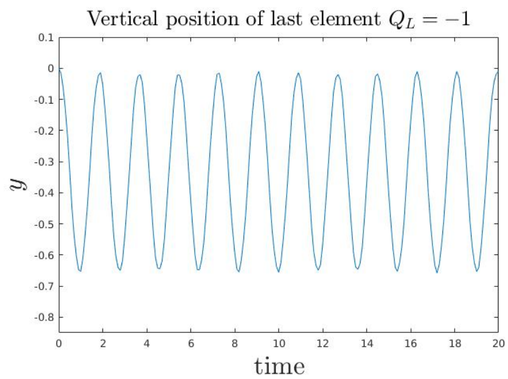
(a) Positions at $t = 0.1$



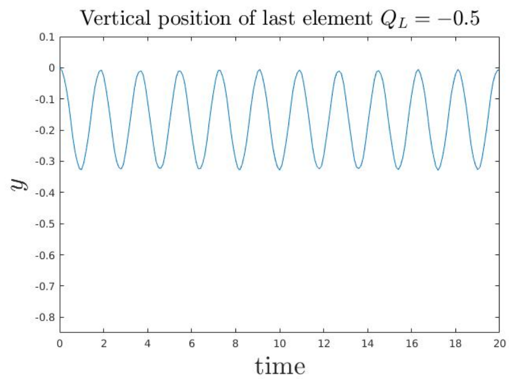
(b) Positions at $t = 1.0$

Figure 1: Results of starting the simulation from horizontal position for different shear forces.

Then, a comparison is done between different forces $Q_L = -1$ and $Q_L = -0.5$ at figures 2(a) and 2(b) correspondingly. We can notice at that no matter how big is the force, the beam returns to the horizontal position and with the oscillations have the same frequency. The only difference is that for double force, there is a double maximum (negative) displacement at each time step and each point.



(a) Position of last node over time with a shear force $Q_L = -1$

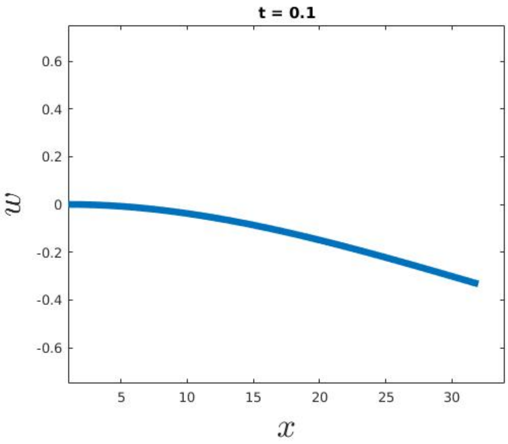


(b) Position of last node over time with a shear force $Q_L = -0.5$

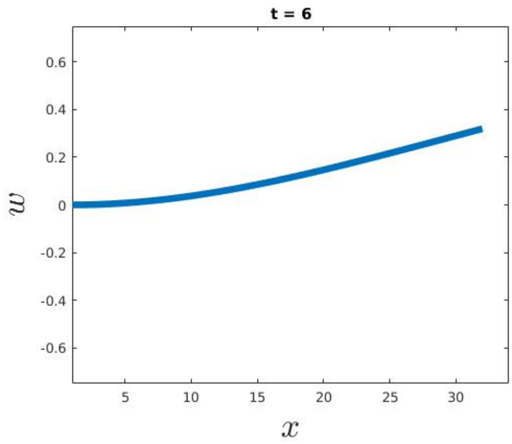
Figure 2: Results of starting the simulation from horizontal position for different shear forces.

2.1.2 Starting from position of stationary solution

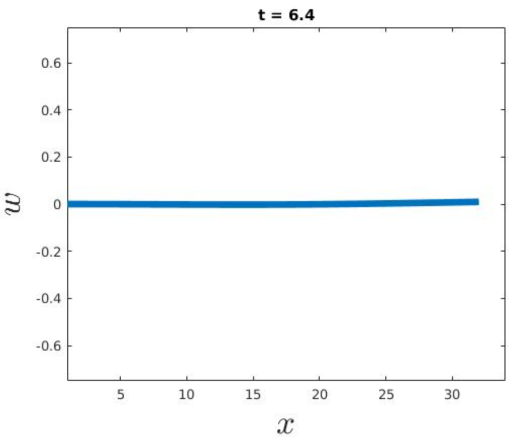
In the second configuration the initial position of the beam is the position of the stationary solution 3(a). A shear force $Q_L = -1$ is applied at point x=L up to t=5 and keeps the beam static. The exact solution of the displacement of the beam at stationary solution can be calculated and is at point x=L equal with w(t<5,x=L)=-$\frac{1}{3}$. At t=5 we remove the force and then the beam is moving upwards until it reaches an upper limit 3(b) . The movement is without damping so it will oscillate forever and the displacement at each node of the beam at the lowest limit will be algebraic equal in absolute value with the displacement of the same point at the upper limit. At figure 3(d) we can see the frequency of those oscillations and the maximum and minimum displacement which are equal for last node the beam.



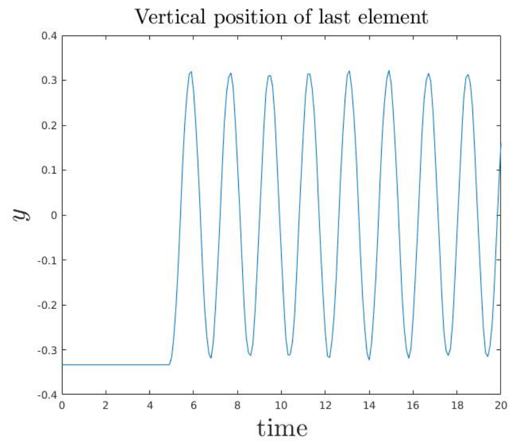
(a) Positions at $t = 0.1$



(b) Positions at $t = 6.0$



(c) Positions at $t = 6.4$

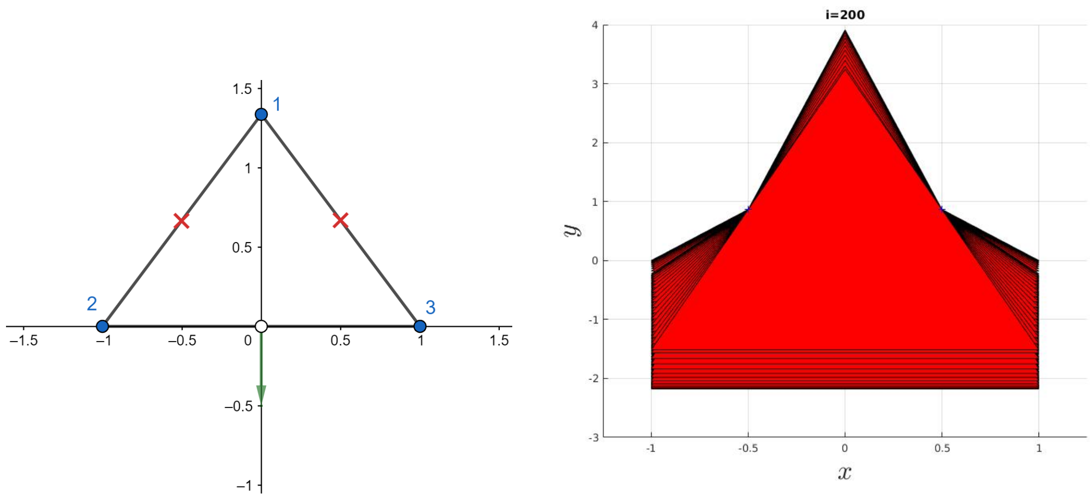


(d) Position of last node over time

Figure 3: Results of starting the simulation from the stationary state after applying a force and letting it go at $t = 5$. The movement after its release is clearly harmonic.

3 2D Prism

Both the Dirichlet u^D and Neumann τ^N boundary conditions are acting in the middle points of the boundary edges and only horizontally(in x or y-direction). There are no forces acting on the upper and lower surface, so we have ignored the z-coordinate.
At figure 4(a) we have the case of single prism. There are two points with a fixed zero displacement at the middle of two edges and a constant traction density τ acting on the middle point of the third edge which pulls the triangle to the negative y-direction.



(a) Prism with a fixed point half way between 1 and 2 as well as 1 and 3 and a vertical force between 2 and 3

(b) Result of time evolution of single prism. The shape in each time step is plotted over again.

Figure 4: Comparison of the formulation of reference prism and the results of the simulation.

The external force f (e.g. gravity) acts on the whole prism. The Volume of the triangle prism is given by the following equation:

$$vol(B) = \frac{h}{2} \det \left| \begin{pmatrix} p_1 & p_2 & p_3 \\ 1 & 1 & 1 \end{pmatrix} \right| \tag{11}$$

Our aim is to derive a differential equation for the displacements via the principle of virtual work. We implement the finite element method and together with the boundary conditions mentioned above we get the following extended system which has been solved via Newmark method:

$$\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \ddot{w} + \begin{bmatrix} S & C \\ C^T & 0 \end{bmatrix} w = \begin{bmatrix} T\tau^N + Vol(B)f \\ \overline{w}^D \end{bmatrix} \tag{12}$$

At figure 4(b) we can see the effect of the traction density τ on our prism after 200 iterations. It can be seen that the triangle deforms and the comes back to its original shape several times. Th final position is not the steady state.

4 2D Beam

4.1 Method

The next step in the simulation is to create our 2D beam out of a collection of prisms.The displacement at vertex p_i is u_i and is described by two coordinates x,y. We assume that common edges of two triangles which intersect in more than one point are identical. The traction densities τ at the common edge are equal in absolute values from Newton's third law.

Therefore, on the right hand side of the extended system 13 are present only traction densities τ^N acting on middle points of the boundary edges and not internal traction densities τ. On all boundary edges either Dirichlet or Neumann boundary conditions are applied. Boundary edges with Dirichlet boundary conditions $u^D = 0$ applied in all configurations on the left side of the beam, which have all horizontal coordinate $x = 0$. The $\tilde{f}$ is applied in the whole structure, so in all prisms.

$$\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \ddot{w} + \begin{bmatrix} S & C \\ C^T & 0 \end{bmatrix} w = \begin{bmatrix} T\tau^N + V\tilde{f} \\ \bar{w}^D \end{bmatrix} \tag{13}$$

The symmetric global and mass M and stiffness S matrices are constructed by finding which block entries of the local matrices contribute to each of the block entries of the global. This is achieved by looking for all surrounding triangles of each node $M_{i,j}$, $S_{i,j}$ and then adding all the block entries from the corresponding triangles. The matrix C shows in which triangles acts each Dirichlet boundary condition u^D and is also sparse.

The triangulation of the beam was done using the open source library `mesh2D` for MATLAB® (`https://github.com/dengwirda/mesh2d`). The calculations of the displacements were done from the reference positions of triangles.

4.2 Results

We adjusted the parameters so that the results were as close as possible to a real beam:

- The density $\rho = 1000$.
- The Lamé parameters $\lambda = 1000$ and $\mu = 100000$.
- The width of each triangular element $h = 1$.
- Evolved the system from $t = 0$ up to $t = 100$, using 500 time steps of length $\delta t = 0.2$.

The beam was localized with its left side in $x = 0$ and centered along the axis $y = 0$ The beam had always width equal to 2 and the length could be variable These parameters remained constant for the next simulations.

4.2.1 Without holes

For the first simulation, we created a beam that was very long ($L = 50$) compared to its width. With the automatic triangulation of `mesh2D`, we generated a lattice of 2674 elements. We set a "gravity" parameter of $f = -0.01$ and an external load τ = -10 located at the very right top end of the beam.

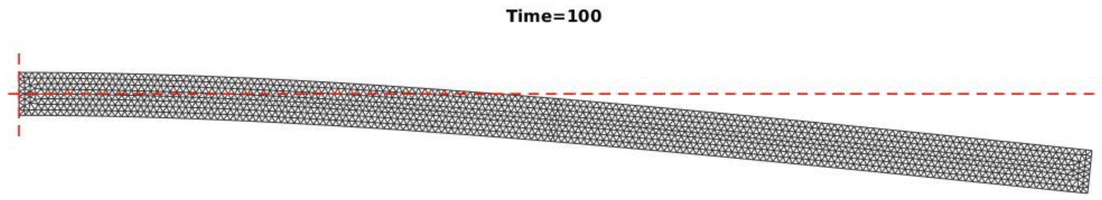


Figure 5: Long beam in time $t = 100$ after 500 iterations. The red dotted lines represent the coordinate axes.

We can see that the beam deflects at the end where the load is applied and transmits the force along its body, moving adjacent elements subject to the constraint that the left end is fixed.
To show in a clearer way the transmission of force between triangles, we generated another case, in which the beam has a length $L = 10$ and we apply a load $\tau = -75$ in $x = L, y = 1$ with a gravity force of $f = -0.015$. In this case the triangulation generated 540 elements.

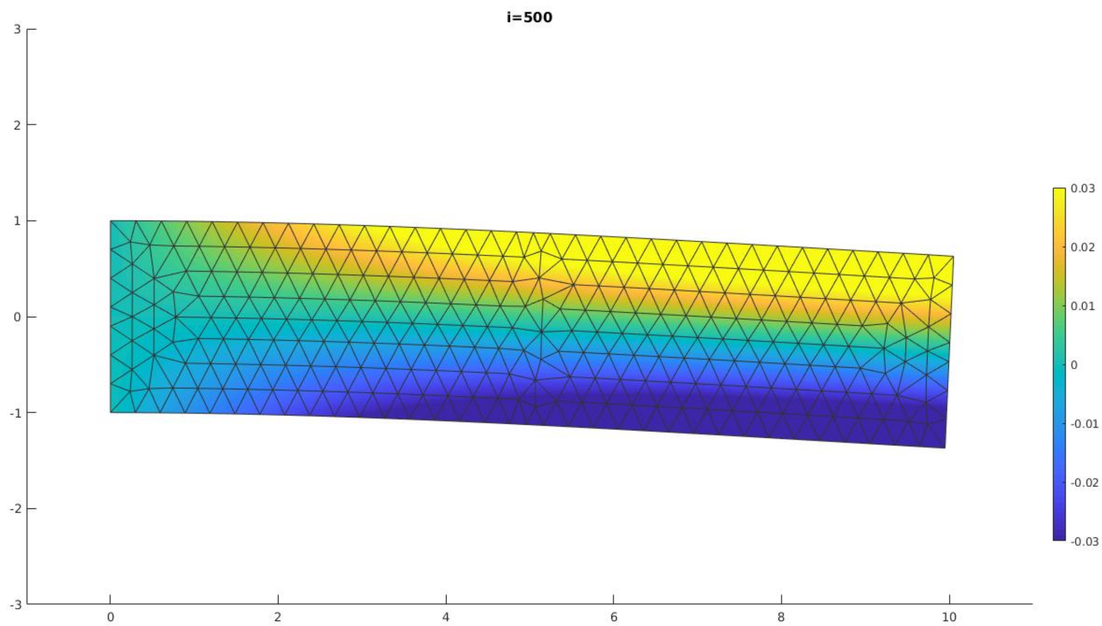


Figure 6: Displacement in the x direction of every element showed with color gradient. The displaced elements almost reach the fixed end of the beam.

We can see in a color gradient how each element deforms in the x axis with a force applied in the y axis. The transmission of force causes a positive deformation (expansion) in the upper part of the beam (yellow) and a negative deformation (contraction) in the lower part of the beam (dark blue). The elements in the middle remained with a small or zero deformation as well as the elements in the constrained end.

4.2.2 With one hole

Now we wanted to experiment with different configurations, to see if the deformations of the body as well as the elements are affected by the shape. In the first case we generated a beam of the same length as the previous one with a hole located from $x = 1.25$ to $x = 3.75$, and from $y = -0.5$ to $y = 0.5$. The number of elements is 412.

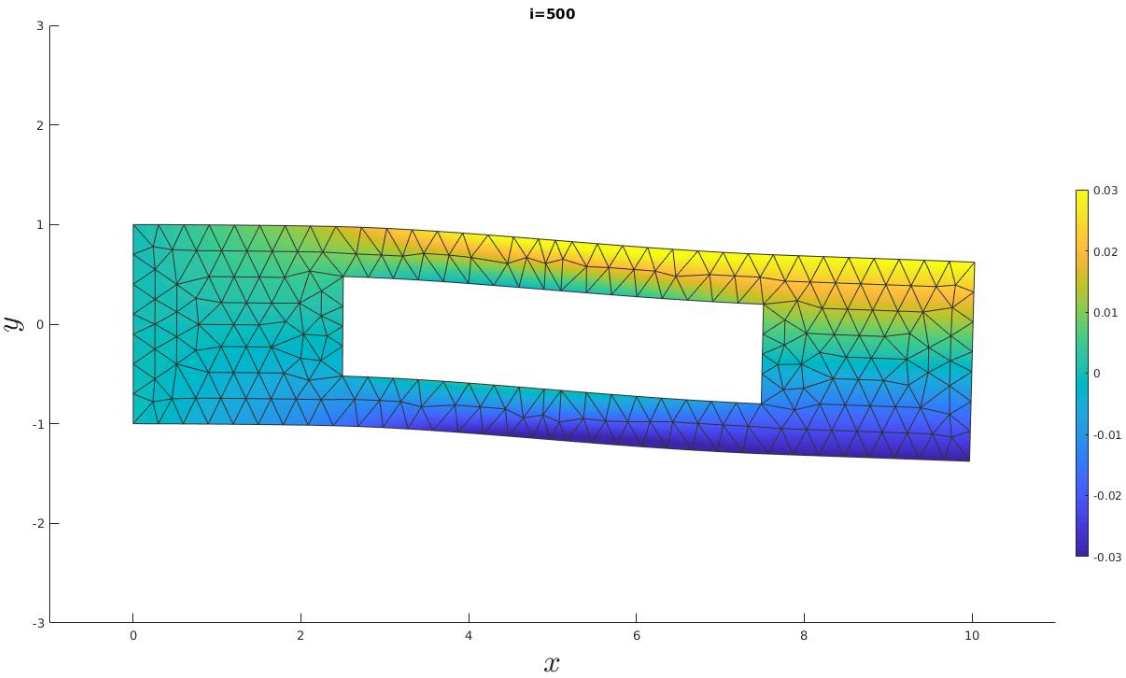


Figure 7: Displacement in the x direction of every element showed with color gradient. The beam seems more deformed than in the previous case but the displaced elements seem fewer.

We can see that there is a slightly greater deformation in the whole beam, and that the deformed elements are less propagated. This behaviour is probably because the empty space in the middle of the beam makes the beam less rigid and less stable, and it makes space for the lower and upper elements to move without the need to affect other elements that are far away.

4.2.3 With two holes

Finally, we created a beam with two equal, non-consecutive holes, located from $x = 1$ to $x = 2$ and from $x = 3$ to $x = 4$, both from $y = -0.5$ to $y = 0.5$. This triangulation gave 444 elements.

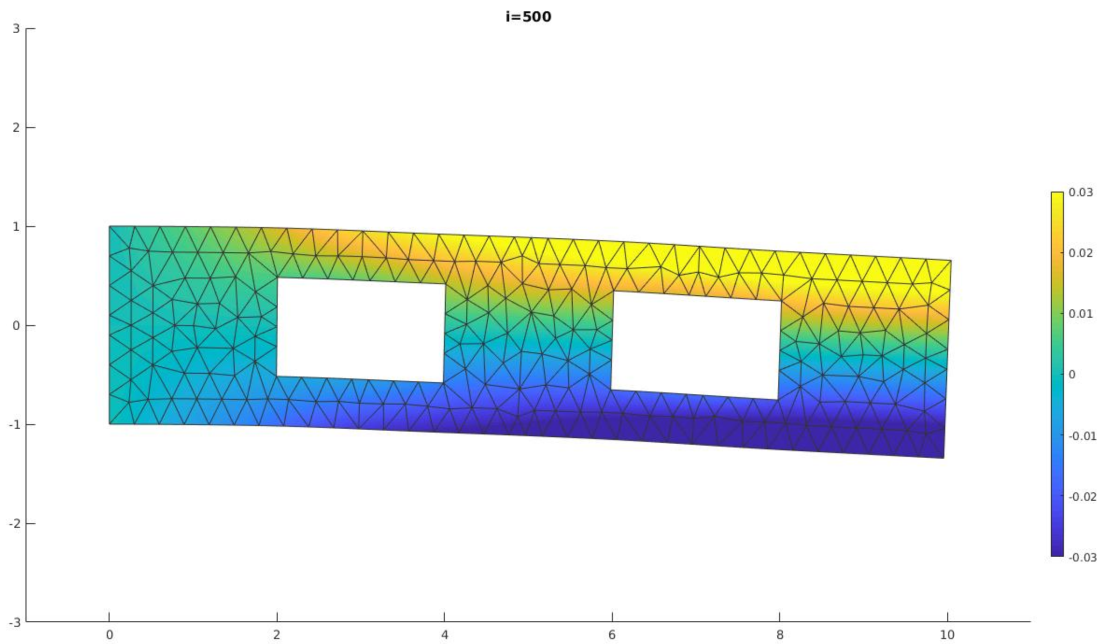


Figure 8: Displacement in the x direction of every element showed with color gradient.The beam looks less deformed than in the previous case, but more elements are displaced. Still fewer than in the first case with no hole.

We can see that the stability of the first beam without holes is recovered and still the elements don't propagate their deformation as far. The separate holes help the beam gain the robustness of the first beam. This type of analysis could be useful to optimize the use of raw materials in construction, as it is desired that the structures are stable with the minimum possible material used.

5 Conclusion

As a final word here are a few of our conclusions from this project.

- The beam is an elastic object and it stores energy. As there is no damping, the energy is conserved. Without damping the simulated system will oscillate forever.
- Starting from a stationary solution gives a steady state if the force stays applied.
- The dimensions of the extended system depends on the amount of Dirichlet boundary conditions.
- Young modulus parameters λ, μ and density ρ affects the simulation behavior.
- With displacement analysis we can identify the advantages and disadvantages of different shapes and configurations for the beam. This could be important for industrial applications, in which we want to get a very stable structure with the minimum material possible.

As future work one could determine when this model holds. That is checking for which combinations of λ and μ at a given force is $w'(x) << 1$ true. With that data you can define for each type of material and shape, as this defines λ and μ in reality, what loads can be modeled with this method.

If you wish to study the code used or data further this entire project is available on GitHub at `https://github.com/traikaras/Project_in_num_analysis`