CSC 225: Spring 2018: Lab 3
Proof Techniques

January 24, 2018

1 Proof by Counterexample

Usually we use this technique to disprove some statement. It should be noted that, it is not sufficient to show the counterexample but it must be showed why it is a counterexample.

1. Disprove: All prime numbers are odd.

   Solution: The only counterexample is 2, which is an even number and a prime number.

2. Disprove: Two circles always cross each other at exactly two points.

   Solution: Take the circles $x^2 + y^2 = 1$ and $x^2 + (y-2)^2 = 1$. To find the intersection points, we solve the two equations as follows. From the first circle equation, we get $x^2 = 1 - y^2$. We substitute x^2 with $1 - y^2$ in the second circle equation and get $1 - y^2 + y^2 - 4y + 4 = 1$ or $4y = 4$ or $y = 1$. Substituting the value of y, we get $x^2 = 1 - 1 = 0$ or $x = 0$. Therefore, the two circles intersect only at the point $(0, 1)$, a counterexample. SO the claim is not true.

2 Proof by Contradiction

Suppose that we have to prove "if A then B". We assume for a contradiction that the proposition is false. Then we arrive at a contradiction, which shows that our initial assumption was not correct and the proposition is actually true.

Let's try to solve the following propositions by contradiction.

1. For all integers n, if $n^3 + 5$ is odd then n is even. `Hint: Assume both are odd.`

2. For any integer a, If $a^2 - 2a + 7$ is even, then a is odd. `Hint: Assume both are even.`

3. There is infinitely many prime numbers.

   Solution: Suppose for a contradiction that the number of prime numbers is finite and the primes are $p_1, p_2, \ldots, p_i$, where i is a finite integer. Now we construct a number $L = p_1 \times p_2 \times \ldots \times p_i + 1$. This is not divisible by any prime number from p_1 to p_i, because there is always the remainder 1 when we divide L by any of the primes. Therefore, L is a prime number and is bigger than the biggest prime according to our assumption, a contradiction. Therefore, the number of prime numbers is infinite.

4. The number $\sqrt{2}$ is irrational. `Hint: Assume` $\sqrt{2} = \frac{a}{b}$.

5. **Ramsey's Theorem.** Prove that out of a party of 6 people, there exists a group of 3 mutual friends or a group of 3 mutual non-friends.

   Solution: Suppose for a contradiction that there are odd number of odd degree vertices. We take the sum of the degrees of each vertex. Since each edge is counted twice, the sum $\sum deg(v)$ should be a even number. Now we subtract all the eve degrees, then the number should still be even. Therefore, $\sum deg(v)$, where $deg(v)$ is odd, is an even number. Since an odd number of odd integers sum up to an odd integer, the number of odd degree vertices cannot be odd, a contradiction. $\square$

3 Proof by Contrapositive

If we have to prove "if A then B", we instead prove "if $\neg B$ then $\neg A$". Try to prove the following propositions by proving the contrapositive.

1. If n is a positive integer such that $n \mod (4)$ is 2 or 3, then n is not a perfect square.

   Solution: Suppose $A =$"$n \mod (4)$ is 2 or 3" and $B =$"n is not a perfect square". Then the contrapositive of the given statement would be "If n is a perfect square then $n \mod (4)$ is 0 or 1", since the only outcomes of n mod (4) are $0, 1, 2$ and 3.

   If n is a perfect square then $n = (2k)(2k)$ or $n = (2k+1)(2k+1)$ for some $k \geq 0$. In the first case $n = 4k^2$ which is divisible by 4, so $n \mod (4) = 0$. In the second case $n = 4k^2+4k+1 = 4(k^2+k)+1$, and $n \mod (4) = 1$. $\square$

2. If x and y are two integers whose product is odd, then both must be odd.

3. If x and y are two integers for which $x + y$ is even, then x and y have the same parity (odd or even).

4. For any integers a and b, $a + b \geq 15$ implies that $a \geq 8$ or $b \geq 8$.

4 Proof by Induction

See the attached pdf.

References

1. http://cgm.cs.mcgill.ca/ godfried/teaching/dm-reading-assignments/Contradiction-Proofs.pdf

2. http://www.math-cs.gordon.edu/courses/mat231/notes/proof-contradiction.pdf

3. http://zimmer.csufresno.edu/ larryc/proofs/proofs.contrapositive.html

4. https://math.dartmouth.edu/ m22x17/misc/LaLonde2012_proof_by_contrapositive.pdf

5. https://brilliant.org/wiki/contradiction/

6. http://math.mit.edu/ fox/MAT307-lecture05.pdf