Proof by Induction Examples

Exercise 1

$$\sum_{i=0}^{n} i = \frac{n(n+1)}{2} \qquad \forall n \geq 0$$

Base Case. For $n = 0$ we have 0 for the LHS and $\frac{0 \cdot 1}{1} = 0$ for the RHS.
Induction Hypothesis. Assume that $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$ for some $n \geq 0$.
Induction Step. Consider $n + 1$.

$$\begin{aligned}
\sum_{i=0}^{n+1} i &= \Big(\sum_{i=0}^{n} i\Big) + (n+1) \\
&= \frac{n(n+1)}{2} + (n+1) \qquad \text{(by I.H.)} \\
&= \frac{n(n+1)}{2} + \frac{2(n+1)}{2} \\
&= \frac{n(n+1) + 2(n+1)}{2} \\
&= \frac{(n+2)(n+1)}{2} \\
&= \frac{((n+1)+1)(n+1)}{2}
\end{aligned}$$

Therefore, the identity holds for $n + 1$, and by induction, the identity holds for all $n \geq 0$.

Exercise 2

$$\sum_{i=1}^{n} (2i - 1) = n^2 \qquad \forall n \geq 0$$

Base Case. For $n = 1$ we have $\sum_{i=1}^{n}(2i - 1) = 2 - 1 = 1$ for the LHS and $1^2 = 1$ for the RHS.
Induction Hypothesis. Assume that $\sum_{i=1}^{n}(2i - 1) = n^2$ for some $n \geq 1$.
Induction Step. Consider $n + 1$.

$$\begin{aligned}
\sum_{i=1}^{n+1}(2i-1) &= \Big(\sum_{i=1}^{n}(2i-1)\Big) + (2(n+1) - 1) \\
&= n^2 + (2(n+1) - 1) \qquad \text{(by I.H.)} \\
&= n^2 + 2n + 1 \\
&= (n+1)^2
\end{aligned}$$

Therefore, the identity holds for $n + 1$, and by induction, the identity holds for all $n \geq 0$.

An alternative proof technique is to rearrange the identity algebraically. Each line below is equivalent to the original identity. Note that arithmetic operations do not change the underlying statement to be proven.

$$\begin{aligned}
\sum_{i=1}^{n+1}(2i-1) &= n^2 \\
\Big(\sum_{i=1}^{n} 2i\Big) - \Big(\sum_{i=1}^{n} 1\Big) &= n^2 \\
2\Big(\sum_{i=1}^{n} i\Big) - (n) &= n^2 \\
2\Big(\sum_{i=1}^{n} i\Big) &= n^2 + n \\
\sum_{i=1}^{n} i &= \frac{n^2+n}{2} \\
\sum_{i=1}^{n} i &= \frac{n(n+1)}{2}
\end{aligned}$$

The last version is equivalent to the identity in example 1. For some identities, manipulating the identity algebraically before proving it can make the proof significantly easier.

Exercise 3

$$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1 \qquad \forall n \geq 0$$

Base Case. For $n = 0$ we have $\sum_{i=0}^{n} 2^i = 2^0 = 1$ for the LHS and $2^1 - 1 = 1$ for the RHS.
Induction Hypothesis. Assume that $\sum_{i=1}^{n} 2^i = 2^{n+1} - 1$ for some $n \geq 0$.
Induction Step. Consider $n + 1$.

$$\begin{aligned}
\sum_{i=0}^{n+1} 2^i &= \Big(\sum_{i=0}^{n} 2^i\Big) + 2^{n+1} \\
&= (2^{n+1} - 1) + 2^{n+1} \qquad \text{(by I.H.)} \\
&= 2 \cdot 2^{n+1} - 1 \\
&= 2^{n+2} - 1
\end{aligned}$$

Therefore, the identity holds for $n + 1$, and by induction, the identity holds for all $n \geq 0$.

Exercise 4

$$\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \forall n \geq 0$$

Base Case. For $n = 0$ we have $0^2 = 0$ for the LHS and $\frac{0\cdot(1)\cdot(1)}{6} = 0$ for the RHS.
Induction Hypothesis. Assume that $\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ for some $n \geq 0$.
Induction Step. Consider $n + 1$.

$$\begin{aligned}
\sum_{i=0}^{n+1} i^2 &= \Big(\sum_{i=0}^{n} i^2\Big) + (n+1)^2 \\
&= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \qquad \text{(by I.H.)} \\
&= \frac{6(n+1)^2 + n(n+1)(2n+1)}{6} \\
&= \frac{(n+1)(6(n+1) + n(2n+1))}{6} \\
&= \frac{(n+1)(6n + 6 + 2n^2 + n)}{6} \\
&= \frac{(n+1)(2n^2 + 7n + 6)}{6}
\end{aligned}$$

To finish the proof, we expand the right hand side of the identity until it matches the form above. Although it is possible to continue the algebra above to obtain the right hand side, doing so requires factoring, while starting with the right hand side only requires expanding. As usual, as long as the proof is concise and mathematically correct, it will receive full marks on an assignment or test (so there is no requirement that it be done a certain way).
For $n + 1$, the right hand side of the identity is

$$\begin{aligned}
\frac{(n+1)(n+2)(2(n+1)+1)}{6} &= \frac{(n+1)(n+2)(2n+3)}{6} \\
&= \frac{(n+1)(2n^2 + 7n + 6)}{6}
\end{aligned}$$

Thus, we have shown that, for $n+1$, the two sides of the identity can be transformed into the same formula using only algebra and the induction hypothesis. Therefore, the identity holds for $n + 1$, completing the proof. As a result, by induction, the identity holds for all $n \geq 0$.