1 Semantic Tableaux

a) From the open leaves of the tableau we see that there are the following models that satisfy the formula $(p \to q) \to r$: {

$\{(p \to F), (q, \to F), (r \to T)\}$,
$\{(p \to F), (q, \to T), (r \to T)\}$,
$\{(p \to T), (q, \to F), (r \to F)\}$,
$\{(p \to T), (q, \to F), (r \to T)\}$,
$\{(p \to T), (q, \to T), (r \to T)\}$

}. In the set of propositions $P = \{p, q, r\}$, the formula $(\neg p \to r) \land (q \to r)$ has the same models $M((p \to q) \to r) = M((\neg p \to r) \land (q \to r))$. Therefore, we conclude that $(p \to q) \to r$ and $(\neg p \to r) \land (q \to r)$ are logically equivalent.

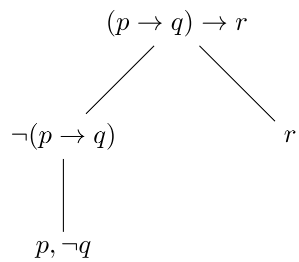


Figure 1: Left-hand side

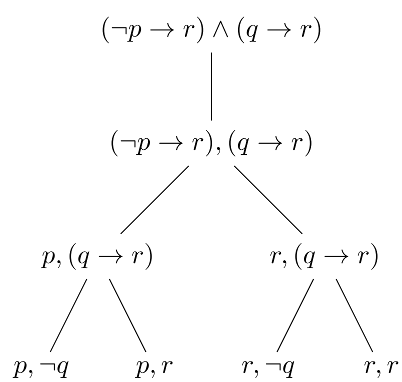


Figure 2: Right-hand side

b) To show that $(((p \to (q \land r)) \lor ((q \land r) \to p)) \to p)$ is falsifiable, we must show that there exist models for the inverse, namely $\neg(((p \to (q \land r)) \lor ((q \land r) \to p)) \to p)$. The semantic tableaux of the inverse can be found in Figure 3. Two examples of interpretations which yield false are: $\{\{(p \to F), (q \to T), (r \to T)\}, \{(p \to F), (q \to F), (r \to F)\}\}$

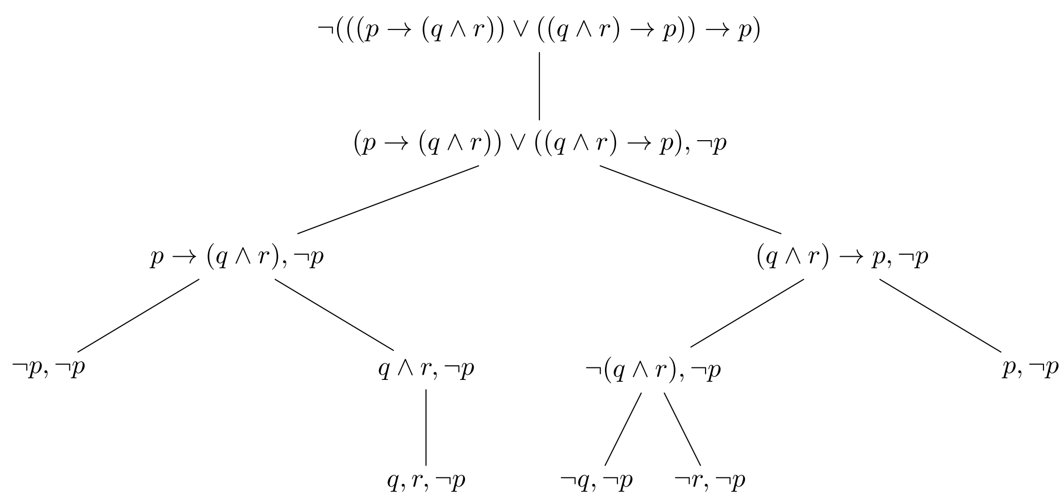


Figure 3: Semantic Tableaux for $\neg(((p \to (q \wedge r)) \vee ((q \wedge r) \to p)) \to p)$

2 Logical Equivalence

a)
We prove by induction:
basis $n = 1$: $p_1 \equiv p_1$
induction $n > 1$
Assumption: $\neg(p_1 \wedge ... \wedge p_{n-1}) \equiv \neg p_1 \vee ... \vee \neg p_{n-1}$
To prove: $\neg(p_1 \wedge ... \wedge p_{n-1} \wedge p_n) \equiv \neg p_1 \vee ... \vee \neg p_{n-1} \vee \neg p_n$
Deduction:
$\neg p1 \vee ... \vee \neg p_{n-1} \vee \neg p_n$
$\equiv (\neg p1 \vee ... \vee \neg p_{n-1}) \vee \neg p_n$
$\equiv \neg(p_1 \wedge ... \wedge p_{n-1}) \vee \neg p_n$ (substitute assumption)
$\equiv \neg((p_1 \wedge ... \wedge p_{n-1}) \wedge p_n)$ (De Morgen's law)
$\equiv \neg(p_1 \wedge ... \wedge p_{n-1} \wedge p_n)$

Conclusion: $\neg(p_n \wedge ... \wedge p_0) \equiv \neg p_n \vee ... \vee \neg p_1$
b)
We again prove by induction.
Basis $n = 1$: $p_1 \to p_0 \equiv p_1 \to p_1$
Induction $n > 1$:
Assumption: $p_{n-1} \to (... \to (p_1 \to p_0)...) \equiv (p_{n-1} \wedge ... \wedge p_1) \to p_0$
To prove: $p_n \to (p_{n-1} \to (... \to (p_1 \to p_0)...)) \equiv (p_n \wedge p_{n-1} \wedge ... \wedge p_1) \to p_0$
Deduction:
$p_n \to (p_{n-1} \to (... \to (p_1 \to p_0)...))$
$p_n \to ((p_{n-1} \wedge ... \wedge p_1) \to p_0)$ (substitute assumption)
$\neg p_n \vee ((p_{n-1} \wedge ... \wedge p_1) \to p_0)$

$\neg p_n \vee \neg(p_{n-1} \wedge ... \wedge p_1) \vee p_0$
$\neg(p_n \wedge (p_{n-1} \wedge ... \wedge p_1) \vee p_0$ (De Morgan's law)
$\neg(p_n \wedge p_{n-1} \wedge ... \wedge p_1) \vee p_0$
$(p_n \wedge p_{n-1} \wedge ... \wedge p_1) \rightarrow p_0$

Conclusion: $p_n \rightarrow (p_{n-1} \rightarrow (... \rightarrow (p_1 \rightarrow p_0)...)) \equiv (p_n \wedge p_{n-1} \wedge ... \wedge p_1) \rightarrow p_0$

3 Gentzen

$$\frac{\Gamma, \phi \rightarrow \psi, \psi \rightarrow \phi \vdash \Delta}{\Gamma, \phi \leftrightarrow \psi \vdash \Delta} \ (L_{\leftrightarrow})$$

$$\frac{\Gamma \vdash \Delta, \phi \rightarrow \psi \quad \Gamma \vdash \Delta, \psi \rightarrow \phi}{\Gamma \vdash \Delta, \phi \leftrightarrow \psi} \ (R_{\leftrightarrow})$$

4 Hilbert

In order to solve the theorem in H we first transform the formula eliminating all operators except for $\neg$ and $\rightarrow$.

$$(p \rightarrow (q \rightarrow r)) \rightarrow ((p \wedge q) \rightarrow r) \equiv p \rightarrow (q \rightarrow r)) \rightarrow (\neg(p \rightarrow \neg q) \rightarrow r)$$

To shorten the eventual proof we first prove that $\vdash \neg(p \rightarrow q) \rightarrow p$ and $\vdash (p \rightarrow q) \rightarrow q$.

Theorem 4.1:

$$\cfrac{\cfrac{\cfrac{p, q \vdash p}{p \wedge q \vdash p}}{\neg(p \rightarrow \neg q) \vdash p}}{\vdash \neg(p \rightarrow \neg q) \rightarrow p}$$

Theorem 4.2:

$$\cfrac{\cfrac{\cfrac{p, q \vdash q}{p \wedge q \vdash q}}{\neg(p \rightarrow \neg q) \vdash q}}{\vdash \neg(p \rightarrow \neg q) \rightarrow q}$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{r, (p \to q), (q \to p), (q \to r) \vdash p, r \qquad \dfrac{r, q, (p \to q), (q \to r) \vdash p, q \qquad r, q, p, (p \to q), (q \to r) \vdash p}{r, q, (p \to q), (q \to p), (q \to r) \vdash p}\, L_{\to}}{r, (p \to q), (q \to p), (q \to r), (r \to q) \vdash p}\, L_{\to}}{(p \to q), (q \to p), (q \to r), (r \to q) \vdash (r \to p)}\, R_{\to} \qquad \dfrac{\dfrac{p, (q \to p), (q \to r), (r \to q) \vdash r, p \qquad \dfrac{p, q, (q \to p), (r \to q) \vdash r, q \qquad p, q, r, (q \to p), (r \to q) \vdash r}{p, q, (q \to p), (q \to r), (r \to q) \vdash r}\, L_{\to}}{p, (p \to q), (q \to p), (q \to r), (r \to q) \vdash r}\, L_{\to}}{(p \to q), (q \to p), (q \to r), (r \to q) \vdash (p \to r)}\, R_{\to}}{(p \to q), (q \to p), (q \to r), (r \to q) \vdash (r \leftrightarrow p)}\, R_{\leftrightarrow}}{(p \to q), (q \to p), (q \leftrightarrow r) \vdash (r \leftrightarrow p)}\, L_{\leftrightarrow}}{(p \leftrightarrow q), (q \leftrightarrow r) \vdash (r \leftrightarrow p)}\, L_{\leftrightarrow}}{((p \leftrightarrow q) \wedge (q \leftrightarrow r)) \vdash (r \leftrightarrow p)}\, L_{\wedge}}{\vdash ((p \leftrightarrow q) \wedge (q \leftrightarrow r)) \to (r \leftrightarrow p)}\, R_{\to}
$$

Figure 4: Proof for $\vdash ((p \leftrightarrow q) \wedge (q \leftrightarrow r)) \to (r \leftrightarrow p)$ in G

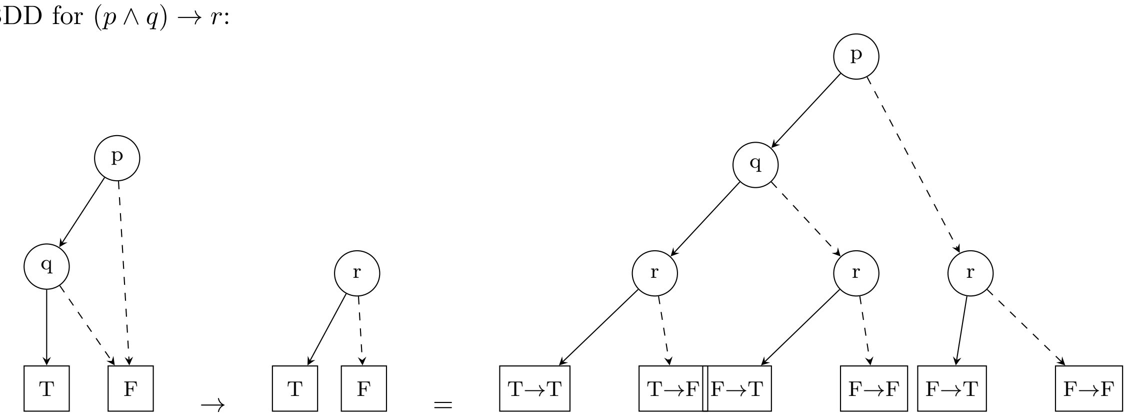
The proof of $(p \to (q \to r)) \to (\neg(p \to \neg q) \to r)$ is listed in figure 5.

Conclusion: $\vdash (p \to (q \to r) \to ((p \land q) \to r)$.

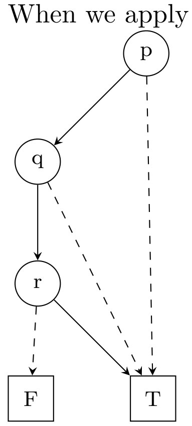
5 Clausal From

6 BDDs

When we use the Apply algorithm on the sub-BDD for subformula $p \land q$, combined with the sub-BDD for the formula r using the $\to$ operator, we get the BDD for $(p \land q) \to r$:



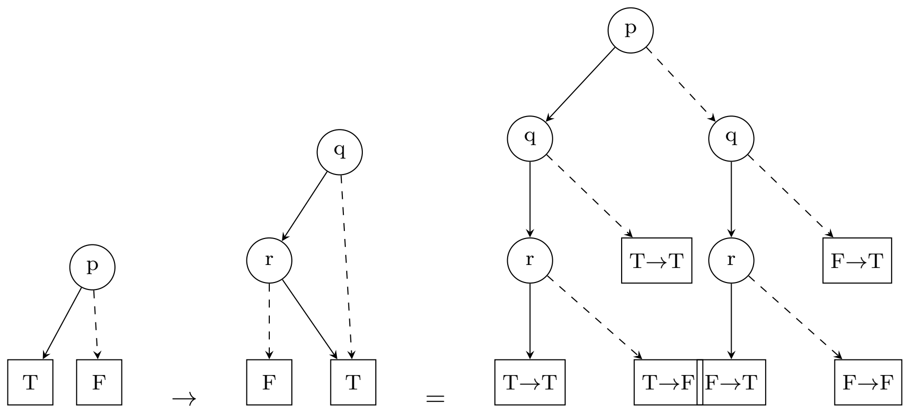
When we apply the Reduce algorithm on this result we get:



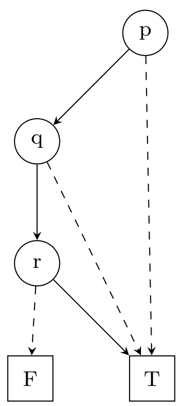
We will now use the Apply algorithm on the sub-BDD for subformula p, combined with the sub-BDD for the formula $q \to r$ using the $\to$ operator, to get the BDD for $p \to (q \to r)$:

$$
\dfrac{\dfrac{\dfrac{\dfrac{p \to (q \to r), \neg(p \to \neg q) \vdash \neg(p \to \neg q) \qquad \dfrac{Theorem4.2}{\Gamma \vdash \neg(p \to \neg q) \to q}}{p \to (q \to r), \neg(p \to \neg q) \vdash q}\,MP \qquad \dfrac{\dfrac{p \to (q \to r), \neg(p \to \neg q) \vdash \neg(p \to \neg q) \qquad \dfrac{Theorem4.1}{\Gamma \vdash \neg(p \to \neg q) \to p}}{p \to (q \to r), \neg(p \to \neg q) \vdash p}\,MP \qquad p \to (q \to r), \neg(p \to \neg q) \vdash p \to (q \to r)}{p \to (q \to r), \neg(p \to \neg q) \vdash q \to r}\,MP}{p \to (q \to r), \neg(p \to \neg q) \vdash r}\,MP}{p \to (q \to r) \vdash \neg(p \to \neg q) \to r}\,DR}{\vdash (p \to (q \to r)) \to (\neg(p \to \neg q) \to r)}\,DR
$$

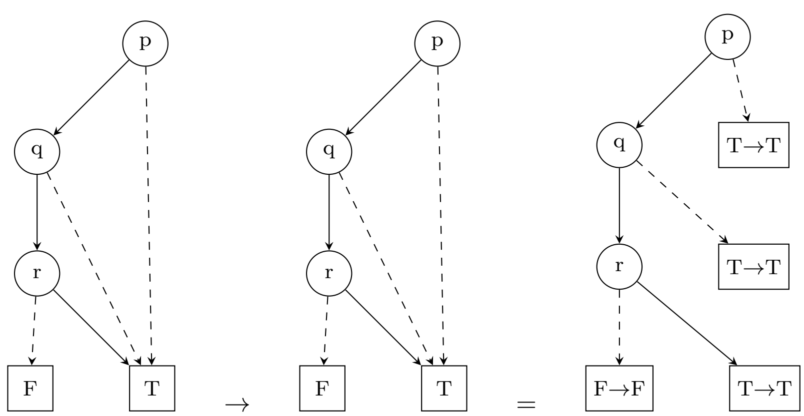
Figure 5: Proof for $(p \to (q \to r)) \to (\neg(p \to \neg q) \to r))$ in H



When we apply the Reduce algorithm on this result we (again) get:



If we now use the Apply algorithm using the previously obtained sub-BDDs and the $\rightarrow$ operator, we get the BDD for $(p \rightarrow (q \rightarrow r)) \rightarrow ((p \wedge q) \rightarrow r)$:



When we apply the Reduce algorithm on this result we get $\boxed{\mathrm{T}}$ which means that $\models (p \to (q \to r)) \to ((p \wedge q) \to r)$ holds, since it yields *True* for all values of p, q and r.

7 ...

8 The Lady, or the Tiger?