1 Semantic Tableaux

a) From the open leaves of the tableau we see that there are the following models that satisfy the formula $(p \to q) \to r$: $\{\{(p \to F), (q, \to F), (r \to T)\}, \{(p \to F), (q, \to T), (r \to T)\}, \{(p \to T), (q, \to F), (r \to F)\}, \{(p \to T), (q, \to F), (r \to T)\}, \{(p \to T), (q, \to T), (r \to T)\}\}$. In the set of propositions $P = \{p, q, r\}$, the formula $(\neg p \to r) \wedge (q \to r)$ has the same models $M((p \to q) \to r) = M((\neg p \to r) \wedge (q \to r))$. Therefore, we conclude that $(p \to q) \to r$ and $(\neg p \to r) \wedge (q \to r)$ are logically equivalent.

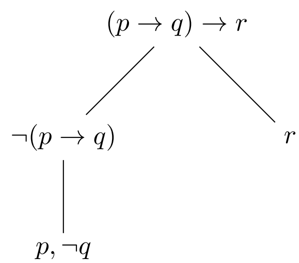

Figure 1: Left-hand side

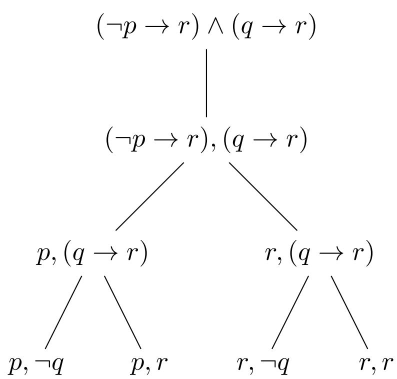

Figure 2: Right-hand side

b) To show that $(((p \to (q \wedge r)) \vee ((q \wedge r) \to p)) \to p)$ is falsifiable, we must show that there exist models for the inverse, namely $\neg(((p \to (q \wedge r)) \vee ((q \wedge r) \to p)) \to p)$. The semantic tableaux of the inverse can be found in Figure 3. Two examples of interpretations which yield false are: $\{\{(p \to F), (q \to T), (r \to T)\}, \{(p \to F), (q \to F), (r \to F)\}\}$

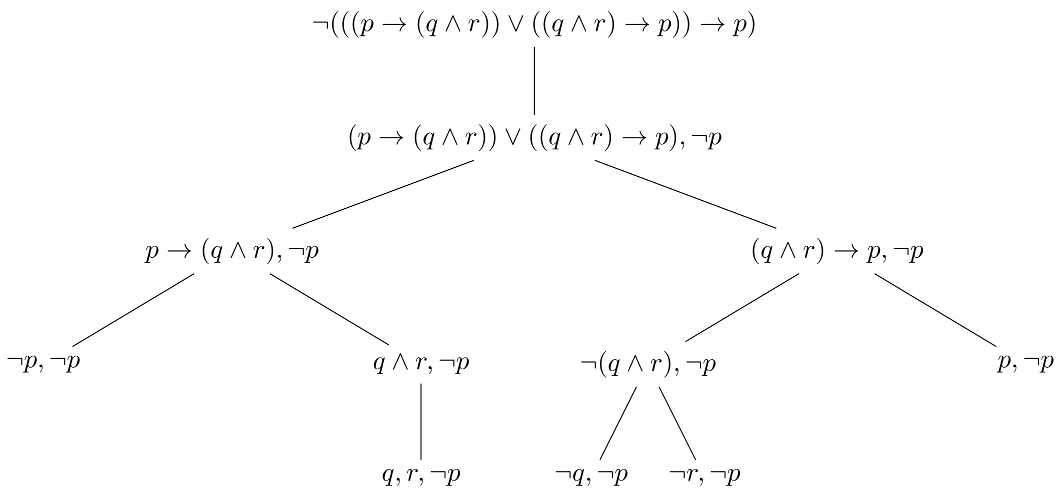

Figure 3: Semantic Tableaux for $\neg(((p \to (q \wedge r)) \vee ((q \wedge r) \to p)) \to p)$

2 Logical Equivalence

a)
We prove by induction:
basis $n = 1$: $p_1 \equiv p_1$
induction $n > 1$
Assumption: $\neg(p_1 \wedge ... \wedge p_{n-1}) \equiv \neg p_1 \vee ... \vee \neg p_{n-1}$
To prove: $\neg(p_1 \wedge ... \wedge p_{n-1} \wedge p_n) \equiv \neg p_1 \vee ... \vee \neg p_{n-1} \vee \neg p_n$
Deduction:
$\neg p1 \vee ... \vee \neg p_{n-1} \vee \neg p_n$
$\equiv (\neg p1 \vee ... \vee \neg p_{n-1}) \vee \neg p_n$
$\equiv \neg(p_1 \wedge ... \wedge p_{n-1}) \vee \neg p_n$ (substitute assumption)
$\equiv \neg((p_1 \wedge ... \wedge p_{n-1}) \wedge p_n)$ (De Morgen's law)
$\equiv \neg(p_1 \wedge ... \wedge p_{n-1} \wedge p_n)$

Conclusion: $\neg(p_n \wedge ... \wedge p_0) \equiv \neg p_n \vee ... \vee \neg p_1$
b)
We again prove by induction.
Basis $n = 1$: $p_1 \to p_0 \equiv p_1 \to p_1$
Induction $n > 1$:
Assumption: $p_{n-1} \to (... \to (p_1 \to p_0)...) \equiv (p_{n-1} \wedge ... \wedge p_1) \to p_0$
To prove: $p_n \to (p_{n-1} \to (... \to (p_1 \to p_0)...)) \equiv (p_n \wedge p_{n-1} \wedge ... \wedge p_1) \to p_0$
Deduction:
$p_n \to (p_{n-1} \to (... \to (p_1 \to p_0)...))$
$p_n \to ((p_{n-1} \wedge ... \wedge p_1) \to p_0)$ (substitute assumption)
$\neg p_n \vee ((p_{n-1} \wedge ... \wedge p_1) \to p_0)$

$\neg p_n \vee \neg(p_{n-1} \wedge ... \wedge p_1) \vee p_0$
$\neg(p_n \wedge (p_{n-1} \wedge ... \wedge p_1) \vee p_0$ (De Morgan's law)
$\neg(p_n \wedge p_{n-1} \wedge ... \wedge p_1) \vee p_0$
$(p_n \wedge p_{n-1} \wedge ... \wedge p_1) \to p_0$

Conclusion: $p_n \to (p_{n-1} \to (... \to (p_1 \to p_0)...)) \equiv (p_n \wedge p_{n-1} \wedge ... \wedge p_1) \to p_0$

3 Gentzen

$$\frac{\Gamma, \phi \to \psi, \psi \to \phi \vdash \Delta}{\Gamma, \phi \leftrightarrow \psi \vdash \Delta}\ (L_{\leftrightarrow})$$

$$\frac{\Gamma \vdash \Delta, \phi \to \psi \quad \Gamma \vdash \Delta, \psi \to \phi}{\Gamma \vdash \Delta, \phi \leftrightarrow \psi}\ (R_{\leftrightarrow})$$

4 Hilbert

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r, (p \to q), (q \to p), (q \to r) \vdash p, r
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\dfrac{r, q, (p \to q), (q \to r) \vdash p, q \qquad r, q, p, (p \to q), (q \to r) \vdash p}{r, q, (p \to q), (q \to p), (q \to r) \vdash p}\,L_{\to}
}{r, (p \to q), (q \to p), (q \to r), (r \to q) \vdash p}\,L_{\to}
}{(p \to q), (q \to p), (q \to r), (r \to q) \vdash (r \to p)}\,R_{\to}
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p, (q \to p), (q \to r), (r \to q) \vdash r, p
\qquad
\dfrac{p, q, (q \to p), (r \to q) \vdash r, q \qquad p, q, r, (q \to p), (r \to q) \vdash r}{p, q, (q \to p), (q \to r), (r \to q) \vdash r}\,L_{\to}
}{p, (p \to q), (q \to p), (q \to r), (r \to q) \vdash r}\,L_{\to}
}{(p \to q), (q \to p), (q \to r), (r \to q) \vdash (p \to r)}\,R_{\to}
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\dfrac{(p \to q), (q \to p), (q \to r), (r \to q) \vdash (r \leftrightarrow p)}{(p \to q), (q \to p), (q \leftrightarrow r) \vdash (r \leftrightarrow p)}\,L_{\leftrightarrow}
}{(p \leftrightarrow q), (q \leftrightarrow r) \vdash (r \leftrightarrow p)}\,L_{\leftrightarrow}
}{((p \leftrightarrow q) \wedge (q \leftrightarrow r)) \vdash (r \leftrightarrow p)}\,L_{\wedge}
}{\vdash ((p \leftrightarrow q) \wedge (q \leftrightarrow r)) \to (r \leftrightarrow p)}\,R_{\to}
}\,R_{\leftrightarrow}
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Figure 4: Proof for $\vdash ((p \leftrightarrow q) \wedge (q \leftrightarrow r)) \to (r \leftrightarrow p)$ in G