1 Semantic Tableaux

a) From the open leaves of the tableau we see that there are the following models that satisfy the formula $(p \to q) \to r$: {

$$\begin{aligned}
&\{(p \to F), (q, \to F), (r \to T)\},\\
&\{(p \to F), (q, \to T), (r \to T)\},\\
&\{(p \to T), (q, \to F), (r \to F)\},\\
&\{(p \to T), (q, \to F), (r \to T)\},\\
&\{(p \to T), (q, \to T), (r \to T)\}
\end{aligned}$$

}. In the set of propositions $P = \{p, q, r\}$, the formula $(\neg p \to r) \wedge (q \to r)$ has the same models $M((p \to q) \to r) = M((\neg p \to r) \wedge (q \to r))$. Therefore, we conclude that $(p \to q) \to r$ and $(\neg p \to r) \wedge (q \to r)$ are logically equivalent.

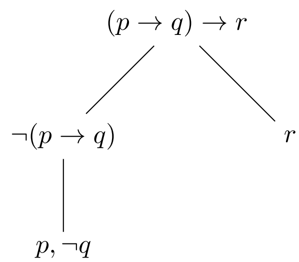


Figure 1: Left-hand side

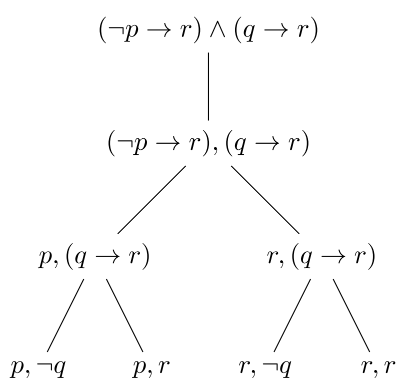


Figure 2: Right-hand side

b) To show that $(((p \to (q \wedge r)) \vee ((q \wedge r) \to p)) \to p)$ is falsifiable, we must show that there exist models for the inverse, namely $\neg(((p \to (q \wedge r)) \vee ((q \wedge r) \to p)) \to p)$. The semantic tableaux of the inverse can be found in Figure 3. Two examples of interpretations which yield false are: $\{\{(p \to F), (q \to T), (r \to T)\}, \{(p \to F), (q \to F), (r \to F)\}\}$

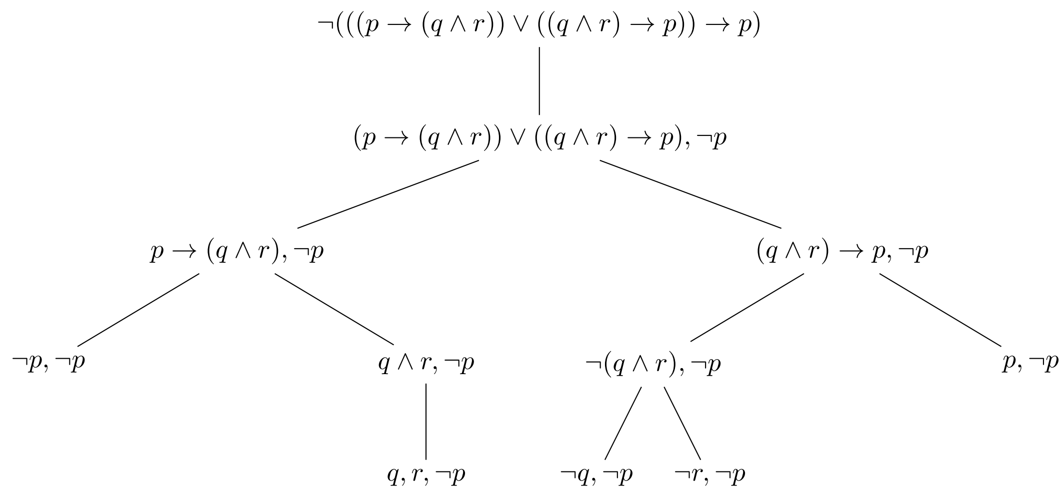


Figure 3: Semantic Tableaux for $\neg(((p \rightarrow (q \wedge r)) \vee ((q \wedge r) \rightarrow p)) \rightarrow p)$

2 Logical Equivalence

a)
We prove by induction:
basis $n = 1$: $p_1 \equiv p_1$
induction $n > 1$
Assumption: $\neg(p_1 \wedge ... \wedge p_{n-1}) \equiv \neg p_1 \vee ... \vee \neg p_{n-1}$
To prove: $\neg(p_1 \wedge ... \wedge p_{n-1} \wedge p_n) \equiv \neg p_1 \vee ... \vee \neg p_{n-1} \vee \neg p_n$
Deduction:
$\neg p1 \vee ... \vee \neg p_{n-1} \vee \neg p_n$
$\equiv (\neg p1 \vee ... \vee \neg p_{n-1}) \vee \neg p_n$
$\equiv \neg(p_1 \wedge ... \wedge p_{n-1}) \vee \neg p_n$ (substitute assumption)
$\equiv \neg((p_1 \wedge ... \wedge p_{n-1}) \wedge p_n)$ (De Morgen's law)
$\equiv \neg(p_1 \wedge ... \wedge p_{n-1} \wedge p_n)$

Conclusion: $\neg(p_n \wedge ... \wedge p_0) \equiv \neg p_n \vee ... \vee \neg p_1$
b)
We again prove by induction.
Basis $n = 1$: $p_1 \rightarrow p_0 \equiv p_1 \rightarrow p_1$
Induction $n > 1$:
Assumption: $p_{n-1} \rightarrow (... \rightarrow (p_1 \rightarrow p_0)...) \equiv (p_{n-1} \wedge ... \wedge p_1) \rightarrow p_0$
To prove: $p_n \rightarrow (p_{n-1} \rightarrow (... \rightarrow (p_1 \rightarrow p_0)...)) \equiv (p_n \wedge p_{n-1} \wedge ... \wedge p_1) \rightarrow p_0$
Deduction:
$p_n \rightarrow (p_{n-1} \rightarrow (... \rightarrow (p_1 \rightarrow p_0)...))$
$p_n \rightarrow ((p_{n-1} \wedge ... \wedge p_1) \rightarrow p_0)$ (substitute assumption)
$\neg p_n \vee ((p_{n-1} \wedge ... \wedge p_1) \rightarrow p_0)$

$\neg p_n \vee \neg(p_{n-1} \wedge ... \wedge p_1) \vee p_0$
$\neg(p_n \wedge (p_{n-1} \wedge ... \wedge p_1) \vee p_0$ (De Morgan's law)
$\neg(p_n \wedge p_{n-1} \wedge ... \wedge p_1) \vee p_0$
$(p_n \wedge p_{n-1} \wedge ... \wedge p_1) \rightarrow p_0$

Conclusion: $p_n \rightarrow (p_{n-1} \rightarrow (... \rightarrow (p_1 \rightarrow p_0)...)) \equiv (p_n \wedge p_{n-1} \wedge ... \wedge p_1) \rightarrow p_0$

3 Gentzen

The Gentzen rules for $L_{\leftrightarrow}$ and $R_{\leftrightarrow}$ are:

$$\frac{\Gamma, \phi \rightarrow \psi, \psi \rightarrow \phi \vdash \Delta}{\Gamma, \phi \leftrightarrow \psi \vdash \Delta} \ (L_{\leftrightarrow})$$

$$\frac{\Gamma \vdash \Delta, \phi \rightarrow \psi \quad \Gamma \vdash \Delta, \psi \rightarrow \phi}{\Gamma \vdash \Delta, \phi \leftrightarrow \psi} \ (R_{\leftrightarrow})$$

They are derived by substituting $\phi \leftrightarrow \psi$ with $\phi \rightarrow \psi \wedge \psi \rightarrow \phi$ and subsequently applying the $L_{\wedge}$ and $R_{\wedge}$ Gentzen rules to obtain the $L_{\leftrightarrow}$ and $R_{\leftrightarrow}$ rules respectively. The proof for Exercise 3 is presented in Figure 4.

4 Hilbert

5 Clausal From

We will first need to convert the formulas to CNF:

- $\phi_1 = (s \rightarrow t) \rightarrow (q \rightarrow (p \vee r)) = \neg(\neg s \vee t) \vee (\neg q \vee (p \vee r))$
  $= (\neg\neg s \wedge \neg t) \vee (\neg q \vee (p \vee r)) = (s \wedge \neg t) \vee (\neg q \vee (p \vee r))$
  $= (s \vee (\neg q \vee (p \vee r))) \wedge (\neg t \vee (\neg q \vee (p \vee r)))$
- $\phi_2 = (\neg q \rightarrow \neg t) \wedge \neg(p \wedge q) = (\neg\neg q \vee \neg t) \wedge \neg(p \wedge q) = (\neg\neg q \vee \neg t) \wedge (\neg p \vee \neg q)$
  $= (q \vee \neg t) \wedge (\neg p \vee \neg q)$
- $\phi_3 = \neg s \vee \neg q$
- $\phi_4 = q \vee t$
- $\phi_5 = (p \rightarrow t) \wedge (\neg r \vee p) = (\neg p \vee t) \wedge (\neg r \vee p)$

If we convert these formulas to clausal form we get:

- $\phi_1 = \{p\bar{q}rs, p\bar{q}r\bar{t}\}$
- $\phi_2 = \{q\bar{t}, \bar{p}\bar{q}\}$
- $\phi_3 = \{\bar{q}\bar{s}\}$
- $\phi_4 = \{qt\}$
- $\phi_5 = \{\bar{p}t, p\bar{r}\}$

$$
\dfrac{
\dfrac{
\dfrac{
r, (p \to q), (q \to p), (q \to r) \vdash p, r
\qquad
\dfrac{
r, q, (p \to q), (q \to r) \vdash p, q
\qquad
r, q, p, (p \to q), (q \to r) \vdash p
}{
r, q, (p \to q), (q \to p), (q \to r) \vdash p
}\; L_{\to}
}{
r, (p \to q), (q \to p), (q \to r), (r \to q) \vdash p
}\; L_{\to}
}{
(p \to q), (q \to p), (q \to r), (r \to q) \vdash (r \to p)
}\; R_{\to}
\qquad
\dfrac{
\dfrac{
p, (q \to p), (q \to r), (r \to q) \vdash r, p
\qquad
\dfrac{
p, q, (q \to p), (r \to q) \vdash r, q
\qquad
p, q, r, (q \to p), (r \to q) \vdash r
}{
p, q, (q \to p), (q \to r), (r \to q) \vdash r
}\; L_{\to}
}{
p, (p \to q), (q \to p), (q \to r), (r \to q) \vdash r
}\; L_{\to}
}{
(p \to q), (q \to p), (q \to r), (r \to q) \vdash (p \to r)
}\; R_{\to}
}{
\dfrac{
(p \to q), (q \to p), (q \to r), (r \to q) \vdash (r \leftrightarrow p)
}{
\dfrac{
(p \to q), (q \to p), (q \leftrightarrow r) \vdash (r \leftrightarrow p)
}{
\dfrac{
(p \leftrightarrow q), (q \leftrightarrow r) \vdash (r \leftrightarrow p)
}{
\dfrac{
((p \leftrightarrow q) \wedge (q \leftrightarrow r)) \vdash (r \leftrightarrow p)
}{
\vdash ((p \leftrightarrow q) \wedge (q \leftrightarrow r)) \to (r \leftrightarrow p)
}\; R_{\to}
}\; L_{\wedge}
}\; L_{\leftrightarrow}
}\; L_{\leftrightarrow}
}\; R_{\leftrightarrow}
$$

Figure 4: Proof for $\vdash ((p \leftrightarrow q) \wedge (q \leftrightarrow r)) \to (r \leftrightarrow p)$ in G

- $\{\phi_1, \phi_2, \phi_3, \phi_4, \phi_5\} = \{p\bar{q}rs, p\bar{q}r\bar{t}, q\bar{t}, \bar{p}\bar{q}, \bar{q}\bar{s}, qt, \bar{p}t, p\bar{r}\ \}$

The resolution refutation is represented as a tree in Figure 5

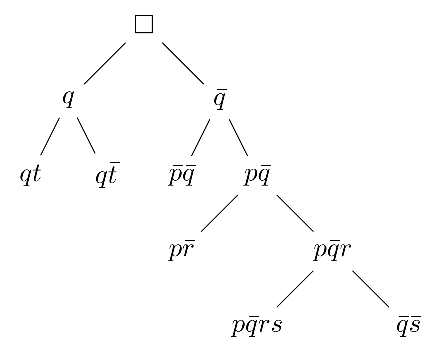
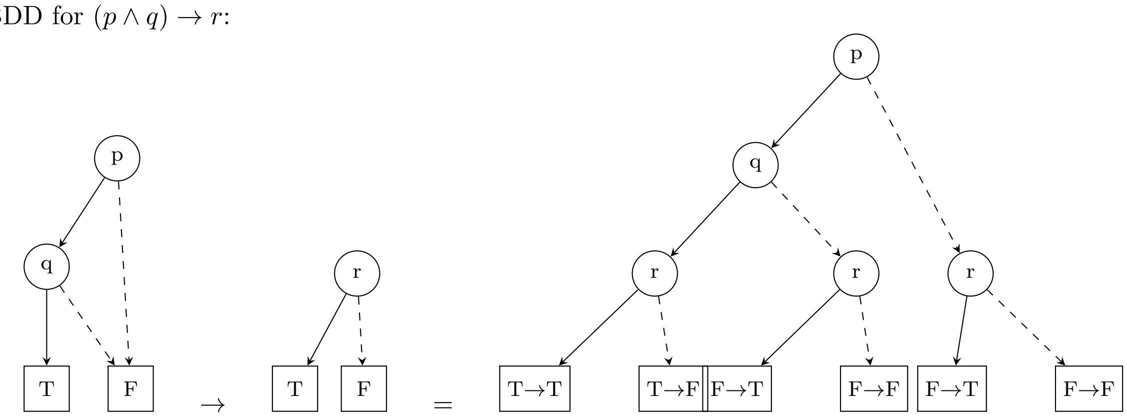


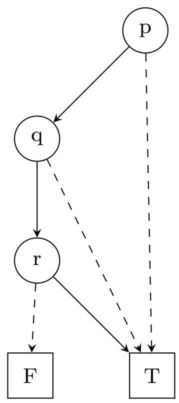
Figure 5: Resolution refutation tree for $\{\phi_1, \phi_2, \phi_3, \phi_4, \phi_5\}$

6 BDDs

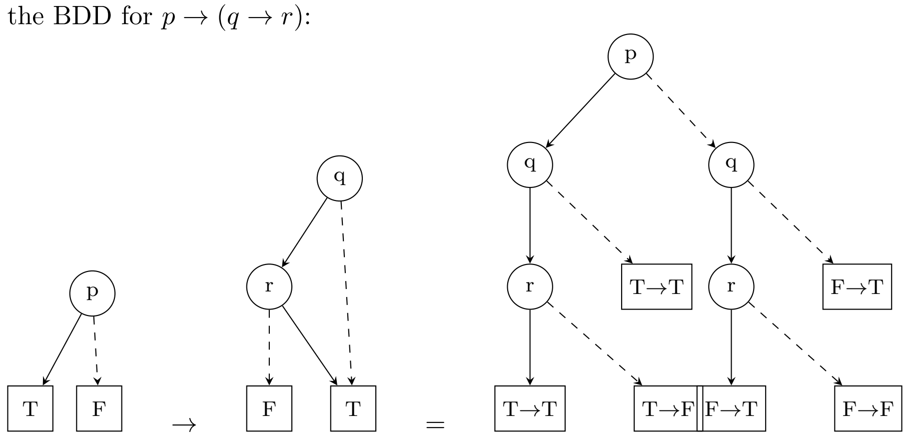
When we use the Apply algorithm on the sub-BDD for subformula $p \wedge q$, combined with the sub-BDD for the formula r using the $\rightarrow$ operator, we get the BDD for $(p \wedge q) \rightarrow r$:



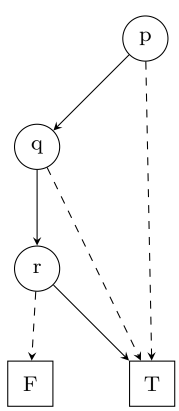
When we apply the Reduce algorithm on this result we get:



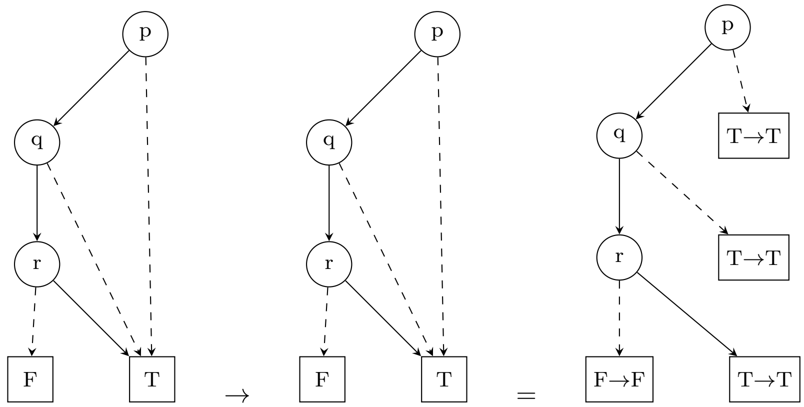
We will now use the Apply algorithm on the sub-BDD for subformula p, combined with the sub-BDD for the formula $q \to r$ using the $\to$ operator, to get the BDD for $p \to (q \to r)$:



When we apply the Reduce algorithm on this result we (again) get:



If we now use the Apply algorithm using the previously obtained sub-BDDs and the $\rightarrow$ operator, we get the BDD for $(p \rightarrow (q \rightarrow r)) \rightarrow ((p \wedge q) \rightarrow r)$:



When we apply the Reduce algorithm on this result we get $\boxed{\mathrm{T}}$ which means that $\models (p \rightarrow (q \rightarrow r)) \rightarrow ((p \wedge q) \rightarrow r)$ holds, since it yields *True* for all values of p, q and r.

7 ...

8 The Lady, or the Tiger?