Flash Tensor Product Attention

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Abstract

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1 Toward Faster Computation Without Materializing $\mathbf{Q}$, $\mathbf{K}$ and $\mathbf{V}$

We now explore whether it is possible to compute the attention scores $\mathbf{Q}\,\mathbf{K}^\top$ of Tensor Product Attention (TPA) (Zhang et al., 2025) *directly* from their factorized forms, thereby reducing floating-point operations.

1.1 Single-Head Factorization Setup Without Materializing $\mathbf{Q}$ and $\mathbf{K}$

Consider a single head i. Each query vector $\mathbf{Q}_t^{(i)} \in \mathbb{R}^{d_h}$ is factorized (with rank R_q):

$$\mathbf{Q}_t^{(i)} \;=\; \sum_{r=1}^{R_q} a_{q,i}^{(r)}(\mathbf{x}_t)\,\mathbf{b}_q^{(r)}(\mathbf{x}_t),$$

and each key vector $\mathbf{K}_\tau^{(i)} \in \mathbb{R}^{d_h}$ is factorized (with rank R_k):

$$\mathbf{K}_\tau^{(i)} \;=\; \sum_{s=1}^{R_k} a_{k,i}^{(s)}(\mathbf{x}_\tau)\,\mathbf{b}_k^{(s)}(\mathbf{x}_\tau).$$

Their dot-product for tokens t, τ is

$$\left[\mathbf{Q}^{(i)}\,(\mathbf{K}^{(i)})^\top\right]_{t,\tau} \;=\; \sum_{r=1}^{R_q}\sum_{s=1}^{R_k} a_{q,i}^{(r)}(\mathbf{x}_t)\,a_{k,i}^{(s)}(\mathbf{x}_\tau)\,\left\langle \mathbf{b}_q^{(r)}(\mathbf{x}_t), \mathbf{b}_k^{(s)}(\mathbf{x}_\tau)\right\rangle. \tag{1.1}$$

A custom kernel could compute this sum directly, though whether it outperforms a conventional GEMM depends on the ratio $\frac{R_q}{d_h}$, $\frac{R_k}{d_h}$, hardware parallelization, etc.

1.2 Multi-Head Case

For multi-head attention with h heads, one repeats the factorization across all heads. The $\mathbf{b}_q^{(r)}, \mathbf{b}_k^{(s)}$ vectors are shared across heads.

1.3 Complexity Analysis

We compare the cost of standard multi-head attention versus TPA under two scenarios:

1. **Naïve:** Materialize $\mathbf{Q}$ and $\mathbf{K}$ from factors, then perform the usual batched GEMM.
2. **Specialized:** Attempt to compute $\mathbf{Q}\,\mathbf{K}^\top$ directly from the rank-(R_q, R_k) factors without explicitly forming $\mathbf{Q}, \mathbf{K}$.

Standard Multi-Head Attention. For batch size B and sequence length n:

- *Projection cost:* $\mathcal{O}\big(B\,N\,d_{\text{model}}^2\big)$ or $\mathcal{O}\big(B\,N\,d_{\text{model}}\,d_h\big)$.
- *Dot-product:* $\mathbf{Q}\,(\mathbf{K})^\top \in \mathbb{R}^{(B\,h)\times N\times N}$ costs $\mathcal{O}\big(B\,N^2\,d_{\text{model}}\big)$.

For large n, the $\mathcal{O}(B\,N^2\,d_{\text{model}})$ term dominates.

TPA: Naïve Implementation.

- *Constructing factors:* $\mathcal{O}\big(B\,N\,d_{\text{model}}\,\times R_q(h+d_h)+R_k(h+d_h)+R_v(h+d_h)\big)$.
- *Materializing* $\mathbf{Q},\mathbf{K}$*:* $\mathcal{O}\big(B\,N\,(R_q\,h\,d_h + R_k\,h\,d_h)\big)$.
- *Dot-product* $\mathbf{Q}\,(\mathbf{K})^\top$*:* $\mathcal{O}\big(B\,N^2\,d_{\text{model}}\big)$.

Typically $R_q, R_k, R_v \ll h$, so the overhead of constructing factors is small relative to $\mathcal{O}(N^2\,d_{\text{model}})$. Meanwhile, we still gain KV caching benefits.

TPA: Specialized Implementation. If we bypass explicitly forming $\mathbf{Q},\mathbf{K}$, each dot product $\mathbf{Q}_t \cdot \mathbf{K}_\tau$ is a double sum over rank indices. Below we detail its complexity.

1.4 Complexity Analysis for the Specialized Implementation

Single-Head Complexity. A single attention head of dimension d_h. For each query:

$$\mathbf{Q}_t^{(i)} \;=\; \sum_{r=1}^{R_q} a_{q,i}^{(r)}(\mathbf{x}_t)\,\mathbf{b}_q^{(r)}(\mathbf{x}_t),$$

and for each key:

$$\mathbf{K}_\tau^{(i)} \;=\; \sum_{s=1}^{R_k} a_{k,i}^{(s)}(\mathbf{x}_\tau)\,\mathbf{b}_k^{(s)}(\mathbf{x}_\tau).$$

Their dot product:

$$\mathbf{Q}_t^{(i)}\cdot\mathbf{K}_\tau^{(i)} \;=\; \sum_{r=1}^{R_q}\sum_{s=1}^{R_k}\Big[a_{q,i}^{(r)}(\mathbf{x}_t)\,a_{k,i}^{(s)}(\mathbf{x}_\tau)\Big]\;\big\langle \mathbf{b}_q^{(r)}(\mathbf{x}_t), \mathbf{b}_k^{(s)}(\mathbf{x}_\tau)\big\rangle.$$

For each pair (r,s), we pay:

1. $\mathcal{O}(1)$ for multiplying two scalars,
2. $\mathcal{O}(d_h)$ for the dot product $\mathbf{b}_q^{(r)}(\mathbf{x}_t)\cdot\mathbf{b}_k^{(s)}(\mathbf{x}_\tau)$.

Since (r,s) runs over $R_q \times R_k$, each token-pair (t,τ) costs roughly

$$\mathcal{O}\Big(R_q\,R_k\,\big(1+d_h\big)\Big) \;\approx\; \mathcal{O}(R_q\,R_k\,d_h).$$

For N queries and N keys, that is $\mathcal{O}(N^2\,R_q\,R_k\,d_h)$ for a single head.

Multi-Head and Batches (Reusing $\mathbf{b}$-Dot Products). When extending to h heads, each head i has its own scalar factors $\mathbf{a}_{q,i}^{(r)}(\mathbf{x}_t)$ and $\mathbf{a}_{k,i}^{(s)}(\mathbf{x}_\tau)$, but the $\mathbf{b}$-vectors $\mathbf{b}_q^{(r)}(\mathbf{x}_t)$ and $\mathbf{b}_k^{(s)}(\mathbf{x}_\tau)$ can still be *shared* across all heads (assuming the same rank-R factors for every head). Hence, one can split the total cost into two stages:

1. **$\mathbf{b}$-Dot-Product Stage:**
   For each token pair (t,τ) and each rank pair (r,s), compute the dot product
   $$\big\langle \mathbf{b}_q^{(r)}(\mathbf{x}_t),\ \mathbf{b}_k^{(s)}(\mathbf{x}_\tau)\big\rangle \;\in\; \mathbb{R}.$$
   Since each dot product is $\mathcal{O}(d_h)$ and there are R_qR_k rank pairs as well as N^2 token pairs, this stage costs:
   $$\mathcal{O}\big(N^2\,R_qR_k\,d_h\big).$$
   Crucially, these $\mathbf{b}$-dot products need only be computed *once* and can be cached for reuse by all heads.

2. **Per-Head Scalar Multiplications:**
After the $\mathbf{b}$-dot products are precomputed (and cached), each head i only needs to multiply each stored dot product by the corresponding scalars $\mathbf{a}_{q,i}^{(r)}(\mathbf{x}_t)\,\mathbf{a}_{k,i}^{(s)}(\mathbf{x}_\tau)$. Since this scalar multiplication is $\mathcal{O}(1)$ per pair, and there are N^2 token pairs and $R_q R_k$ rank pairs for each of the h heads, this step costs:
$$\mathcal{O}\big(h\,N^2\,R_q R_k\big).$$

Putting these together, for batch size B, the total cost is
$$\mathcal{O}\big(B\,N^2\,R_q R_k\,d_h\big) \;+\; \mathcal{O}\big(B\,N^2\,h\,R_q R_k\big) \;=\; \mathcal{O}\Big(B\,N^2\,R_q R_k\big(d_h + h\big)\Big).$$
By contrast, the standard multi-head attention dot-product step is $\mathcal{O}\big(B\,N^2\,h\,d_h\big)$. Hence, for the specialized TPA approach to *reduce* flops,
$$R_q R_k\,(d_h + h) \;\le\; h\,d_h.$$
Thus a practical guideline is to ensure $R_q R_k \;<\; h\,\frac{d_h}{d_h+h}$. When that holds, bypassing explicit materialization of $\mathbf{Q}$ and $\mathbf{K}$ can be beneficial.

1.5 Toward Faster Computation Without Materializing Q, K, V

We have explored a two-step procedure for computing $\mathbf{Q}\,\mathbf{K}^\top$ directly from factorized queries and keys *without* materializing $\mathbf{Q}$ or $\mathbf{K}$. Here, we extend this idea to also avoid explicitly forming $\mathbf{V}$. That is, all three activations $\mathbf{Q}, \mathbf{K}, \mathbf{V}$ remain factorized throughout the attention pipeline. We present a single-head formulation below, and then discuss multi-head and batch extensions.

Extending the Two-Step Approach to Avoid V Materialization. After we obtain $\mathbf{Q}\mathbf{K}^\top$, we apply $\alpha_{t,\tau} = \mathrm{softmax}\big(\frac{1}{\sqrt{d_h}}\,(QK^\top)_{t,\tau}\big)$. The final attention output at token t (single head) is
$$\mathrm{head}(t) \;=\; \sum_{\tau=1}^{N} \alpha_{t,\tau}\;\mathbf{V}_\tau.$$
Using the factorization $\mathbf{V}_\tau = \sum_{u=1}^{R_v} a_v^{(u)}(\mathbf{x}_\tau)\,\mathbf{b}_v^{(u)}(\mathbf{x}_\tau)$, we write:
$$\mathrm{head}(t) \;=\; \sum_{\tau=1}^{N} \alpha_{t,\tau} \sum_{u=1}^{R_v} a_v^{(u)}(\mathbf{x}_\tau)\,\mathbf{b}_v^{(u)}(\mathbf{x}_\tau).$$
Rearrange sums:
$$\mathrm{head}(t) \;=\; \sum_{u=1}^{R_v}\Big[\sum_{\tau=1}^{N}\big(\alpha_{t,\tau}\,a_v^{(u)}(\mathbf{x}_\tau)\big)\,\mathbf{b}_v^{(u)}(\mathbf{x}_\tau)\Big].$$
We *still* do not explicitly form $\mathbf{V}_\tau$. Instead:

Stage 1: Calculating $\mathbf{b}_v^{(u)}(\mathbf{x}_\tau)$ for all tokens. We simply observe that each output $\mathrm{head}(t)$ can be computed by summing vectors $\mathbf{b}_v^{(u)}(\mathbf{x}_\tau) \in \mathbb{R}^{d_h}$ weighted by $\alpha_{t,\tau}\,a_v^{(u)}(\mathbf{x}_\tau)$. The complexity for constructing $\mathbf{b}_v^{(u)}(\mathbf{x}_\tau)\;\forall u, \tau$ is $\mathcal{O}(N\,R_v\,d_h)$.

Stage 2: Weighted Summation by $\alpha_{t,\tau}\,a_v^{(u)}(\mathbf{x}_\tau)$. For each token t, the final attention head output is
$$\sum_{\tau=1}^{N} \alpha_{t,\tau} \sum_{u=1}^{R_v} a_v^{(u)}(\mathbf{x}_\tau)\,\mathbf{b}_v^{(u)}(\mathbf{x}_\tau) \;=\; \sum_{u=1}^{R_v}\Big[\sum_{\tau=1}^{N}\big(\alpha_{t,\tau}\,a_v^{(u)}(\mathbf{x}_\tau)\big)\,\mathbf{b}_v^{(u)}(\mathbf{x}_\tau)\Big].$$
We still never explicitly materialize $\mathbf{V}$. Instead, for each pair (t, u), we must accumulate the sum of N vectors $\mathbf{b}_v^{(u)}(\mathbf{x}_\tau) \in \mathbb{R}^{d_h}$, each scaled by the scalar $\alpha_{t,\tau}\,a_v^{(u)}(\mathbf{x}_\tau)$. Because each vector is d_h-dimensional, each (t, u) summation costs $\mathcal{O}(N\,d_h)$. Summed over $t = 1 \ldots N$ and $u = 1 \ldots R_v$, the total work is $\mathcal{O}(N^2\,R_v\,d_h)$ for the entire sequence.

In practice, one precomputes all $\mathbf{b}_v^{(u)}(\mathbf{x}_\tau)$ for $\tau = 1 \ldots N$, so each accumulation can be implemented as a simple "scalar-times-vector add" in a tight loop. This cost is usually smaller than the $\mathbf{Q}\mathbf{K}^\top$ factorized cost if $R_v \ll d_h$.

1.6 Overall Complexity for Single-Head

Combining the four bullet-point stages from above (ignoring smaller overheads like the softmax) yields:

(i) **QK $\mathbf{b}$-Dot Product Stage:** $\mathcal{O}(N^2\, R_q\, R_k\, d_h)$.

(ii) **QK Scalar-Multiply Stage:** $\mathcal{O}(N^2\, R_q\, R_k)$.

(iii) **Computing $\mathbf{b}_v^{(u)}(\mathbf{x}_\tau)$ for all tokens:** $\mathcal{O}(N\, R_v\, d_h)$.

(iv) **Weighted Summation by $\alpha_{t,\tau}\, a_v^{(u)}(\mathbf{x}_\tau)$:** $\mathcal{O}(N^2\, R_v\, d_h)$.

Hence, for a single head, the total cost is:

$$\mathcal{O}\Big(N^2\, R_q\, R_k\, d_h \;+\; N^2\, R_q\, R_k \;+\; N\, R_v\, d_h \;+\; N^2\, R_v\, d_h\Big).$$

In many cases (especially for large N), the $\mathcal{O}(N^2)$ terms dominate, so one often focuses on

$$\mathcal{O}\Big(N^2\, R_q\, R_k\, d_h \;+\; N^2\, R_q\, R_k \;+\; N^2\, R_v\, d_h\Big).$$

1.7 Multi-Head and Batch Extensions (Reuse of $\mathbf{b}$-Dot Products)

When extending to h heads and batch size B, all sequence-length-dependent terms are multiplied by $\sim B\, h$. However, crucial $\mathbf{b}$-dot products can be partially reused or shared across heads:

QK $\mathbf{b}$-Dot Products. Since each head has distinct scalar factors $a_{q,i},\ a_{k,i}$ but the same $\mathbf{b}_q^{(r)},\ \mathbf{b}_k^{(s)}$ across heads, each pairwise dot product

$$\langle \mathbf{b}_q^{(r)}(\mathbf{x}_t),\ \mathbf{b}_k^{(s)}(\mathbf{x}_\tau)\rangle$$

is computed just once per batch. That cost remains

$$\mathcal{O}\big(B\, N^2\, R_q\, R_k\, d_h\big),$$

not multiplied by h. After caching these dot products, each of the h heads pays $\mathcal{O}\big(B\, N^2\, h\, R_q\, R_k\big)$ total for the head-specific scalar multiplications (the "$\alpha_{t,\tau}$"–like factors).

V $\mathbf{b}$-Evaluations. Likewise, the $\mathbf{b}_v^{(u)}$ factors are shared across heads (i.e. one set of $\mathbf{b}_v$-vectors for all heads). Hence, computing all $\mathbf{b}_v^{(u)}(\mathbf{x}_\tau)$ for $\tau = 1 \ldots N$ (across the batch) is a one-time cost:

$$\mathcal{O}\big(B\, N\, R_v\, d_h\big).$$

Then each head i has its own scalar factors $a_{v,i}^{(u)}(\mathbf{x}_\tau)$, so the final accumulation $\sum_{\tau=1}^{N} \alpha_{t,\tau}\, a_{v,i}^{(u)}(\mathbf{x}_\tau)\, \mathbf{b}_v^{(u)}(\mathbf{x}_\tau)$ costs $\mathcal{O}\big(B\, N^2\, h\, R_v\, d_h\big)$ in total (for all t, u).

Putting it all together, the total flops for multi-head attention with batch size B are:

$$\underbrace{\mathcal{O}\big(B\, N^2\, R_q\, R_k\, d_h\big)}_{\substack{\text{QK } \mathbf{b}\text{-dot products} \\ \text{(shared across heads)}}} + \underbrace{\mathcal{O}\big(B\, N^2\, h\, R_q\, R_k\big)}_{\text{per-head QK scalar mult.}} + \underbrace{\mathcal{O}\big(B\, N\, R_v\, d_h\big)}_{\substack{\text{Compute } \mathbf{b}_v \text{ for all tokens} \\ \text{(shared across heads)}}} + \underbrace{\mathcal{O}\big(B\, N^2\, h\, R_v\, d_h\big)}_{\substack{\text{final accumulations} \\ \text{(per head)}}}.$$

Comparison to Standard Multi-Head. By contrast, standard multi-head attention typically requires $\mathcal{O}\big(B\, N^2\, h\, d_h\big)$ flops for the $\mathbf{Q}\mathbf{K}^\top$ dot product (plus a similar $\mathcal{O}\big(B\, N^2\, h\, d_h\big)$ for multiplying by $\mathbf{V}$). The factorization can yield savings provided $R_q R_k \ll h$ (for QK) and $R_v \ll h$ (for V), though actual speedups depend on how well these multi-stage kernels are implemented and on hardware efficiency.

Discussion. By retaining $\mathbf{Q}, \mathbf{K}$, and $\mathbf{V}$ in factorized form, one can forgo the usual steps:

$$\mathbf{x}_t \;\mapsto\; \mathbf{Q}_t,\ \mathbf{K}_\tau \;\mapsto\; (\mathbf{Q}\,\mathbf{K}^\top) \;\mapsto\; \mathrm{softmax}(\mathbf{Q}\,\mathbf{K}^\top)\,\mathbf{V} \;\mapsto\; \text{final output.}$$

Instead, the large $\mathbf{Q}, \mathbf{K}, \mathbf{V}$ tensors (of size $N \times d_h$) are never materialized. The cost is replaced by rank-based $\mathbf{b}$-dot-product computations plus per-head scalar multiplications. The main challenge is to keep the factor ranks (R_q, R_k, R_v) sufficiently small relative to d_h and to implement the necessary multi-stage kernels efficiently (i.e. without losing the benefits of highly optimized GEMM libraries). When $R_q, R_k, R_v \ll h$, fully factorized QKV attention can yield substantial gains in both computation and memory footprint.

References

Yifan Zhang, Yifeng Liu, Huizhuo Yuan, Zhen Qin, Yang Yuan, Quanquan Gu, and Andrew Chi-Chih Yao. Tensor product attention is all you need. *arXiv preprint arXiv:2501.06425*, 2025.