Primitive Operations

- Assignments (A)
- Comparisons (C)
- Boolean expressions (B)
- Array indexing (I)
- Record selector or object reference (R)
- Arithmetic operations
  - Add, subtract (S)
  - Multiplication, division (D)
- Trigonometric operations (e.g., sin, cos, tan) (T)
- Calling a method, function, procedure, routine (M)

Worst Case Running Time T(n)
Counting Assignments, Comparisons & Indexing

```
currentMax ← A[0]
for k ← 1 to n-1 do
    if currentMax < A[k] then
        currentMax ← A[k]
    end
end
return currentMax
```

- How has the input to be arranged to produce the best case and the worst case?

Worst case

- 1 A + 1 I +
- 1 A + (n-1)*{
  - 1 A + 1 S +
  - 1 C +
  - 1 C + 1 I +
  - 1 A + 1 I +
- }
- 1 C (to terminate loop) +
- 1 A

$T(n) = 5+(n-1)*7$

$T(n) = 7n - 2$

Worst Case Running Time T(n)
Counting Assignments & Comparisons

```
currentMax ← A[0]
for k ← 1 to n-1 do
    if currentMax < A[k] then
        currentMax ← A[k]
    end
end
return currentMax
```

- How has the input to be arranged to produce the best case and the worst case?

Worst case

- 1 A +
- 1 A + (n-1)*{
  - 1 A +
  - 1 C +
  - 1 C +
  - 1 A +
- }
- 1 C (to terminate loop) +
- 1 A

$T(n) = 3+(n-1)*4$

$T(n) = 4n - 1$

Design & Analysis of Algorithms Go Hand in Hand

For almost all computational problems *many* candidate solutions are possible, but most of them we don't want! Sometimes we even want only one.

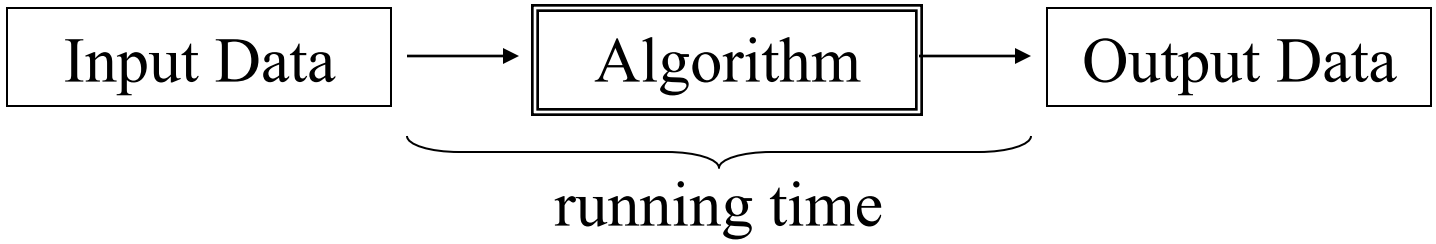
Examples

1. **Sorting** Any sequence of *n* given numbers is a possible candidate solution for a sorted sequence of these numbers.
2. **15-puzzle** Any sequence of moves is a possible candidate solution to solve the puzzle.

Tools for Algorithm Analysis

- Language for describing algorithms
  - *pseudo-code*
- Metric for measuring algorithm running time
  - *primitive operations*
- Most common approach for characterizing running times
  - *Worst-case analysis*

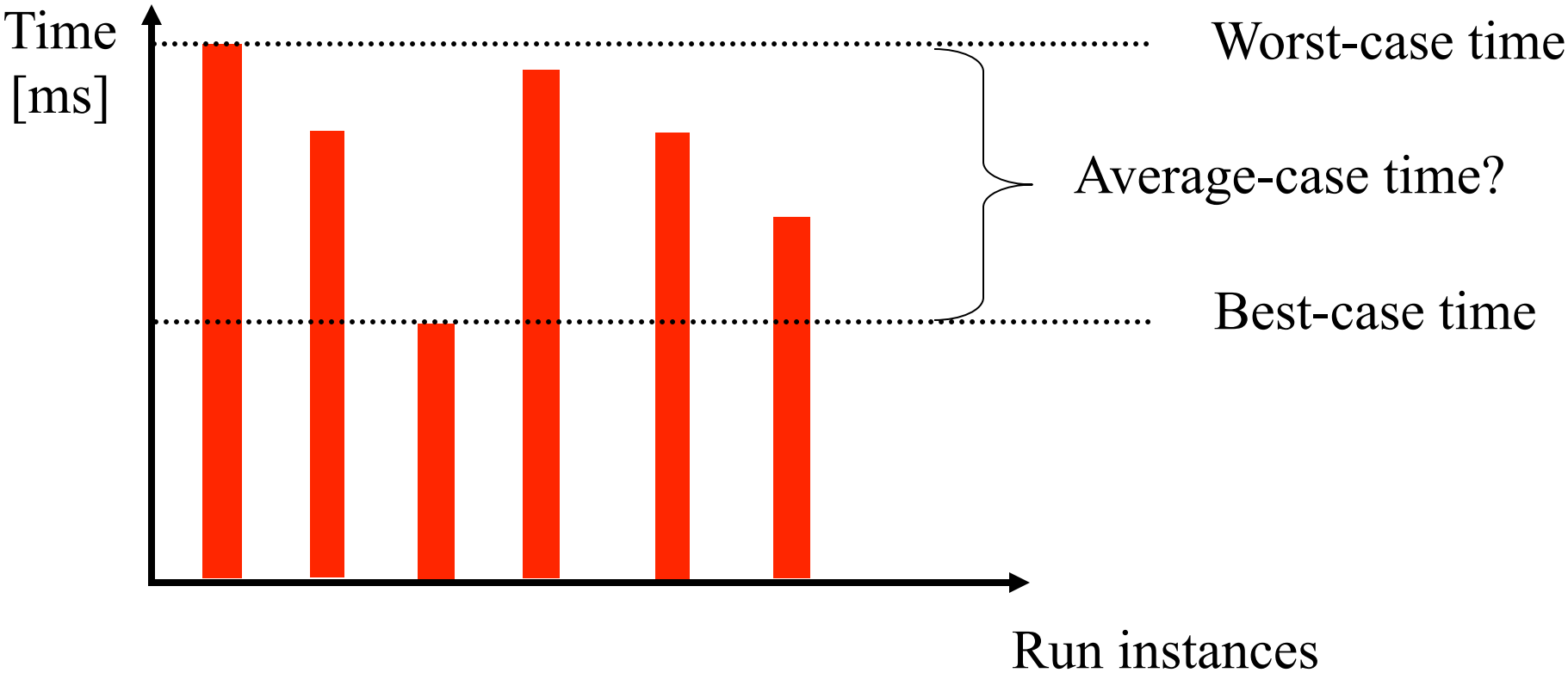
Algorithm analysis



Categories of algorithm running times

- Best case analysis $T_b(n)$
- Average-case analysis $T_a(n)$
- Worst-case analysis $T(n)$

Best-case, Average-case and Worst-case Analysis



Determining the Average-case Time

- Calculate expected running times based on a given input distribution
- Typically the analysis requires heavy math and probability theory

Determining the Best-case and Worst-case Running Time

- Worst-case T(n)
  - What is the maximum number of primitive operations (depending on *n*) executed by the algorithm, taken over all inputs of size *n*?
  - Worst-case analysis is most common and may aid in the design of the algorithm (e.g., Linear Median algorithm)
- Best-case $T_b(n)$
  - What is the minimum number of primitive operations (depending on *n*) executed by the algorithm, given the most advantageous or best input configuration of size *n*?

Examples: Worst-case, Best-case and Average-case Analysis

Basic units: A & C

```
a = 3 * n
cnt = 1
while a > n do
   a = a - 1
   cnt = cnt + 1
end
```

$T(n) = 2 + 2n(3)+1$

$T(n) = 3 + 6n$

$T_b(n) = T(n)$

Basic units: A & C

```
A: array[0..n-1]
k = 0
while k < n do
   if A[k] = key then
      return k
   end
   k = k + 1
end
return "not found"
```

$T(n) = 1 + n(3) + 2$

$T(n) = 3 + 3n$

$T_b(n) = 4$

$T_a(n) = 3 + 3n/2$

Examples: Worst-case, Best-case and Average-case Analysis

Basic units: A & C

```
A: array[0..n-1]
Swap: 3 A per swap
For: 2 A per iteration;
   ignore initial A in loops
for k=0 to n-1 do
   for j=0 to n-1 do
      if A[k]<=A[j] then
         swap(A[k],A[j])
      end
   end
end
```

$T(n) = (6n+2)n$

$T(n) = 6n^2 + 2n$

Basic units: A & C

```
For: 2 A per iteration;
   ignore initial A;
   ignore return A
s = 0
for k=1 to n do
   s = s + k*k
end
return s
```

$T(n) = 1 + 3n$

Structure of a Recursive Algorithm

```
Algorithm recursiveAlgorithm(n)
   if n = 1 then
      base-case
   else
      induction-step
      recursiveAlgorithm(n-1)
   end
```

- Let the worst case running time of `recursiveAlgorithm` be T(n)

- Then $$T(n) = \begin{cases} c_1 & \text{if } n = 1 \\ T(n-1) + c_2 & \text{otherwise} \end{cases}$$

Structure of a Recursive Algorithm

```
Algorithm recursiveMax(A,n):
  Input: An array A storing n ≥ 1  integers.
  Output: The maximum element in A.

if n = 1 then return A[0] end
return max(recursiveMax(A,n-1),A[n-1])
```

- **Base case:** 3 operations (n=1, A[0], return)
- **Induction step:** T(n-1)+7 ops (n=1, A, n-1, n-1,A[n-1], call, max)

Solving Recurrence Equation by Repeated Substitution

$$T(n) = T(n-1) + 7$$
$$T(n-1) = T(n-2) + 7$$
$$T(n-2) = T(n-3) + 7$$
$$\ldots$$
$$T(2) = T(1) + 7$$
$$T(1) = 3$$

$$T(1) = 3$$
$$T(2) = 3 + 7 = 10$$
$$T(3) = 3 + 7 + 7 = 17$$
$$T(4) = 3 + 7 + 7 + 7 = 24$$
$$\ldots$$
$$T(n) = 3 + 7(n-1)$$
$$T(n) = 3 + 7(n-1)$$
$$T(n) = 7n - 4$$
$$T(n) \in O(n)$$

$$T(n) = c_1 + c_2(n-1)$$
$$T(n) \in O(n)$$

Asymptotic Notation

- Evaluating running time in detail as for `arrayMax` and `recursiveMax` is cumbersome
- Fortunately, there are asymptotic notations which allow us to characterize the main factors affecting an algorithm's running time without going into detail
- A good notation for large inputs

- **Big-Oh O(.)**
- Big-Omega Ω(.)
- Big-Theta Θ(.)
- Little-Oh o(.)
- Little-Omega ω(.)

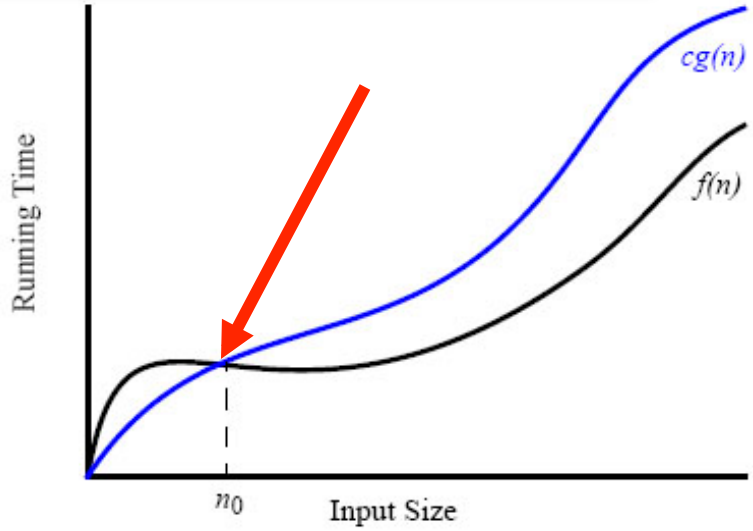
Big-Oh Notation

- Formal definition
- Most frequently used
- Is a good measure for *large* inputs

Formal Definition of Big-Oh Notation

Let f: IN→R and g: IN→R. f(n) is O(g(n)) if and only if
there exists a real constant c > 0
and an integer constant $n_0 > 0$
such that $f(n) \leq c \cdot g(n)$ for all $n \geq n_0$.
IN: non-negative integers
IR: real numbers

- We say
  - f(n) is *order* g(n)
  - f(n) is *big-Oh* of g(n)
- Visually, this says that the f(n) curve must eventually fit under the cg(n) curve.



Big-Oh: Examples

- $f(n) = 4n + 20n^4 + 117$
  $O(f(n))$ is ?
- $f(n) = 1083$
  $O(f(n))$ is ?
- $f(n) = 3\log n$
  $O(f(n))$ is ?
- $f(n) = 3\log n + \log\log n$
  $O(f(n))$ is ?
- $f(n) = 2^{17}$
  $O(f(n))$ is ?
- $f(n) = 33/n$
  $O(f(n))$ is ?
- $f(n) = 2^{\log_2 n}$
  $O(f(n))$ is ?
- $f(n) = 1^n$
  $O(f(n))$ is ?

Big-Oh: Examples

- $f(n) = 4n + 20n^4 + 117$
  O(f(*n*)) is **O(n^4)**
  P: $4n + 20n^4 + 117 \leq 90n^4$
- $f(n) = 1083$
  O(f(*n*)) is **O(1)**
- $f(n) = 3 \log n$
  O(f(*n*)) is **O(log n)**
  P: $3 \log n \leq 4 \log n$
- $f(n) = 3\log n + \log \log n$
  O(f(*n*)) is **O(log n)**
  P: $3\log n + \log \log n \leq 4 \log n$
- $f(n) = 2^{17}$
  O(f(*n*)) is **O(1)**
  P: $2^{17} \leq 1\ 2^{17}$
- $f(n) = 33/n$
  O(f(*n*)) is **O(1/*n*)**
  P: $33/n \leq 33(1/n)$ for $n \geq 1$
- $f(n) = 2^{\log_2 n}$
  O(f(*n*)) is **O(*n*)**
  P: $2^{\log_2 n} = 2$ by log def
- $f(n) = 1^n$
  O(f(*n*)) is **O(1)**
  P: $1^n = 1$ by exponential def

Theorem

- **R1:** If d(n) is O(f(n)), then ad(n) is O(f(n), a > 0
- **R2:** If d(n) is O(f(n)) and e(n) is O(g(n)), then d(n)+e(n) is O(f(n)+g(n))
- **R3:** If d(n) is O(f(n)) and e(n) is O(g(n)), then d(n)e(n) is O(f(n)g(n))
- **R4:** If d(n) is O(f(n)) and f(n) is O(g(n)), then d(n) is O(g(n))
- **R5:** If $f(n) = a_0 + a_1 n + \ldots + a_d n^d$, d and a_k are constants, then f(n) $O(n^d)$
- **R6:** n^x is $O(a^n)$ for any fixed x > 0 and a > 1
- **R7:** log n^x is O(log n) for any fixed x > 0
- **R8:** log ^{x}n is $O(n^y)$ for any fixed constants x > 0 and y > 0

Names of Most Common Big Oh Functions

- Constant $O(1)$
- Logarithmic $O(\log n)$
- Linear $O(n)$
- Quadratic $O(n^2)$
- Polynomial $O(n^k)$ k is a constant

- Exponential $O(2^n)$
- Exponential $O(a^n)$ a is a constant and $a > 1$

L'Hôpital's Rule

$$\lim_{n\to b}\frac{f_1(n)}{f_2(n)} = \lim_{n\to b}\frac{f_1'(n)}{f_2'(n)} = \begin{cases} 0, \text{ then } f_2(n) \text{ grows faster} \\ 0 < x < \infty, \text{ then inconclusive} \\ \infty, \text{ then } f_1(n) \text{ grows faster} \end{cases}$$

Example

$f_1(n) = n^2 \quad f_1'(n) = 2n$

$f_2(n) = e^n \quad f_2'(n) = e^n$

$$\lim_{n\to b}\frac{n^2}{e^n} = \lim_{n\to b}\frac{2n}{e^n} = \lim_{n\to b}\frac{2}{e^n} = \lim_{n\to b}\frac{0}{e^n} = 0$$

thus, $f_2(n)$ grows faster

Order the Functions by Growth!

- $n^2 \quad \log n \quad 2^n \quad \log^2 n \quad n \quad \sqrt{n} \quad n \log n \quad n^3$
- Use L'Hôpital's rule when in doubt
- For example, compare $\log n$ and $\sqrt{n}$ using L'Hôpital's rule

Function Growth Comparison Example

$$f_1(n) = \log n \quad f_1'(n) = \frac{1}{n \ln 2} = \frac{n^{-1}}{\ln 2} = c_1 n^{-1} \quad f_1''(n) = -c_1 n^{-2}$$

$$f_2(n) = \sqrt{n} = n^{\frac{1}{2}} \quad f_2'(n) = \frac{n^{-\frac{1}{2}}}{2} = c_2 n^{-\frac{1}{2}} \quad f_2''(n) = \frac{-n^{-\frac{3}{2}}}{4} = c_3 n^{-\frac{3}{2}}$$

$$\lim_{n \to b} \frac{\log n}{\sqrt{n}} = \lim_{n \to b} \frac{c_1 n^{-1}}{c_2 n^{-\frac{1}{2}}} = \lim_{n \to b} \frac{-c_1 n^{-2}}{c_3 n^{-\frac{3}{2}}} = \lim_{n \to b} c_4 n^{-\frac{1}{2}} = \lim_{n \to b} \frac{c_4}{\sqrt{n}} = 0$$

thus, $f_2(n)$ grows faster

Reading

- Read Section 1.4 in textbook

Quiz
Which statement is True?

1. 2^n is $O(n!)$?

2. $n!$ is $O(2^n)$?

Quiz: 2^n is $O(n!)$ is true

Proof. Let $n \geq 4$. Then $n! > 2^n$.

Therefore, for $n_0 = 4$, $c = 1$, and $n \geq n_0$

Prove the claim by induction.

Quiz: 2^n is $O(n!)$ is true

Induction on *n*. We need to show that

$n! > 2^n$ for all $n \geq 4$ **(hypothesis)**

Base case: $n = 4$

$4! > 2^4 \Leftrightarrow 4 \cdot 3 \cdot 2 \cdot 1 > 2 \cdot 2 \cdot 2 \cdot 2$ **ok 3 > 2**

$n \rightarrow n+1$: Show: $(n+1)! > 2^{n+1}$ for all $n \geq 4$

$(n+1)! = (n+1)\, n! > (n+1)2^n > 2 \cdot 2^n = 2^{n+1}$

$\Leftrightarrow$

$(n+1)! > 2^{n+1}$

Hypothesis: $n! > 2^n$

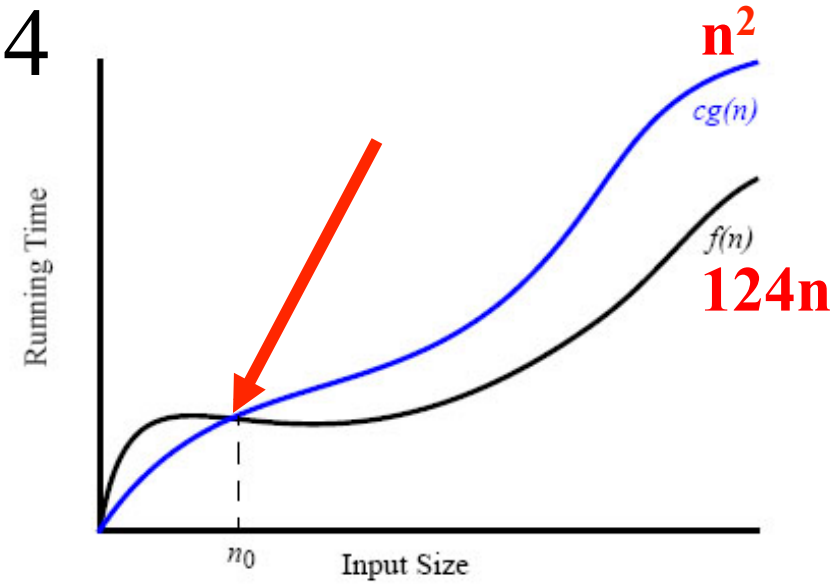
***n*+1 > 4**

Quiz

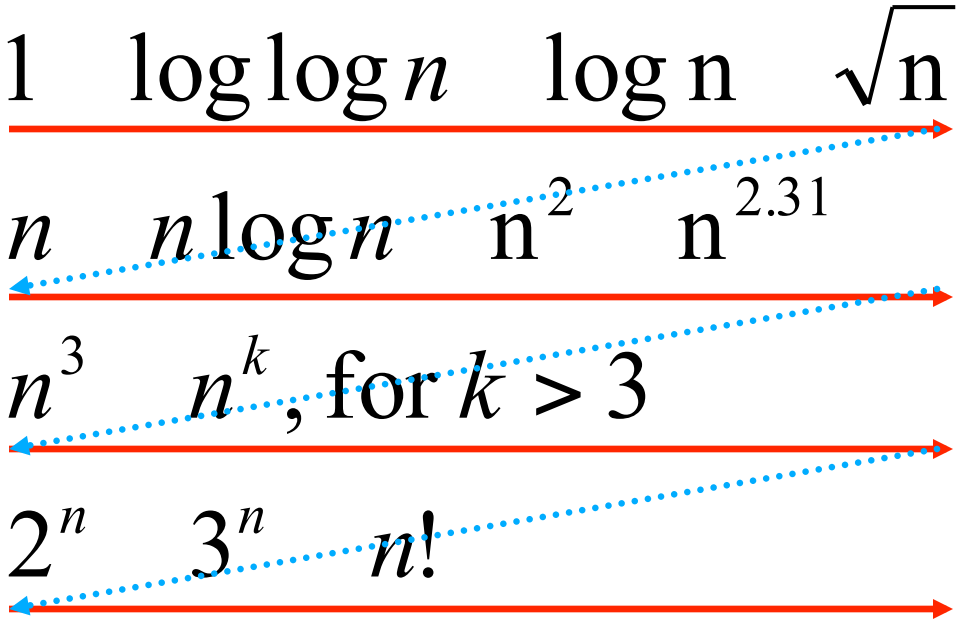
- Are there two functions such that O(n^2) is smaller ("faster") than O(n) for *small* inputs?

Quiz: O(n^2) can be "faster" than O(n) for small inputs

- $124n > n^2$ for $n = 1..123$
- $124n = n^2$ for $n = 124$
- $124n < n^2$ for $n > 124$



Most Common Functions in Algorithm Analysis Ordered by Growth

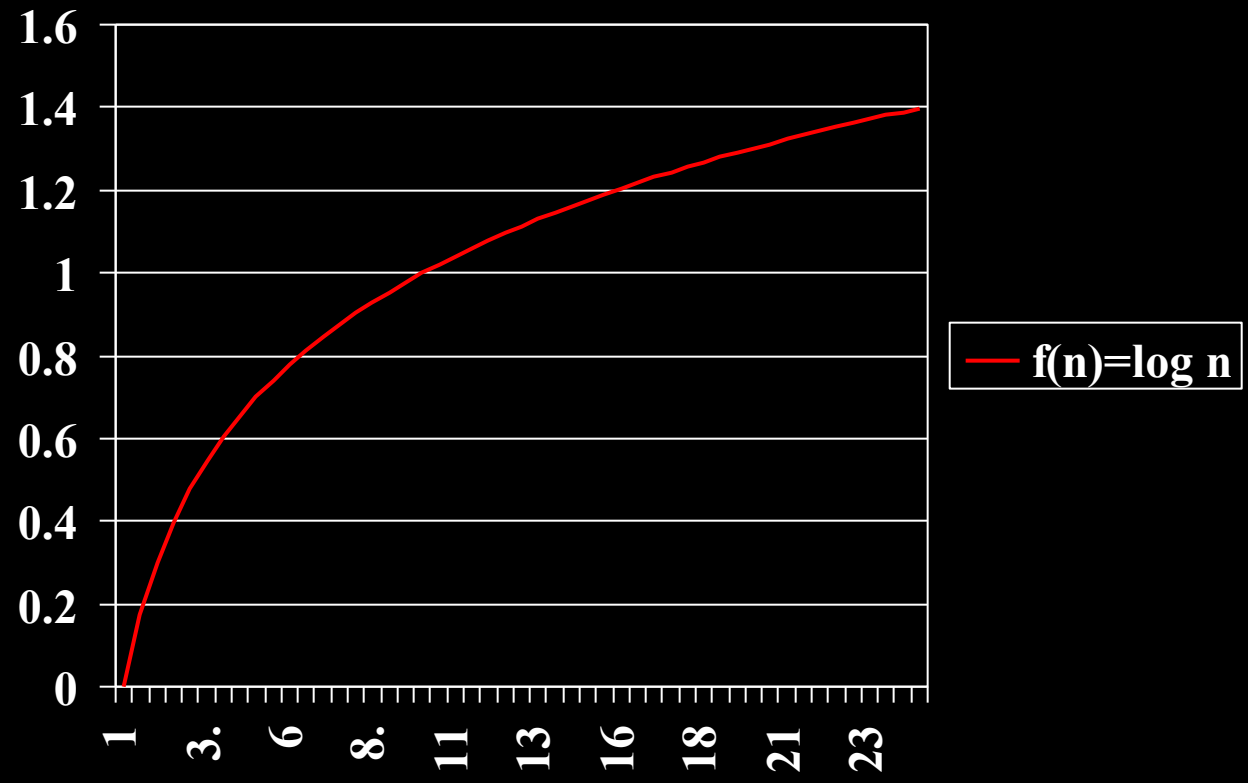


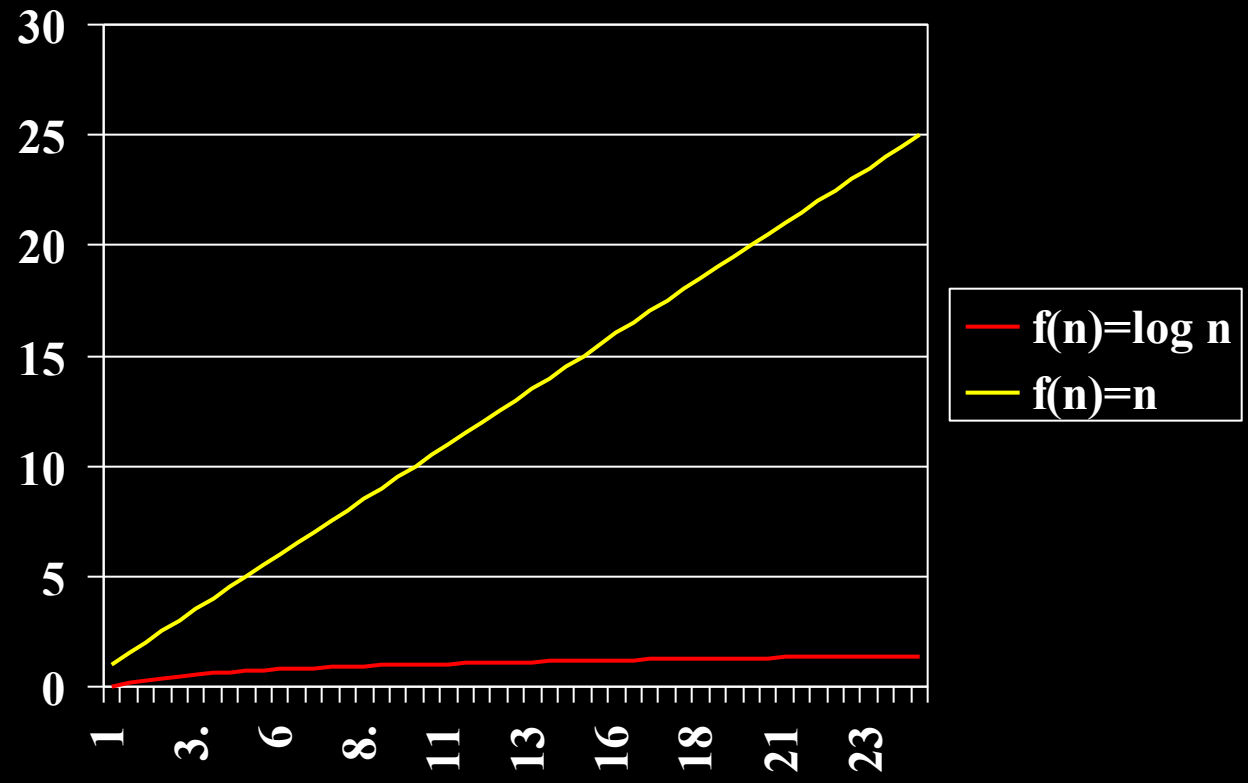
Examples of Most Common Big Oh Functions

- Constant time $O(1)$
- Logarithmic $O(\log n)$
- Linear $O(n)$
- Quadratic $O(n^2)$
- Polynomial $O(n^k)$
- Exponential $O(a^n)$

Examples of Most Common Big Oh Functions

Constant	$O(1)$	Hash search
Logarithmic	$O(\log n)$	Tree search
Linear	$O(n)$	Linear search Linear median
	$O(n \log n)$	Heapsort
Quadratic	$O(n^2)$	Insertionsort
Cubic	$O(n^3)$	Transitive closure
Polynomial	$O(n^k)$	
Exponential	$O(2^n)$	Graph colouring
Exponential	$O(a^n)$	NP-hard problems


1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
1
3.
6
8.
11
13
16
18
21
23
f(n)=log n


30
25
20
15
10
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0
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3.
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8.
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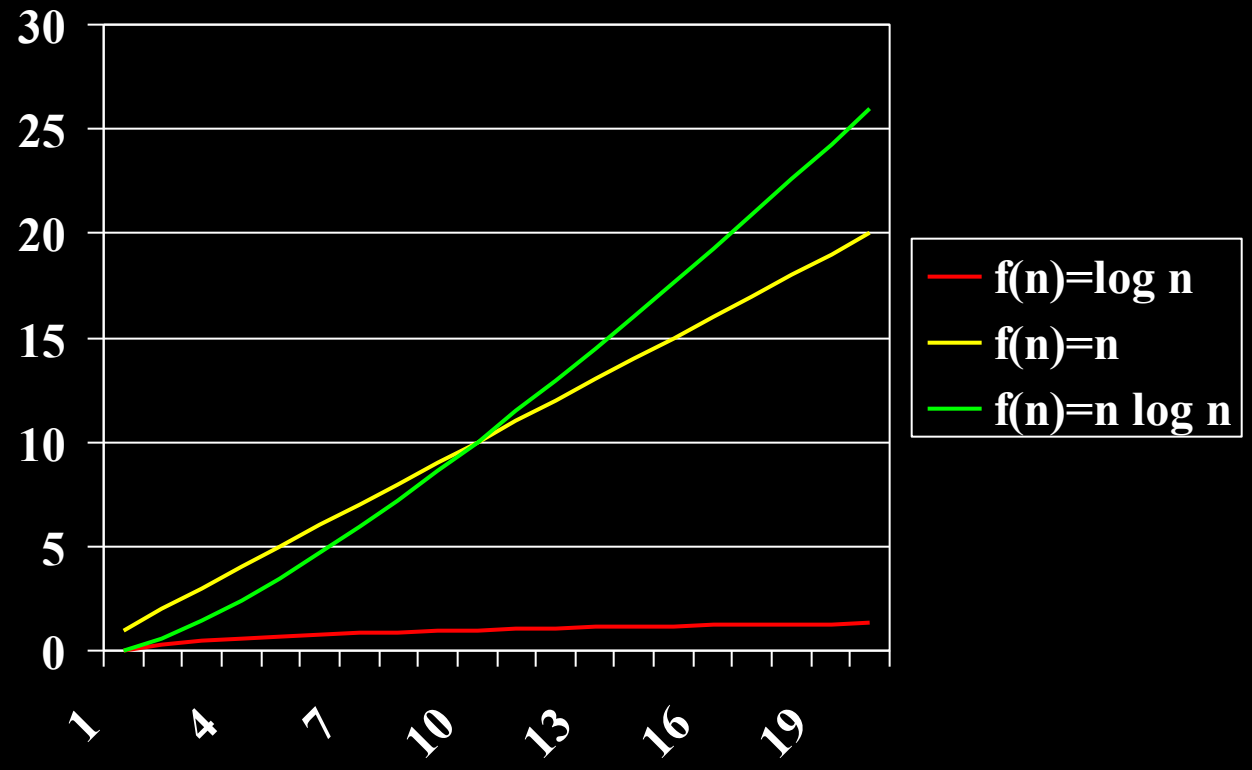
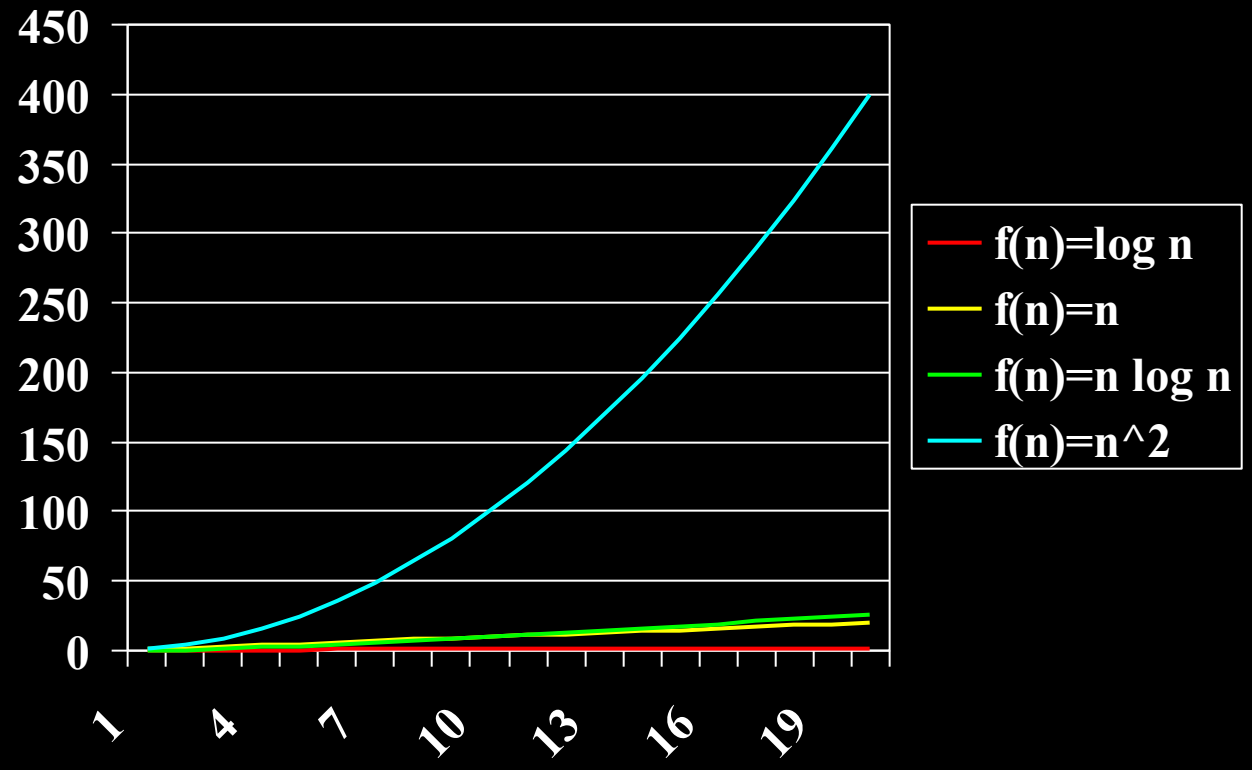
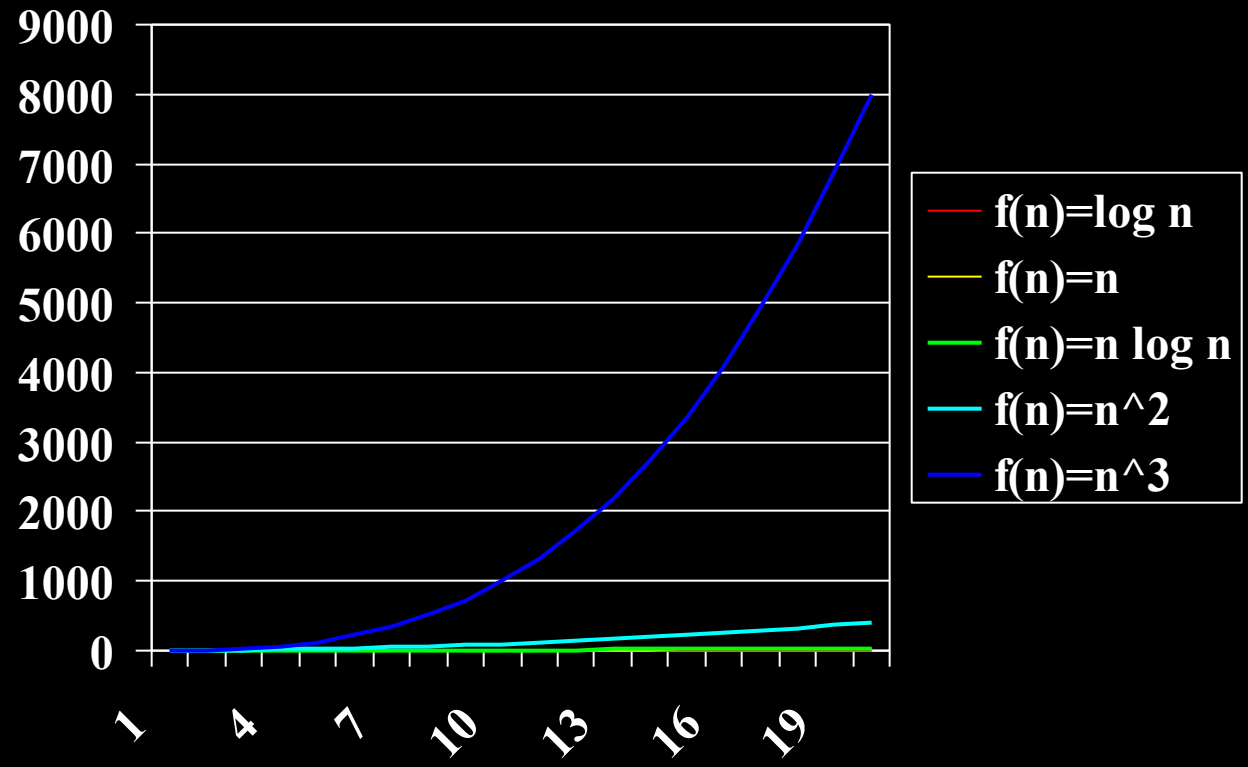
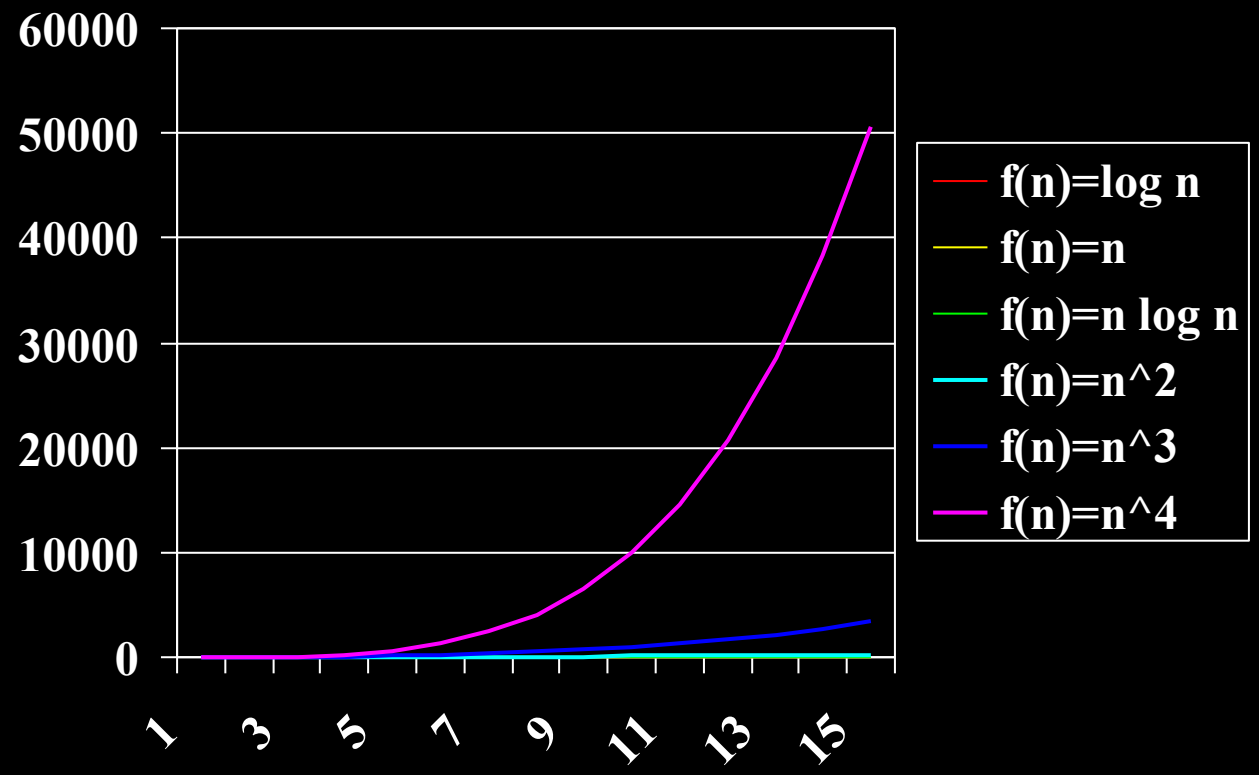
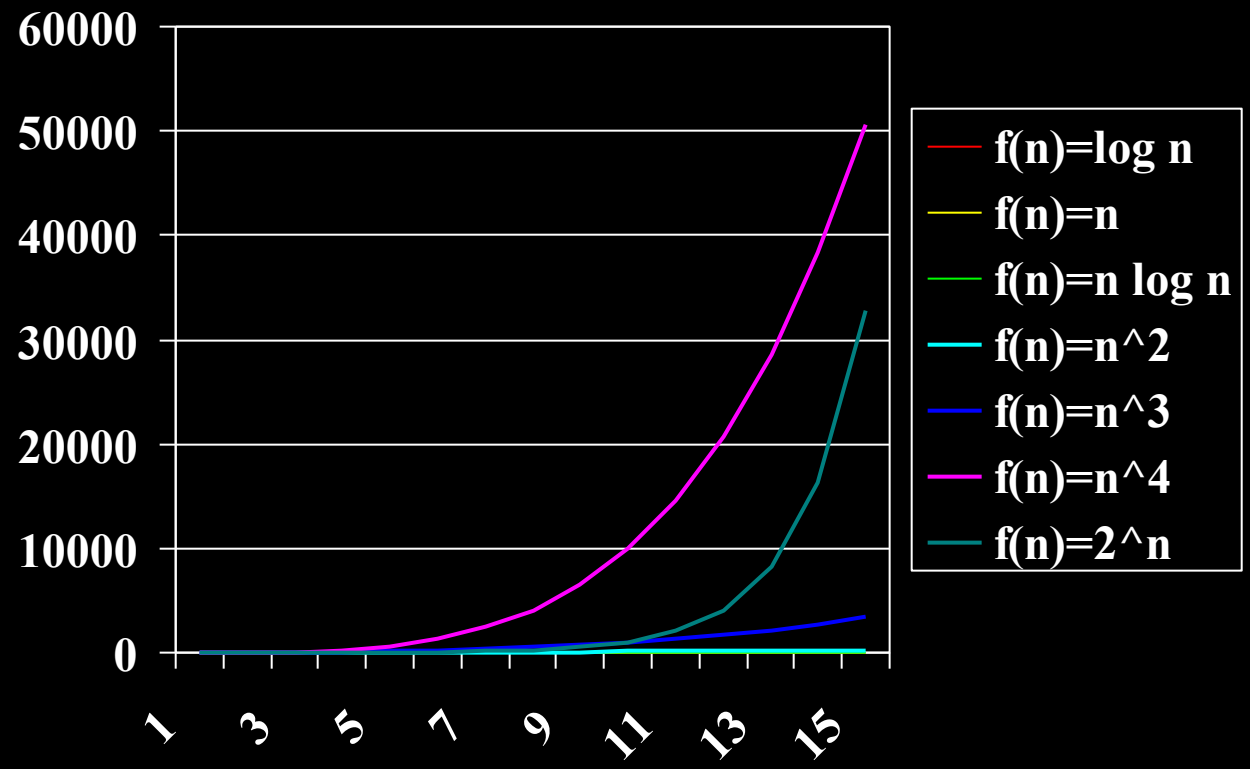
18
21
23
f(n)=log n
f(n)=n

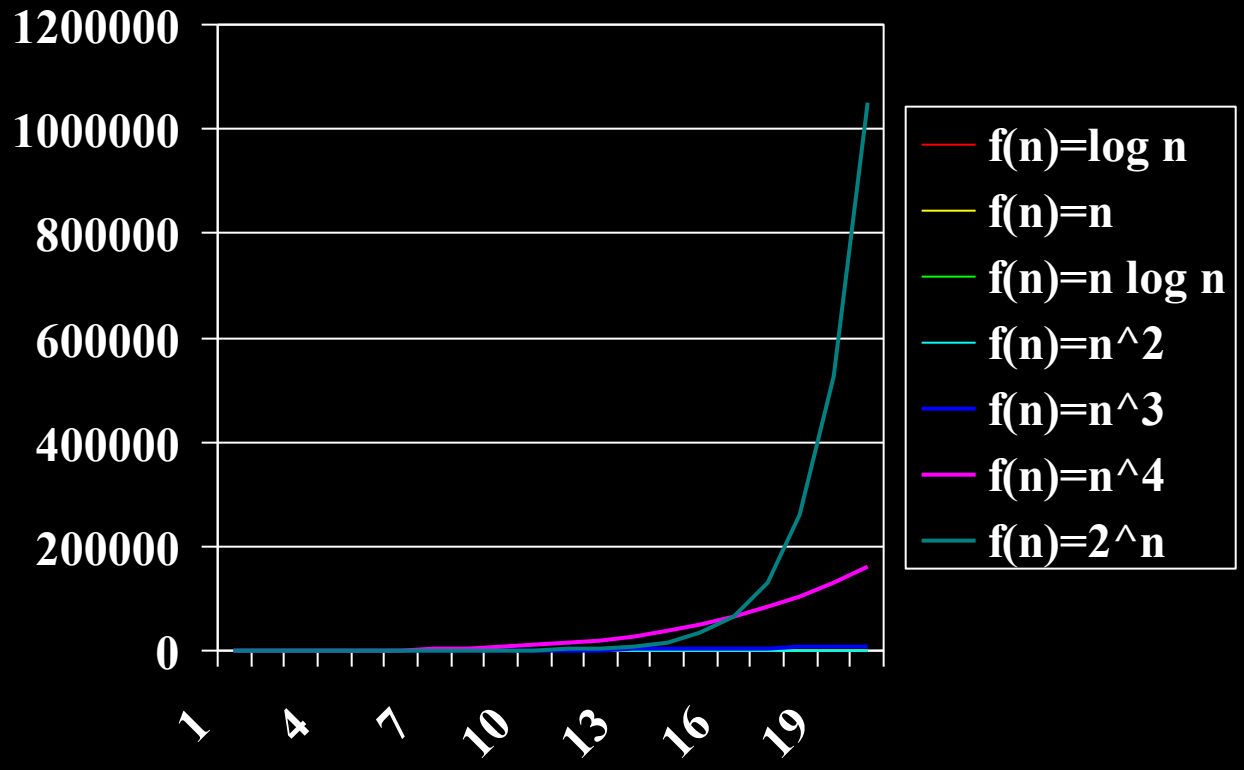

30
25
20
15
10
5
0
1
4
7
10
13
16
19
f(n)=log n
f(n)=n
f(n)=n log n


450
400
350
300
250
200
150
100
50
0
1
4
7
10
13
16
19
f(n)=log n
f(n)=n
f(n)=n log n
f(n)=n^2


9000
8000
7000
6000
5000
4000
3000
2000
1000
0
1
4
7
10
13
16
19
f(n)=log n
f(n)=n
f(n)=n log n
f(n)=n^2
f(n)=n^3


60000
50000
40000
30000
20000
10000
0
1
3
5
7
9
11
13
15
f(n)=log n
f(n)=n
f(n)=n log n
f(n)=n^2
f(n)=n^3
f(n)=n^4


60000
50000
40000
30000
20000
10000
0
1
3
5
7
9
11
13
15
f(n)=log n
f(n)=n
f(n)=n log n
f(n)=n^2
f(n)=n^3
f(n)=n^4
f(n)=2^n


1200000
1000000
800000
600000
400000
200000
0
1
4
7
10
13
16
19
f(n)=log n
f(n)=n
f(n)=n log n
f(n)=n^2
f(n)=n^3
f(n)=n^4
f(n)=2^n

Functions Ordered by Growth and Rate

n	log n	n	n log n	n^2	n^3	2^n	n!
10	3.3	10	33	10^2	10^3	10^3	10^6
10^2	6.6	10^2	$6.6\ 10^2$	10^4	10^6	10^{30}	10^{158}
10^3	10	10^3	$1.0\ 10^3$	10^6	10^9		
10^4	13	10^4	$1.3\ 10^4$	10^8	10^{12}		
10^5	17	10^5	$1.7\ 10^5$	10^{10}	10^{15}		
10^6	20	10^6	$2.0\ 10^6$	10^{12}	10^{18}		

Assume a computer executing 10^{12} operations per second.
To executive 2^{100} operations takes $4\ 10^{10}$ years.
To executive 100! operations takes much longer still.

Functions Ordered by Growth and Rate

- $\log n$
- $\log^2 n$
- $\sqrt{n}$
- n
- $n \log n$
- n^2
- n^3

P = class of polynomial time algorithms

- 2^n

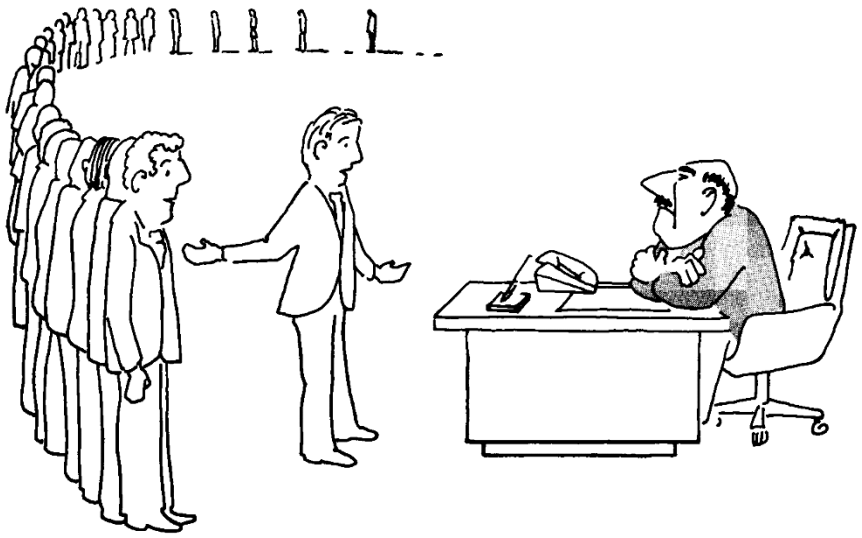
NP = class of *nondeterministic* polynomial time algorithms

A million (US-) dollar question

http://www.claymath.org/millennium/P_vs_NP/

- $P = NP$?
- Obviously $P \subseteq NP$.
- There is a bunch of problems (so called NP-complete problems) for which one assumes that none of those can be solved in polynomial time.
- Examples: Graph coloring, Independent Set, Generalized 15-Puzzle
- Widely assumed: $P \neq NP$

P = NP?



"I can't find an efficient algorithm, but neither can all these famous people."

Garey and Johnson. *Computers and Intractability: A guide to the theory of NP-completeness* Freeman; 1979. ISBN 0-7167-1044-7.

Logarithms and Exponential Functions

- Review properties of Logarithms and exponents

$$\log_b a = c \text{ if } a = b^c$$

$$\log ac = \log a + \log c$$

$$\log a / c = \log a - \log c$$

$$\log a^c = c \log a$$

$$\log_b a = \frac{\log_c a}{\log_c b}$$

$$b^{\log_c a} = a^{\log_c b}$$

$$(b^a)^c = b^{ac}$$

$$b^a b^c = b^{a+c}$$

$$b^a / b^c = b^{a-c}$$

Useful Summations Formulas

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \quad \text{(Gauss)}$$

$$\sum_{k=1}^{n} k^2 = \frac{2n^3 + 3n^2 + n}{6} = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2$$

$$\int_{a-1}^{b} f(x)dx \le \sum_{k=a}^{b} f(k) \le \int_{a}^{b+1} f(x)dx$$

Big-Omega Notation

Let f: IN→IR and g: IN→IR.

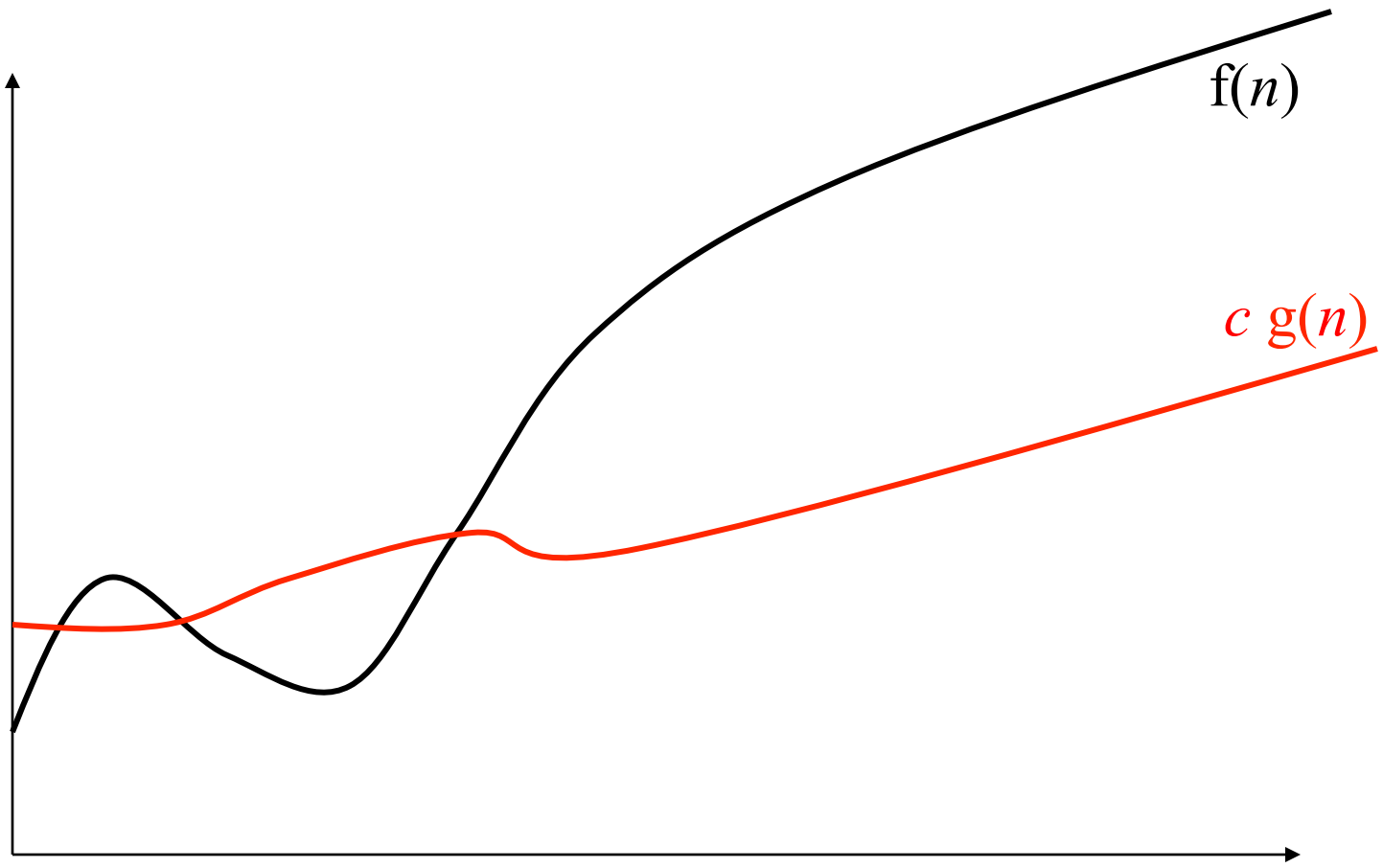
f(n) is $\Omega(g(n))$

if and only if

g(n) is O(f(n))

IN: non-negative integers
IR: real numbers

$f(n) = \Omega(g(n))$



Quiz: What is true, what is false?

f(n) is Ω(g(n))
iff
g(n) is O(f(n))

1. 2^n is $\Omega(n!)$

Previous results
2^n is $O(n!)$ is true
$n!$ is not $O(2^n)$

2. $n!$ is $\Omega(2^n)$

2^n is $\Omega(n!)$ is false

f(n) =2^n
g(n) = *n*!

f(n) is Ω(g(n))
iff
g(n) is O(f(n))

We know 2^n is $O(n!)$ but $n!$ is not $O(2^n)$.

Since 2^n is $\Omega(n!)$ iff $n!$ is $O(2^n)$, the claim is false.

$n!$ is $\Omega(2^n)$ is true

f(n) = *n*!
g(n) = 2^n

f(n) is Ω(g(n))
iff
g(n) is O(f(n))

We know 2^n is $O(n!)$ but $n!$ is not $O(2^n)$.

Since $n!$ is $\Omega(2^n)$ iff 2^n is $O(n!)$, the claim is true.

Big-Theta Notation

Let f: IN→IR and g: IN→IR.

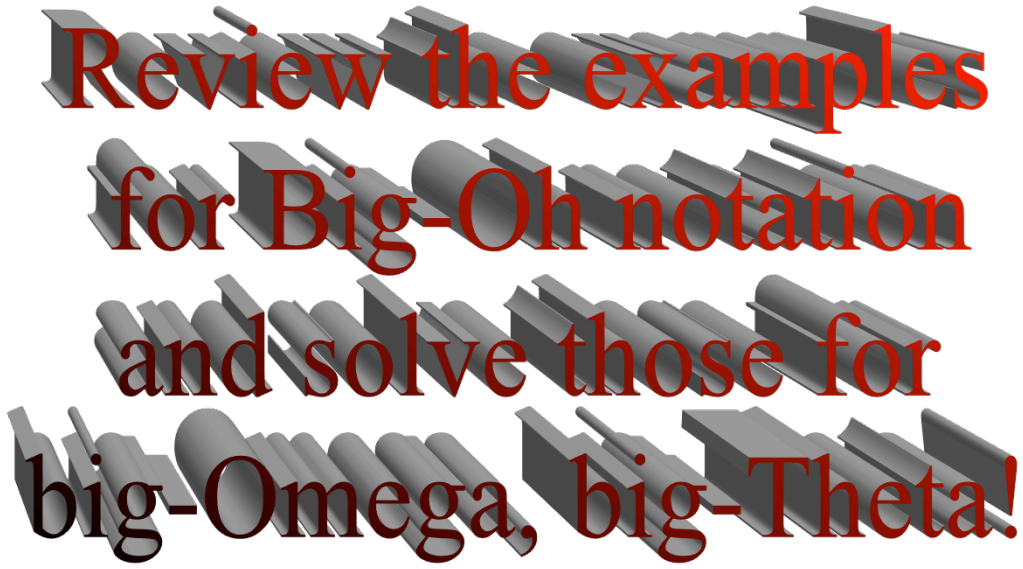
f(n) is $\Theta(g(n))$

if and only if

f(n) is $O(g(n))$ and f(n) is $\Omega(g(n))$.

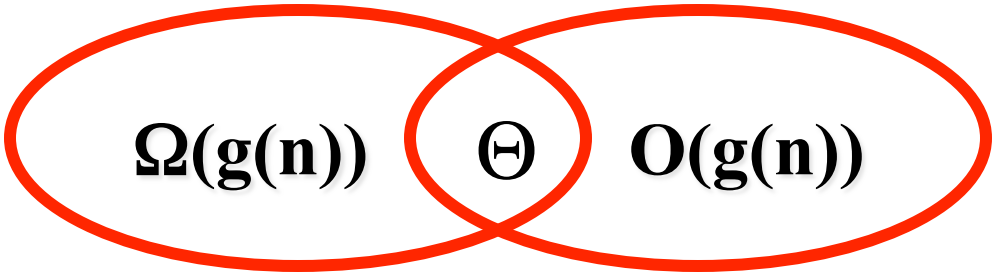
Big-Theta: Examples

- $3n + 1$ is $\Theta(n)$
- $2n^2 + 3n + 1$ is $\Theta(n^2)$



Intuition of Asymptotic Terminology

- Big-Oh: $O(g(n))$ upper bound; functions that grow no faster than g(n)
- Big-Omega: $\Omega(g(n))$ lower bound; functions that grow at least as fast than g(n)
- Big-Theta: $\Theta(g(n))$ asymptotic equivalence; functions that grow at the same rate as g(n)



Asymptotic Notation

- Big-Oh O(.)
- Big-Omega Ω(.)
- Big-Theta Θ(.)
- **Little-Oh o(.)**
- **Little-Omega ω(.)**

Little-Oh Notation

Let f: IN→IR and g: IN→IR.

f(n) is o(g(n))

if and only if

for any constant $c > 0$ there is a constant $n_0 > 0$ such that $f(n) \leq c \cdot g(n)$ for $n \geq n_0$.

Examples: Little-Oh

- $2n$ is $o(n^2)$
- $2n^2$ is ***not*** $o(n^2)$! [but $2n^2$ is $O(n^2)$]

Intuition of Asymptotic terminology

- Big-Oh: upper bound
- Big-Omega: lower bound
- Big-Theta: asymptotic equivalence
- Little-Oh: less than (in asymptotic sense). The bound is not asymptotically tight.

Little-Omega Notation

Let f: IN→IR and g: IN→IR.

f(n) is $\omega(g(n))$

if and only if

g(n) is o(f(n)).

Little-Omega

- $2n^2$ is $\omega(n)$
- If $f(n)$ is $\omega(g(n))$ then $\lim_{n \to \infty} \frac{f(n)}{g(n)} = \infty$.

Intuition of Asymptotic Terminology

- Big-Oh: upper bound
- Big-Omega: lower bound
- Big-Theta: asymptotic equivalence
- Little-Oh: less than (in asymptotic sense).
  The bound is not asymptotically tight.
- Little-Omega: greater than (in asymptotic sense).
  The bound is not asymptotically tight.

Properties of Algorithms

- Correctness
  - Without correctness all other properties are irrelevant
- Efficiency
  - Time complexity
  - Space complexity
  - Running time
  - Program/code complexity
    - Easy to read and understand
    - Modifiability
    - Extensibility
    - Maintainability
- Robustness
  - Collinearity
  - Cocircularity
  - Degenerate input
  - Duplicate input
  - Small input sets
  - Special cases
  - Numerical stability
  - No round off errors
- Termination
  - Can we prove that the algorithm terminates?

Find an Algorithm to solve Prefix Averages Problem

- Note: Efficiency and Design go hand in hand!

Prefix Averages

Input: An n-element array X of numbers.

Output: An n-element array A of numbers such that $A[k]$ is the average of elements $X[0], \ldots, X[k]$

X	12	3	7	24	4	1	1
A	12	7.5	7.3	11.5	10	8.5	7.4

Algorithm PrefixAverages

Input: An n-element array X of numbers.

Output: An n-element array A of numbers such that $A[k]$ is the average of elements $X[0], \ldots, X[k]$

```
Let A be an array of n numbers.
for k←0 to n-1 do
   a←0
   for j←0 to k do
      a←a+X[j]
   end
   A[k]←a/(k+1)
end
return A
```

$k + 1$ times

$1 + 2 + 3 + \ldots + n = ?$

Worst-Case Running Time of Algorithm PrefixAverages

$$1 + 2 + 3 + \ldots + n =$$

$$\sum_{i=1}^{n} k = \frac{1}{2} n(n+1) \text{ is } O(n^2)$$

Quiz: Can we solve the problem faster?

Prefix Averages

Input: An n-element array X of numbers.

Output: An n-element array A of numbers such that $A[k]$ is the average of elements $X[0], \ldots, X[k]$

Quiz

- $A[0] = ?$
- $A[1] = ?$
- $A[2] = ?$

  ⋮

- $A[k\text{-}1] = ?$
- $A[k] = ?$

Quiz

- $A[0] = X[0]$
- $A[1] = ?$
- $A[2] = ?$

⋮

- $A[k\text{-}1] = ?$
- $A[k] = ?$

Quiz

- $A[0] = X[0]$
- $A[1] = (X[0] + X[1])/2$
- $A[2] = ?$

- $A[k\text{-}1] = ?$
- $A[k] = ?$

Quiz

- $A[0] = X[0]$
- $A[1] = (X[0] + X[1])/2$
- $A[2] = (X[0] + X[1] + X[2])/3$

⋮

- $A[k\text{-}1] = ?$
- $A[k] = ?$

Quiz

- $A[0] = X[0]$
- $A[1] = (X[0] + X[1])/2$
- $A[2] = (X[0] + X[1] + X[2])/3$

⋮

- $A[k\text{-}1] = (X[0] + X[1] + \ldots + X[k\text{-}1])/k$
- $A[k] = (X[0] + X[1] + \ldots + X[k\text{-}1] + X[k])/(k+1)$

Quiz

- $A[0] = X[0]$
- $A[1] = (X[0] + X[1])/2$
- $A[2] = (X[0] + X[1] + X[2])/3$

⋮

- $A[k\text{-}1] = (X[0] + X[1] + \ldots + X[k\text{-}1])/\mathrm{k}$
- $A[k] = (X[0] + X[1] + \ldots + X[k\text{-}1] + \mathrm{X}[k])/(k+1)$

Quiz

- $A[0] = X[0]$
- $A[1] = (X[0] + X[1])/2$
- $A[2] = (X[0] + X[1] + X[2])/3$

⋮

- $A[k\text{-}1] = (X[0] + X[1] + \ldots + X[k\text{-}1])/\mathrm{k}$
- $A[k] = (X[0] + X[1] + \ldots + X[k\text{-}1] + \mathrm{X}[k])/(k+1)$

Quiz

Idea: Use part of the computation for $A[k\text{-}1]$ when computing $A[k]$!

- $A[0] = X[0]$
- $A[1] = (X[0] + X[1])/2$
- $A[2] = (X[0] + X[1] + X[2])/3$

⋮

- $A[k\text{-}1] = (X[0] + X[1] + \ldots + X[k\text{-}1])/k$
- $A[k] = (X[0] + X[1] + \ldots + X[k\text{-}1] + X[k])/(k+1)$

Algorithm PrefixAverages2

Input: An n-element array X of numbers.

Output: An n-element array A of numbers such that $A[k]$ is the average of elements $X[0], \ldots, X[k]$

```
Let A be an array of n numbers.
s←0
for k←0 to n-1 do
   s←s+X[k]
   A[k]← s/(k+1)
end
return A
```

} n times

$O(n)$

PrefixAverages vs. PrefixAverages2

- PrefixAverages runs in *quadratic* time $O(n^2)$
- PrefixAverages2 runs in *linear* time $O(n)$

- Thus, PrefixAverages2 is more efficient!
- The analysis drove the design of PrefixAverages2 to a certain extent

Correctness of Algorithms

- Once an algorithm has been properly specified, one can prove its *correctness*.
- That is the algorithm yields a required result for every legitimate input in a finite amount of time (i.e., it halts with the correct output).
  For some algorithms, a correctness proof is easy to obtain; for others it is very complex
- Mathematical induction is a common approach in proofing correctness
- Tracing the performance as in computing asymptotic complexity does not proof correctness

Implementing an Algorithm

- Most algorithms are eventually implemented as computer programs written in a programming language (e.g., coding in Java, C++, C#)
- In the transition from pseudo-code to code, correctness, efficiency and ease of understanding may be lost
- Formal verification is limited to small programs
- The validity of an implementation of an algorithm is established by testing
- Program testing can be used very effectively to show the presence of bugs but never to show their absence.

  E.W. Dijkstra, EWD 303
  http://www.cs.utexas.edu/~EWD/transcriptions/EWD03xx/EWD303.html

Optimality

- When is an algorithm optimal with respect to efficiency, complexity of implementation, maintainability, modifiability, extensibility, etc.?
- There are tradeoffs among all these criteria when implementing an algorithm
- Efficiency: What is the minimum amount of effort *any* algorithm will need to solve the given problem?
- For some problems we know the answer
  - Sorting $\Omega(n \log n)$
  - Searching $\Omega(1)$
  - Selection $\Omega(n)$

How can we prove correctness of an algorithm?

- Using standard proof techniques like
  - Induction
  - Contra attack
  - Counterexample
  - Loop invariants

How can we prove correctness of an algorithm?

- Using standard proof techniques like
  - Induction
  - Contra attack
  - Counterexample
  - Loop invariants

Counterexample

- Used to justify a claim is false.

- *Claim*: $a^2 + b^2 = (a+b)^4$ for all a,b
  - Let a, $b = 1$. Then $1+1 = 2^4$ is false.
  - Therefore $a^2 + b^2 \neq (a+b)^4$.