# 1 Semantic Tableaux

**a)** From the open leaves of the tableau we see that there are the following models that satisfy the formula $(p \to q) \to r$: $\{$

$$\begin{aligned}
&\{(p \to F), (q, \to F), (r \to T)\},\\
&\{(p \to F), (q, \to T), (r \to T)\},\\
&\{(p \to T), (q, \to F), (r \to F)\},\\
&\{(p \to T), (q, \to F), (r \to T)\},\\
&\{(p \to T), (q, \to T), (r \to T)\}
\end{aligned}$$

$\}$. In the set of propositions $P = \{p, q, r\}$, the formula $(\neg p \to r) \wedge (q \to r)$ has the same models $M((p \to q) \to r) = M((\neg p \to r) \wedge (q \to r))$. Therefore, we conclude that $(p \to q) \to r$ and $(\neg p \to r) \wedge (q \to r)$ are logically equivalent.

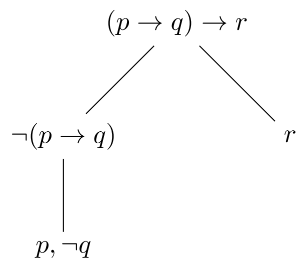



Figure 1: Left-hand side

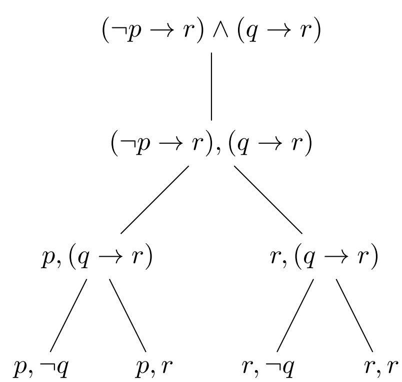



Figure 2: Right-hand side

**b)** To show that $(((p \to (q \wedge r)) \vee ((q \wedge r) \to p)) \to p)$ is falsifiable, we must show that there exist models for the inverse, namely $\neg(((p \to (q \wedge r)) \vee ((q \wedge r) \to p)) \to p)$. The semantic tableaux of the inverse can be found in Figure 3. Two examples of interpretations which yield false are: $\{\{(p \to F), (q \to T), (r \to T)\}, \{(p \to F), (q \to F), (r \to F)\}\}$

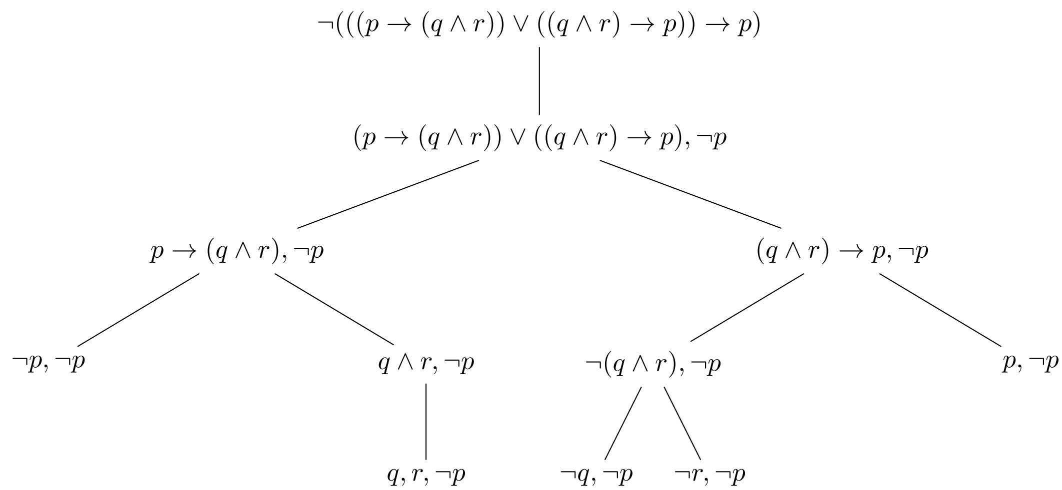



Figure 3: Semantic Tableaux for $\neg(((p \to (q \wedge r)) \vee ((q \wedge r) \to p)) \to p)$

## 2 Logical Equivalence

**a)**
We prove by induction:
**basis** $n = 1$: $p_1 \equiv p_1$
**induction** $n > 1$
Assumption: $\neg(p_1 \wedge ... \wedge p_{n-1}) \equiv \neg p_1 \vee ... \vee \neg p_{n-1}$
To prove: $\neg(p_1 \wedge ... \wedge p_{n-1} \wedge p_n) \equiv \neg p_1 \vee ... \vee \neg p_{n-1} \vee \neg p_n$
Deduction:
$\neg p1 \vee ... \vee \neg p_{n-1} \vee \neg p_n$
$\equiv (\neg p1 \vee ... \vee \neg p_{n-1}) \vee \neg p_n$
$\equiv \neg(p_1 \wedge ... \wedge p_{n-1}) \vee \neg p_n$ (substitute assumption)
$\equiv \neg((p_1 \wedge ... \wedge p_{n-1}) \wedge p_n)$ (De Morgen's law)
$\equiv \neg(p_1 \wedge ... \wedge p_{n-1} \wedge p_n)$

**Conclusion**: $\neg(p_n \wedge ... \wedge p_0) \equiv \neg p_n \vee ... \vee \neg p_1$
**b)**
We again prove by induction.
**Basis** $n = 1$: $p_1 \to p_0 \equiv p_1 \to p_1$
**Induction** $n > 1$:
Assumption: $p_{n-1} \to (... \to (p_1 \to p_0)...) \equiv (p_{n-1} \wedge ... \wedge p_1) \to p_0$
To prove: $p_n \to (p_{n-1} \to (... \to (p_1 \to p_0)...)) \equiv (p_n \wedge p_{n-1} \wedge ... \wedge p_1) \to p_0$
Deduction:
$p_n \to (p_{n-1} \to (... \to (p_1 \to p_0)...))$
$p_n \to ((p_{n-1} \wedge ... \wedge p_1) \to p_0)$ (substitute assumption)
$\neg p_n \vee ((p_{n-1} \wedge ... \wedge p_1) \to p_0)$

$\neg p_n \vee \neg(p_{n-1} \wedge ... \wedge p_1) \vee p_0$
$\neg(p_n \wedge (p_{n-1} \wedge ... \wedge p_1) \vee p_0$ (De Morgan's law)
$\neg(p_n \wedge p_{n-1} \wedge ... \wedge p_1) \vee p_0$
$(p_n \wedge p_{n-1} \wedge ... \wedge p_1) \to p_0$

**Conclusion**: $p_n \to (p_{n-1} \to (... \to (p_1 \to p_0)...)) \equiv (p_n \wedge p_{n-1} \wedge ... \wedge p_1) \to p_0$

# 3 Gentzen

The Gentzen rules for $L_\leftrightarrow$ and $R_\leftrightarrow$ are:

$$\frac{\Gamma, \phi \to \psi, \psi \to \phi \vdash \Delta}{\Gamma, \phi \leftrightarrow \psi \vdash \Delta}\ (L_\leftrightarrow)$$

$$\frac{\Gamma \vdash \Delta, \phi \to \psi \quad \Gamma \vdash \Delta, \psi \to \phi}{\Gamma \vdash \Delta, \phi \leftrightarrow \psi}\ (R_\leftrightarrow)$$

They are derived by substituting $\phi \leftrightarrow \psi$ with $\phi \to \psi \wedge \psi \to \phi$ and subsequently applying the $L_\wedge$ and $R_\wedge$ Gentzen rules to obtain the $L_\leftrightarrow$ and $R_\leftrightarrow$ rules respectively. The proof for Exercise 3 is presented in Figure 4.

# 4 Hilbert

# 5 Clausal From

We will first need to convert the formulas to CNF:

- $\phi_1 = (s \to t) \to (q \to (p \vee r)) = \neg(\neg s \vee t) \vee (\neg q \vee (p \vee r))$
  $= (\neg\neg s \wedge \neg t) \vee (\neg q \vee (p \vee r)) = (s \wedge \neg t) \vee (\neg q \vee (p \vee r))$
  $= (s \vee (\neg q \vee (p \vee r))) \wedge (\neg t \vee (\neg q \vee (p \vee r)))$
- $\phi_2 = (\neg q \to \neg t) \wedge \neg(p \wedge q) = (\neg\neg q \vee \neg t) \wedge \neg(p \wedge q) = (\neg\neg q \vee \neg t) \wedge (\neg p \vee \neg q)$
  $= (q \vee \neg t) \wedge (\neg p \vee \neg q)$
- $\phi_3 = \neg s \vee \neg q$
- $\phi_4 = q \vee t$
- $\phi_5 = (p \to t) \wedge (\neg r \vee p) = (\neg p \vee t) \wedge (\neg r \vee p)$

If we convert these formulas to clausal form we get:

- $\phi_1 = \{p\bar{q}rs, p\bar{q}r\bar{t}\}$
- $\phi_2 = \{q\bar{t}, \bar{p}\bar{q}\}$
- $\phi_3 = \{\bar{q}\bar{s}\}$
- $\phi_4 = \{qt\}$
- $\phi_5 = \{\bar{p}t, p\bar{r}\}$

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{
        r, (p \to q), (q \to p), (q \to r) \vdash p, r
        \qquad
        \dfrac{
          r, q, (p \to q), (q \to r) \vdash p, q
          \qquad
          r, q, p, (p \to q), (q \to r) \vdash p
        }{
          r, q, (p \to q), (q \to p), (q \to r) \vdash p
        }\; L_{\to}
      }{
        r, (p \to q), (q \to p), (q \to r), (r \to q) \vdash p
      }\; L_{\to}
    }{
      (p \to q), (q \to p), (q \to r), (r \to q) \vdash (r \to p)
    }\; R_{\to}
    \qquad
    \dfrac{
      \dfrac{
        p, (q \to p), (q \to r), (r \to q) \vdash r, p
        \qquad
        \dfrac{
          p, q, (q \to p), (r \to q) \vdash r, q
          \qquad
          p, q, r, (q \to p), (r \to q) \vdash r
        }{
          p, q, (q \to p), (q \to r), (r \to q) \vdash r
        }\; L_{\to}
      }{
        p, (p \to q), (q \to p), (q \to r), (r \to q) \vdash r
      }\; L_{\to}
    }{
      (p \to q), (q \to p), (q \to r), (r \to q) \vdash (p \to r)
    }\; R_{\to}
  }{
    \dfrac{
      \dfrac{
        \dfrac{
          (p \to q), (q \to p), (q \to r), (r \to q) \vdash (r \leftrightarrow p)
        }{
          (p \to q), (q \to p), (q \leftrightarrow r) \vdash (r \leftrightarrow p)
        }\; L_{\leftrightarrow}
      }{
        (p \leftrightarrow q), (q \leftrightarrow r) \vdash (r \leftrightarrow p)
      }\; L_{\leftrightarrow}
    }{
      ((p \leftrightarrow q) \land (q \leftrightarrow r)) \vdash (r \leftrightarrow p)
    }\; L_{\land}
  }\; R_{\leftrightarrow}
}{
  \vdash ((p \leftrightarrow q) \land (q \leftrightarrow r)) \to (r \leftrightarrow p)
}\; R_{\to}
$$

Figure 4: Proof for $\vdash ((p \leftrightarrow q) \land (q \leftrightarrow r)) \to (r \leftrightarrow p)$ in $G$

- $\{\phi_1, \phi_2, \phi_3, \phi_4, \phi_5\} = \{p\bar{q}rs, p\bar{q}r\bar{t}, q\bar{t}, \bar{p}\bar{q}, \bar{q}\bar{s}, qt, \bar{p}t, p\bar{r}\ \}$

The resolution refutation is represented as a tree in Figure 5

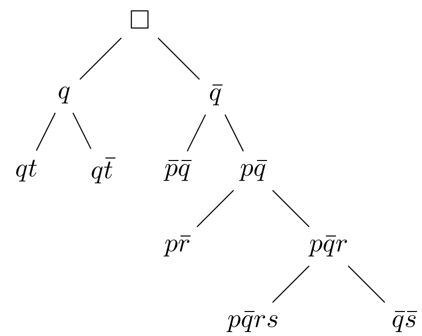
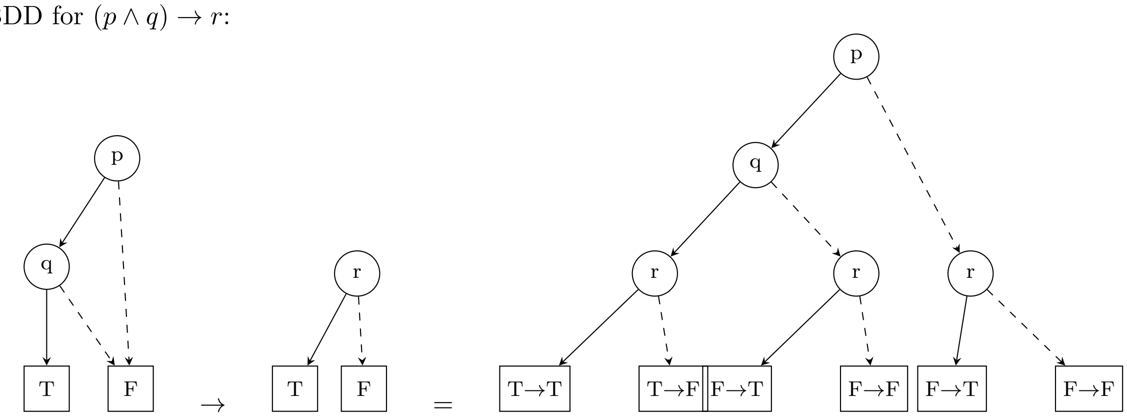


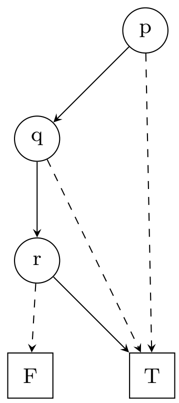

Figure 5: Resolution refutation tree for $\{\phi_1, \phi_2, \phi_3, \phi_4, \phi_5\}$

# 6 BDDs

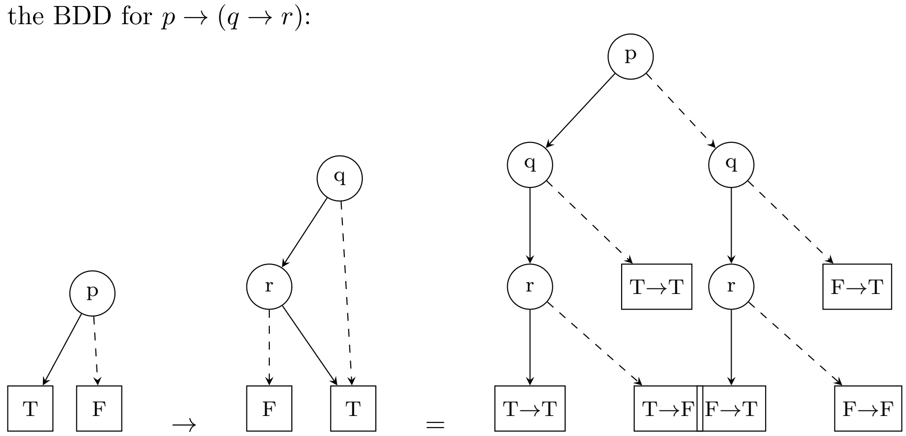
When we use the Apply algorithm on the sub-BDD for subformula $p \wedge q$, combined with the sub-BDD for the formula $r$ using the $\rightarrow$ operator, we get the BDD for $(p \wedge q) \rightarrow r$:



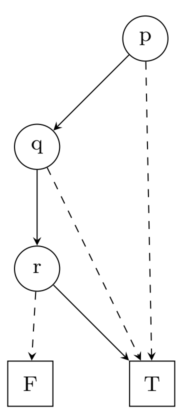

When we apply the Reduce algorithm on this result we get:



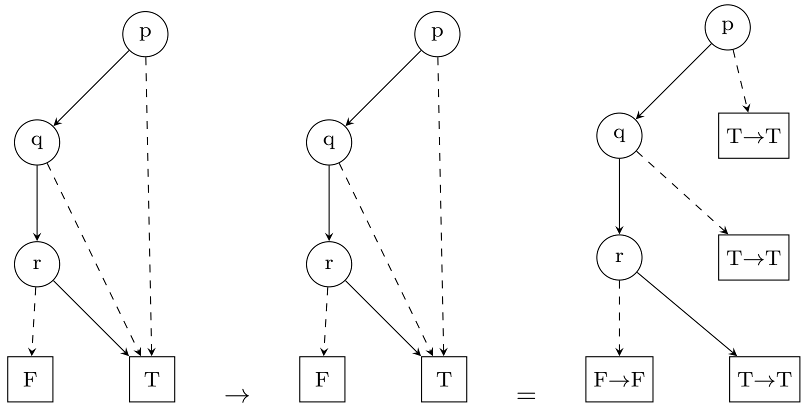

We will now use the Apply algorithm on the sub-BDD for subformula $p$, combined with the sub-BDD for the formula $q \to r$ using the $\to$ operator, to get the BDD for $p \to (q \to r)$:




When we apply the Reduce algorithm on this result we (again) get:



If we now use the Apply algorithm using the previously obtained sub-BDDs and the $\rightarrow$ operator, we get the BDD for $(p \rightarrow (q \rightarrow r)) \rightarrow ((p \wedge q) \rightarrow r)$:




When we apply the Reduce algorithm on this result we get $\boxed{\mathrm{T}}$ which means that $\models (p \rightarrow (q \rightarrow r)) \rightarrow ((p \wedge q) \rightarrow r)$ holds, since it yields *True* for all values of $p$, $q$ and $r$.

## 7 ...

## 8 The Lady, or the Tiger?

Let's start with defining the necessary propositions:

$L_1$: there is a lady in room 1
$L_2$: there is a lady in room 2
$T_1$: there is a tiger in room 1
$T_2$: there is a tiger in room 2

We now that the following must hold due to the facts on the signs:

$T_1 \oplus L_2$ (*Either a tiger is in this room, or a lady is in the other room*)
$\wedge L_1$ (*A lady is in the other room*)

We also now that a room will open up to a lady *or* a tiger, not both, which means the following must also hold:

$(T_1 \oplus L_1) \wedge (T_2 \oplus L_2)$

Since we know $L_1$ is true, we can conclude that $T_1$ must be false, since there are not both a tiger and a lady in room 1. From this we can derive that $L_2$ must be true, since $T_1 \oplus L_2$ must hold. Lastly we can conclude that $T_2$ must also be false, since there are not both a tiger and a lady in room 2. This gives us the (only) model $\{(L_1 \to T), (T_1 \to F), (L_2 \to T), (T_2 \to F)\}$ for the formula $(T_1 \oplus L_1) \wedge (T_2 \oplus L_2) \wedge (T_1 \oplus L_2) \wedge L_1$.

The solution to the puzzle is thus that there is a lady in both room 1 and room 2.