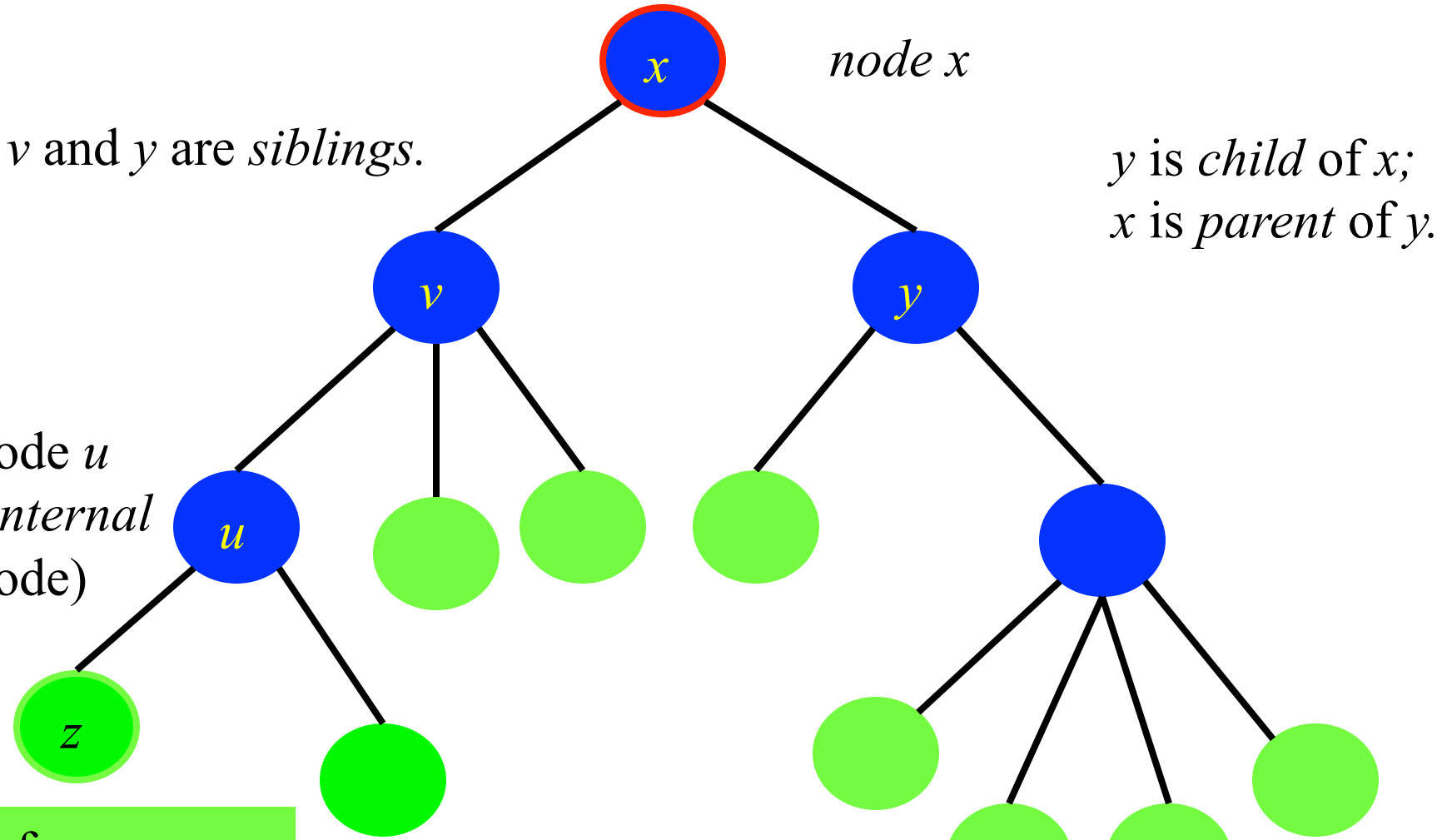
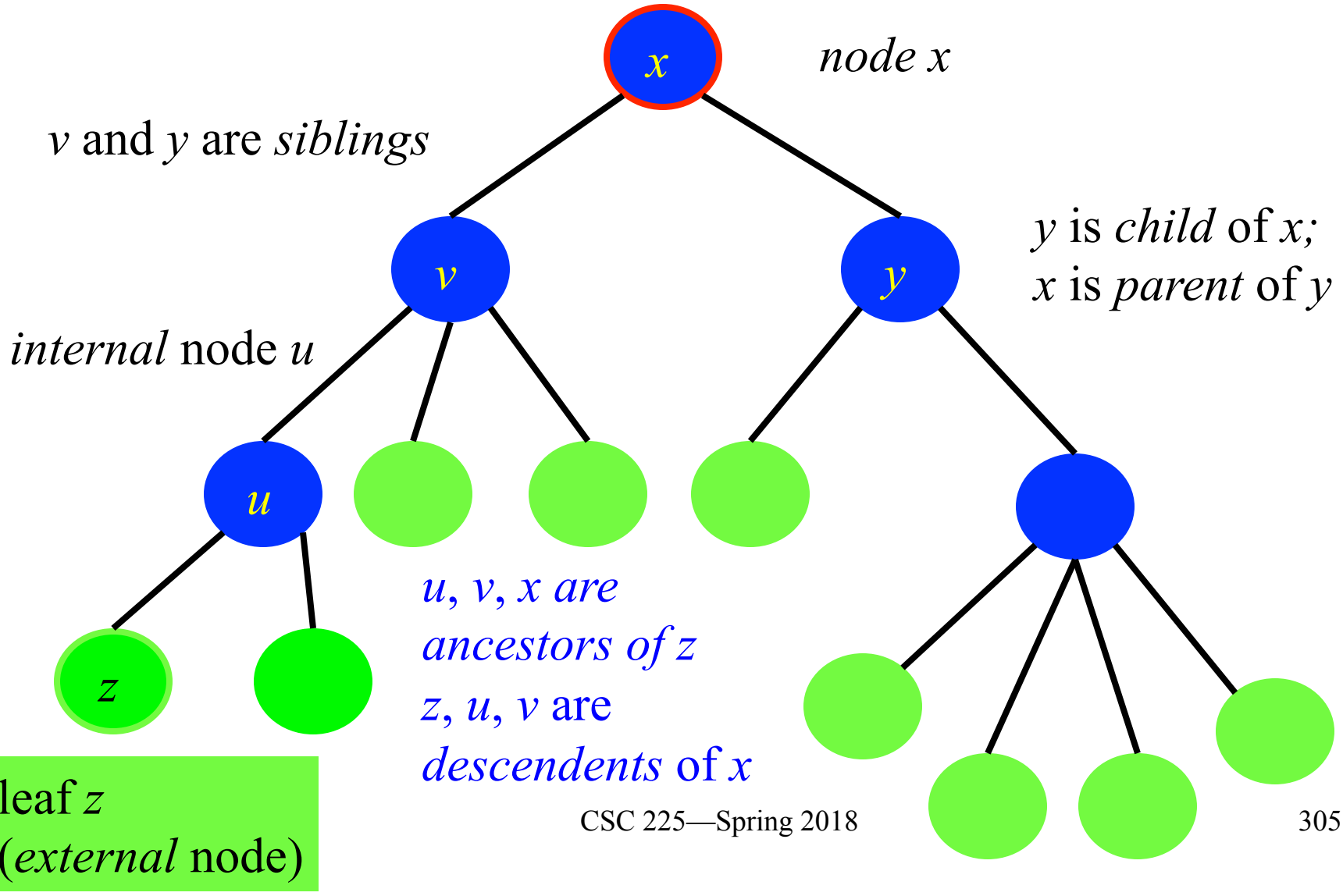
# Trees

- A (*rooted*) *tree T* is a set of *nodes* in a *parent-child relationship* with the following properties:
  - *T* has a special node *r*, called the *root* of *T*
  - Each node *v* of *T* different from *r* has a *parent* node *u*

# Trees

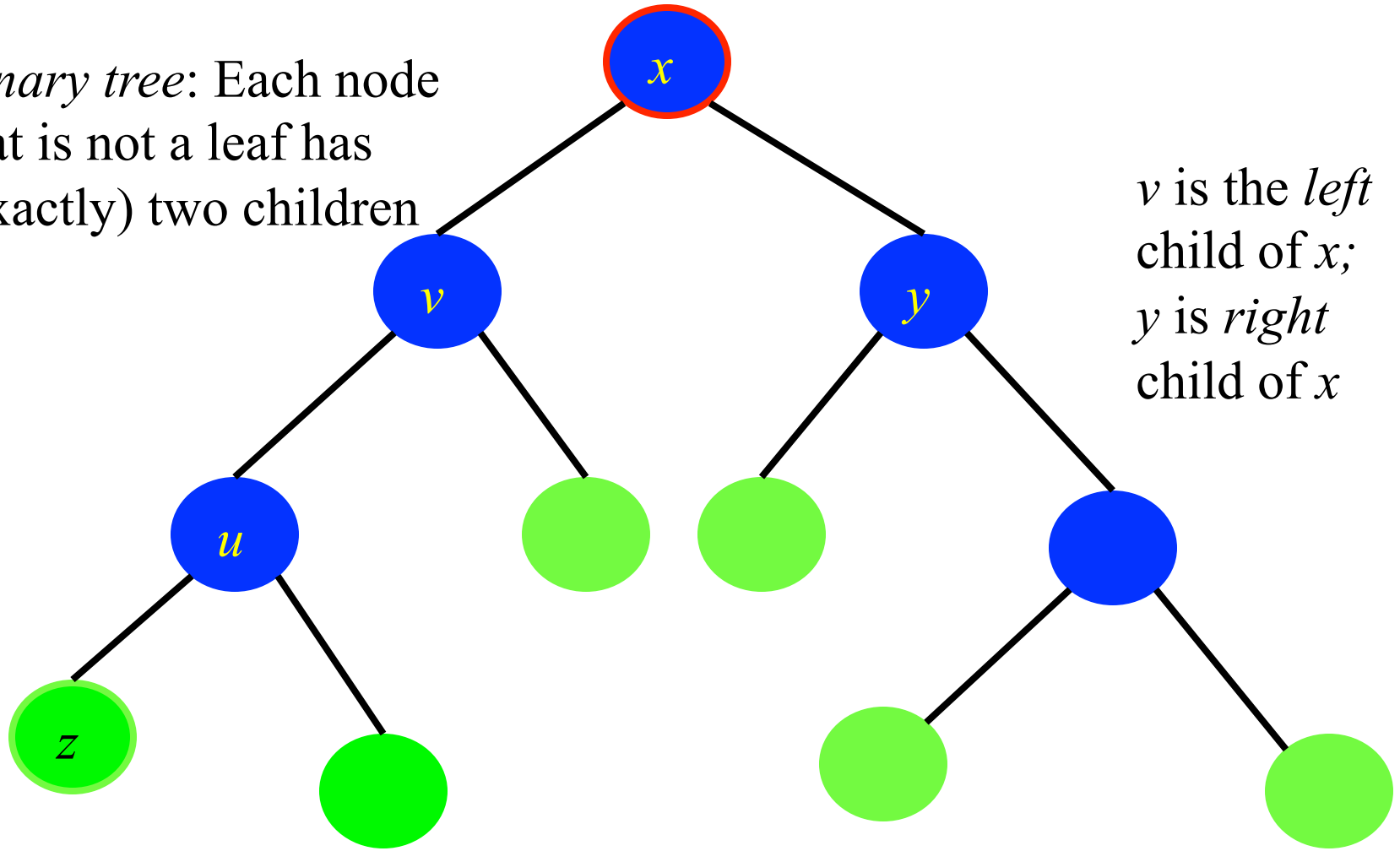

x
node x
v and y are siblings.
y is child of x;
x is parent of y.
v
y
node u
(internal
node)
u
z
leaf z
(external node)

# Trees


x
node x
v and y are siblings
v
y
y is child of x;
x is parent of y
internal node u
u
z
u, v, x are
ancestors of z
z, u, v are
descendents of x
leaf z
(external node)

# Binary Trees

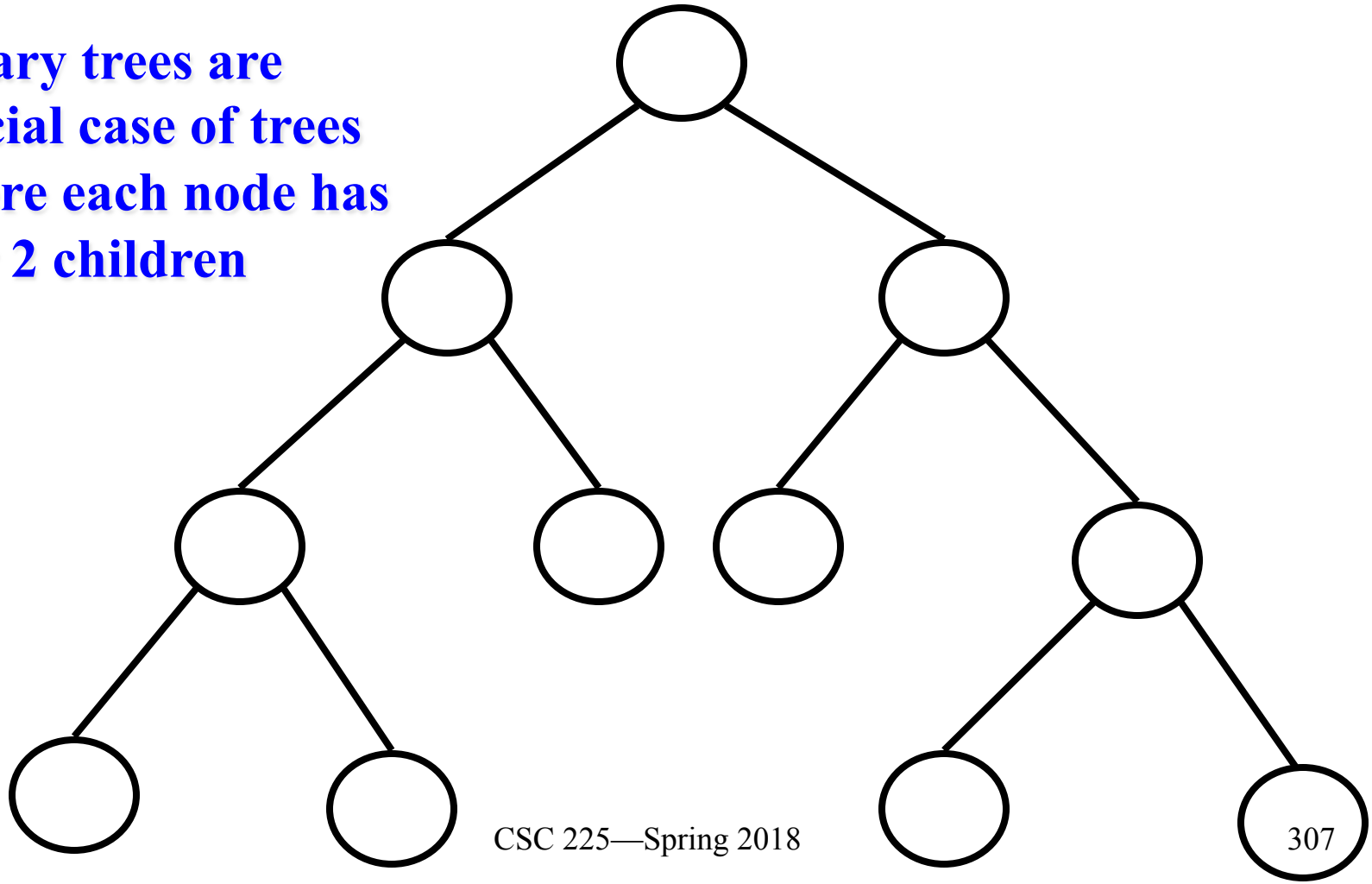
*Binary tree*: Each node that is not a leaf has (exactly) two children




*v* is the *left* child of *x;* *y* is *right* child of *x*

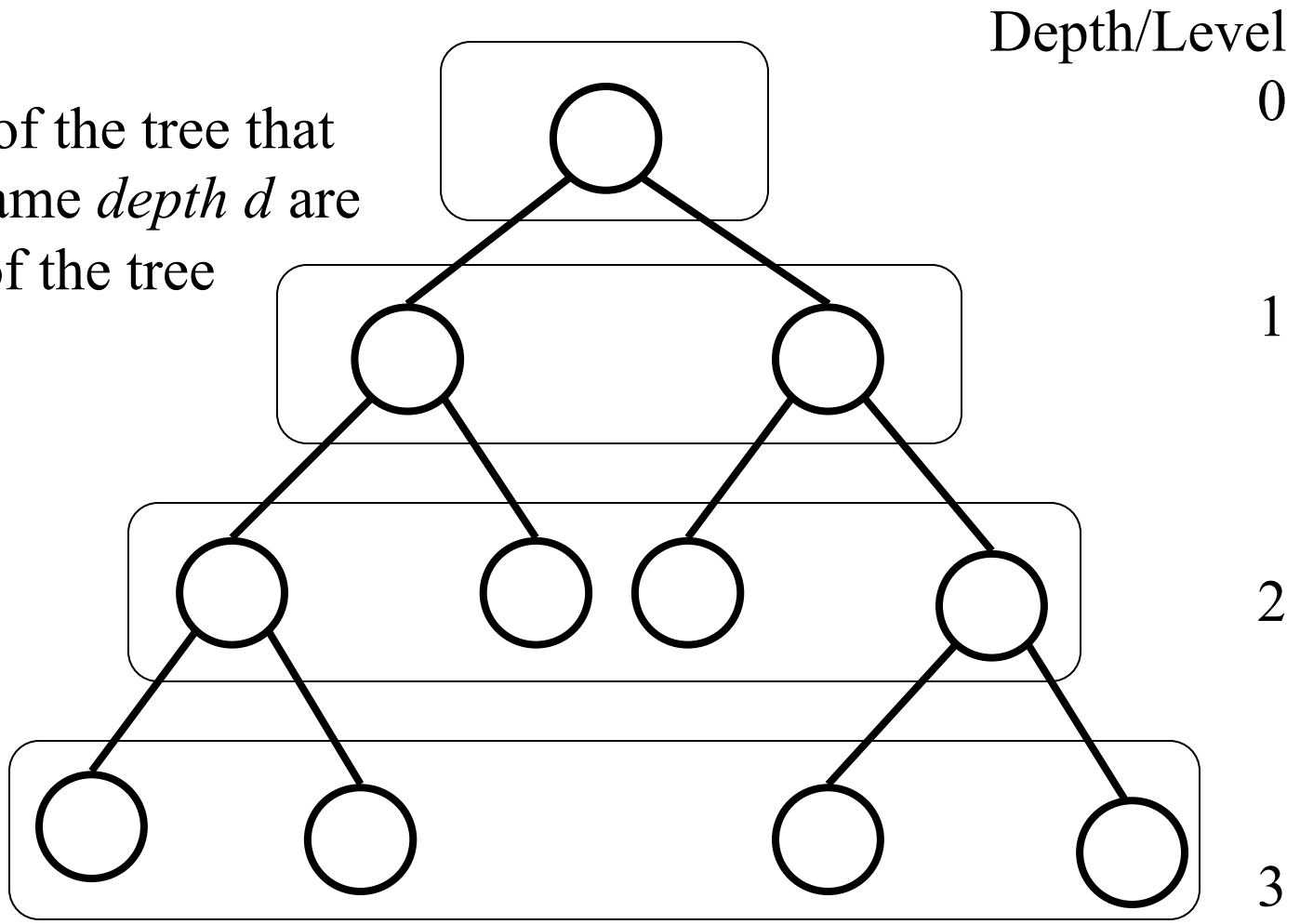
# Binary Trees

**Binary trees are special case of trees where each node has 0 or 2 children**



# Depth and Levels in Trees

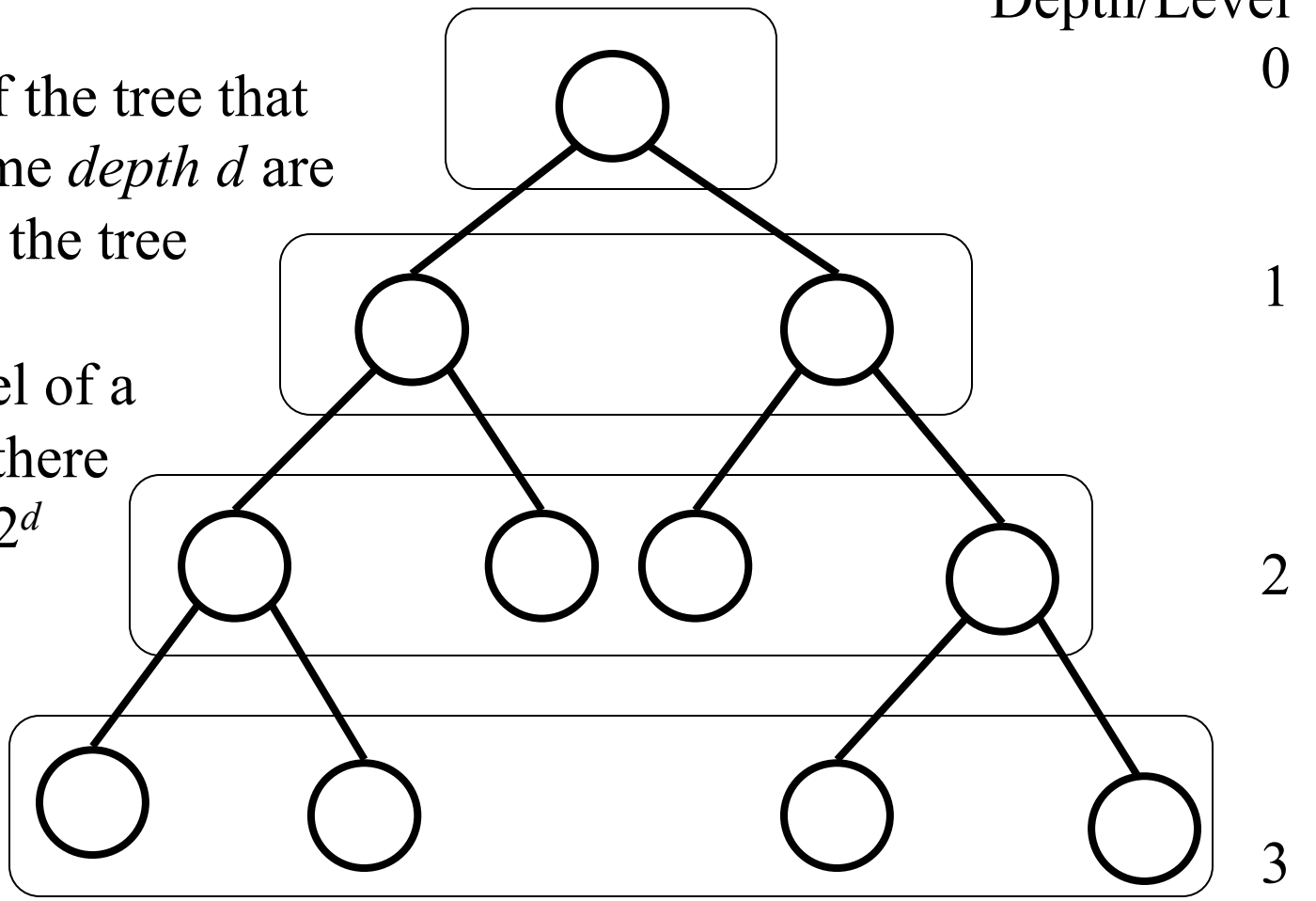
All nodes of the tree that have the same *depth d* are at *level d* of the tree



# Depth and Levels in Trees

All nodes of the tree that have the same *depth d* are at *level d* of the tree

At each level of a *binary* tree there are at most $2^d$ nodes



# Height of a Tree

**Definition:** The *height* of a a tree *T* rooted at node *v* is (recursively) defined to be

- The height is 0 if *v* is a leaf node
- The height is equal to 1 plus the maximum height of any child of *v*, otherwise.

# Properties of Binary Trees

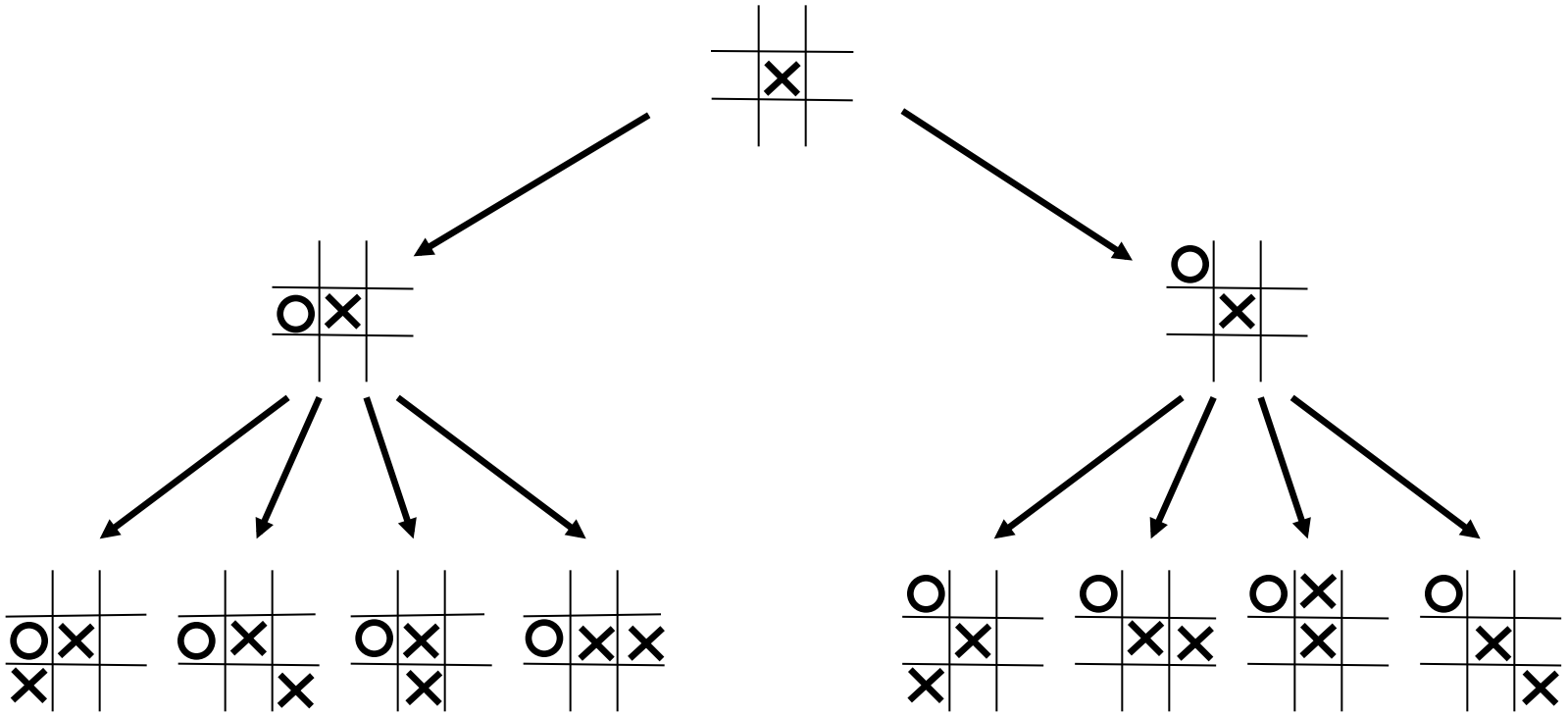
- Let $T$ be a binary tree with $n$ nodes and let $h$ denote the height of $T$
  - The number of *leaves* in $T$ is at least $h + 1$
  - The number of internal nodes in $T$ is at least $h$ and at most $2^h - 1$
  - $n \leq 2^{h+1} - 1$
  - $\log(n+1) - 1 \leq h \leq (n-1)/2$
  - # of leaves $=$ 1 + # of internal nodes

# Applications of Binary Trees

- Phylogenetic trees
- Data structure (search trees)
- Search trees for exponential algorithms (branch-and-bound techniques, fixed-parameter tractable algorithms)
- Visualization of algorithms (and tool for complexity analysis)
- Decision trees
- Parse trees
- Expression trees
- Forests

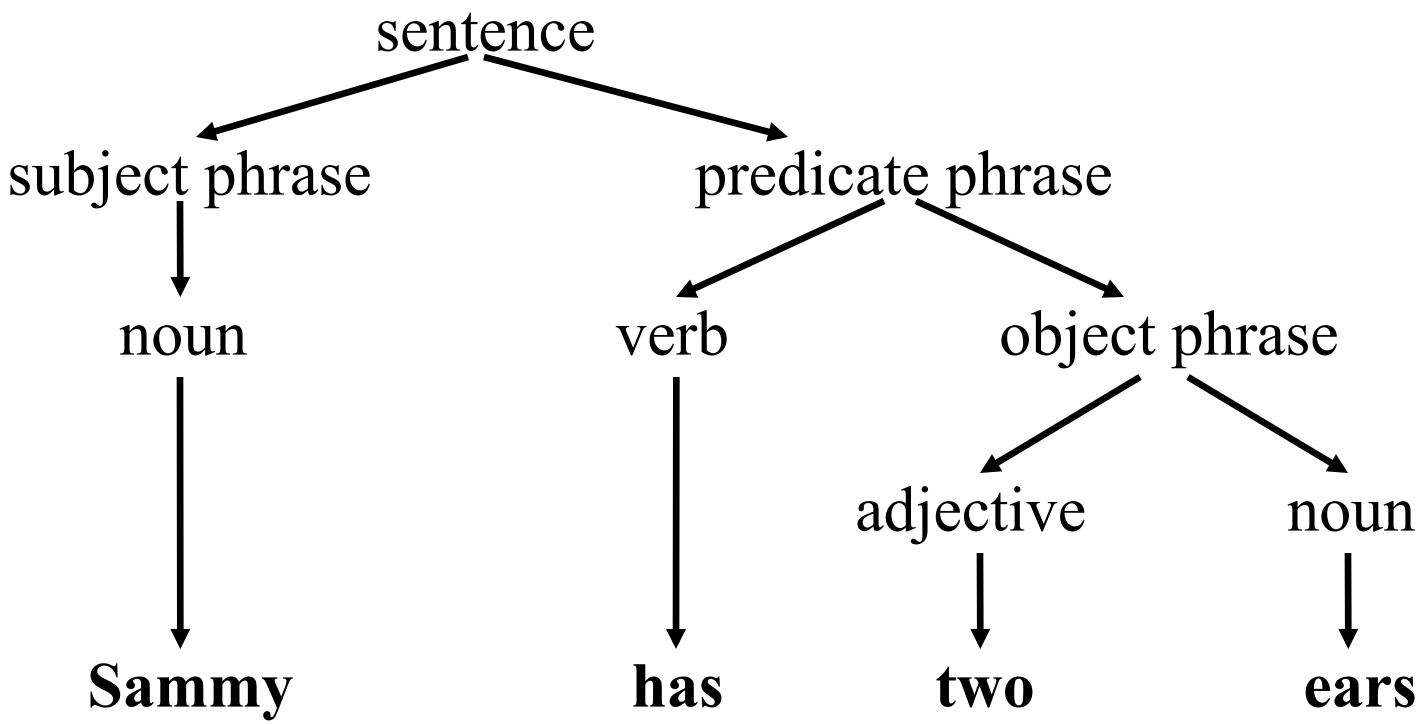
# Decision Trees

(after Gross & Yellen, 1999, p. 93)



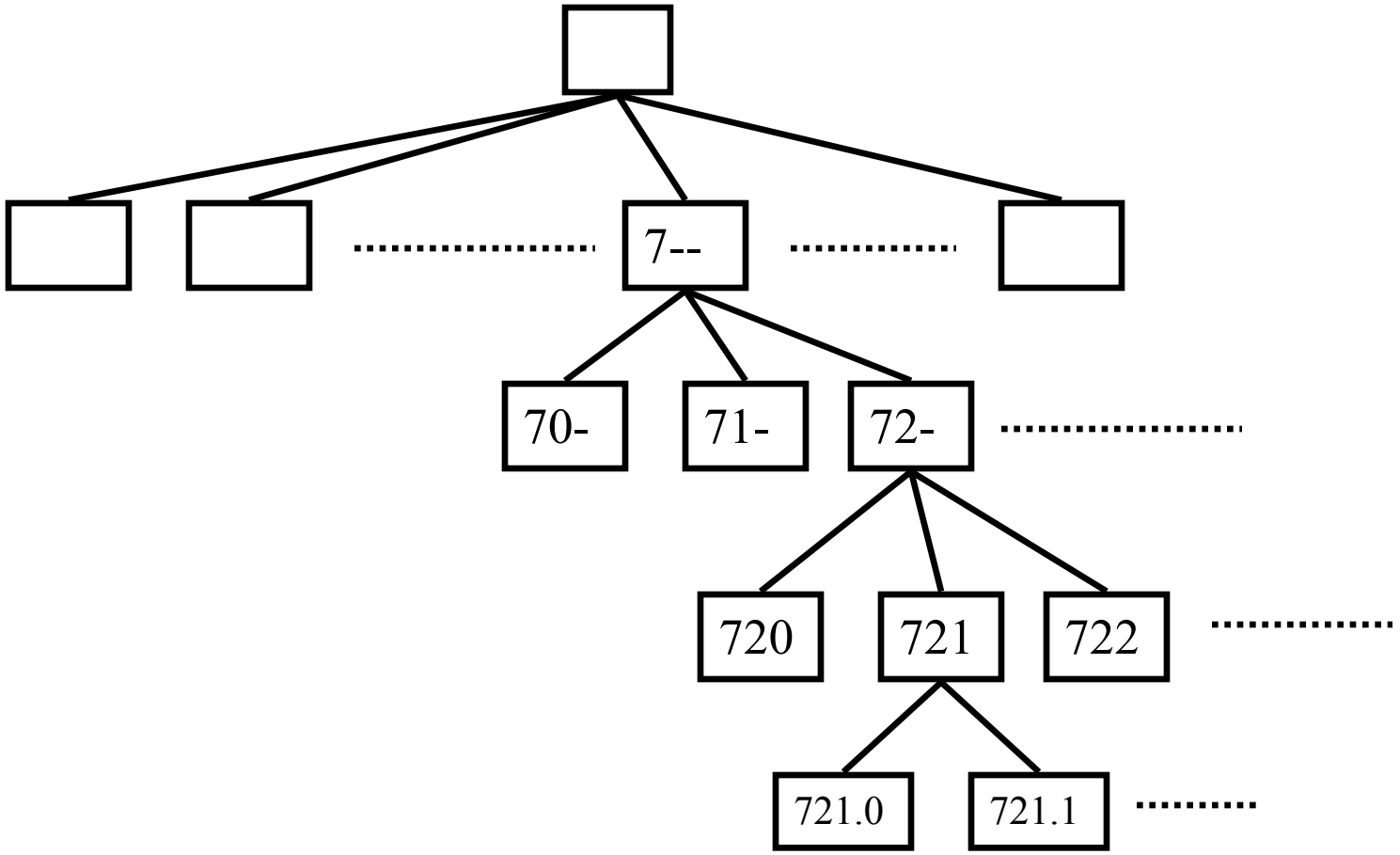
The first three moves of tic-tac-toe

# Sentence Parsing
(after Gross & Yellen, 1999, p. 93)



# Data Organization: DDCS for libraries

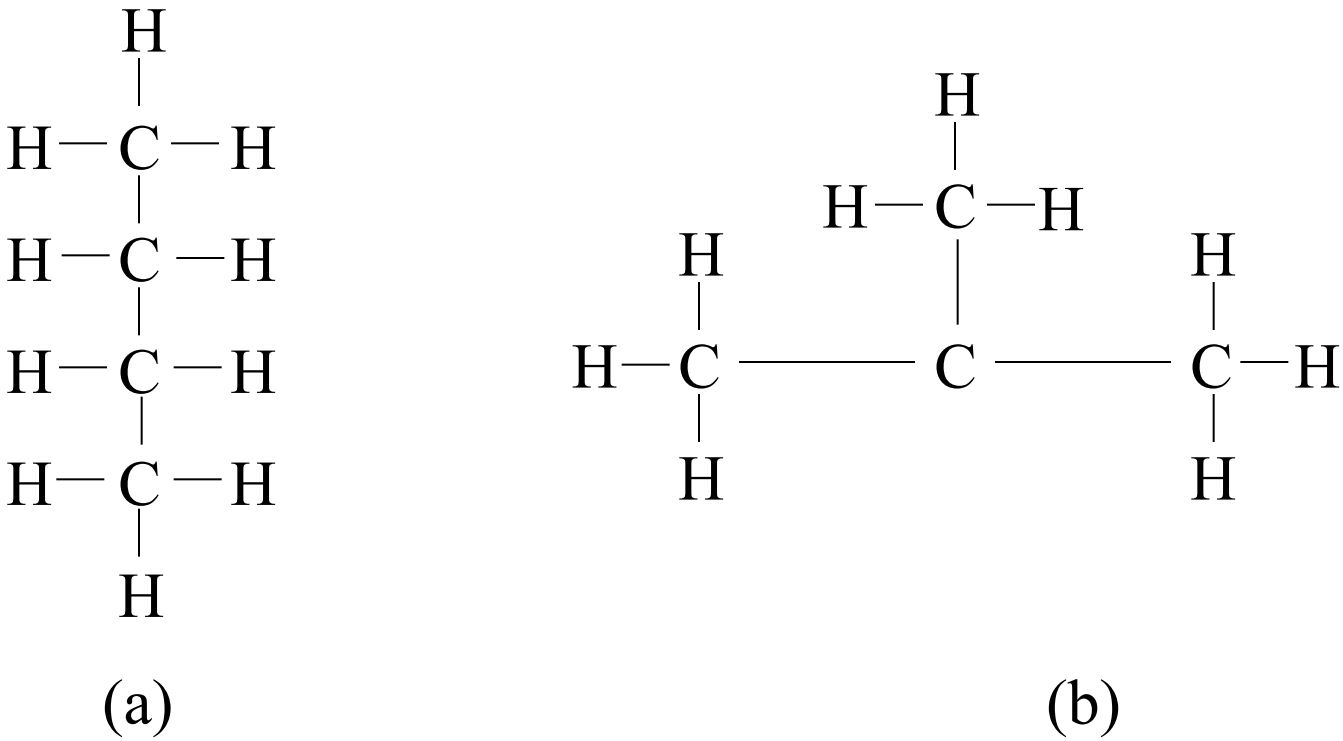
(after Gross & Yellen, 1999, p. 93)




DDDC = *Dewey Decimal Classification System*

# Chemical Isomers

(after Grimaldi, 1994, p. 610)




Two trees, each with 14 vertices (labeled with C's and H's) and 13 edges.
Each vertex has degree 4 (C, carbon atom) or degree 1 (H, hydrogen atom).

# ADT Tree

- Stores *elements* at positions, which are defined relative to their neighbour positions (i.e., parents, children, siblings)
- The *positions* in a tree are its nodes
- Supported methods by a node/position object:
  - **root():** returns the root of the tree
  - **element(t):** returns the object at this position
  - **parent(*v*):** returns the parent of node *v;* an error occurs if *v* is root
  - **children(*v*):** returns an iterator to enumerate the children of node *v*

# ADT Tree …

- **isInternal():** Test whether node *v* is internal
- **isExternal():** Test whether node *v* is external (i.e., leaf)
- **isRoot(*v*):** Test whether node *v* is root.
- **size():** returns the number of nodes in the tree
- **elements():** returns an iterator of all the elements stored at nodes of the tree (i.e., pre-, in-, post-, level-order)
- **positions():** returns an iterator of all the positions of the tree (i.e., pre-, in-, post-, level-order)
- **swapElements(*v*,*w*):** Swap the elements stored at nodes *v* and *w*
- **replaceElements(*v*,*e*):** Replace the element stored at node *v* with *e* and return the original element stored at node *v*

# ADT Binary Tree

- Specialization of a tree ADT that supports the accessor methods
  - **leftChild(*v*):** returns the left child of *v;* an error occurs if *v* is a leaf.
  - **rightChild(*v*):** returns the right child of *v;* an error occurs if *v* is a leaf.
  - **sibling(*v*):** returns the sibling of *v;* an error occurs if *v* is the root.

# Tree Algorithms: depth

**Definition:** The *depth* of a node $v$ in a tree $T$ is (recursively) defined to be

- 0, if $v$ is the root of $T$
- The depth of the parent of $v$ + 1, otherwise

**Algorithm** depth($T$,$v$):

*Input*: Tree $T$, node $v$ in $T$

*Output*: the depth of $v$ in $T$ (i.e., the number of ancestors of $v$ in $T$, excluding $v$ itself)

# Tree Algorithms: height

**Definition:** The *height* of a a tree $T$ rooted at node $v$ is (recursively) defined to be

- 0 if $v$ is a leaf.
- 1+ the maximum height of a child of $v$, otherwise.

**Algorithm** height($T$,$v$):

*Input*: Tree $T$, node $v$ in $T$

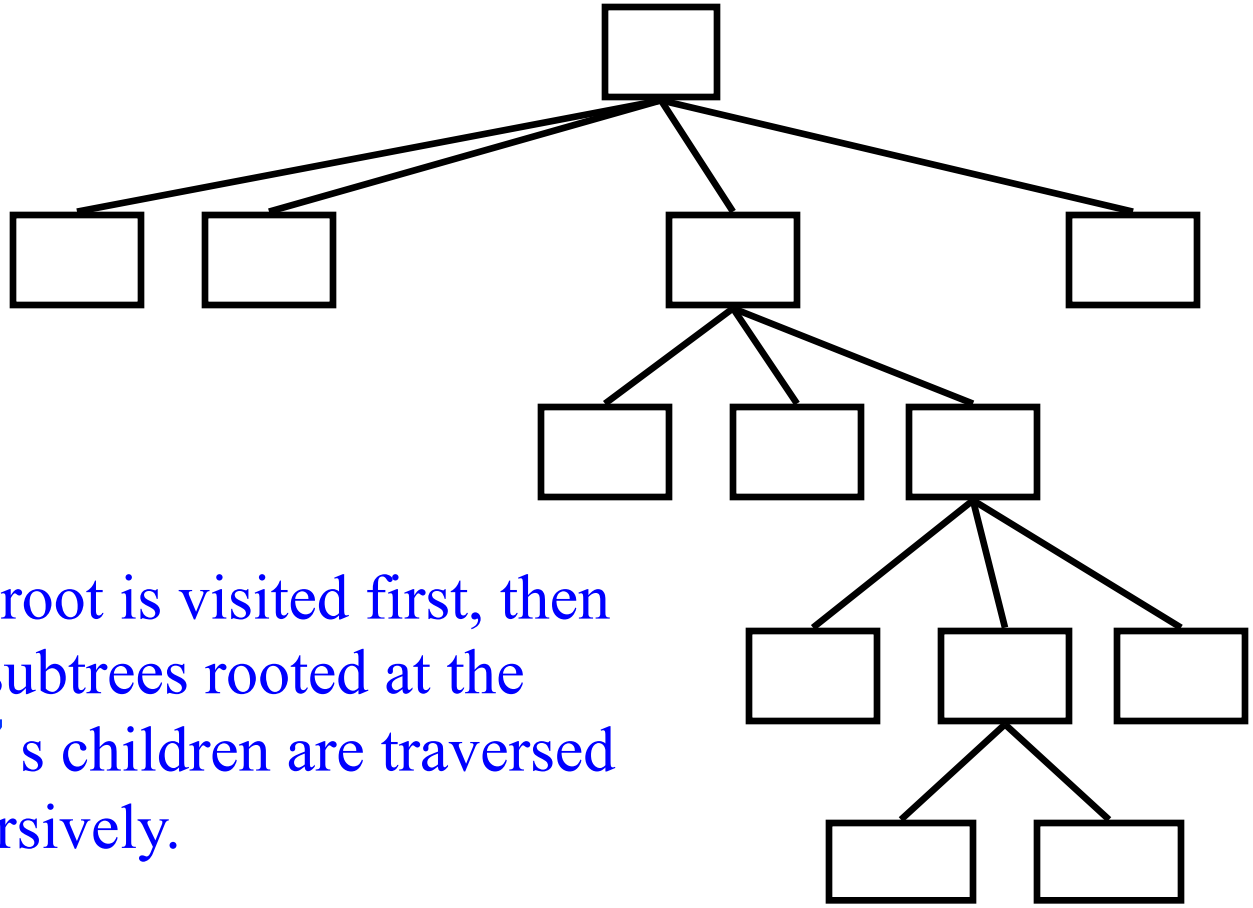
*Output*: the height of tree $T$ rooted at node $v$ (i.e., the maximum depth of a leaf of the tree $T$ rooted by $v$ in $T$)

# Tree Traversals

- n-ary tree traversals
  - Preorder
  - Postorder
  - Level order

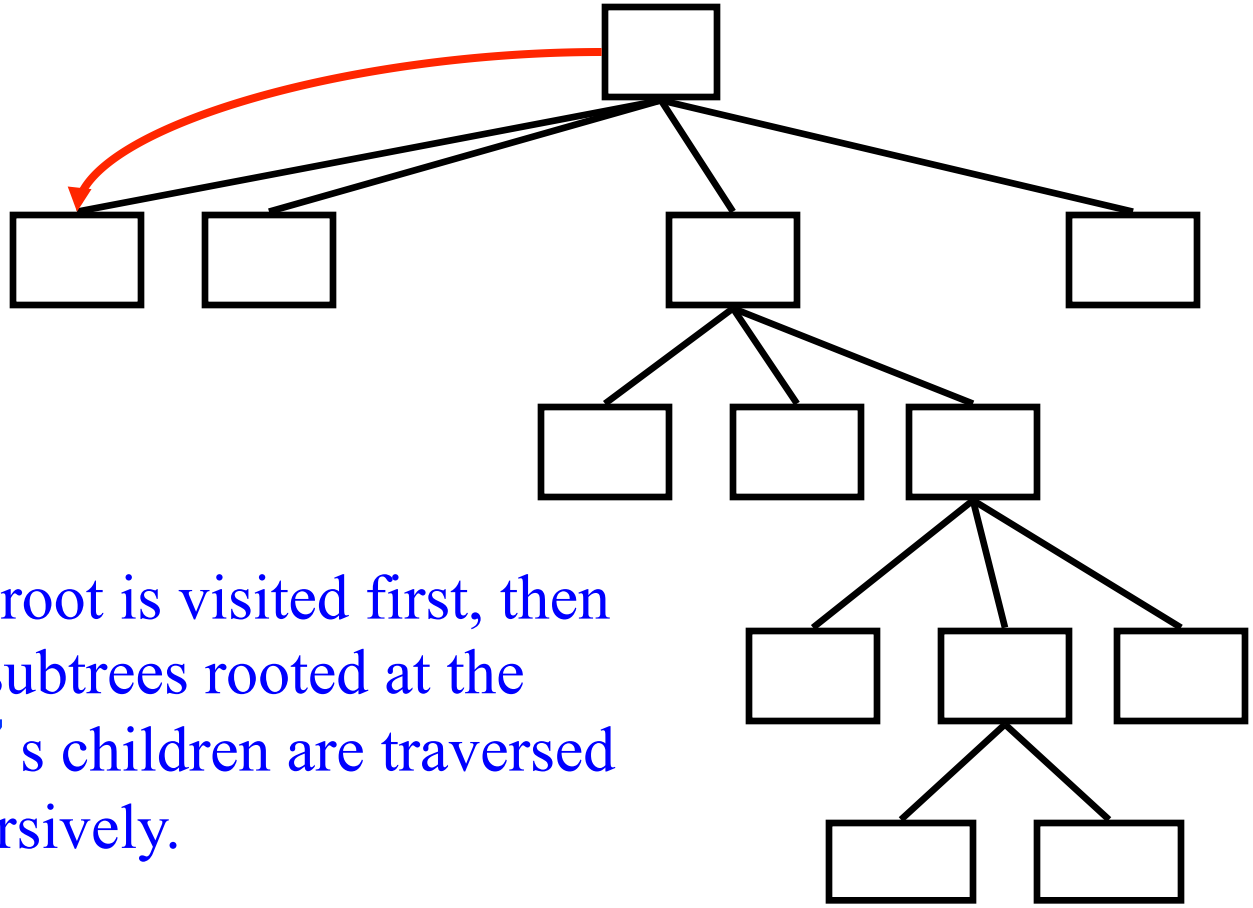
- Binary tree traversals
  - Preorder
  - Postorder
  - Inorder
  - Level order

# Preorder Traversal
# Depth First Search



The root is visited first, then the subtrees rooted at the root's children are traversed recursively.

# Preorder Traversal
# Depth First Search



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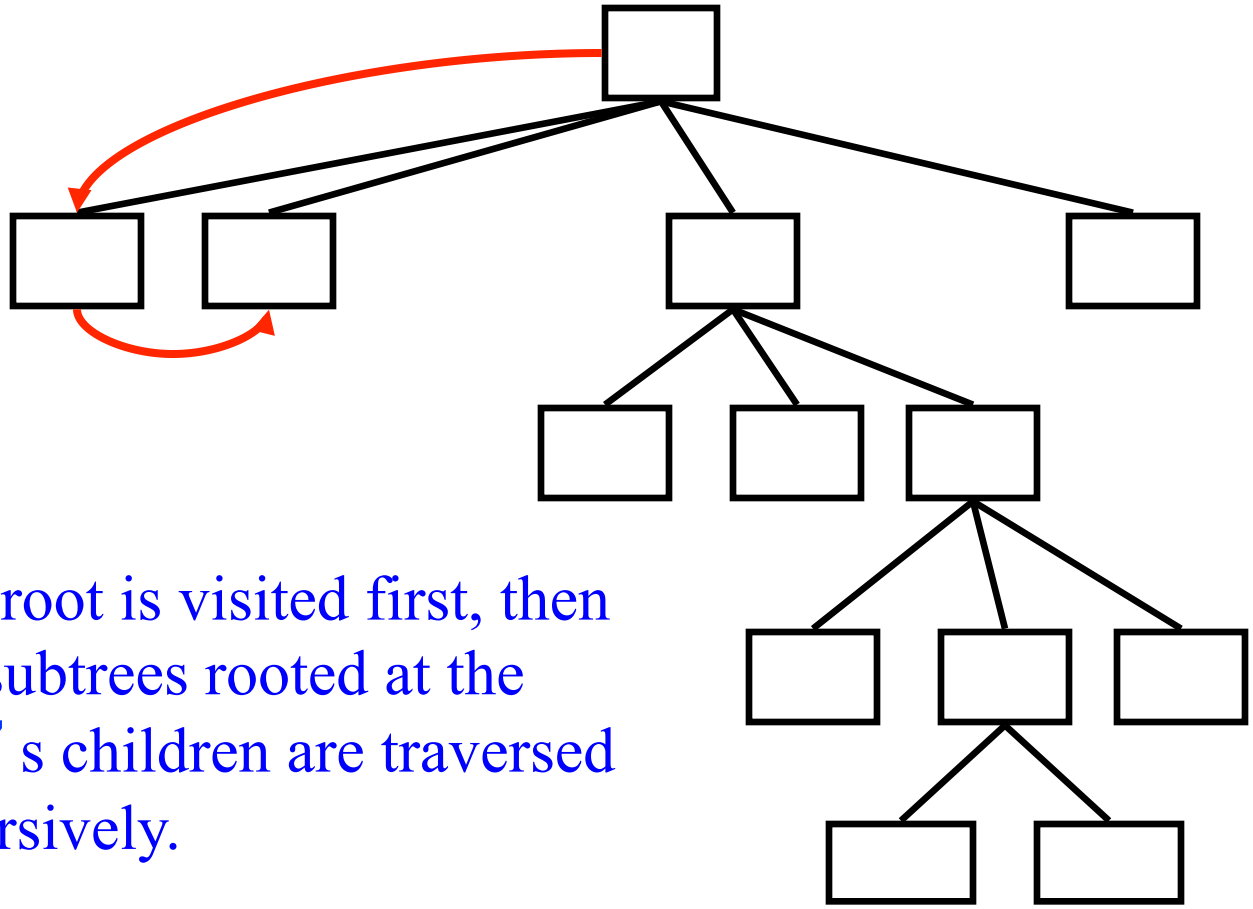

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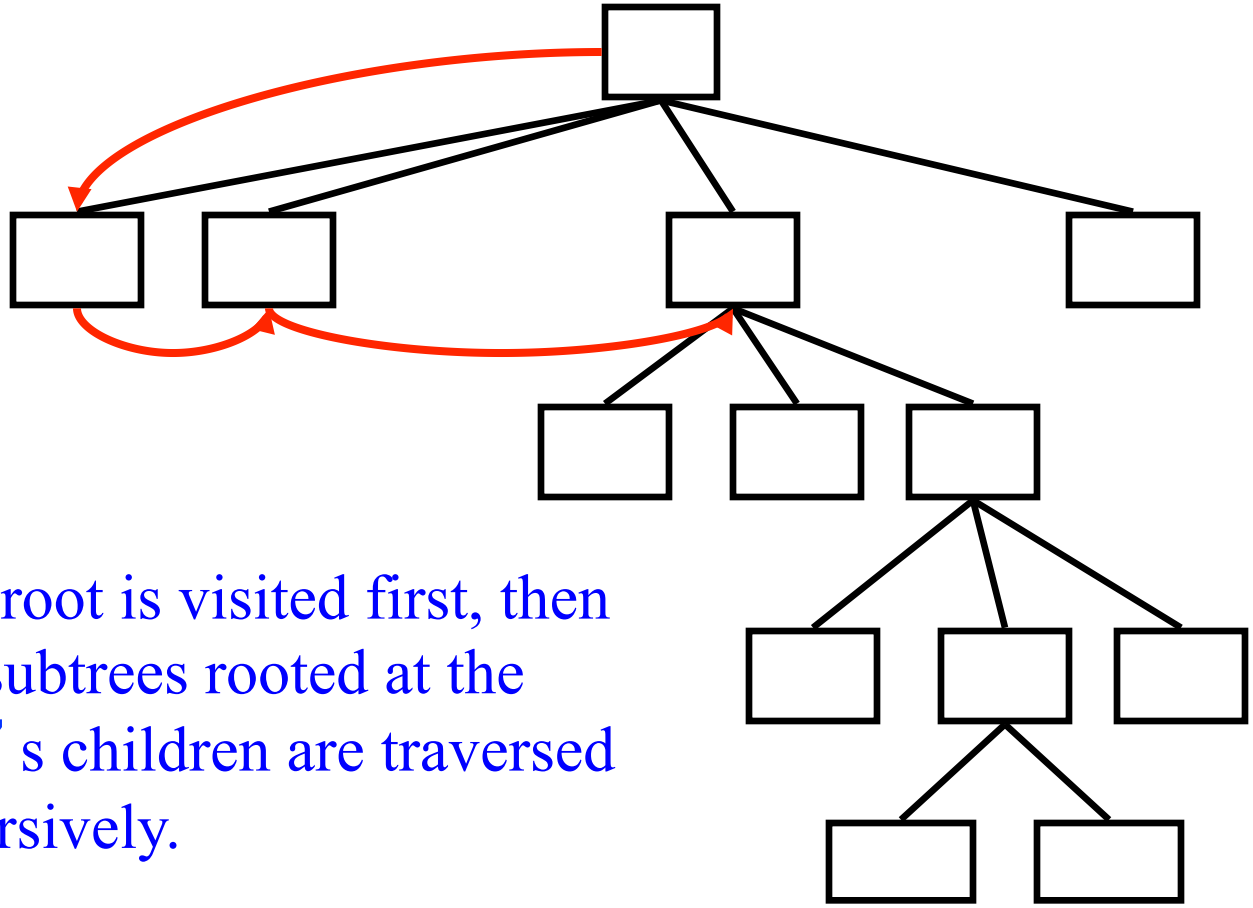

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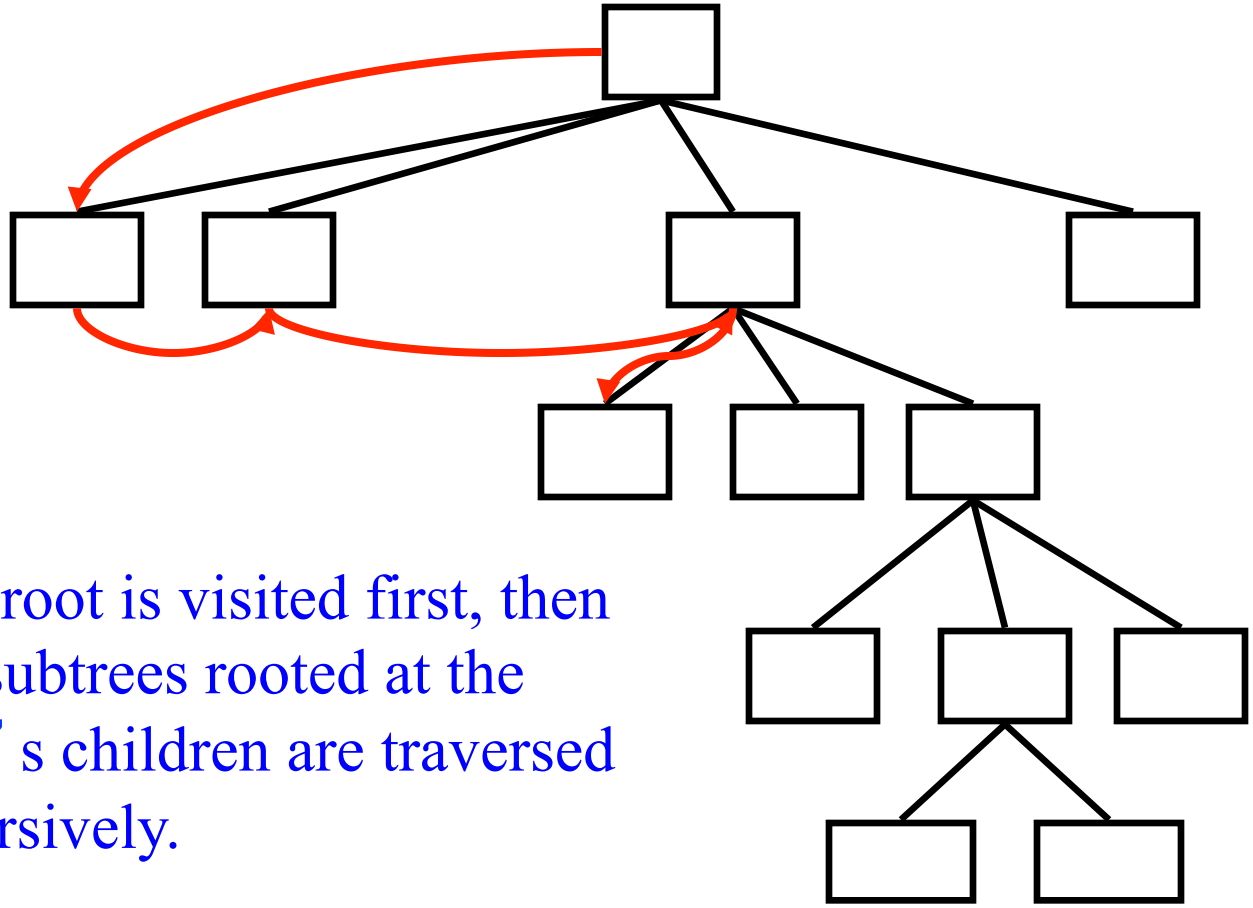

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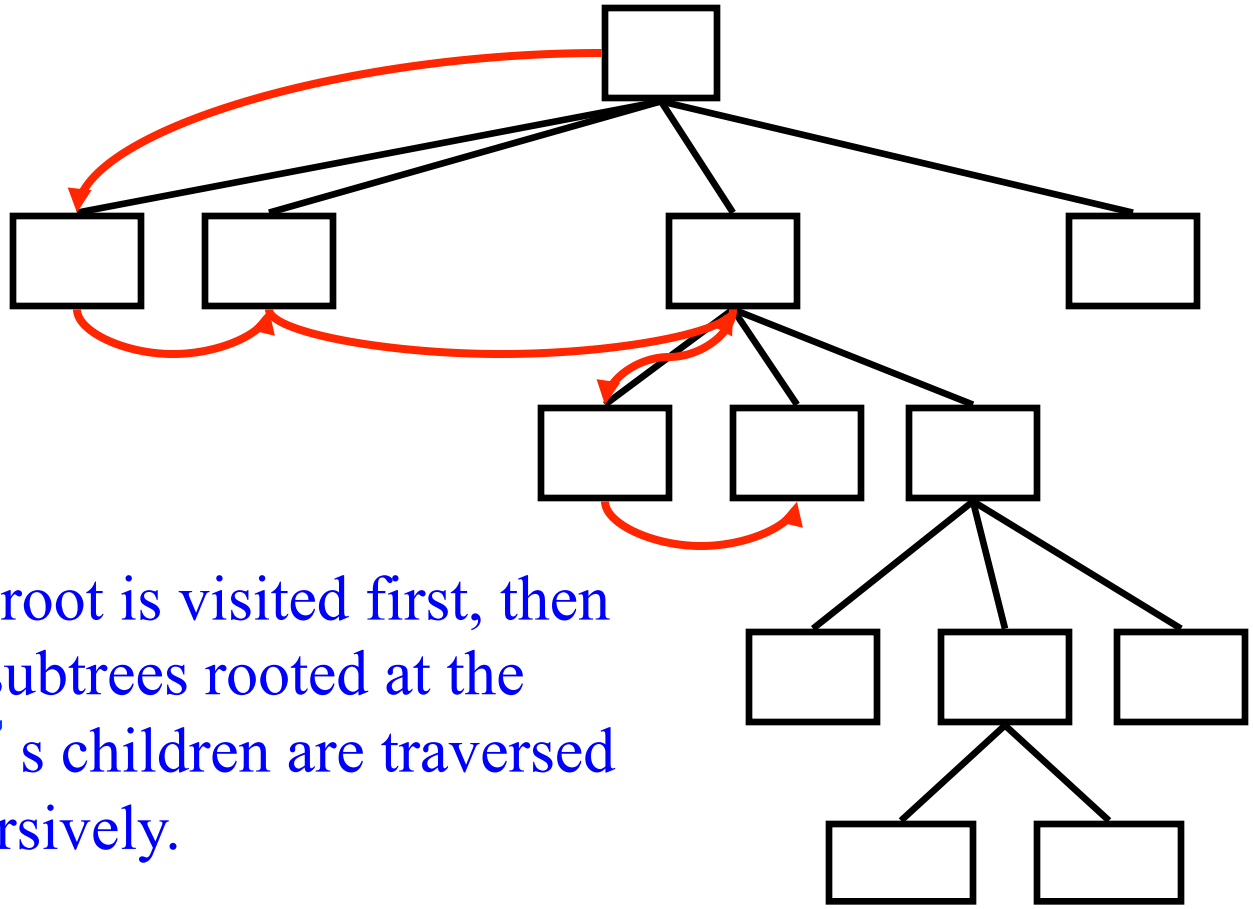

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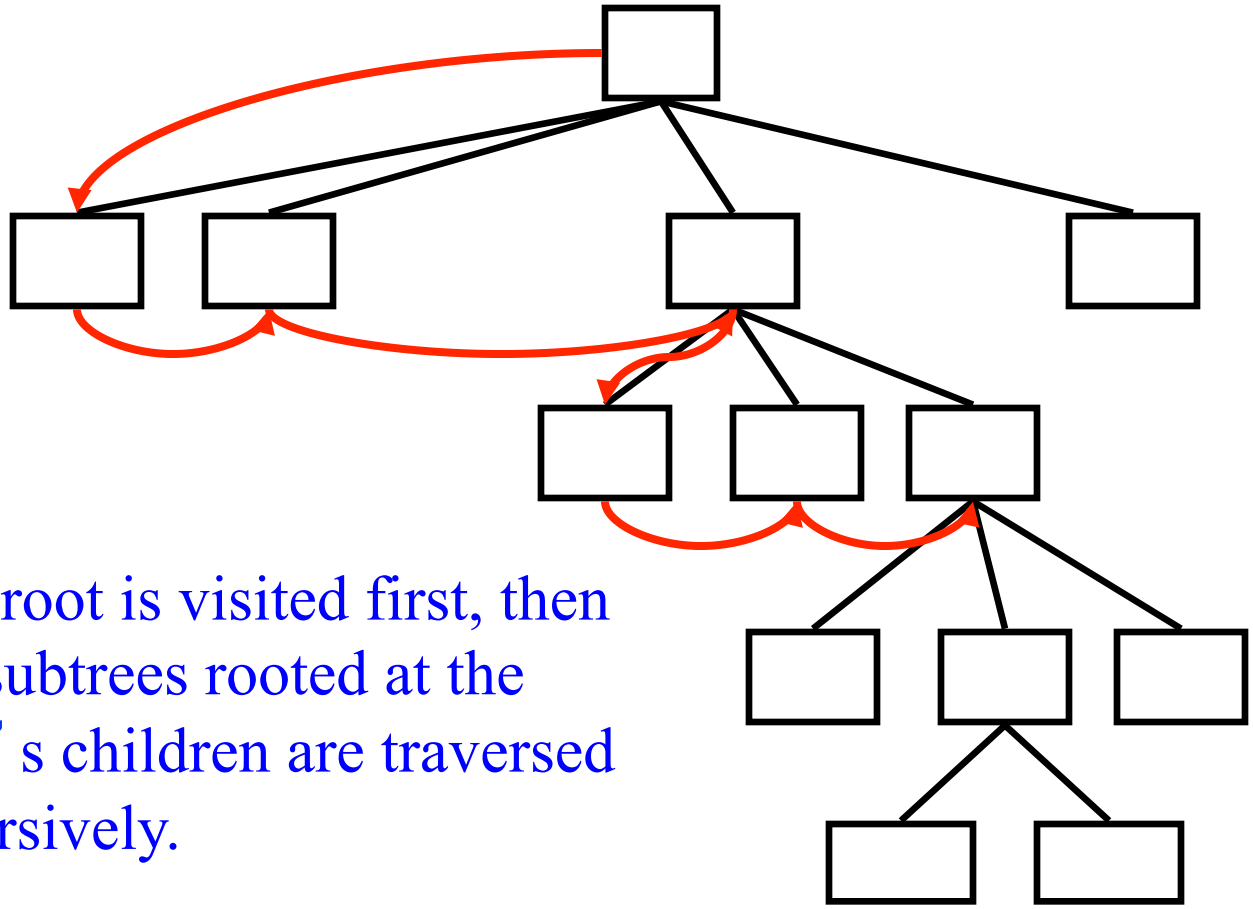

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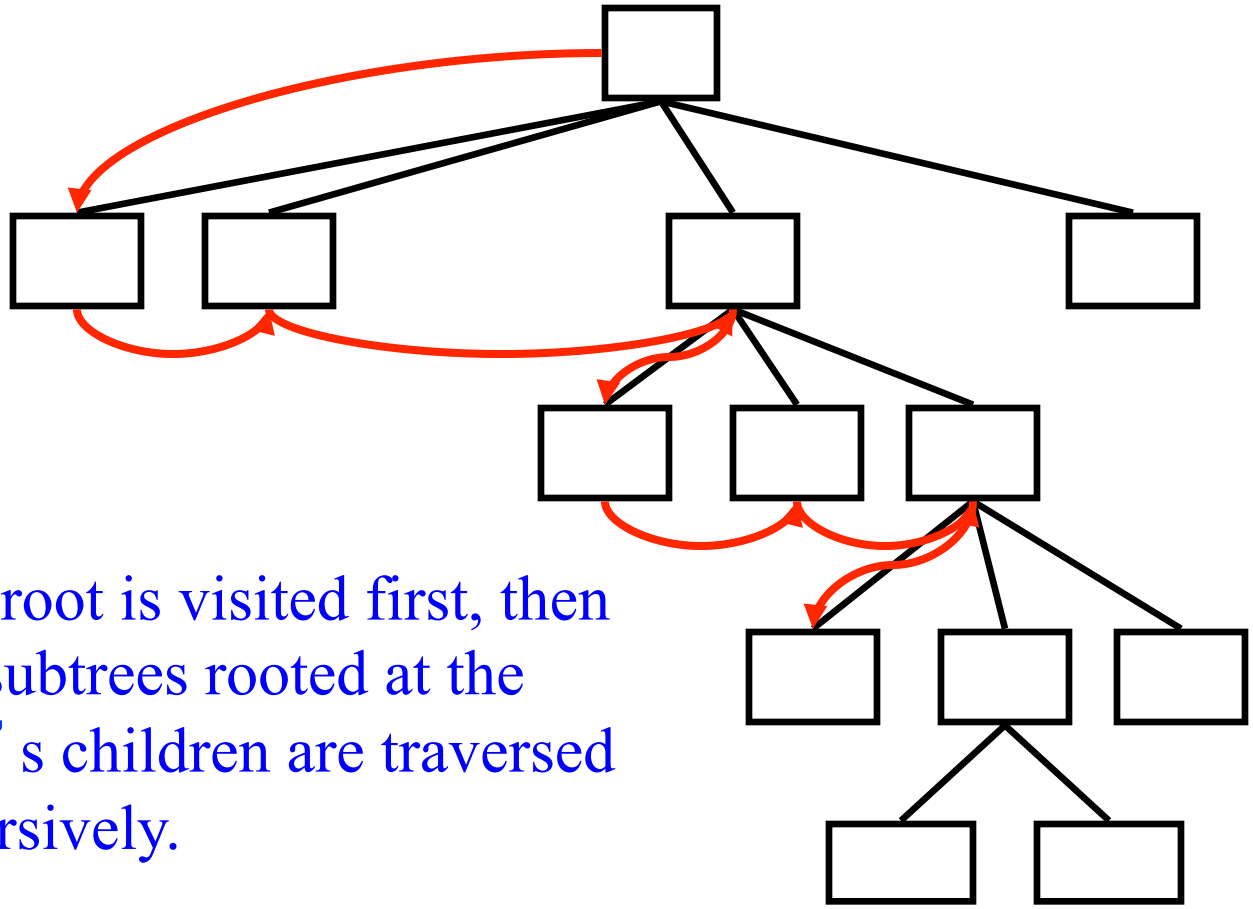

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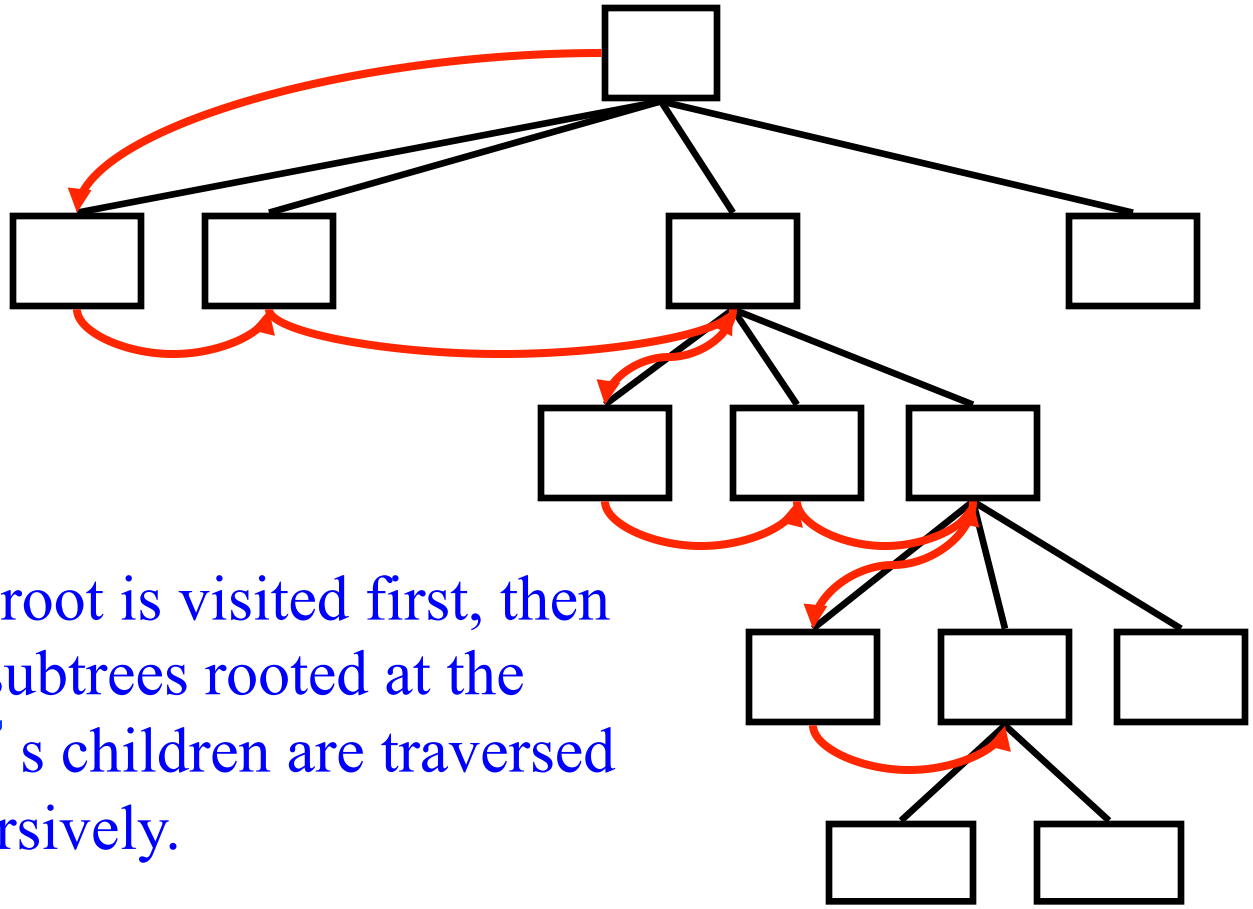
# Depth First Search



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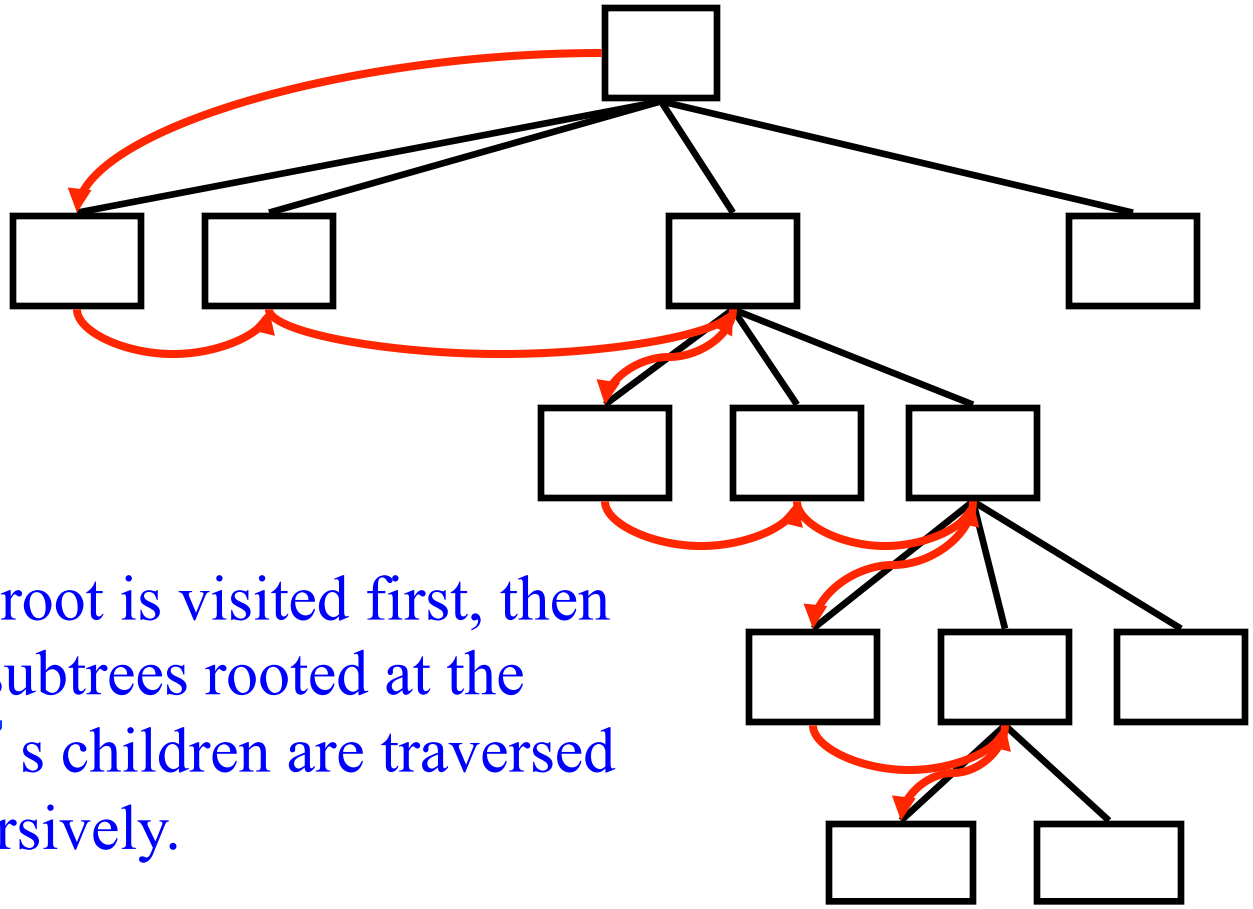
# Depth First Search



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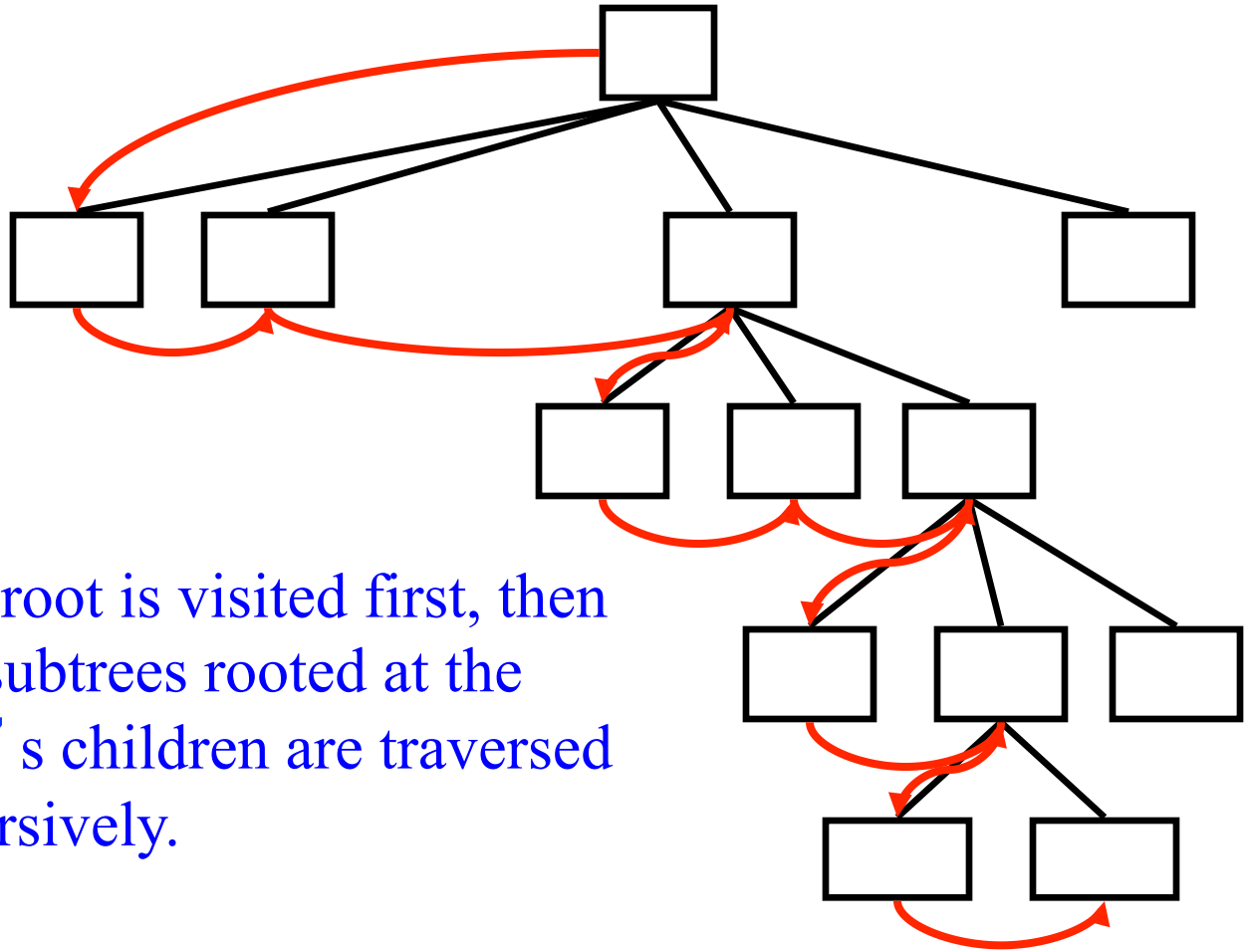

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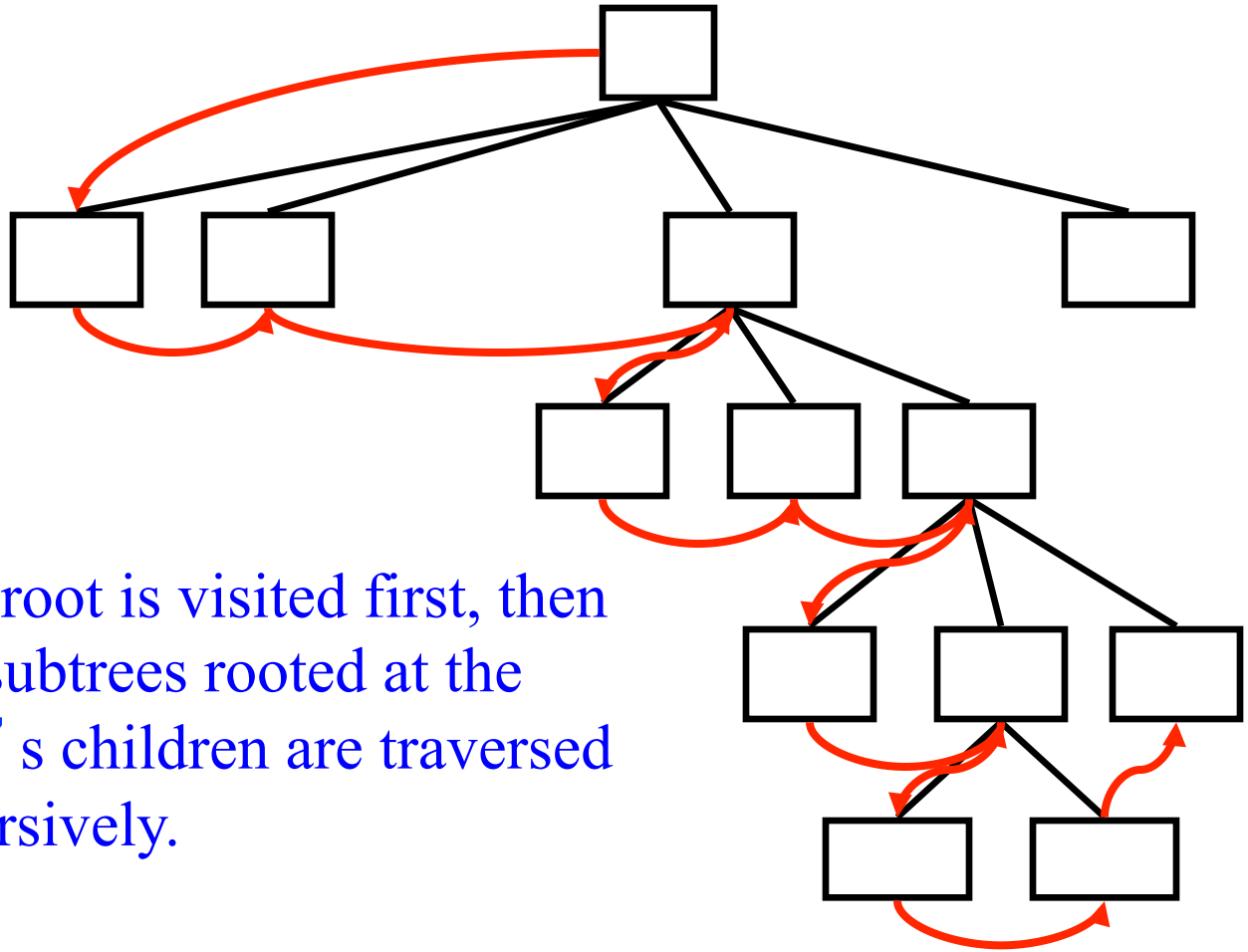

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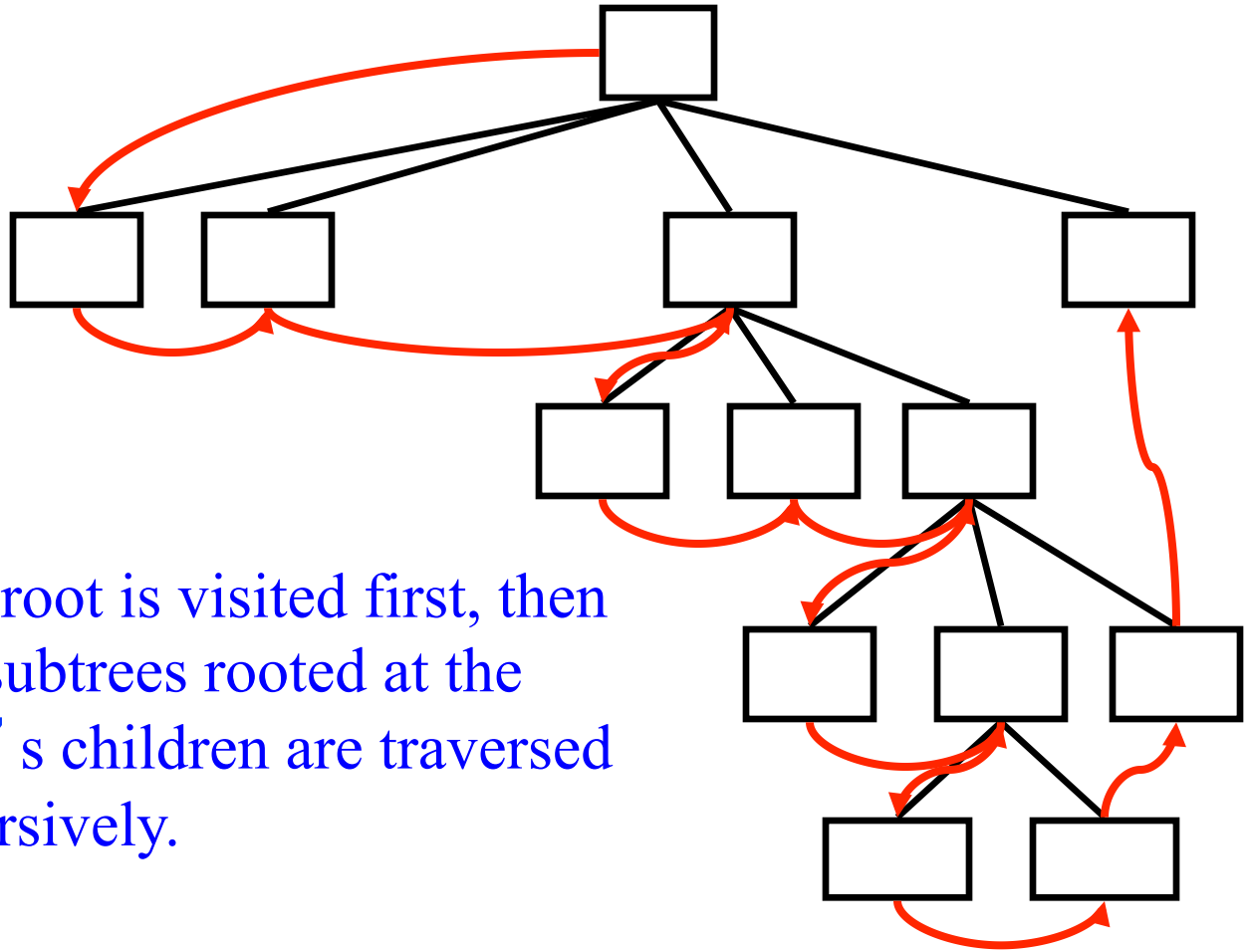
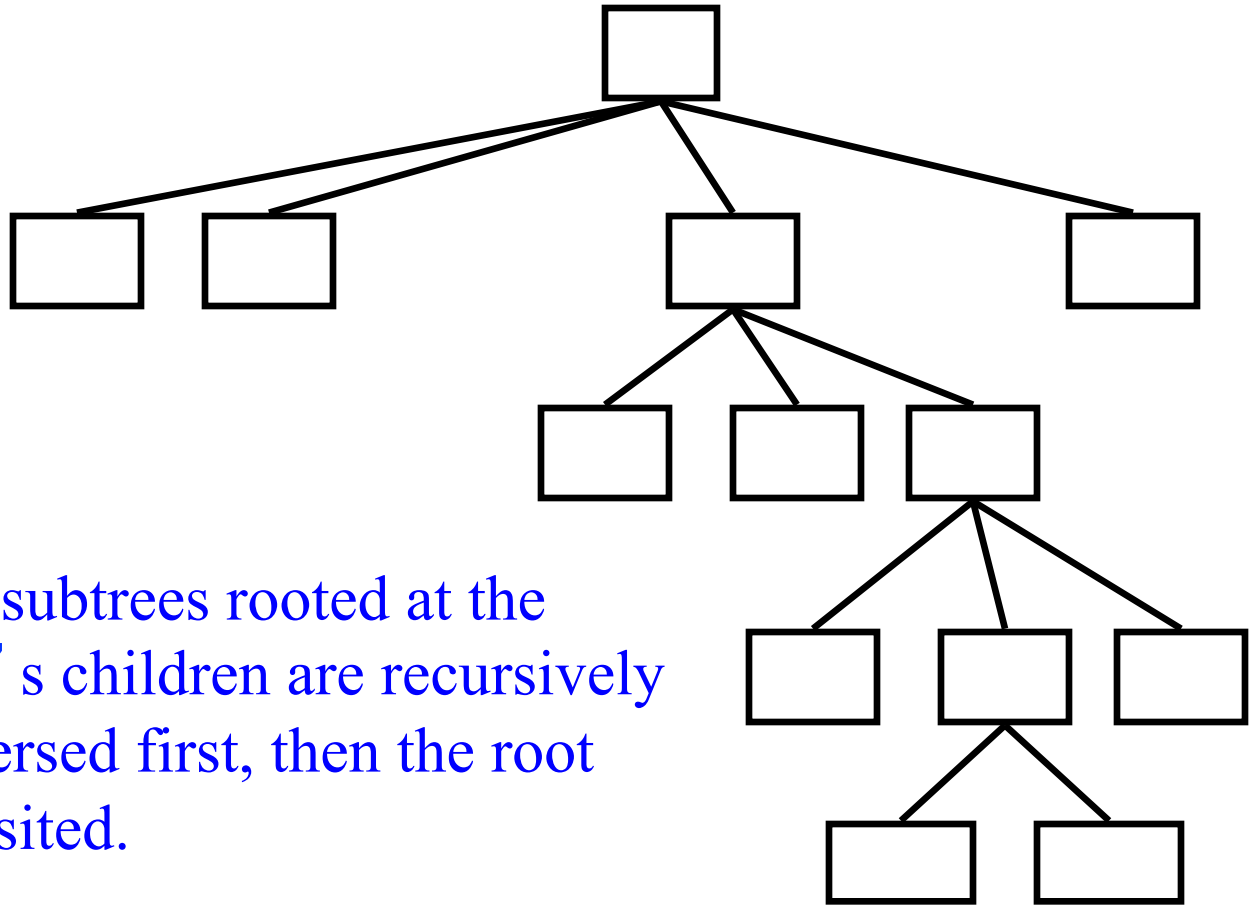
# Depth First Search



The root is visited first, then the subtrees rooted at the root's children are traversed recursively.

# Postorder Traversal



The subtrees rooted at the root's children are recursively traversed first, then the root is visited.

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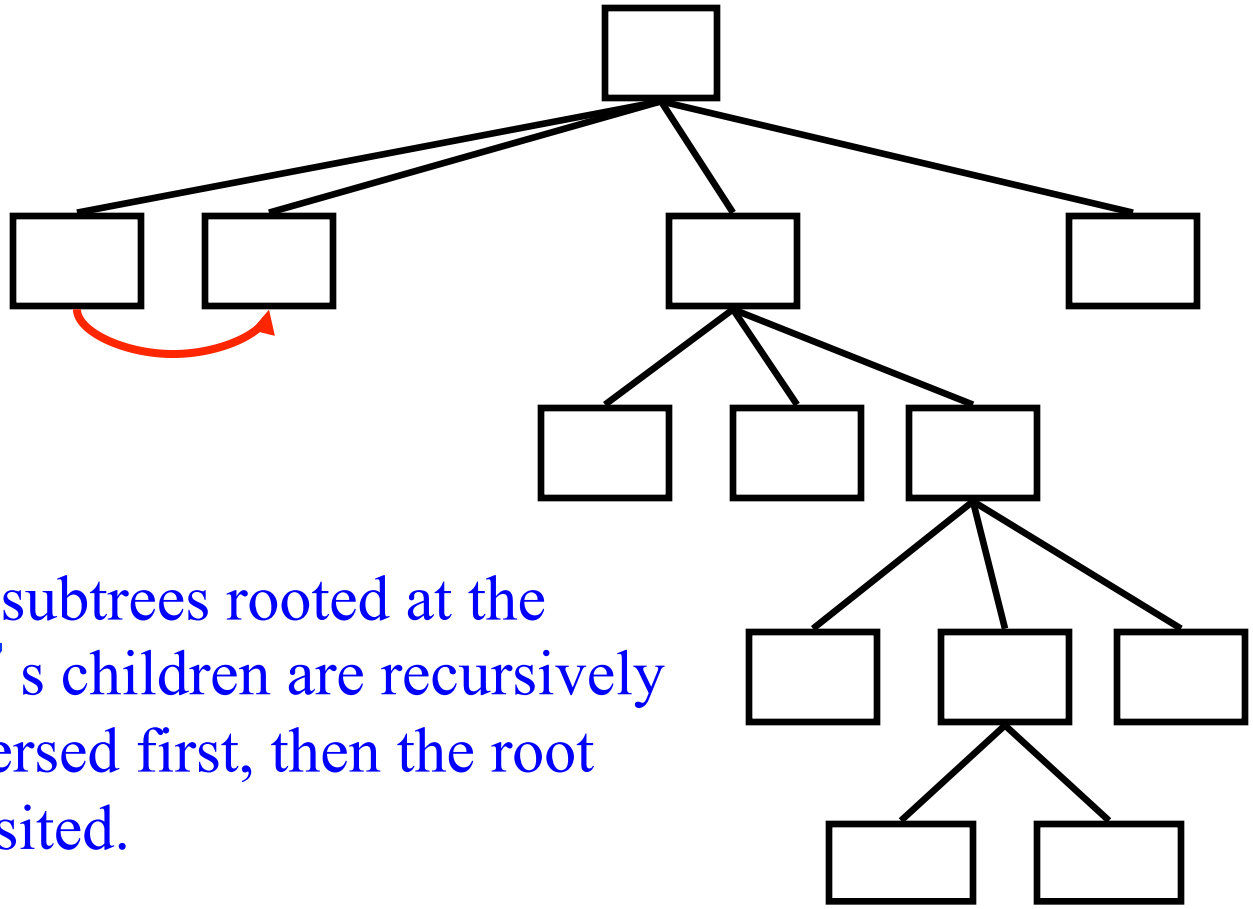
# Postorder Traversal



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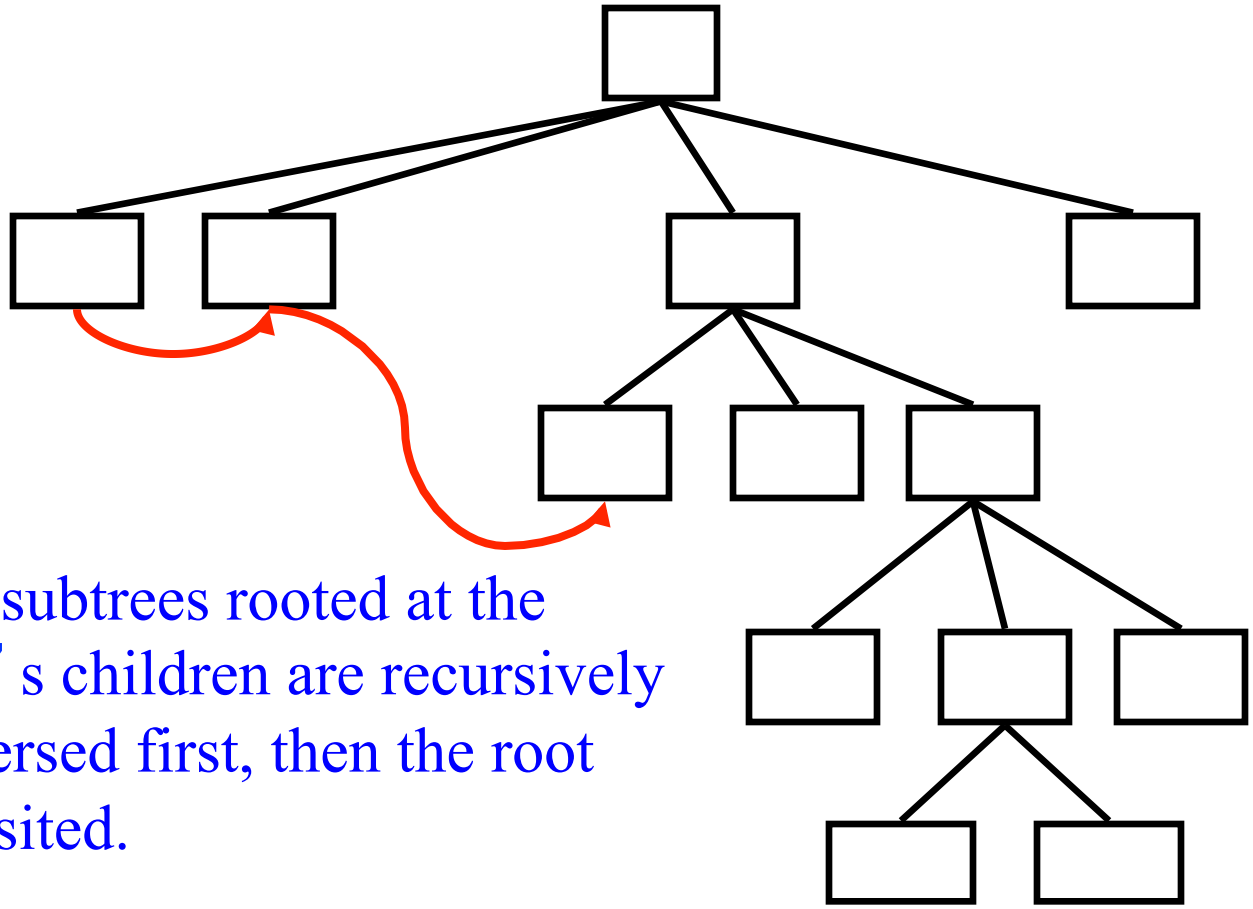
# Postorder Traversal



The subtrees rooted at the root's children are recursively traversed first, then the root is visited.

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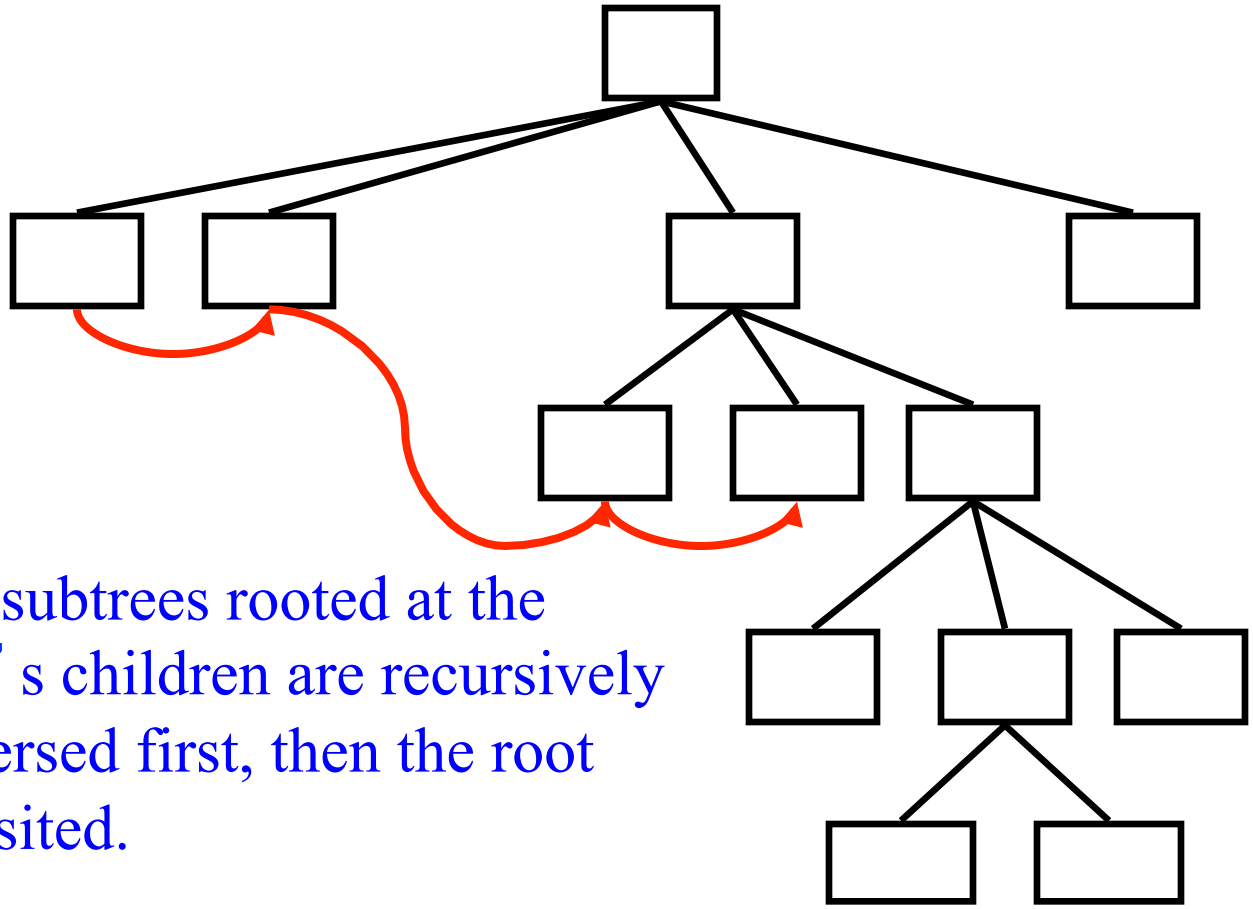
# Postorder Traversal



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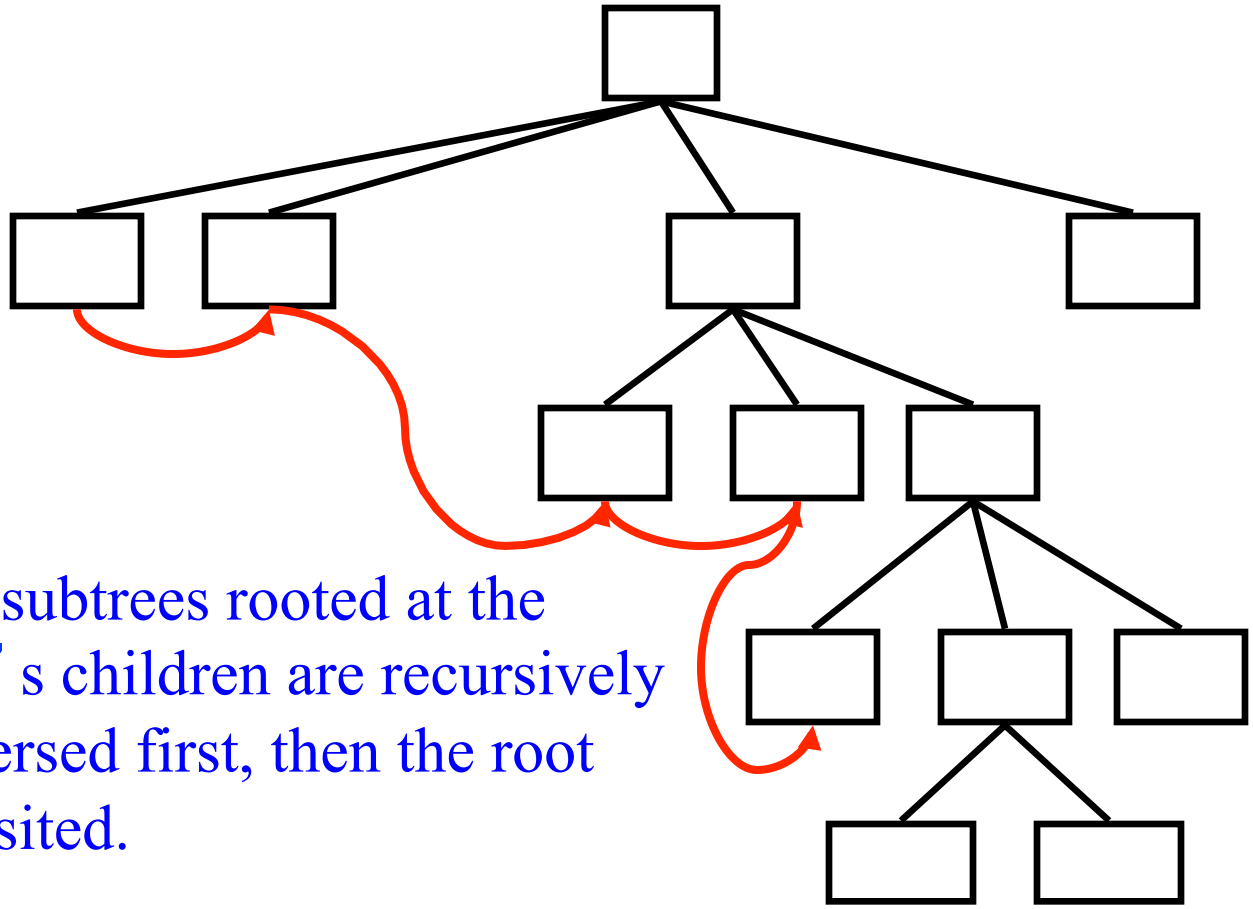

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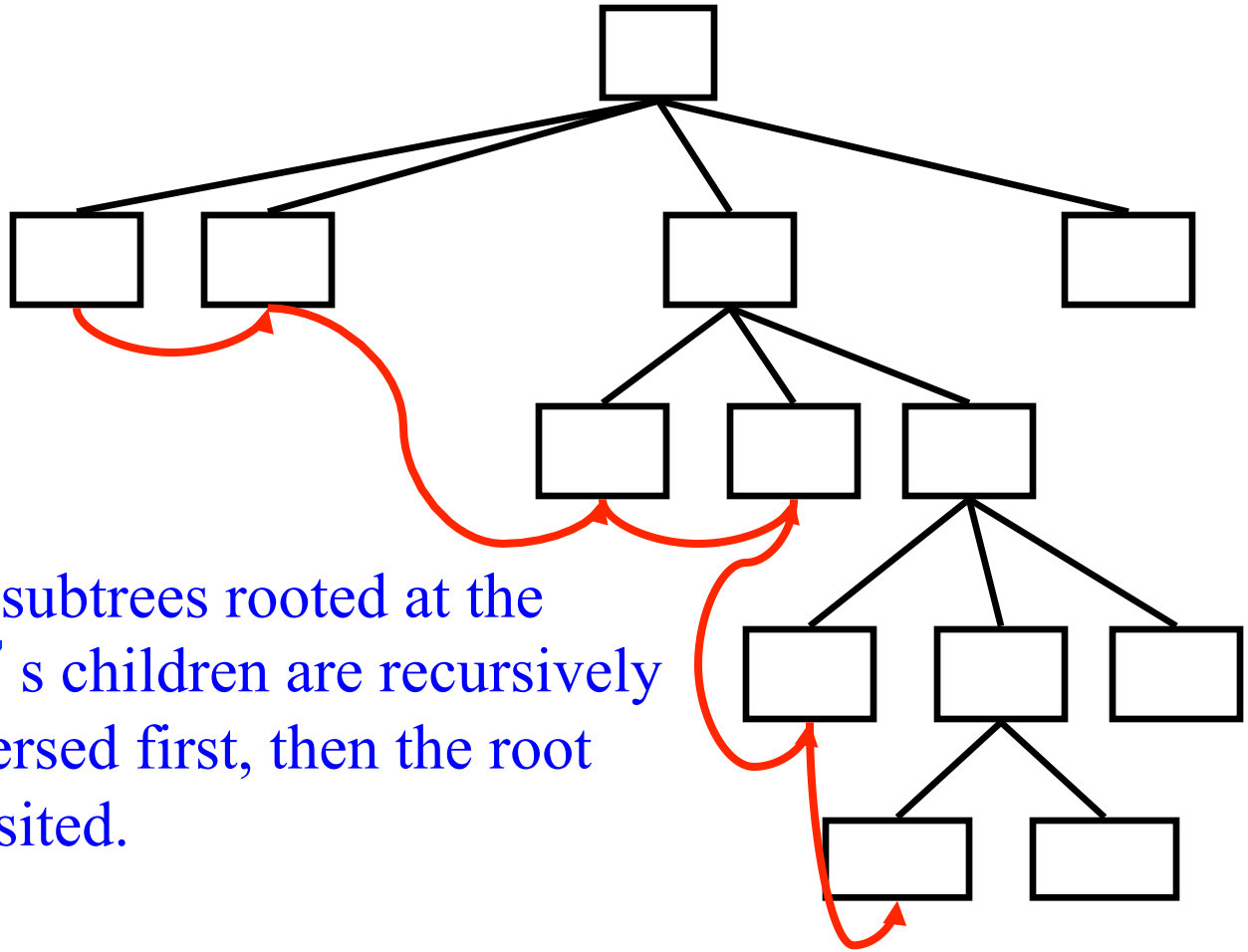

The subtrees rooted at the root's children are recursively traversed first, then the root is visited.

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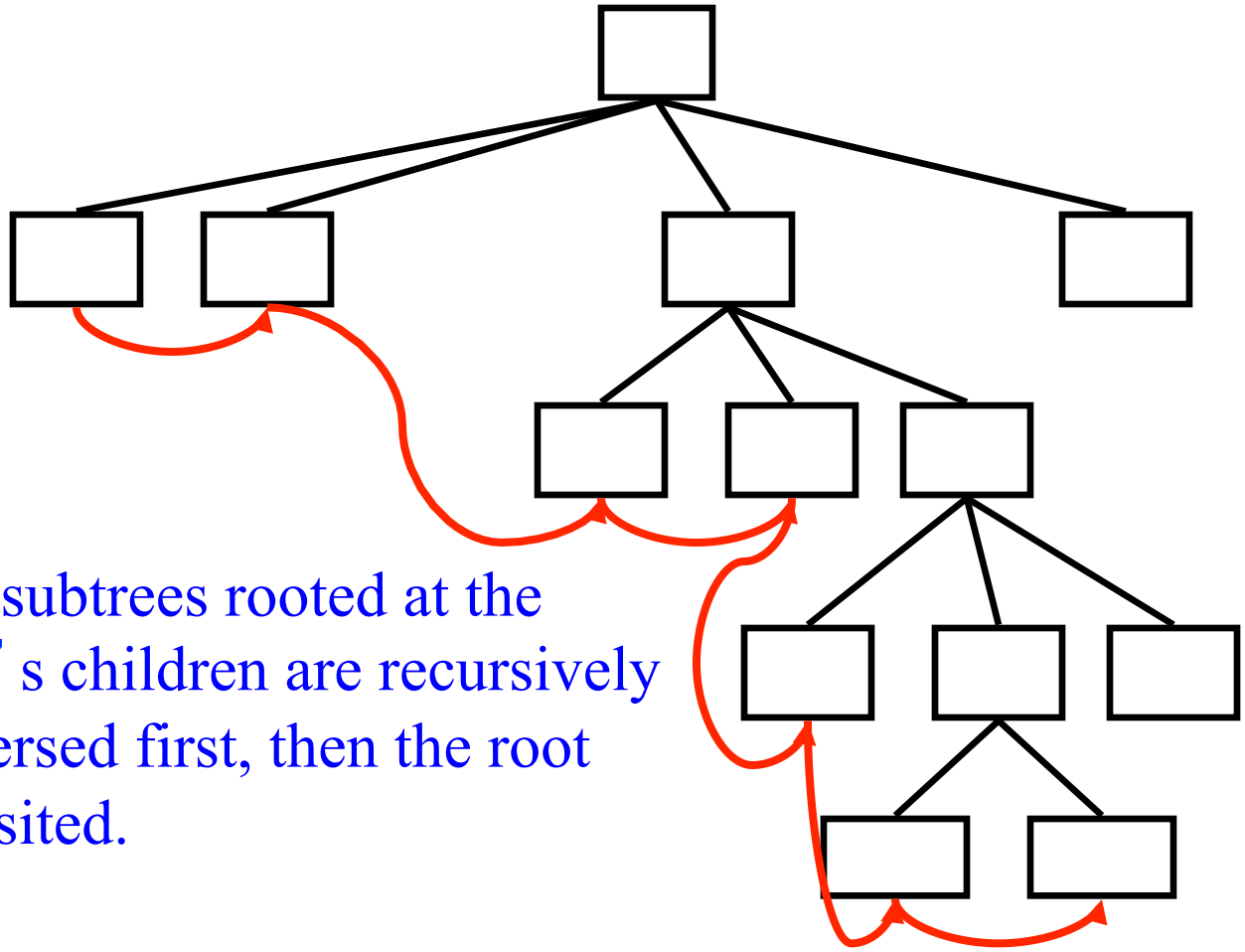

The subtrees rooted at the root's children are recursively traversed first, then the root is visited.

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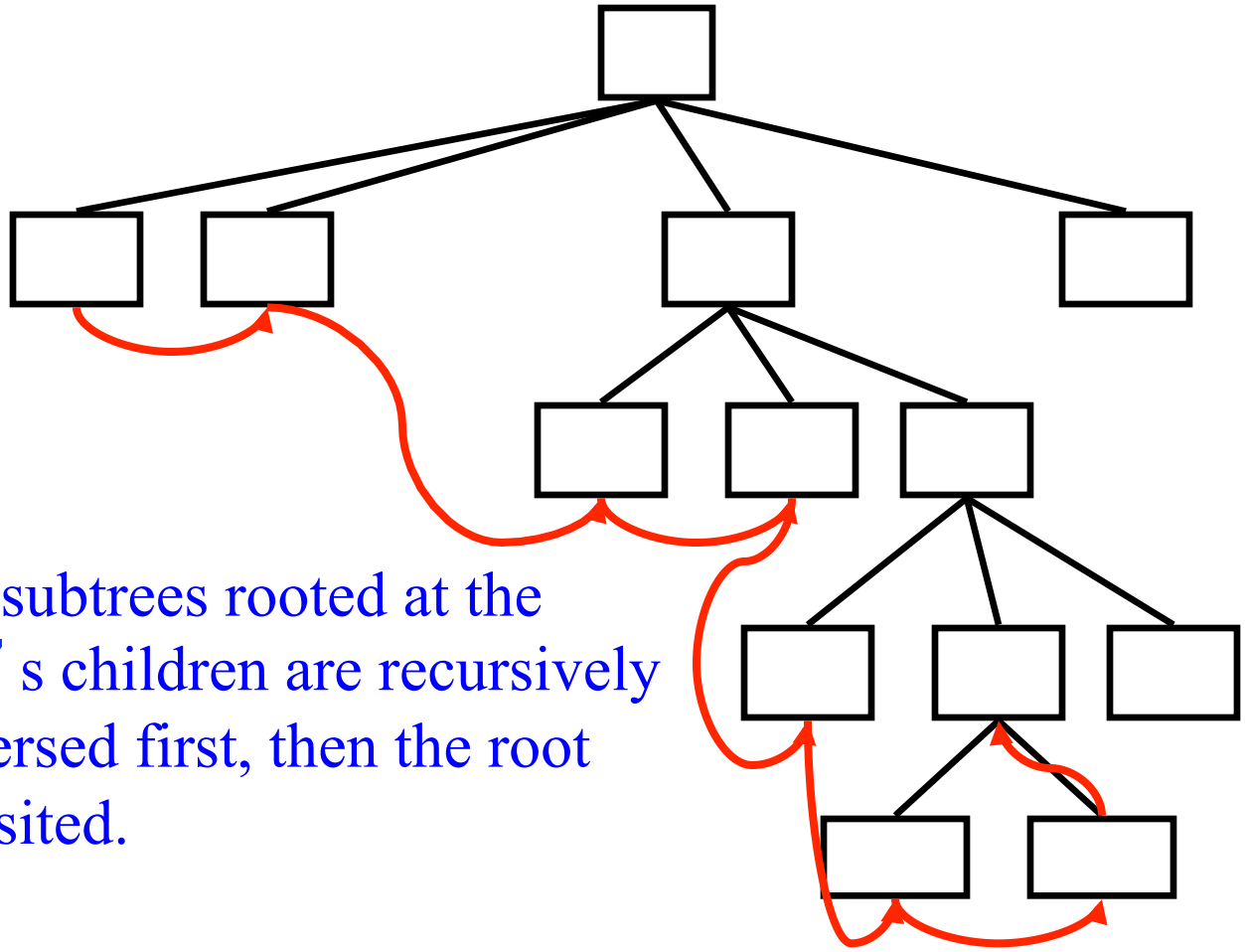

The subtrees rooted at the root's children are recursively traversed first, then the root is visited.

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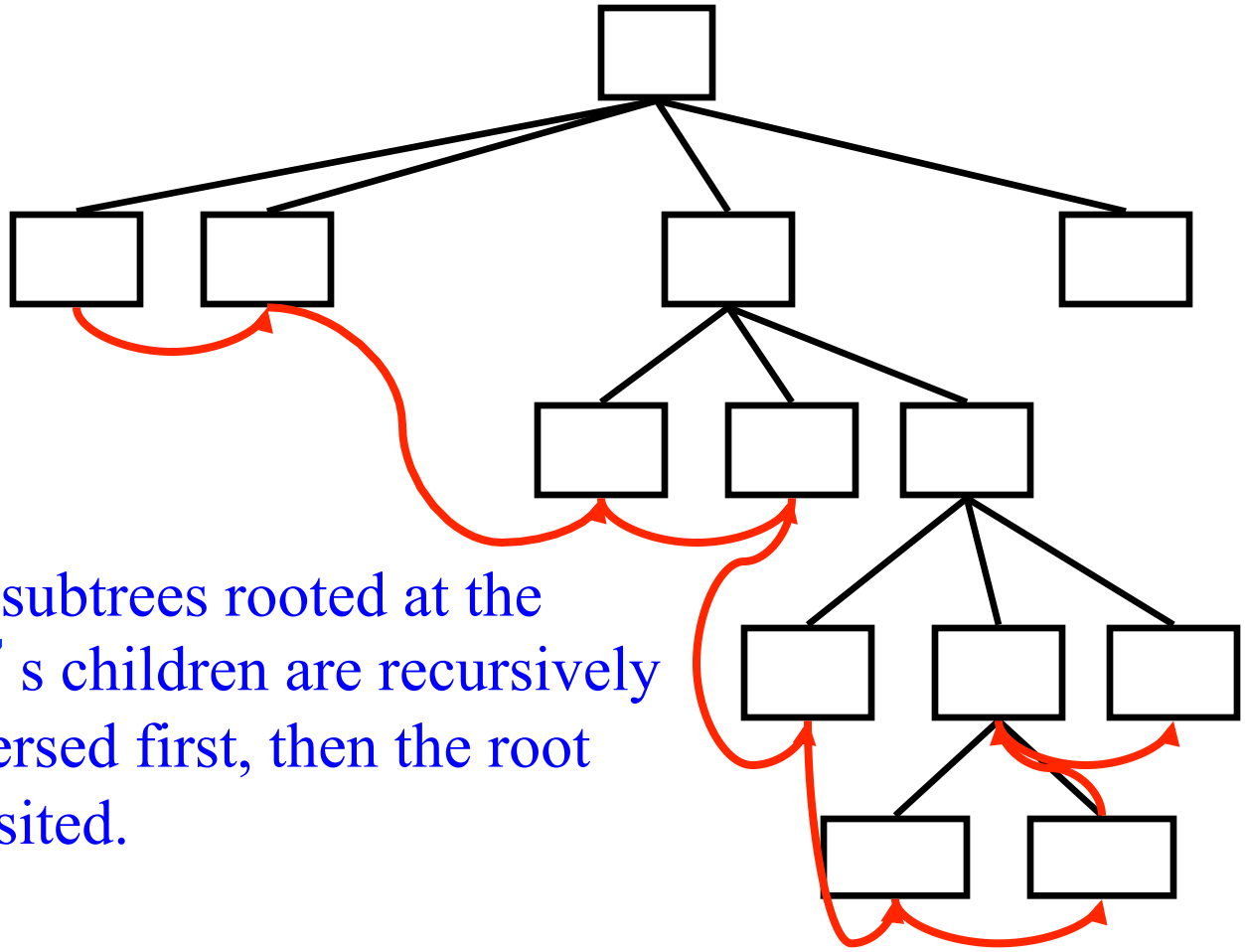

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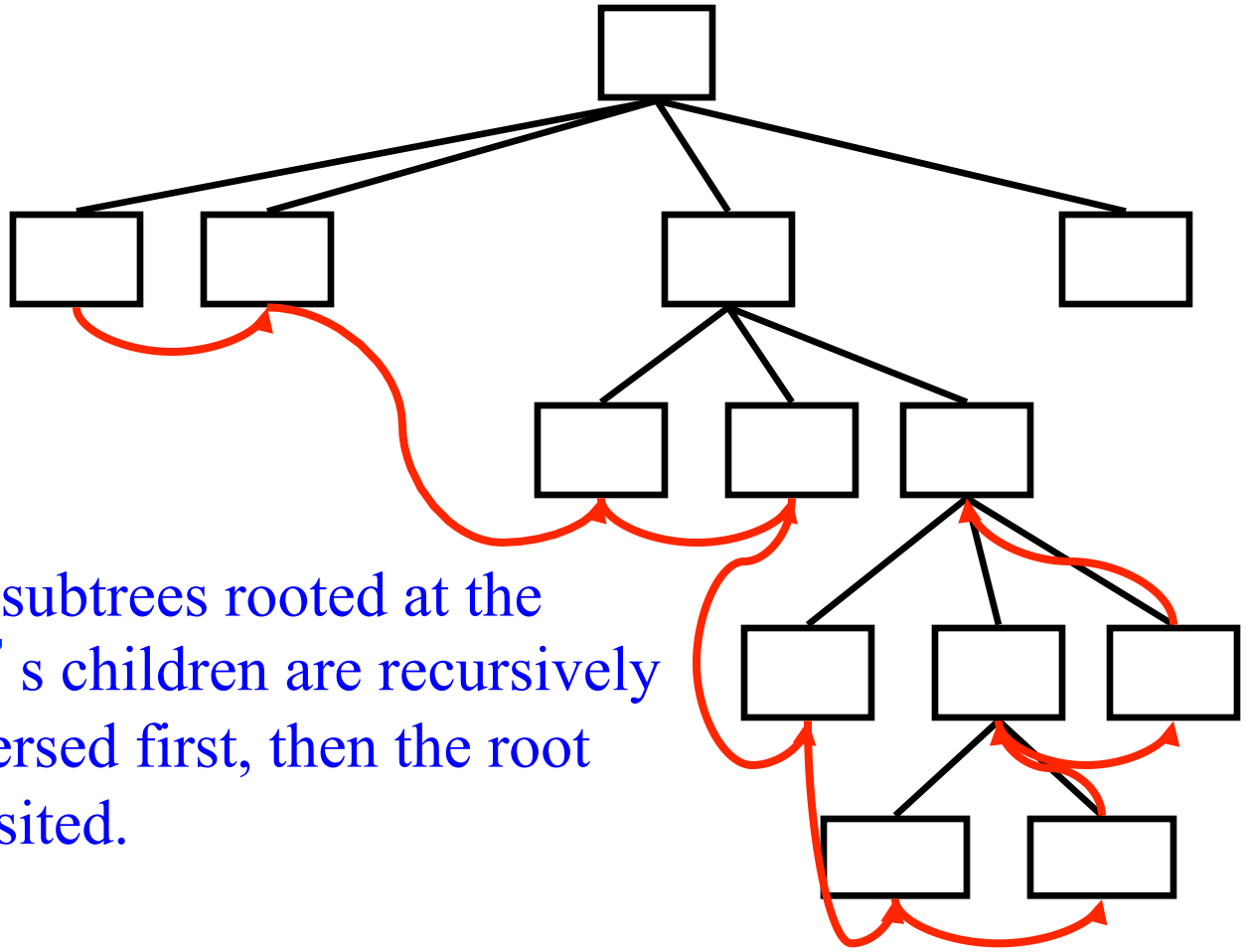

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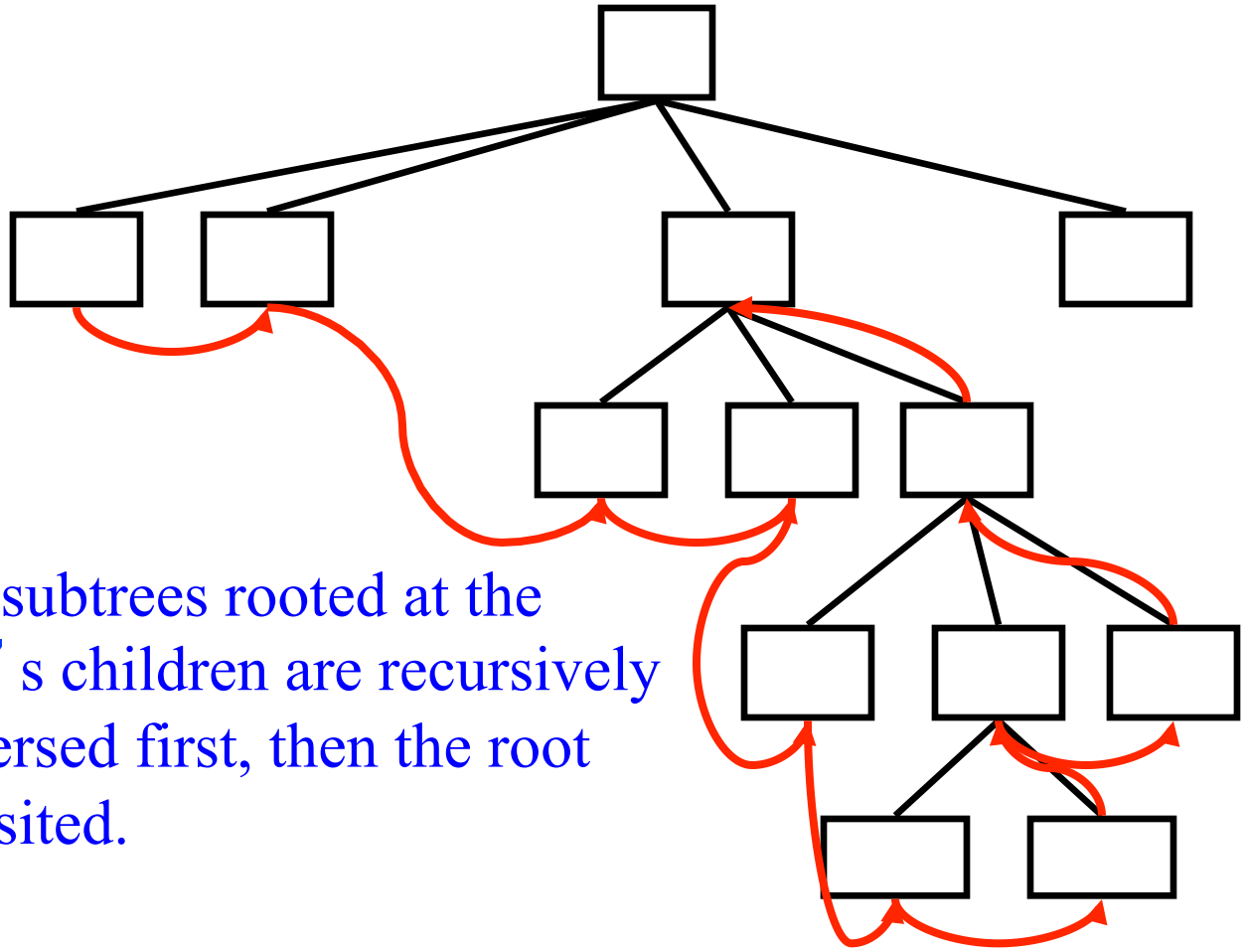

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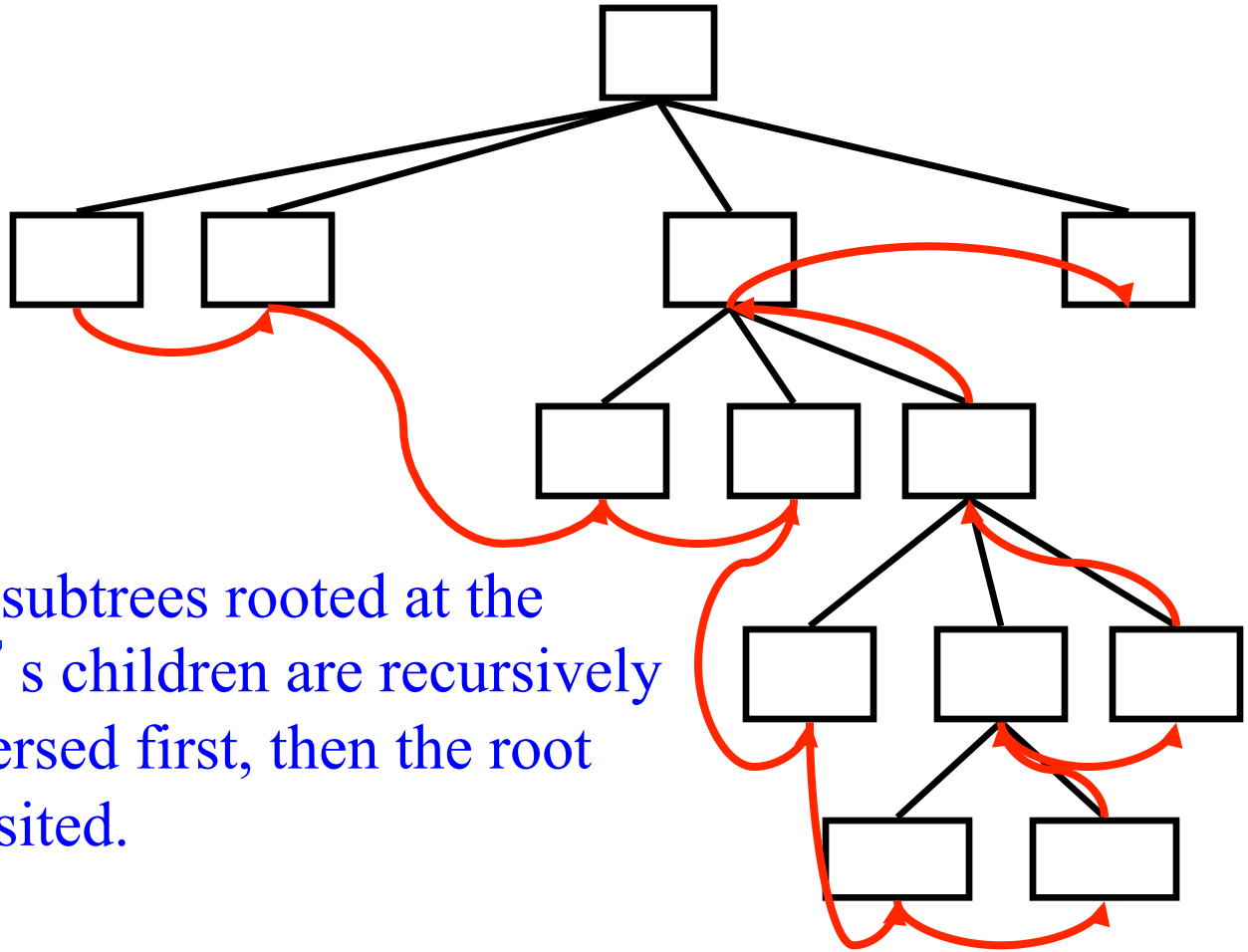

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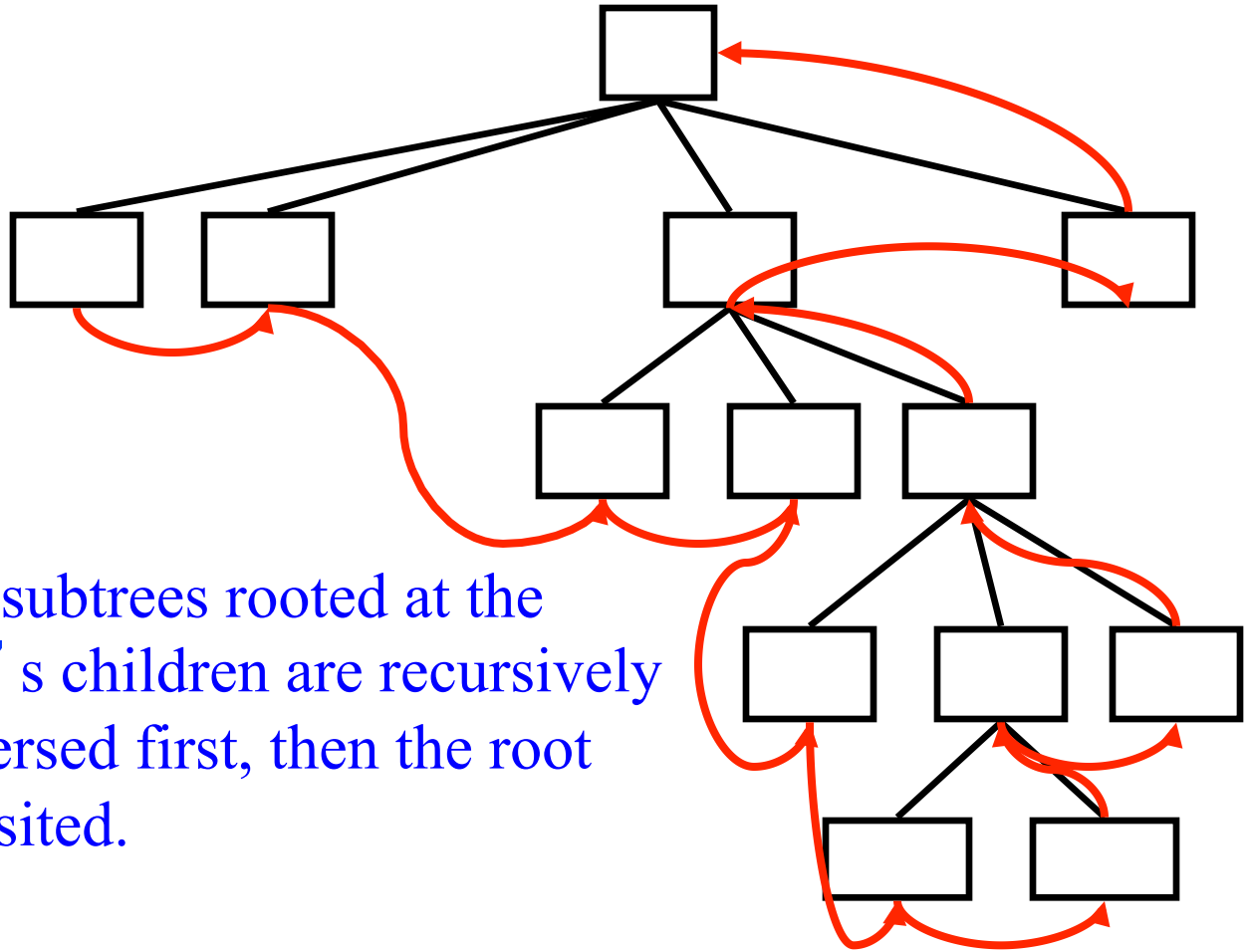

The subtrees rooted at the root's children are recursively traversed first, then the root is visited.

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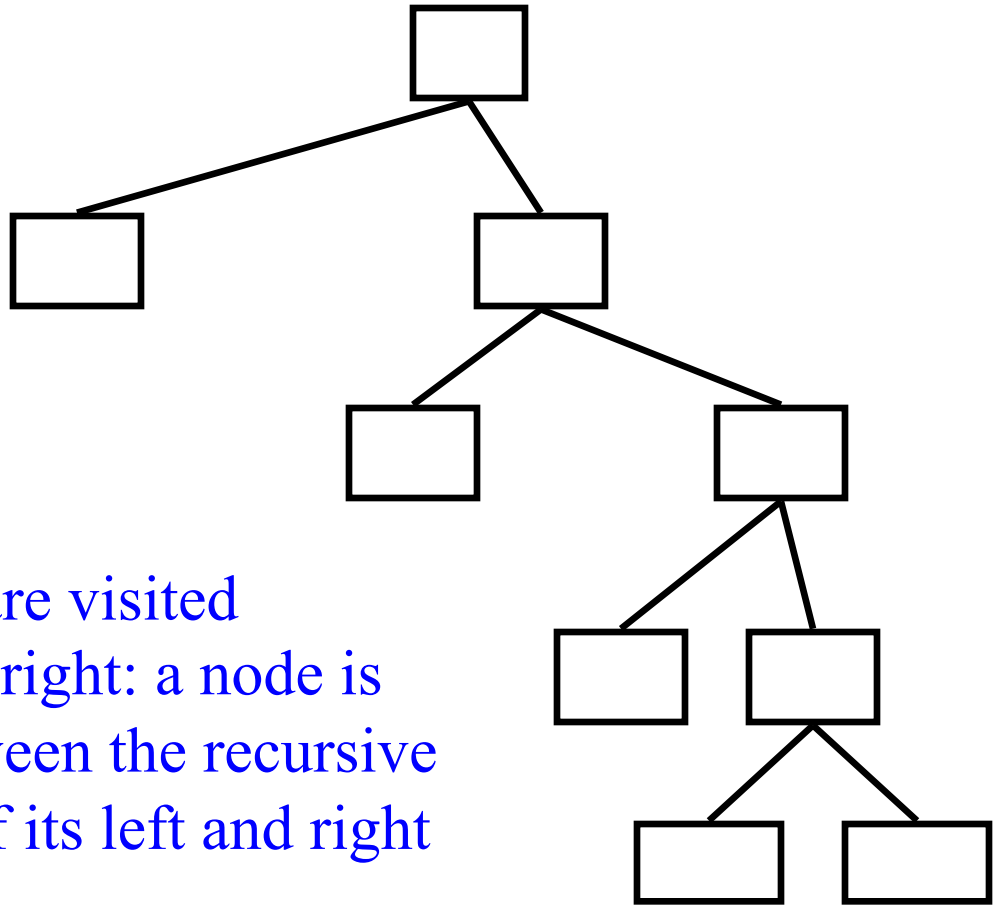
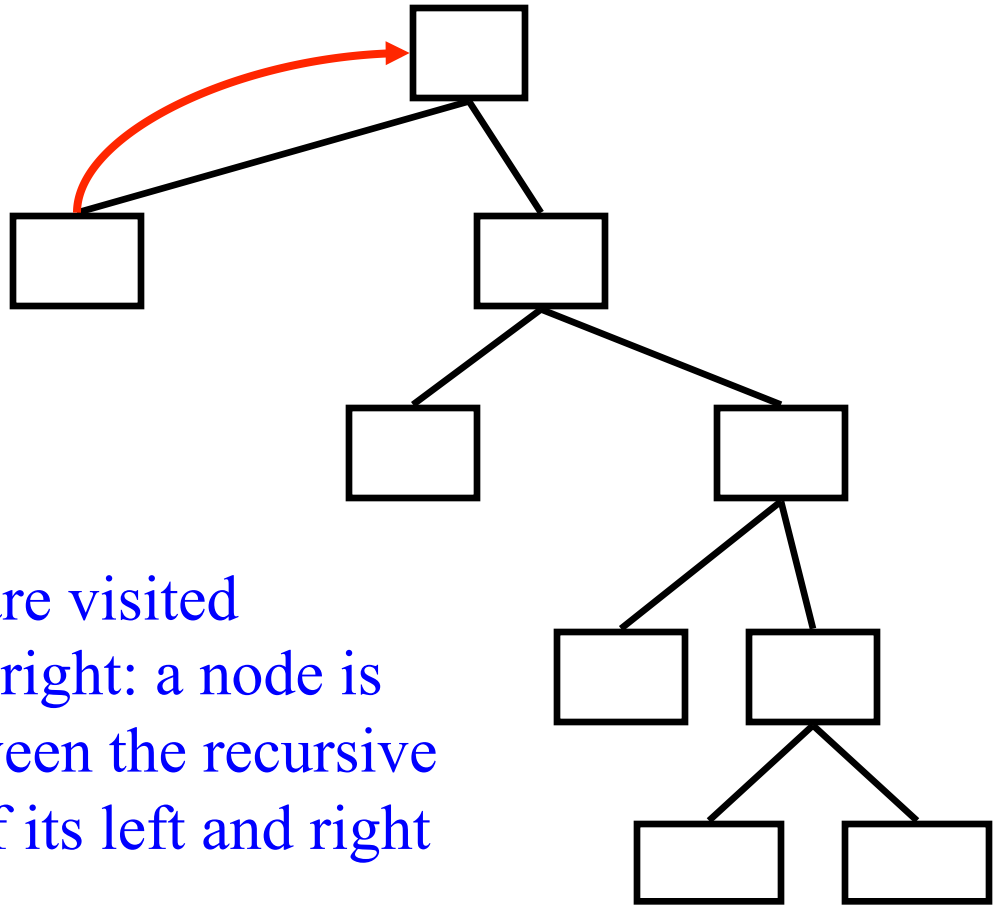
# Inorder Traversal for Binary Trees



The nodes are visited from left to right: a node is visited between the recursive traversals of its left and right subtrees.

# Inorder Traversal for Binary Trees



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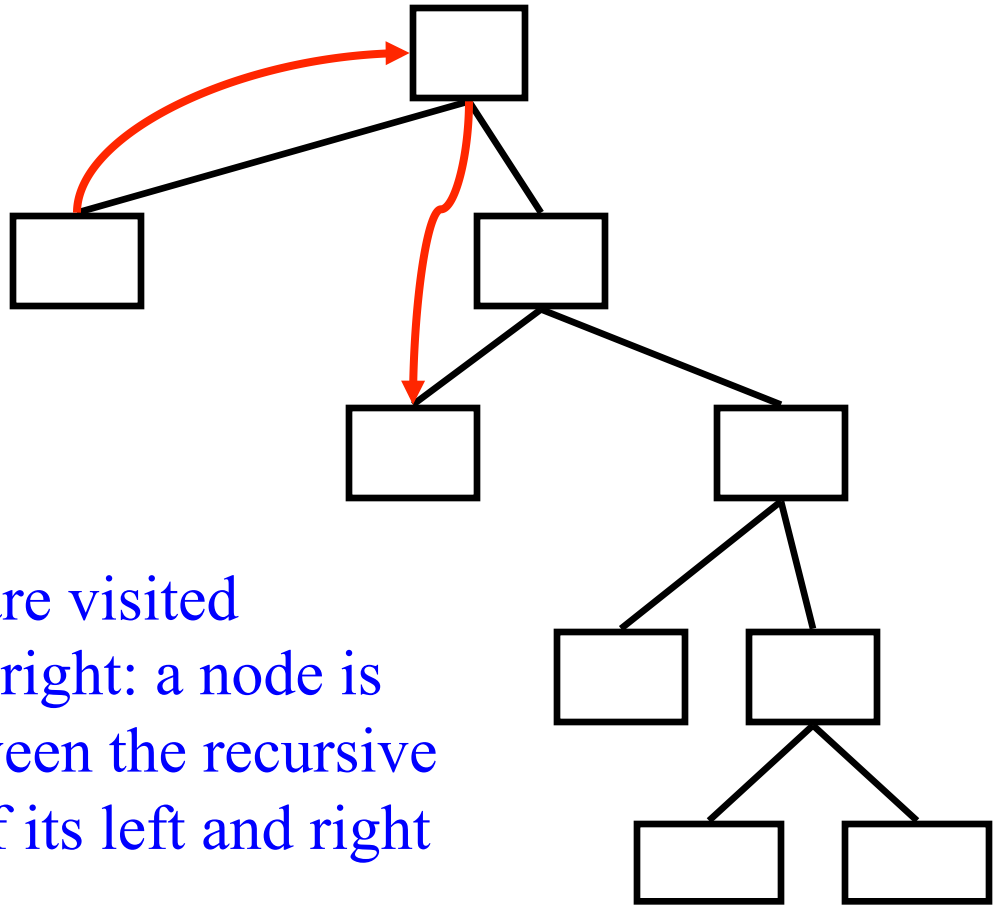

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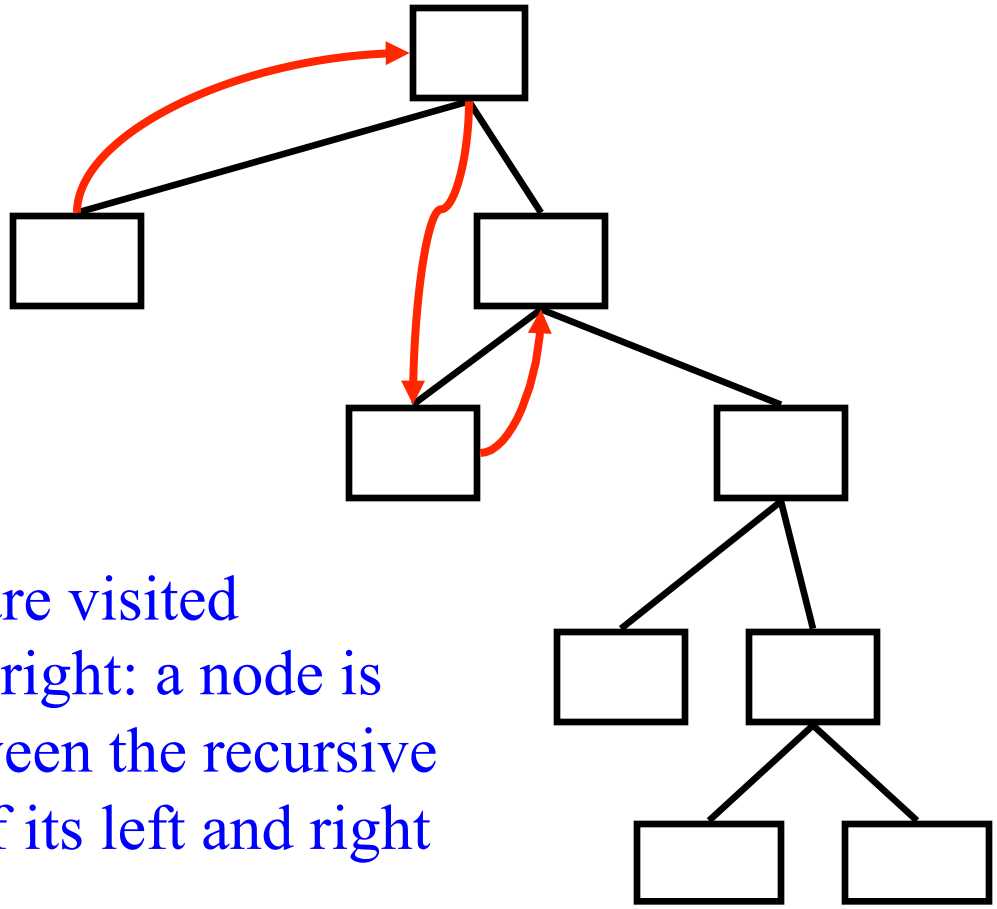

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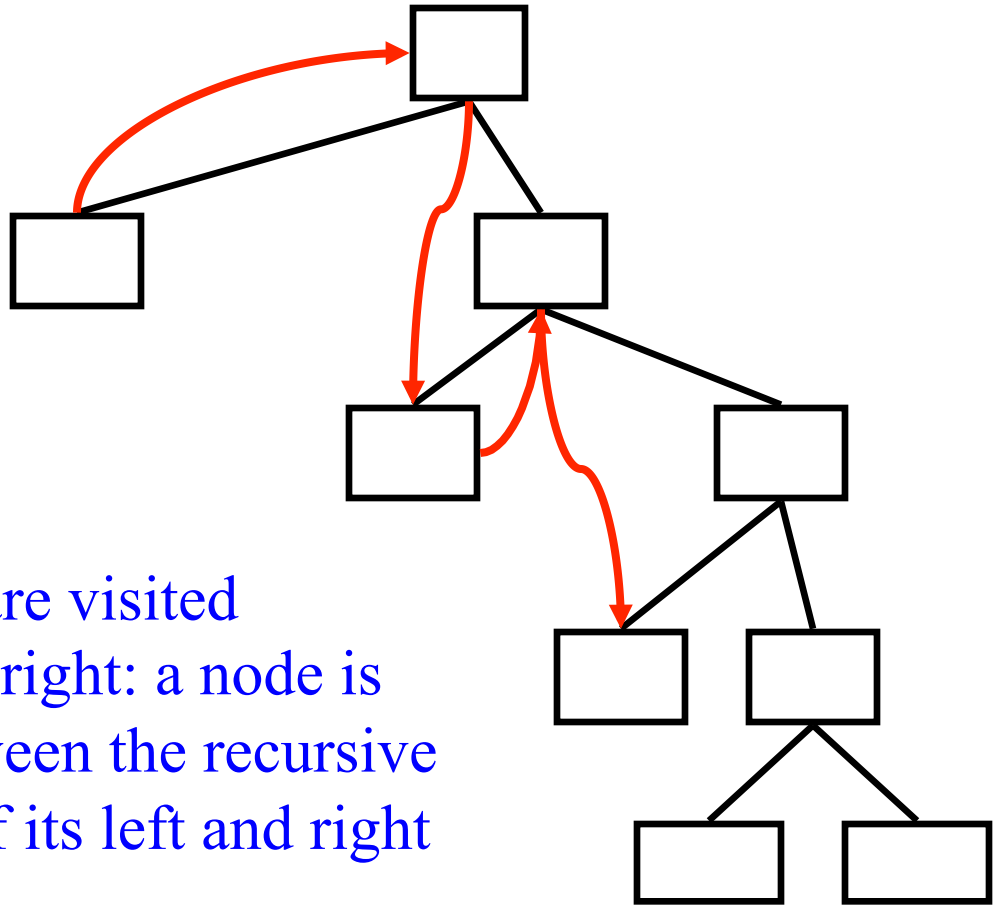

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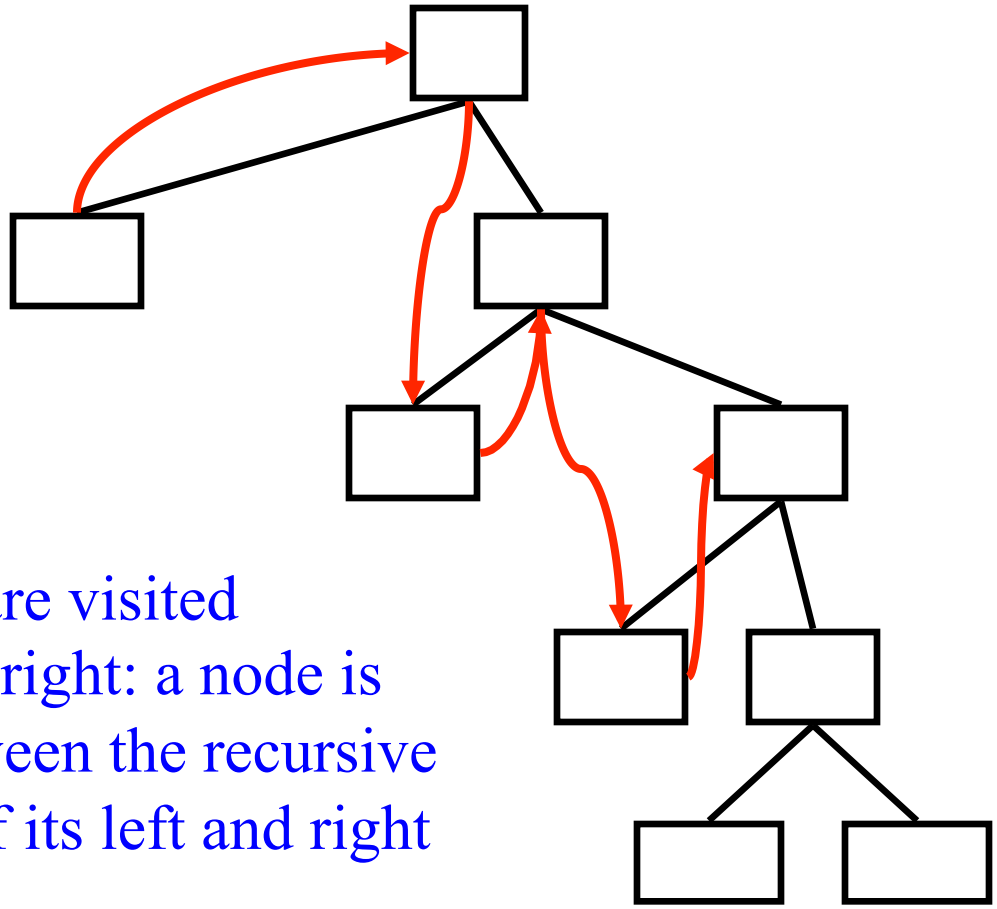
# Inorder Traversal for Binary Trees



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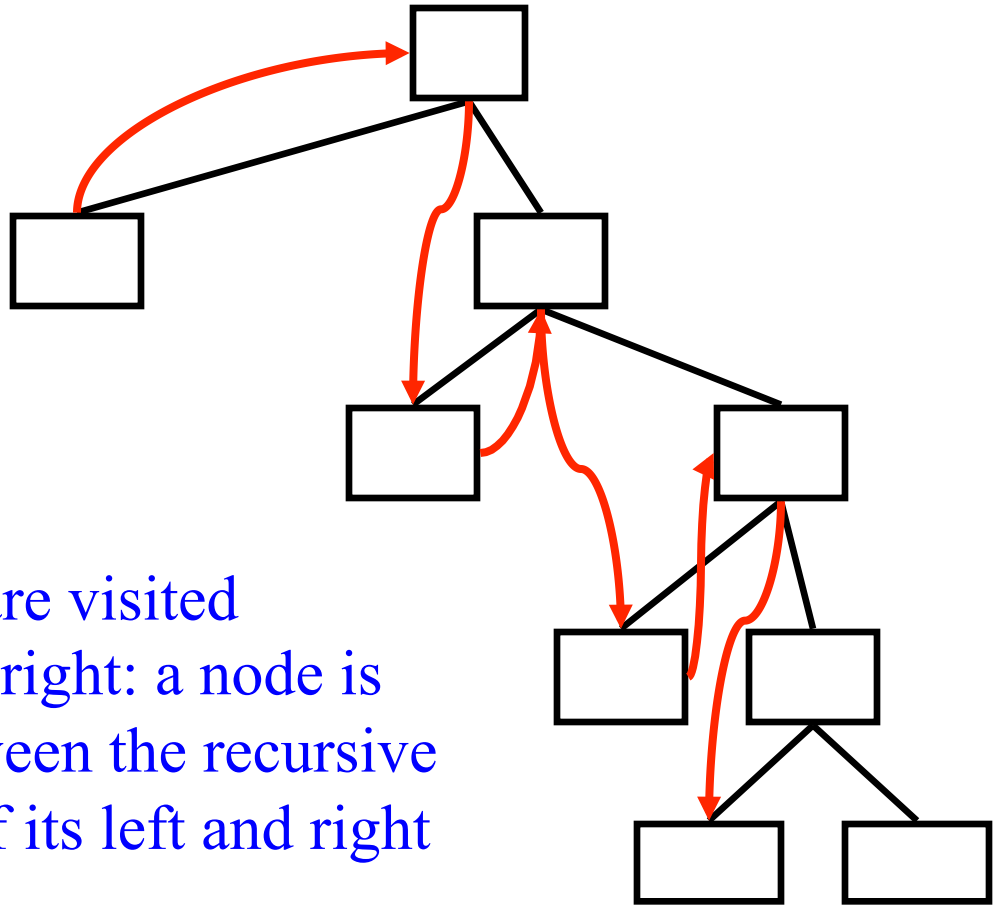
# Inorder Traversal for Binary Trees



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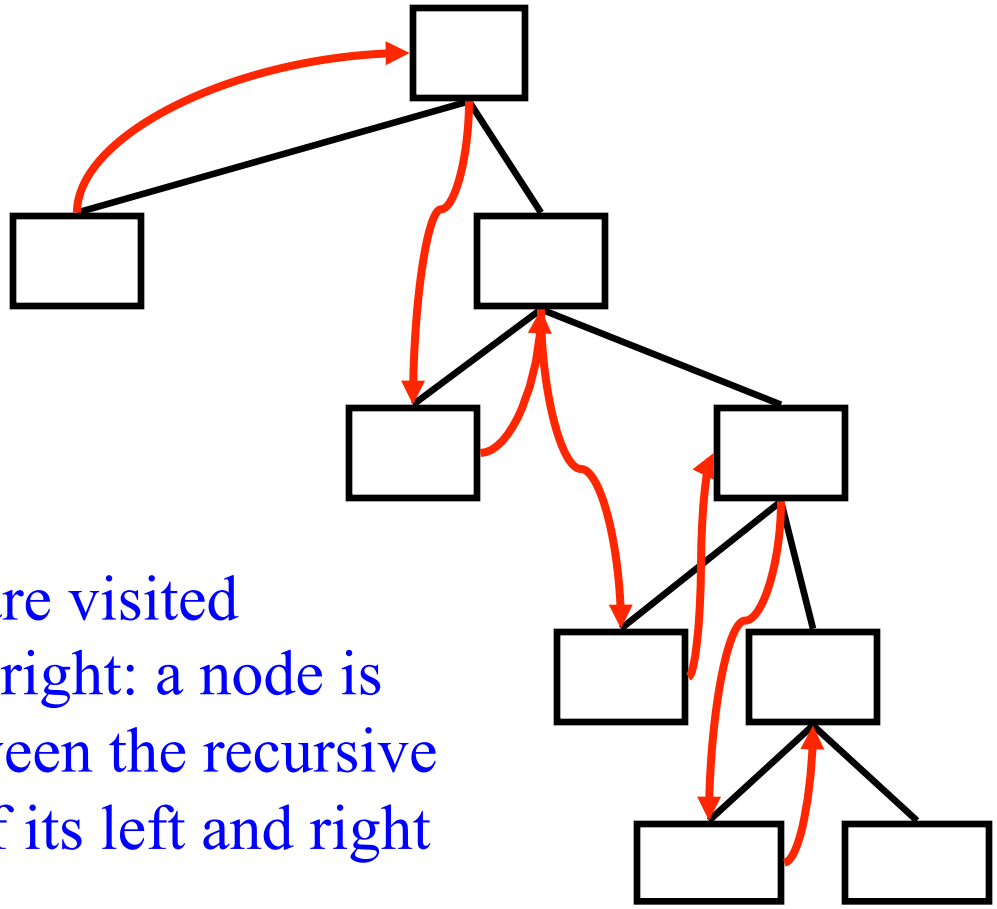
# Inorder Traversal for Binary Trees



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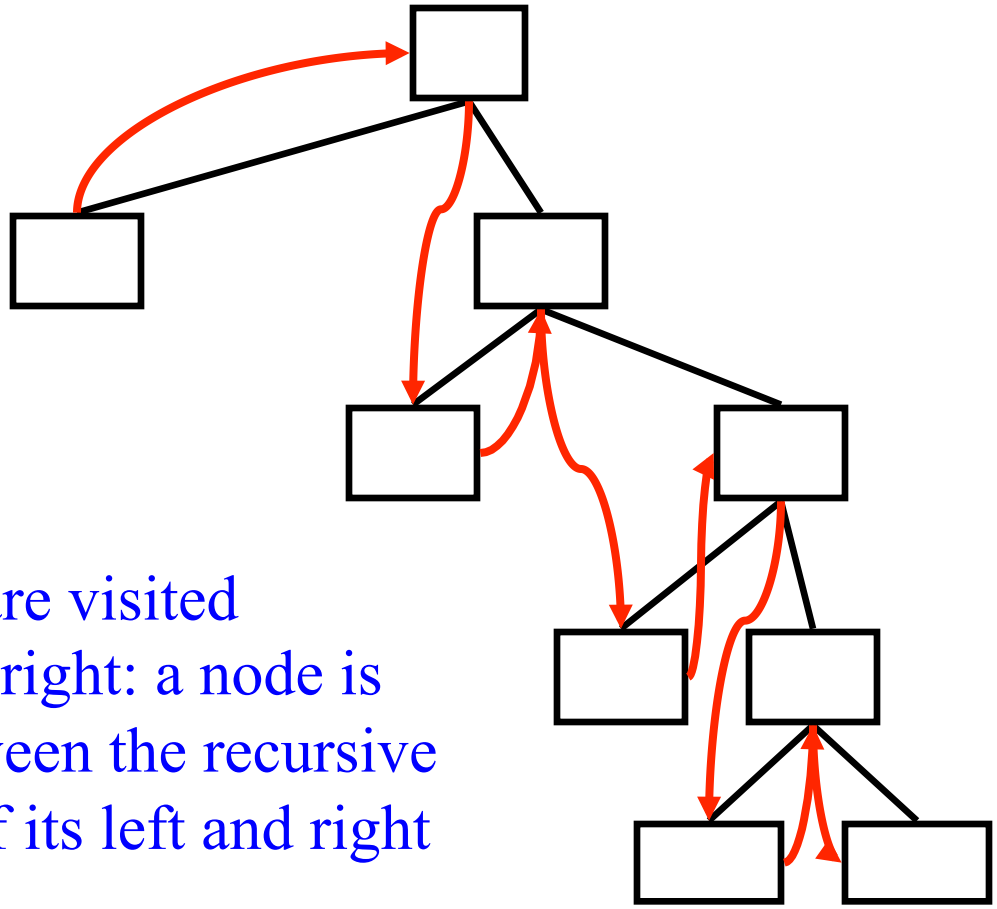
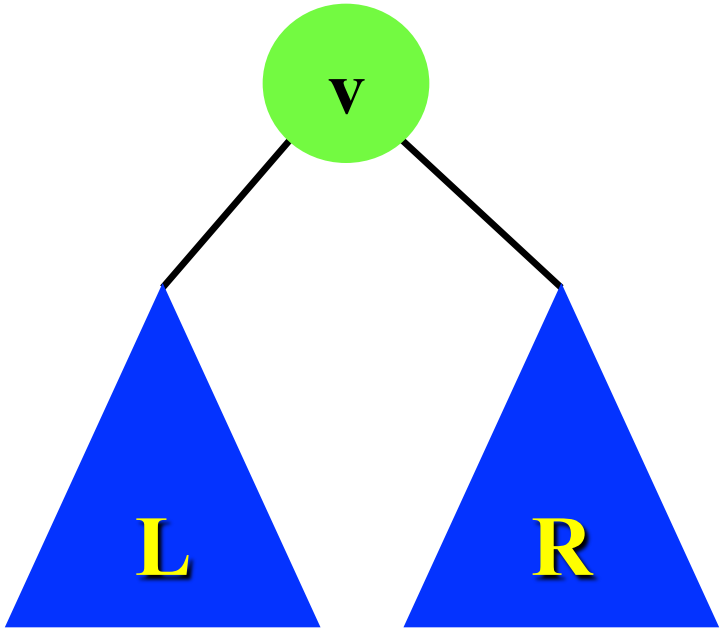
# Inorder Traversal for Binary Trees



The nodes are visited from left to right: a node is visited between the recursive traversals of its left and right subtrees.

# Tree Traversals




- Preorder
  - v, L, R
- Postorder
  - L, R, v
- Inorder
  - L, v, R

# Tree Representations

- Dynamic data structures
  - Using references or pointers
- Heap encoding
  - Using an array

# Dynamic Tree Data Structure

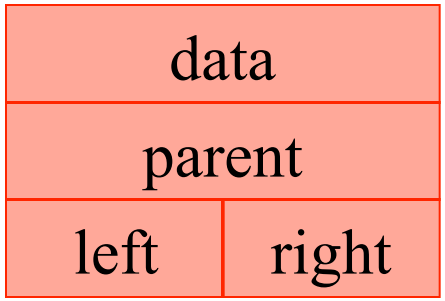
```
public class TreeNode {
      private TreeNode left;
      private TreeNode right;
      private TreeNode parent;
      private int data;

      void setLeft(TreeNode t) { left = t;}
      TreeNode getLeft() { return left; }

      void setRight(TreeNode t) { right = t;}
      TreeNode getRight() { return right; }

      void setParent(TreeNode p) { parent = p;}
      TreeNode getParent() { return parent; }

      void setData(int d) { data = d;}
      int getData() { return data; }
}
```



# Preorder Traversal

```
algorithm preorder(Tree t)
   if t <> null then
        processNode(t.v)
        preorder(t.left)
        preorder(t.right)
    end
end
```

# Inorder Traversal

```
algorithm inorder(Tree t)
    if t <> null then
        inorder(t.left)
        processNode(t.v)
        inorder(t.right)
    end
end
```

# Postorder Traversal

```
algorithm postorder(Tree t)
    if t <> null then
        postorder(t.left)
        postorder(t.right)
        processNode(t.v)
    end
end
```

# Running Time of Tree Traversals

- Each node is visited a fixed number of times
- **Theorem**
  - The time complexity of Preorder, Inorder, Postorder is $O(n)$

# Heap Encoding

Parent of A[k] is at A[k/2]

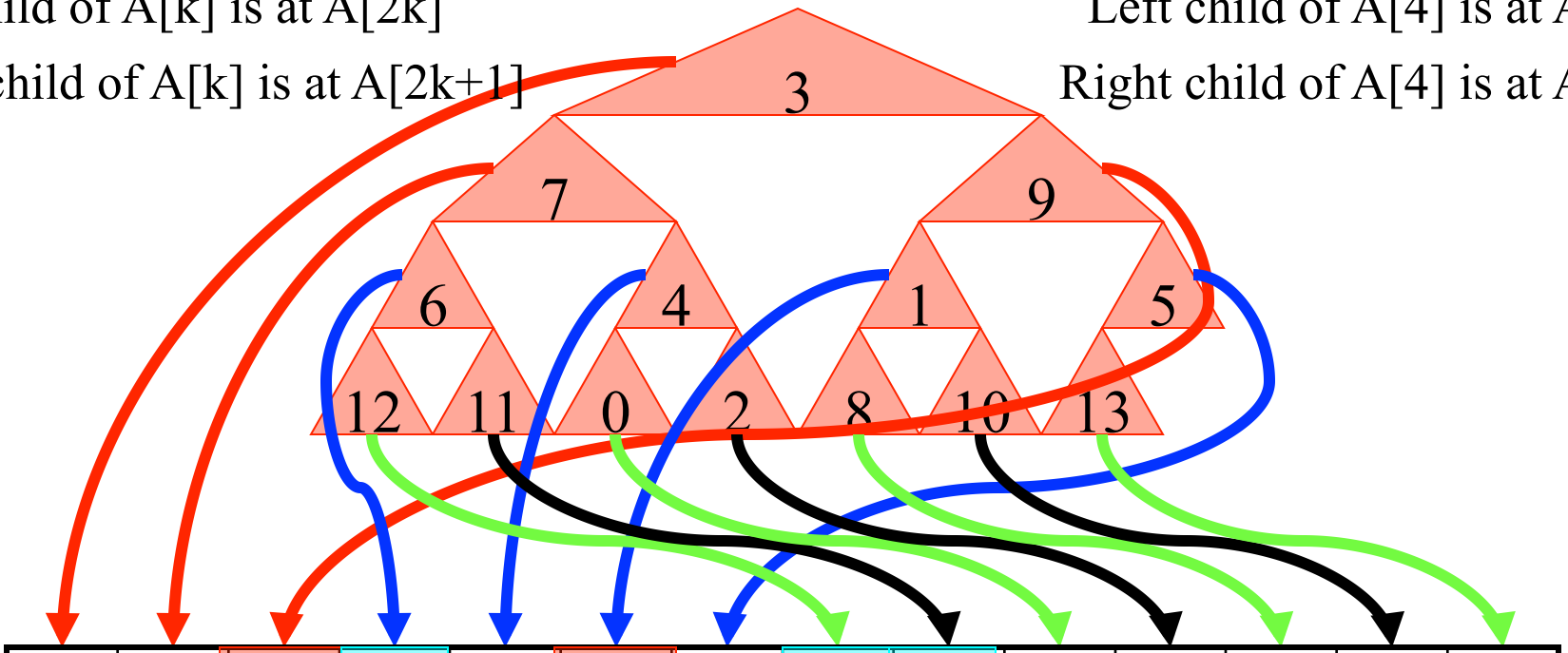
Left child of A[k] is at A[2k]

Right child of A[k] is at A[2k+1]

Parent of A[6] is at A[3]

Left child of A[4] is at A[8]

Right child of A[4] is at A[9]




| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| A | 3 | 7 | 9 | 6 | 4 | 1 | 5 | 12 | 11 | 0 | 2 | 8 | 10 | 13 |
| index | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |

Note index starts at 1

# Data Structures for *binary* Trees

| Operation | Time with array-based structure | Time with linked structure |
|---|---|---|
| positions, elements traversals (iterators): pre-, in-, post-, level-order, Euler tour | $O(n)$ | $O(n)$ |
| size, isEmpty | O(1) | O(1) |
| swapElements, replaceElement | O(1) | O(1) |
| leftChild, rightChild, sibling | O(1) | O(1) |
| isInternal, isExternal, isRoot | O(1) | O(1) |
| root, parent, child | O(1) | O(1) |

# Priority Queues

- A *priority queue (PQ)* is a container of elements, each having an associated *key*
- The keys determine the *priority* used to remove elements (i.e., deleteMin or deleteMax)
- The keys of the elements in a PQ must satisfy the *total order properties*
- The operations of a PQ are
  - insert
  - deleteMin / deleteMax (just one!)

# ADT Priority Queue (PQ)

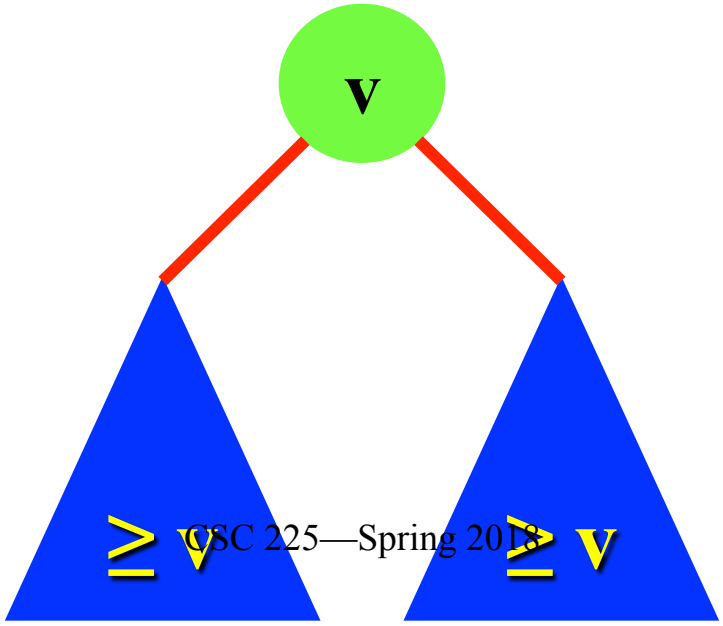
- **insertItem(*k*, *e*):** insert an element *e* with key *k* into PQ
- **removeMin():** return and remove from PQ an element with the smallest key
- **minElement():** return (but do not remove) an elment of PQ with the smallest key
- **minKey():** return (but do not remove) the smallest key in PQ

# The Heap Data Structure

- A *heap* is a realization of a Priority Queue that is efficient for both insertions and removal of minima

# Heap Definition

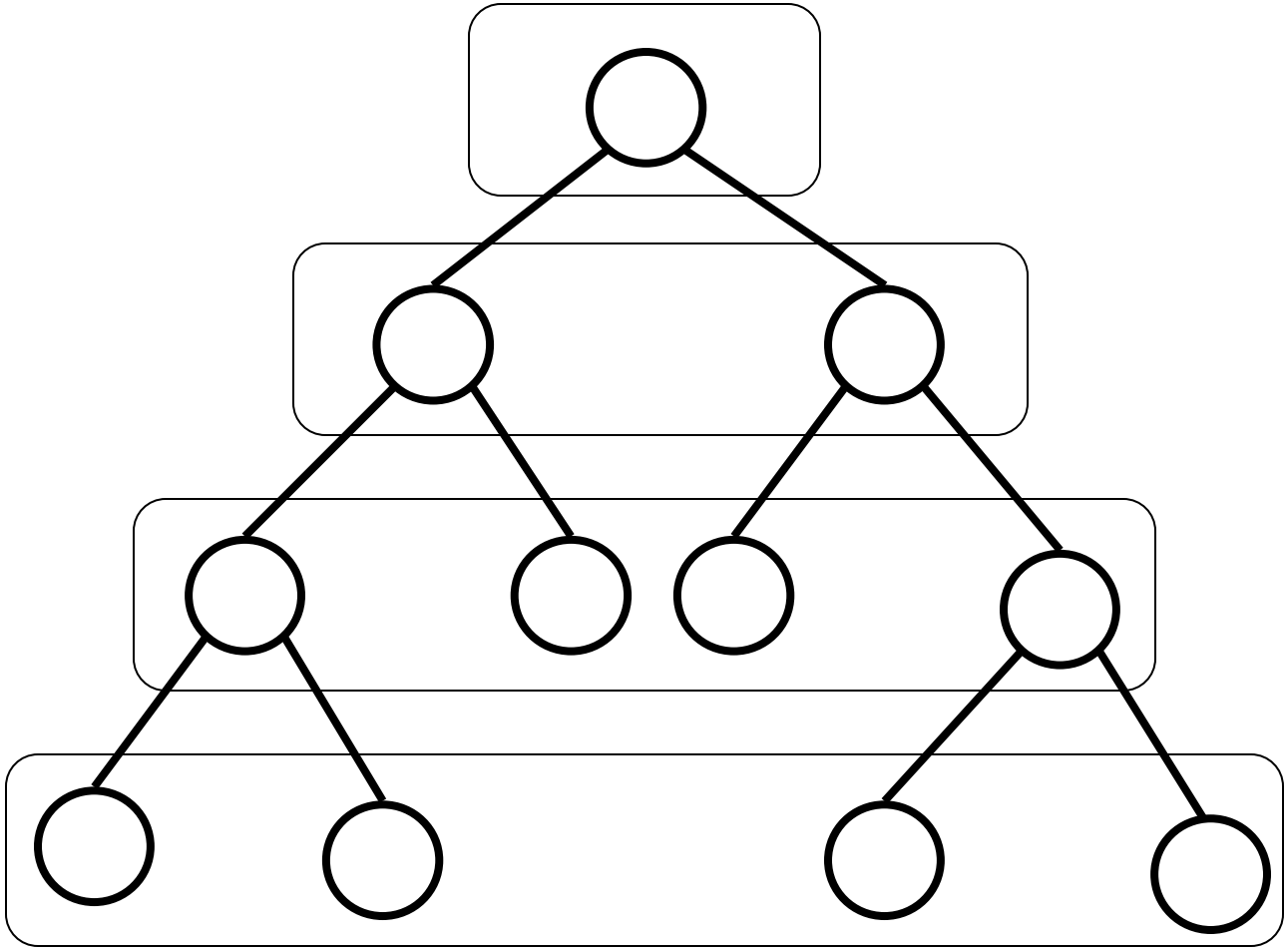
- A *heap* is a complete binary tree that stores a collection of keys at its internal nodes
- A heap satisfies the following properties:
  - *Heap-Order Property*: for every node *v* other than the root, the key stored at *v* is greater than or equal to the key stored at *v*'s parent.



# Complete Binary Tree

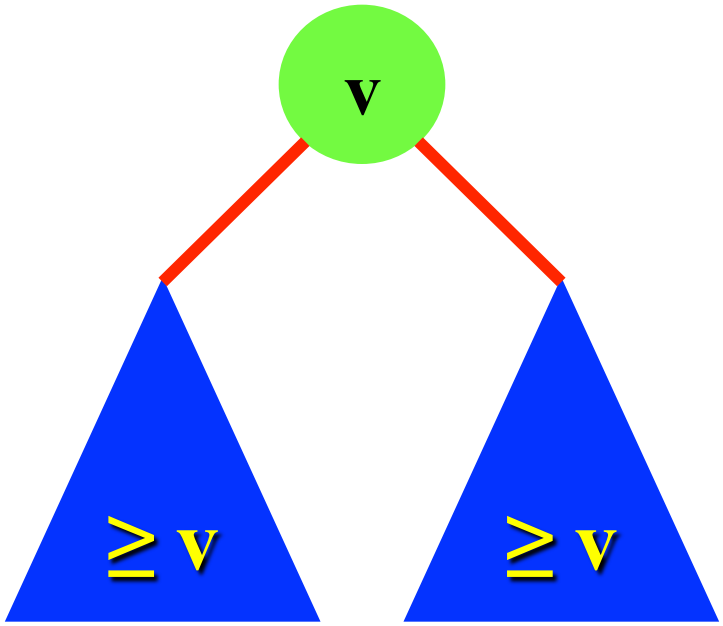
- *Complete Binary Tree*: A binary tree with height *h* is complete if
  - the levels 0, 1, 2,…, *h*-1 have the maximum number of nodes possible and
  - at level *h,* the internal nodes are all to the left of the leaves
- Heap Shape property
  - A heap is a complete binary tree

# Is this a Complete Binary Tree?

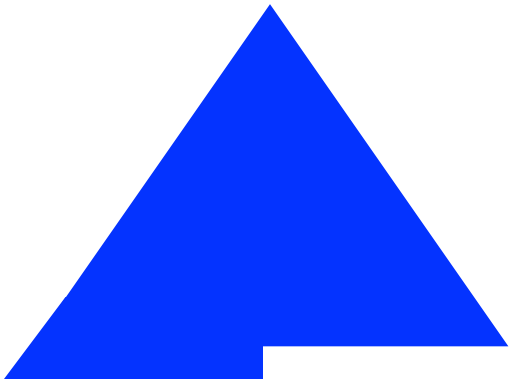


# Heap Properties

- Order property




- Shape property

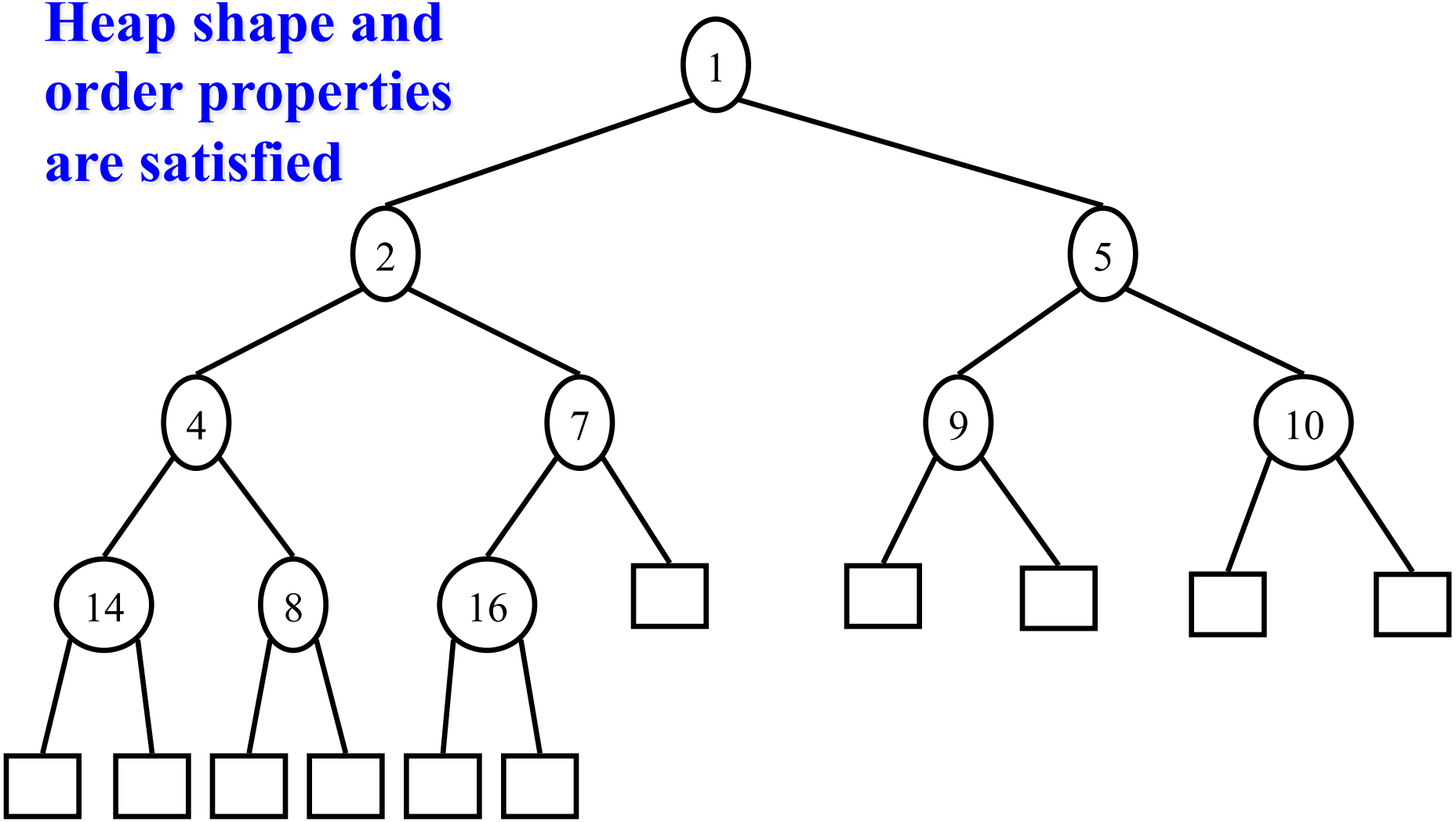


# Inserting an Element into a Heap

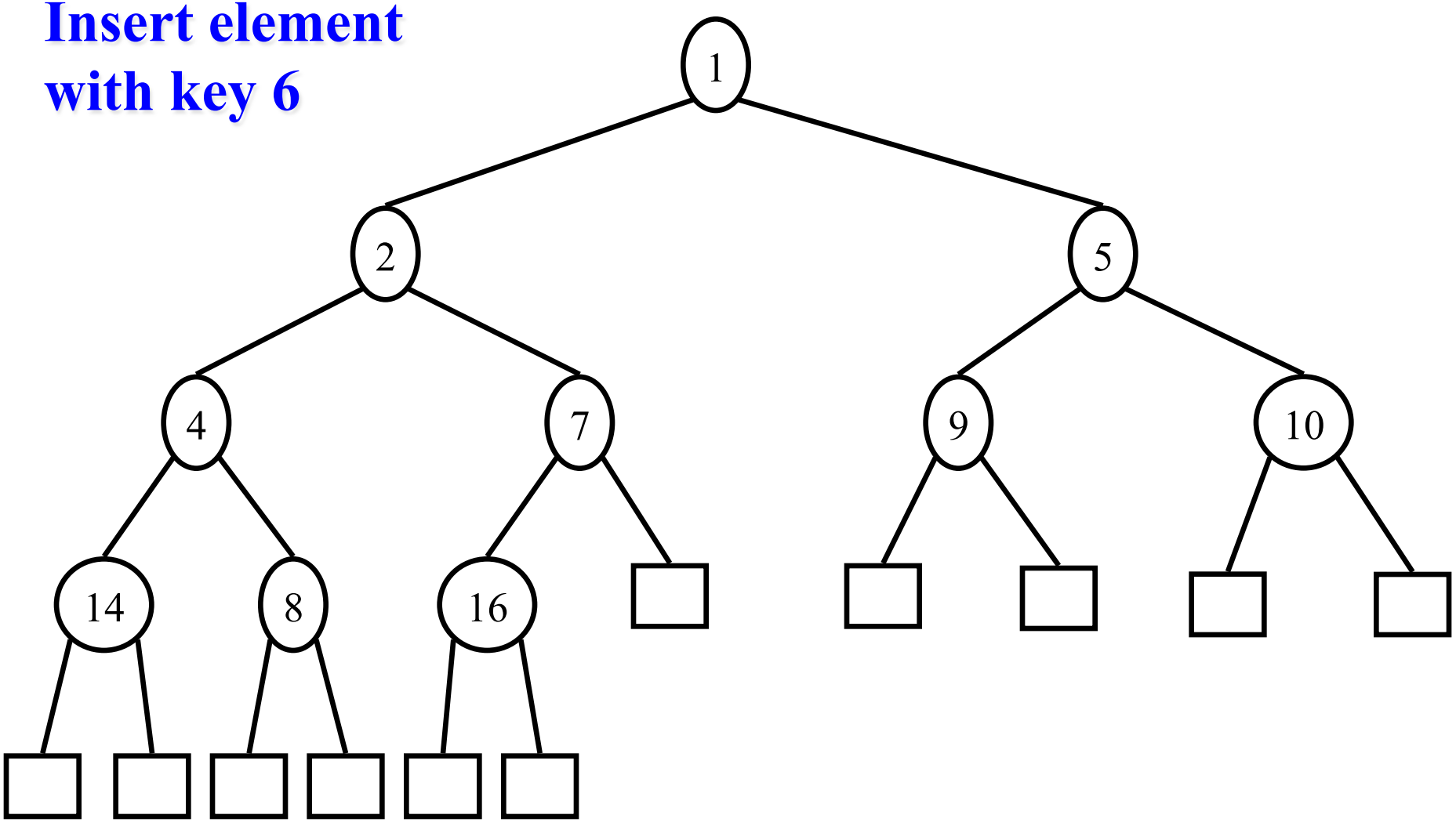
- Satisfy shape and order properties

- Satisfy shape property
  - Insert element at first available spot
- Satisfy order property
  - Bubble the element up the tree until the order property is restored

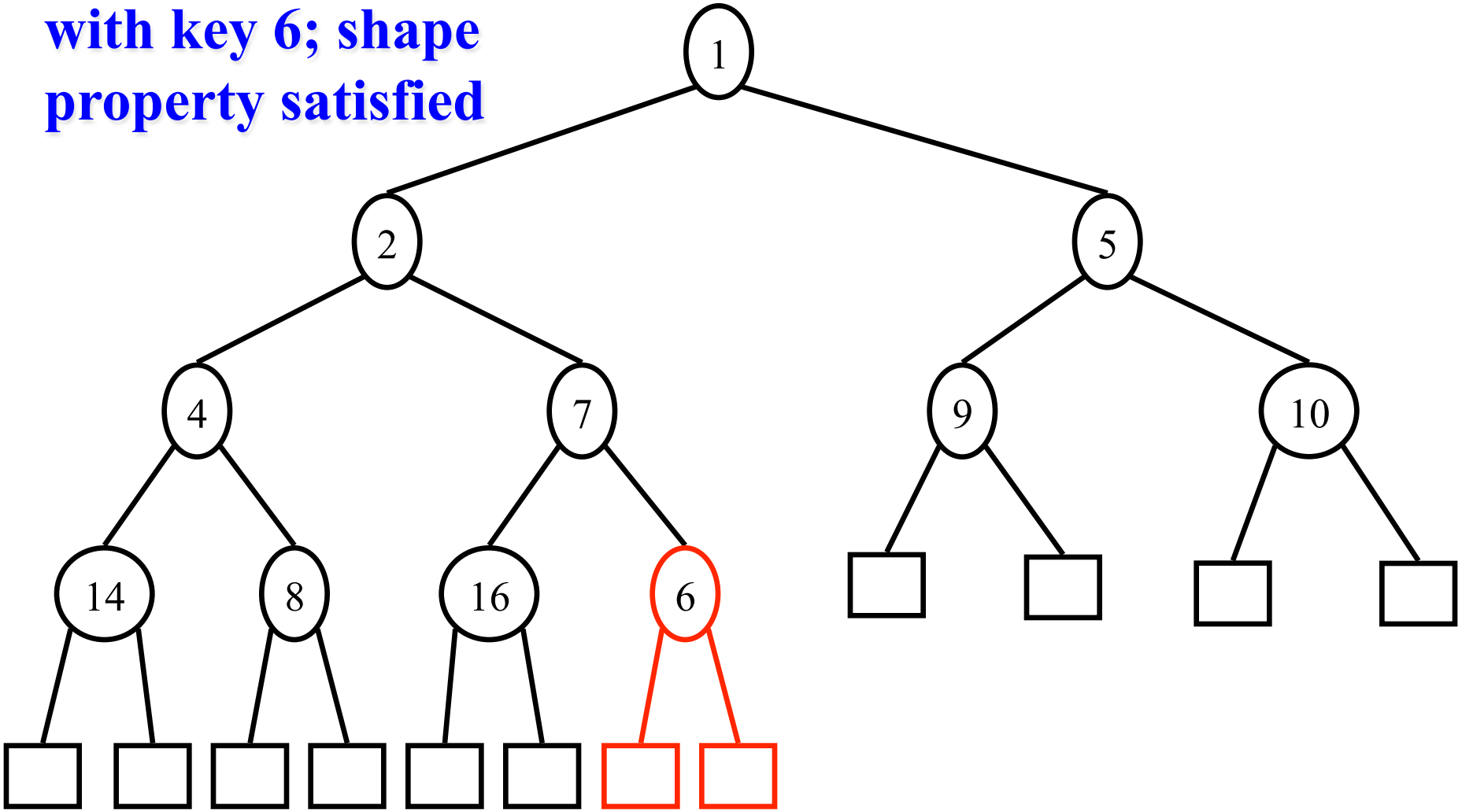
# Heap shape and order properties are satisfied



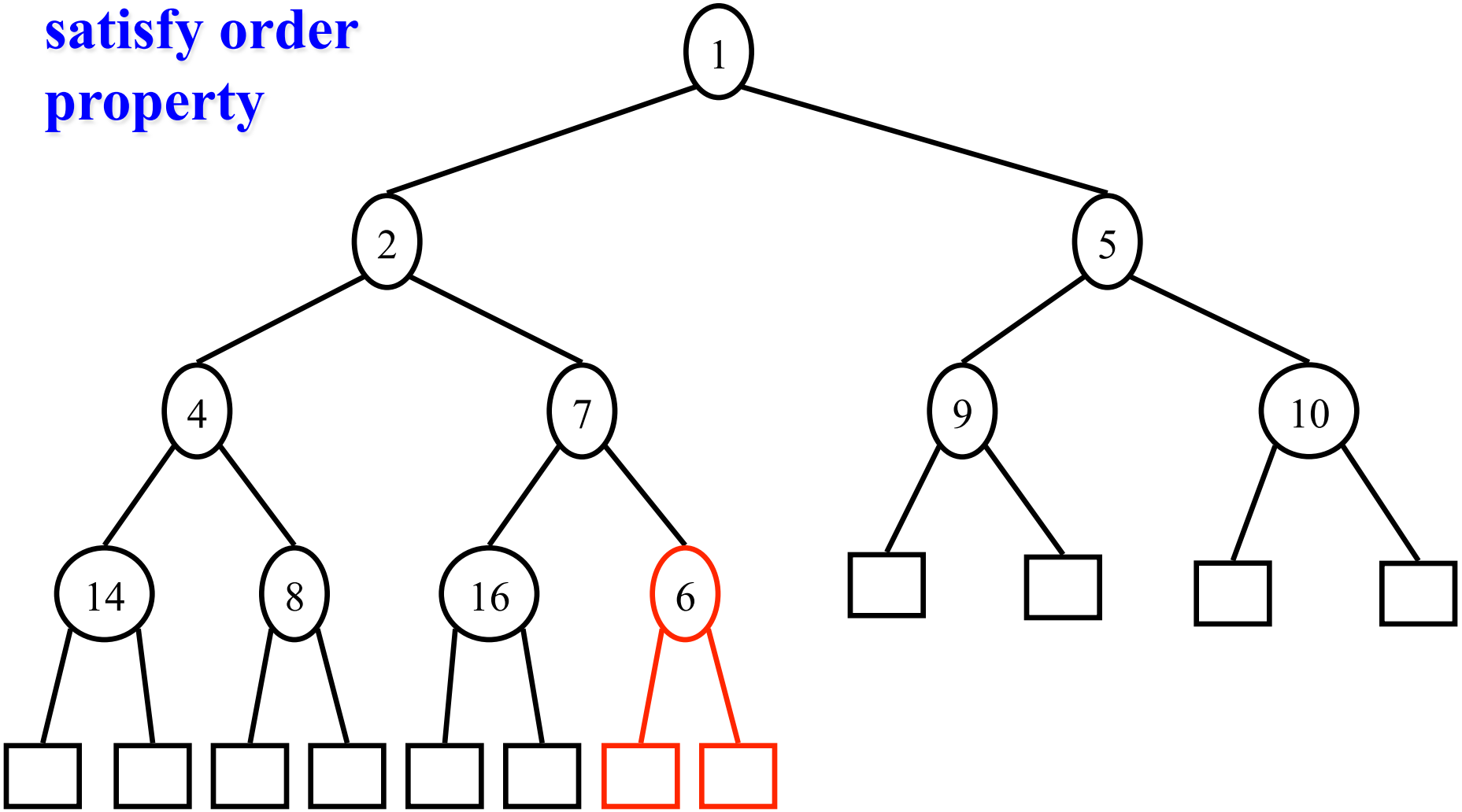
# Insert element with key 6



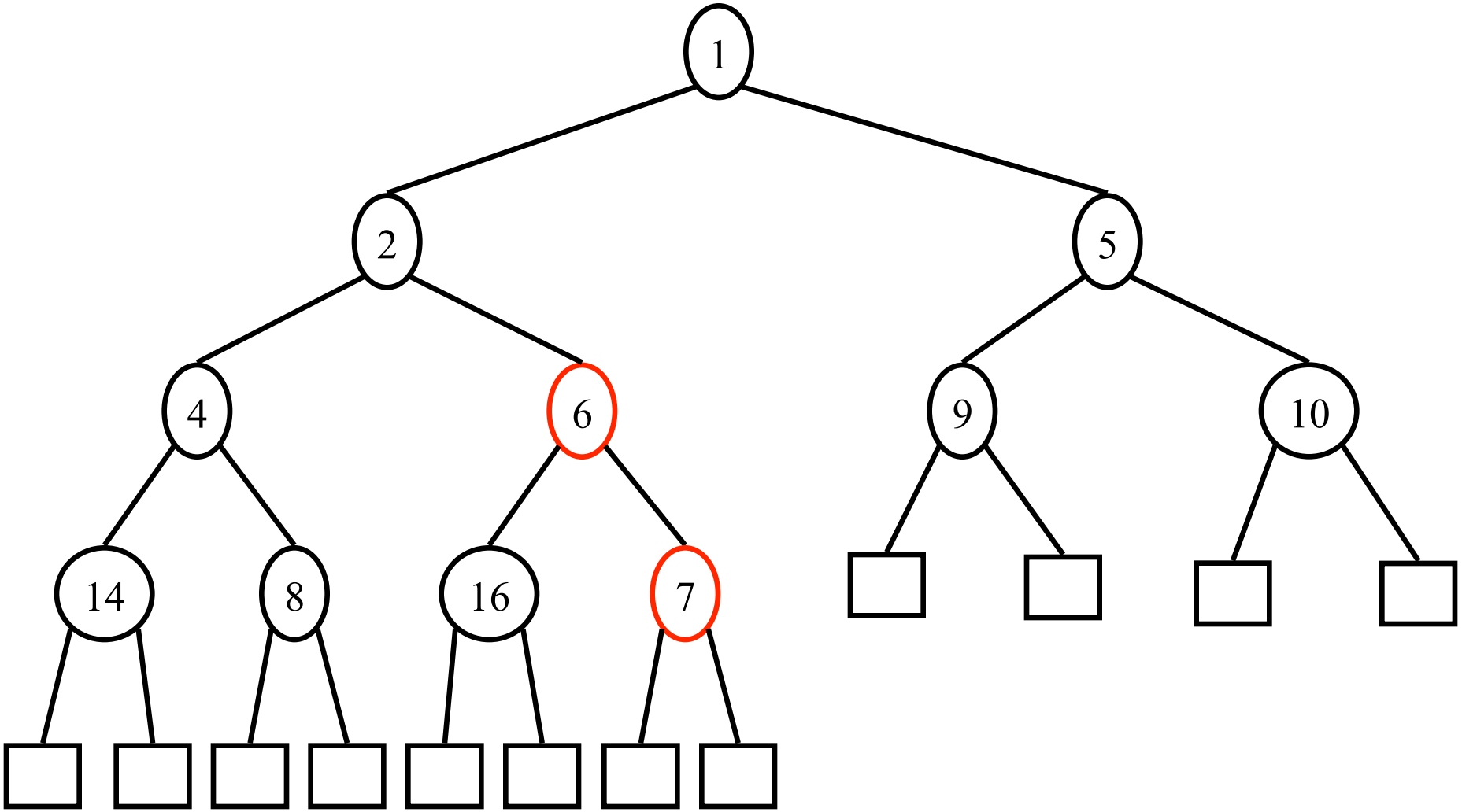
# Insert element with key 6; shape property satisfied



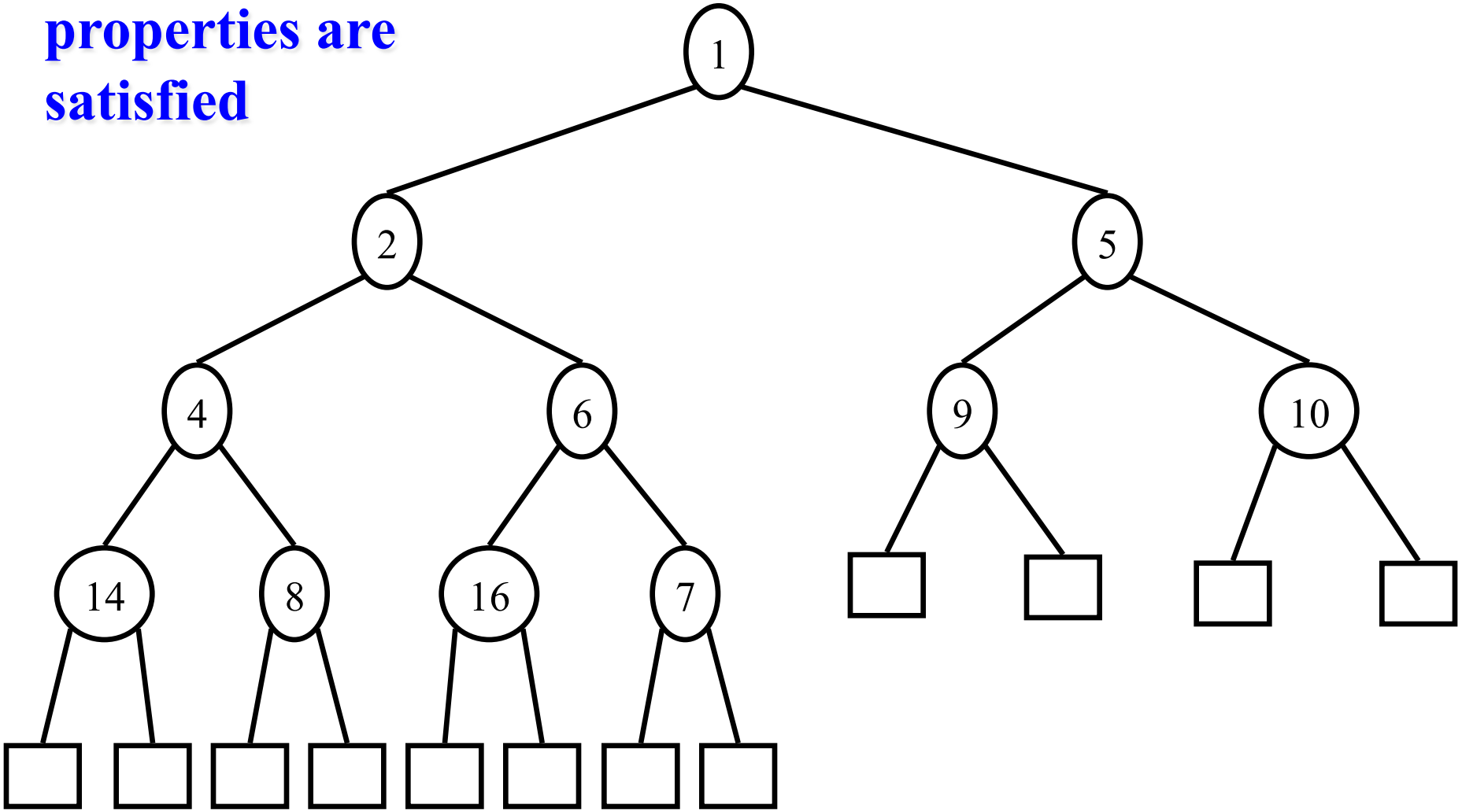
# Bubble up 6 to satisfy order property



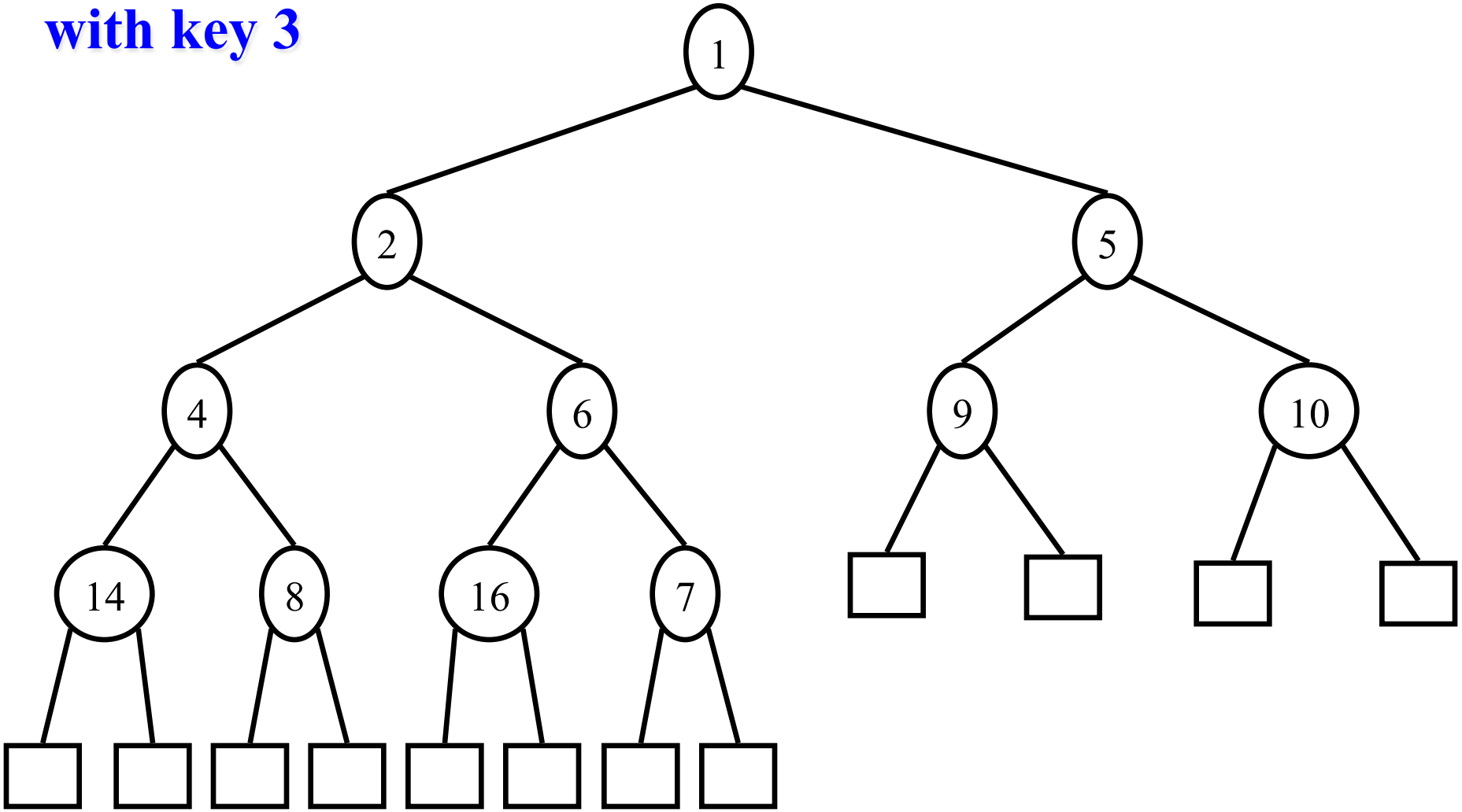
# Order property satisfied



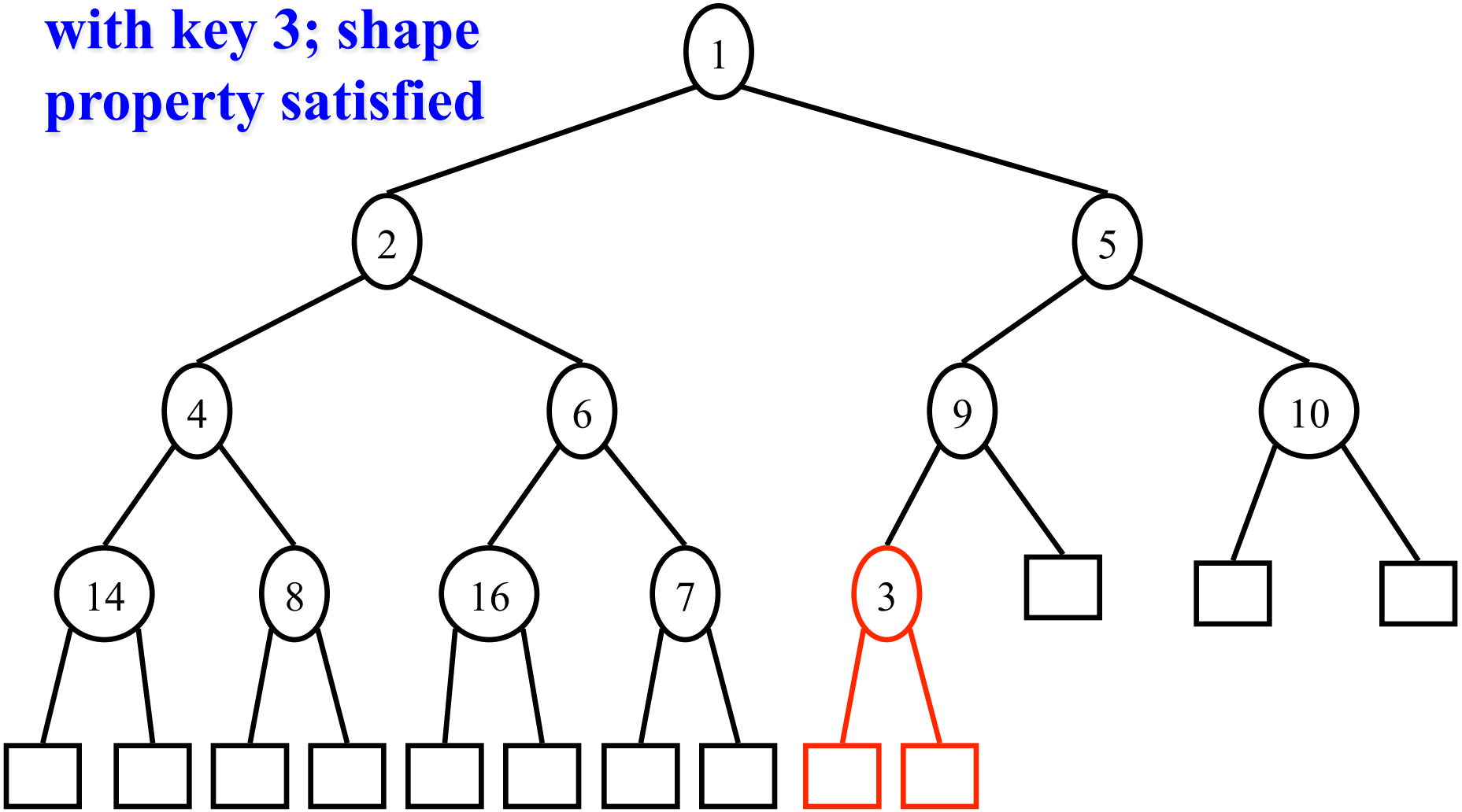
## Shape and order properties are satisfied



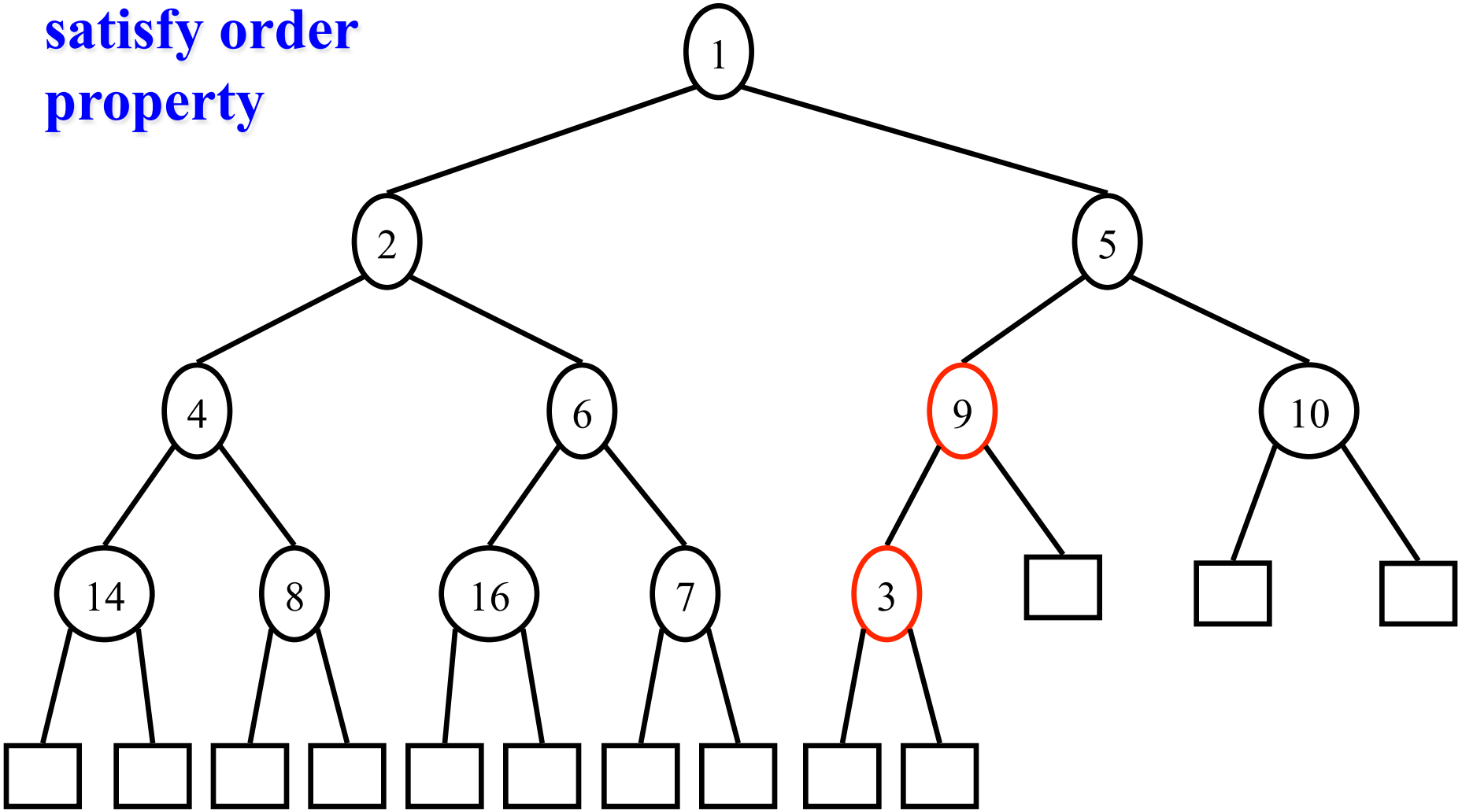
# Insert element with key 3



# Insert element with key 3; shape property satisfied



# Bubble up 3 to satisfy order property



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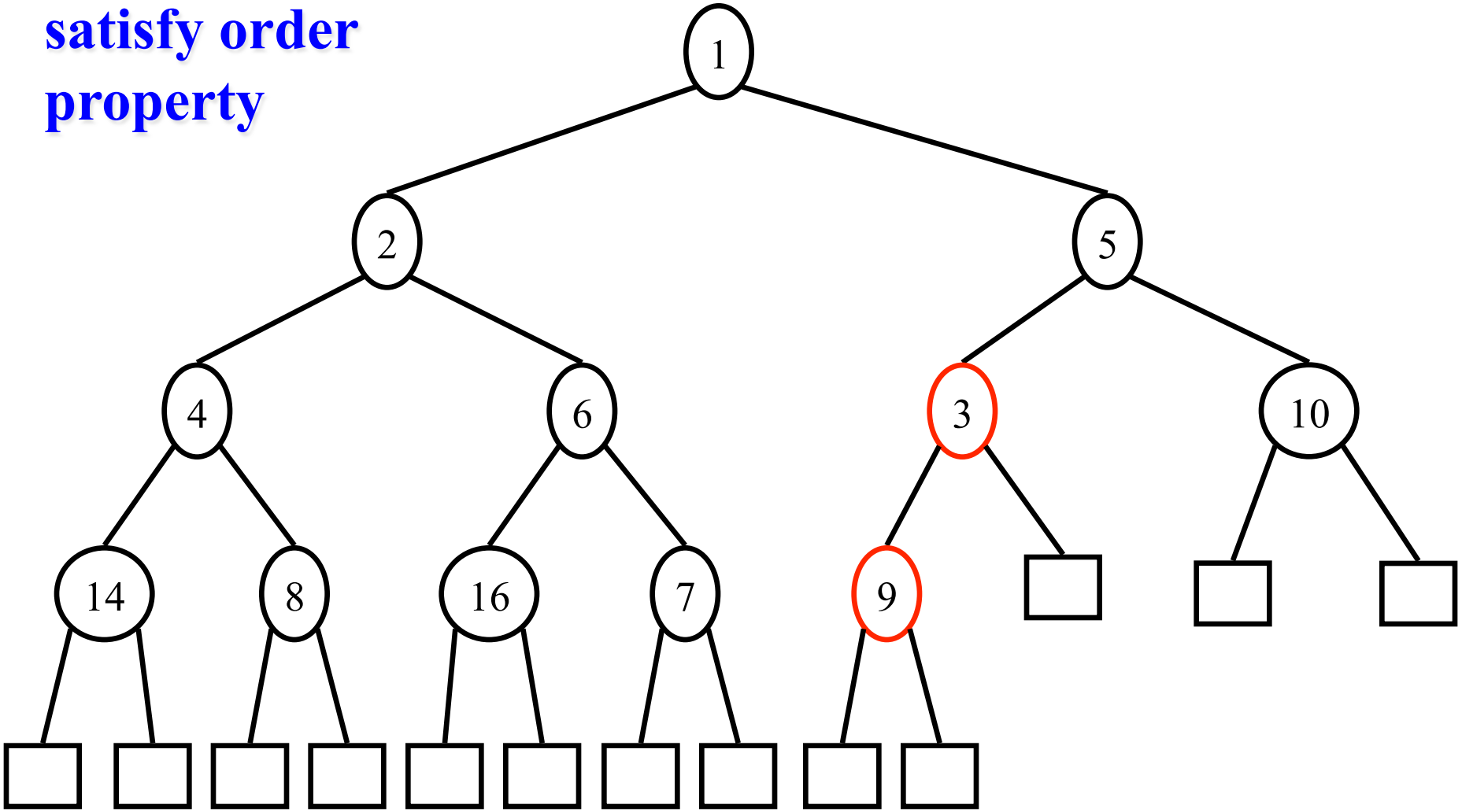

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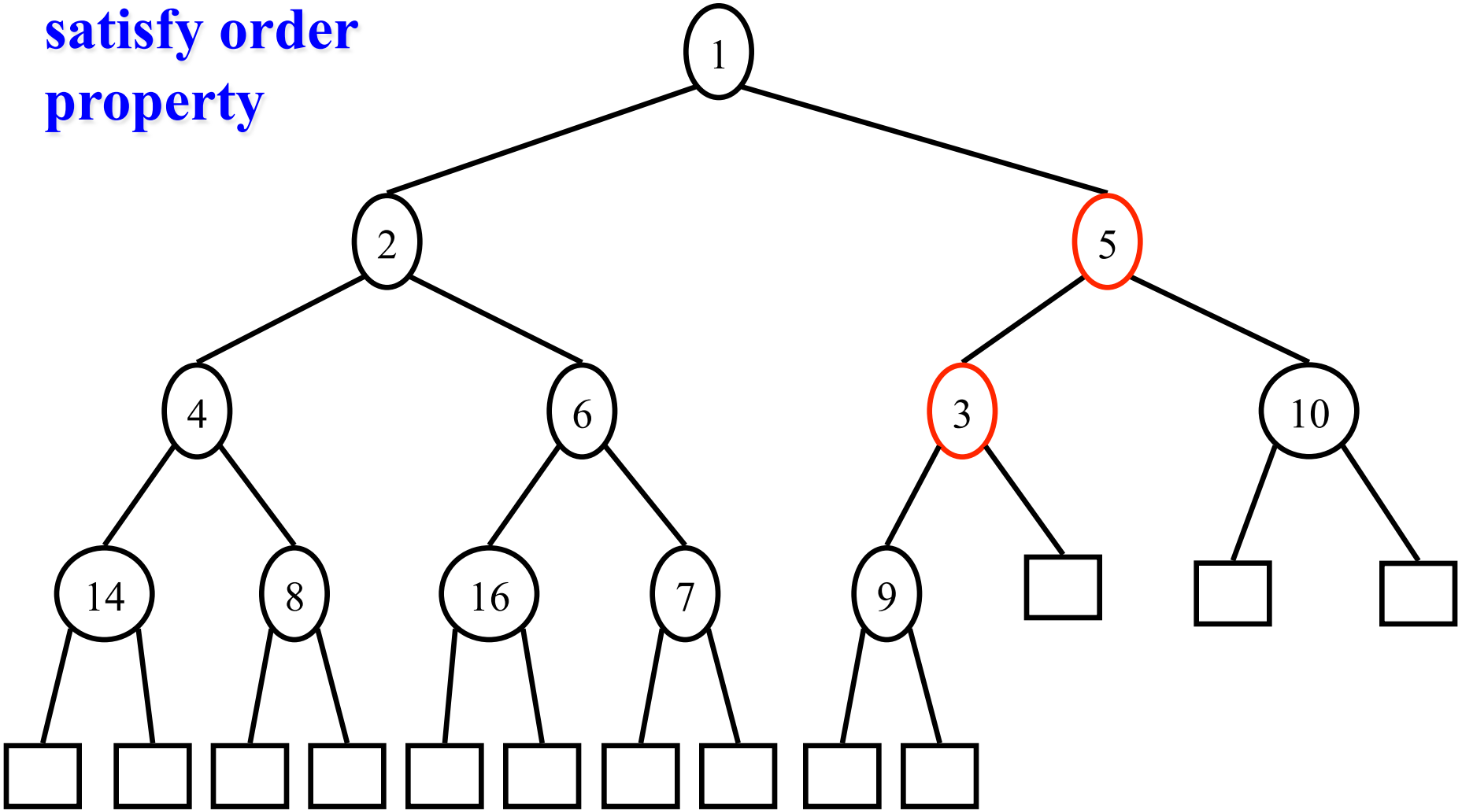

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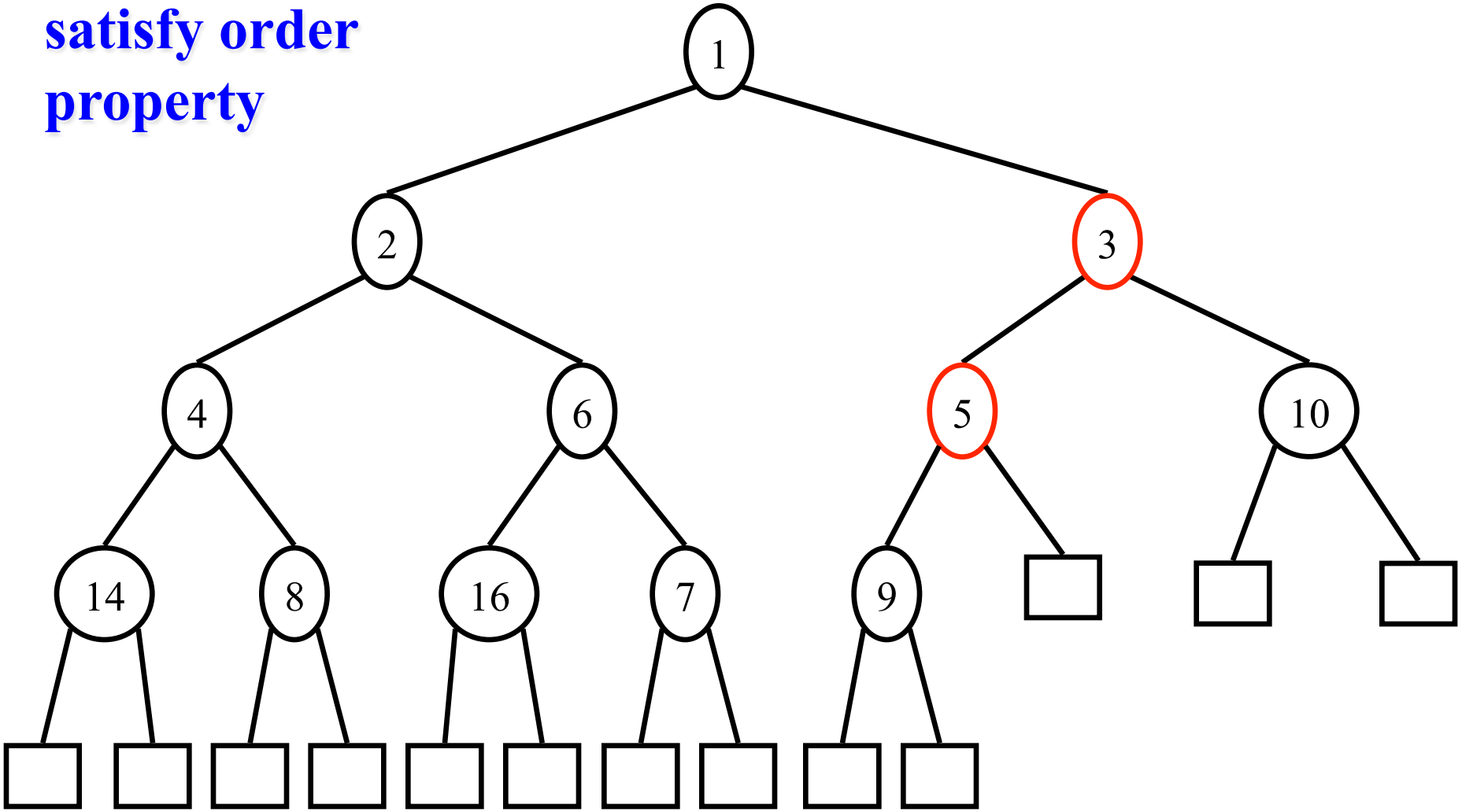
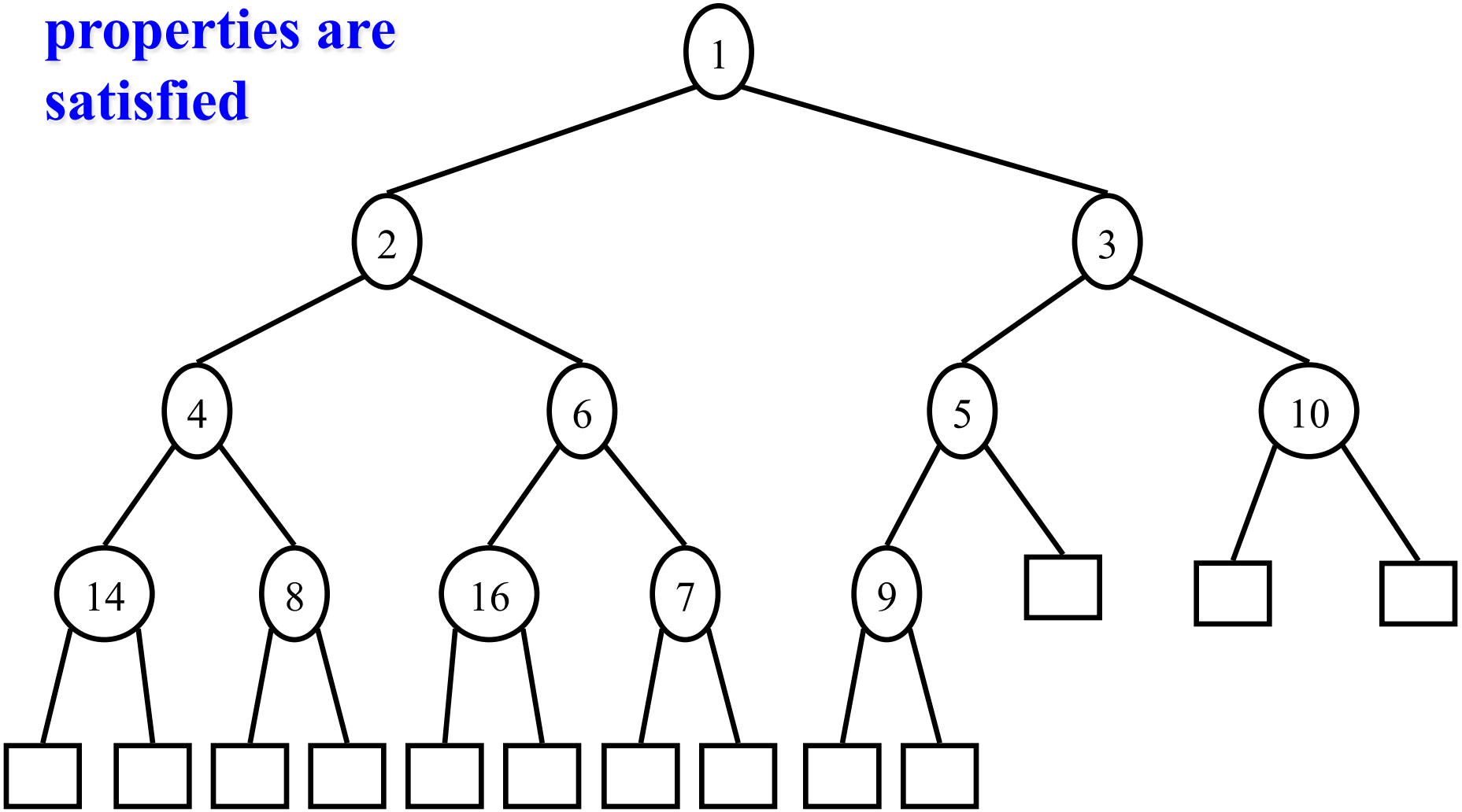
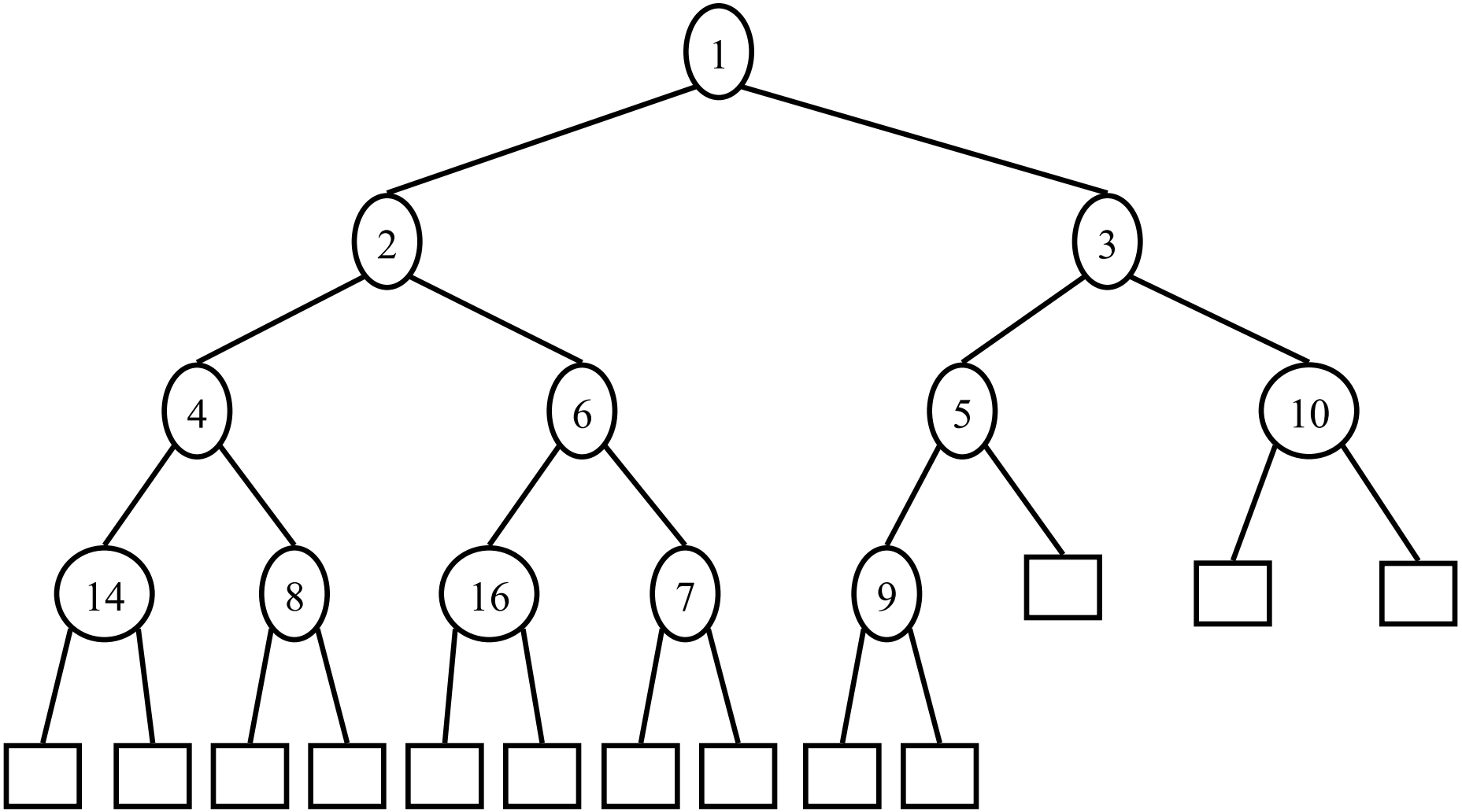


## Shape and order properties are satisfied

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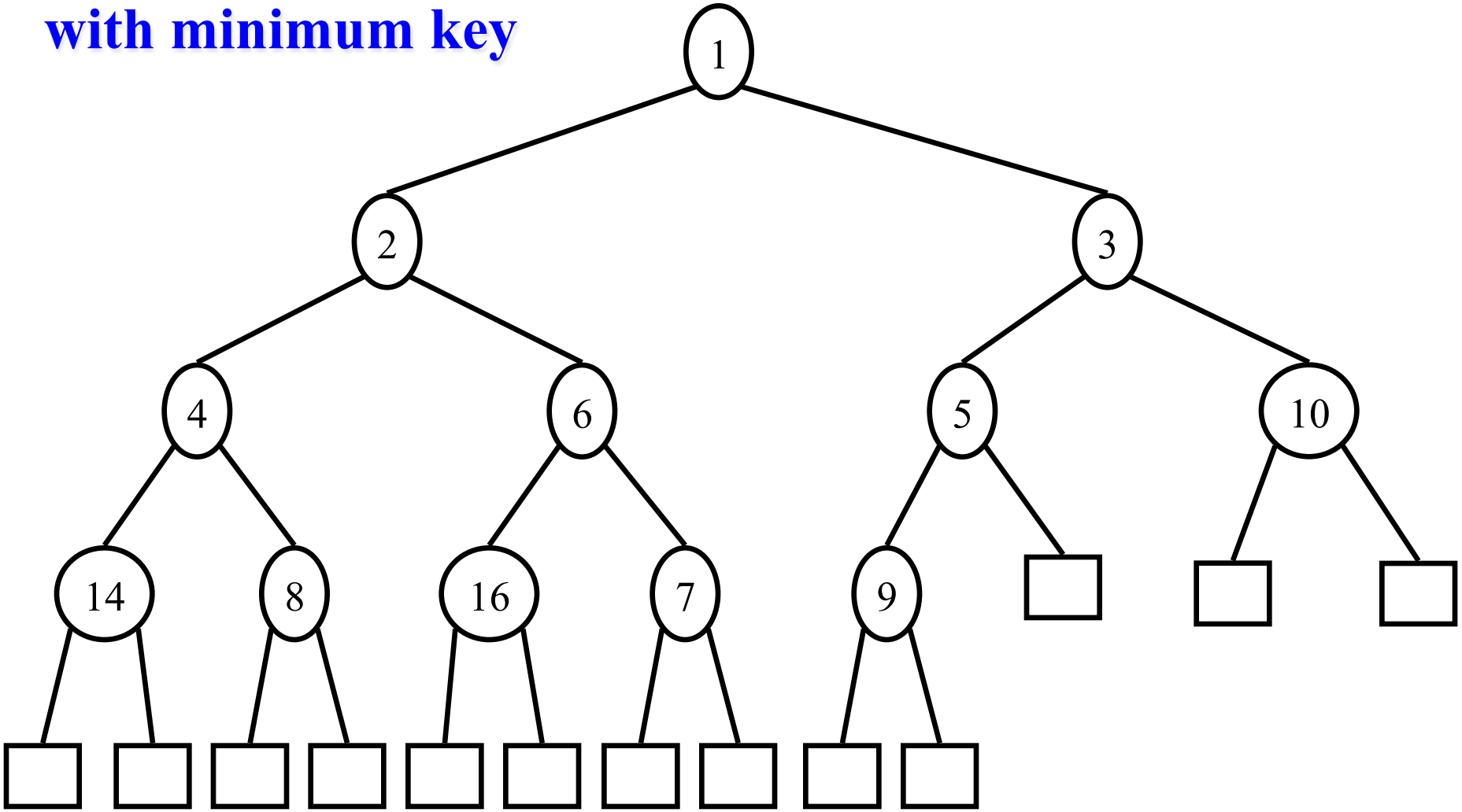
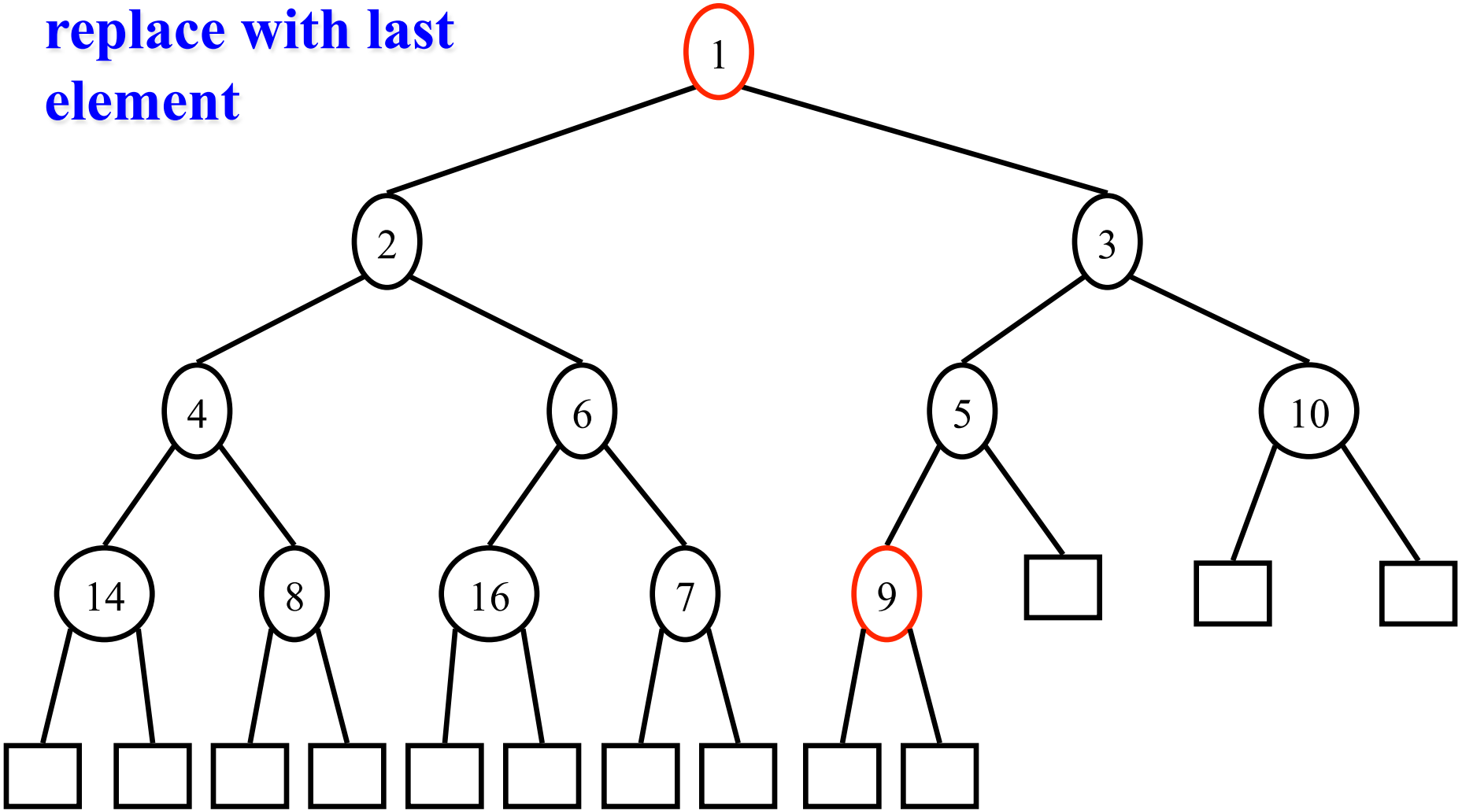
# Removing Min Element from a Heap

- Remove root
- Satisfy shape property
  - Move last element to root spot
- Satisfy order property
  - Bubble root element down the tree until order property is restored

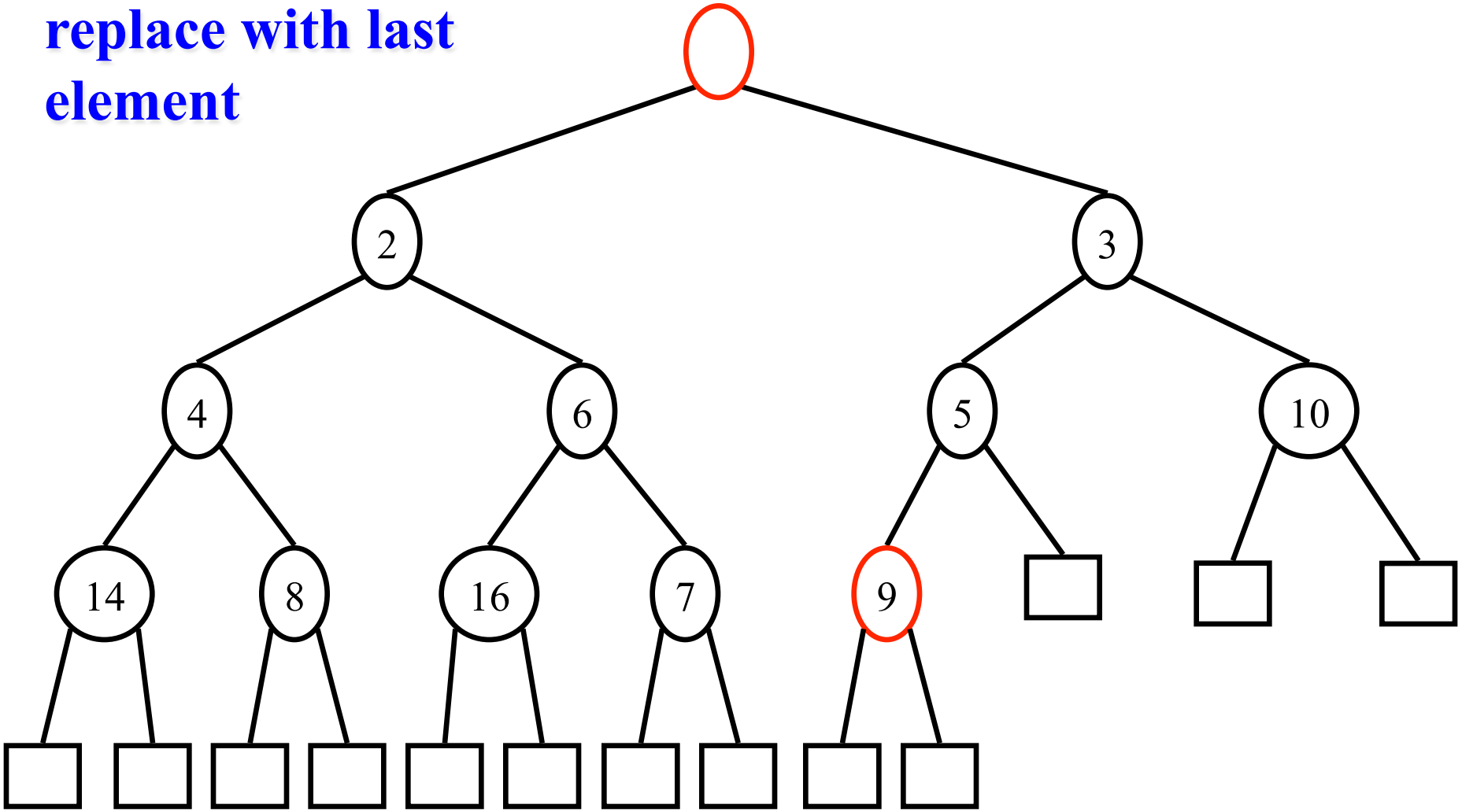
# Remove element with minimum key



# Remove root and replace with last element



# Remove root and replace with last element



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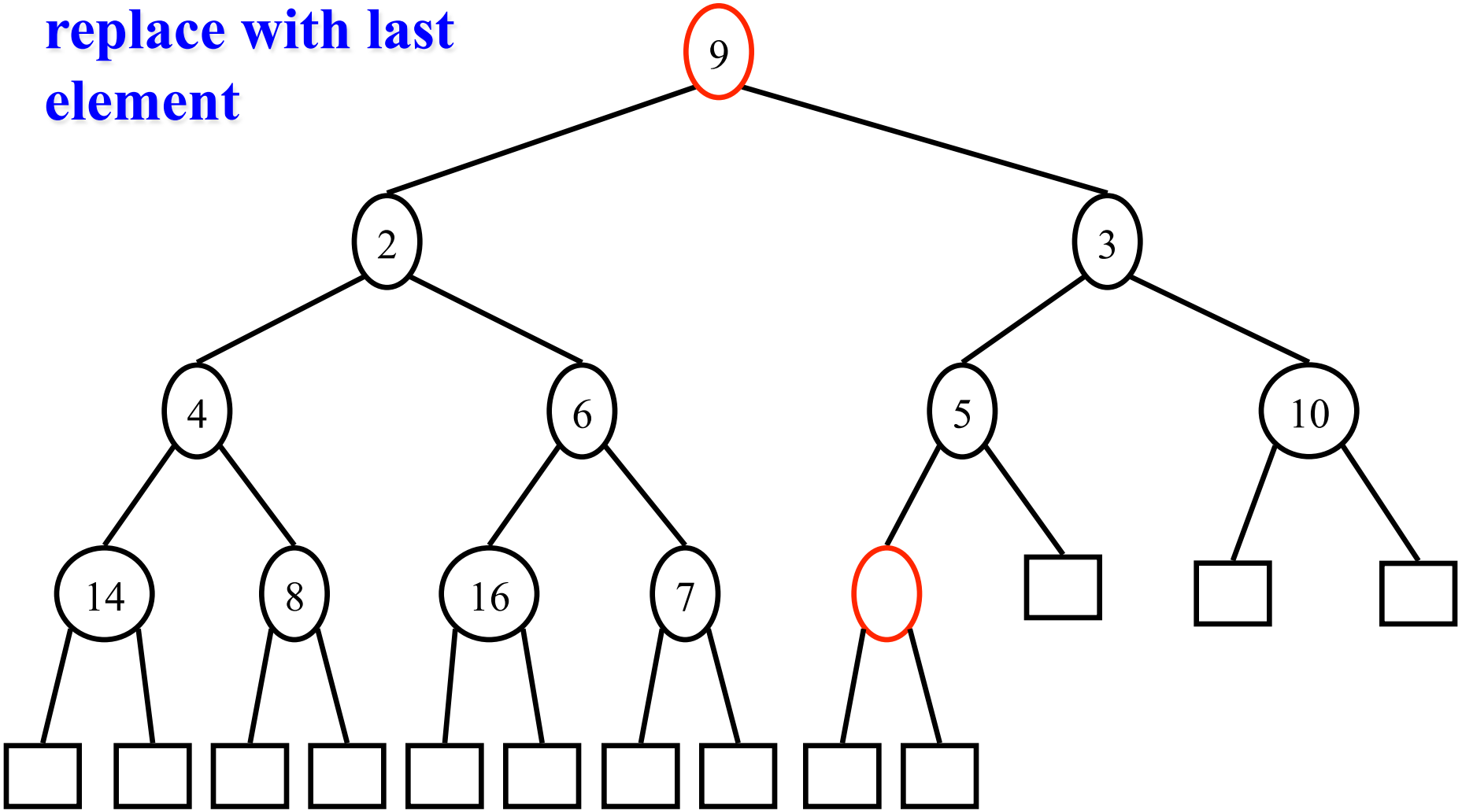
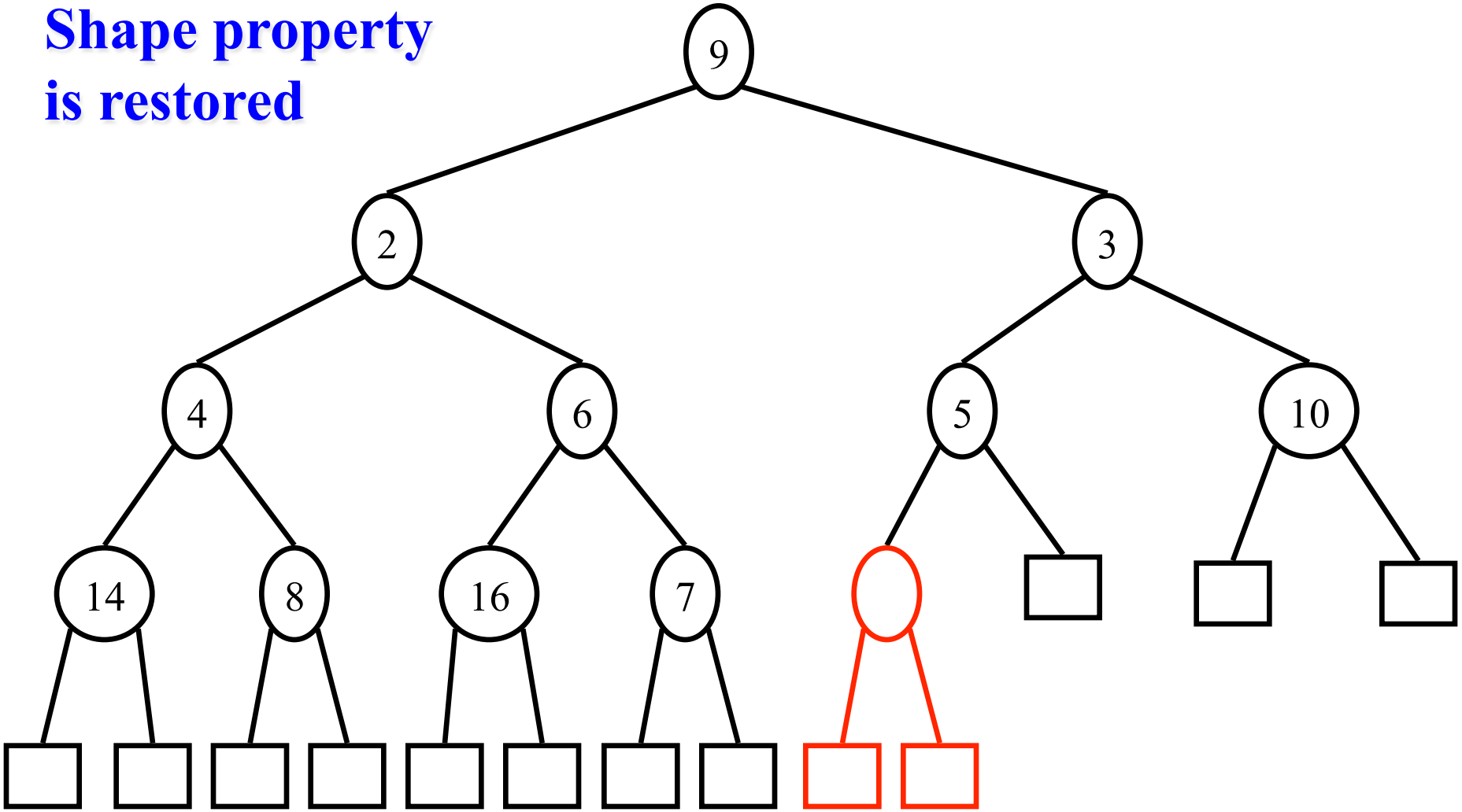
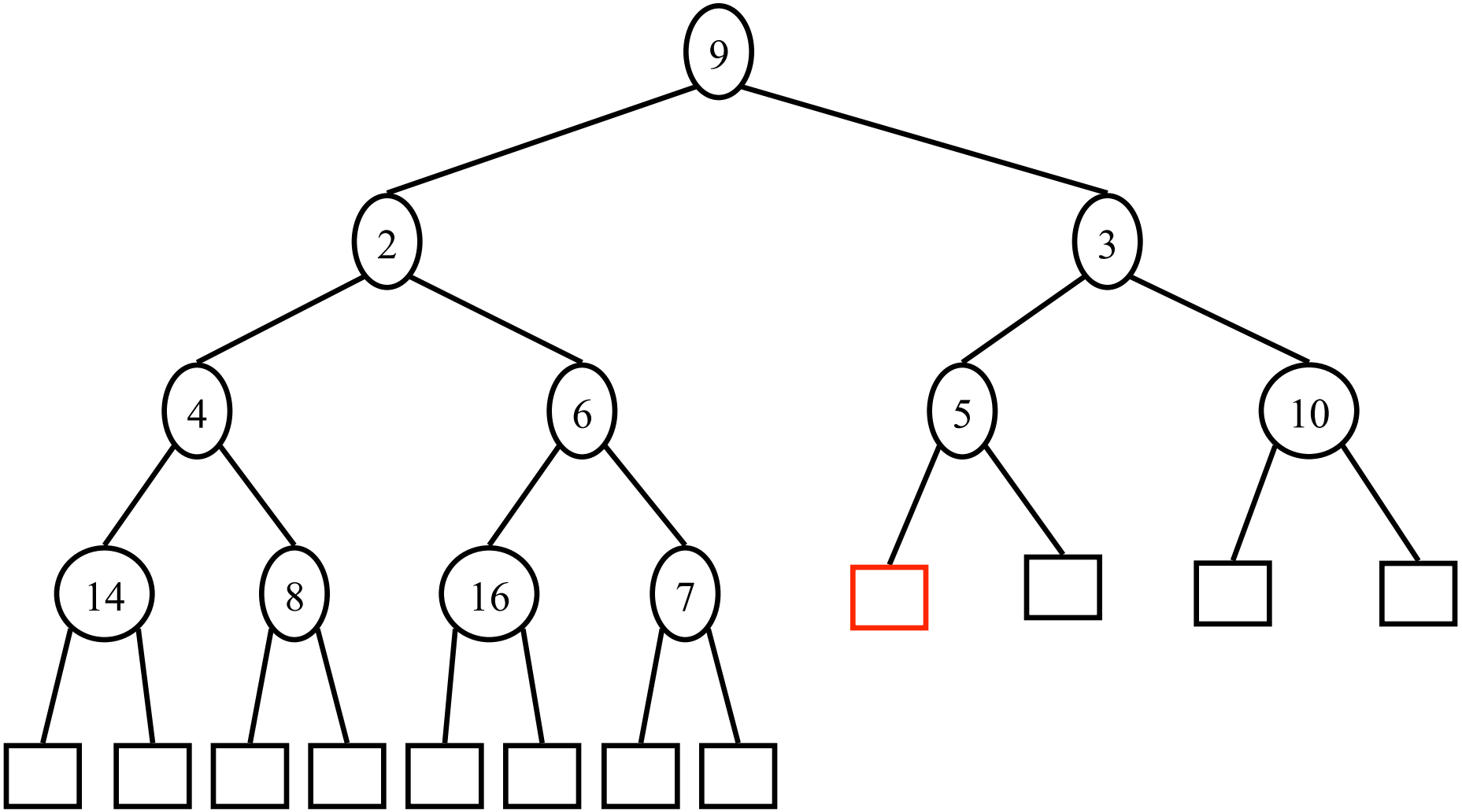


# Delete last element
# Shape property
# is restored

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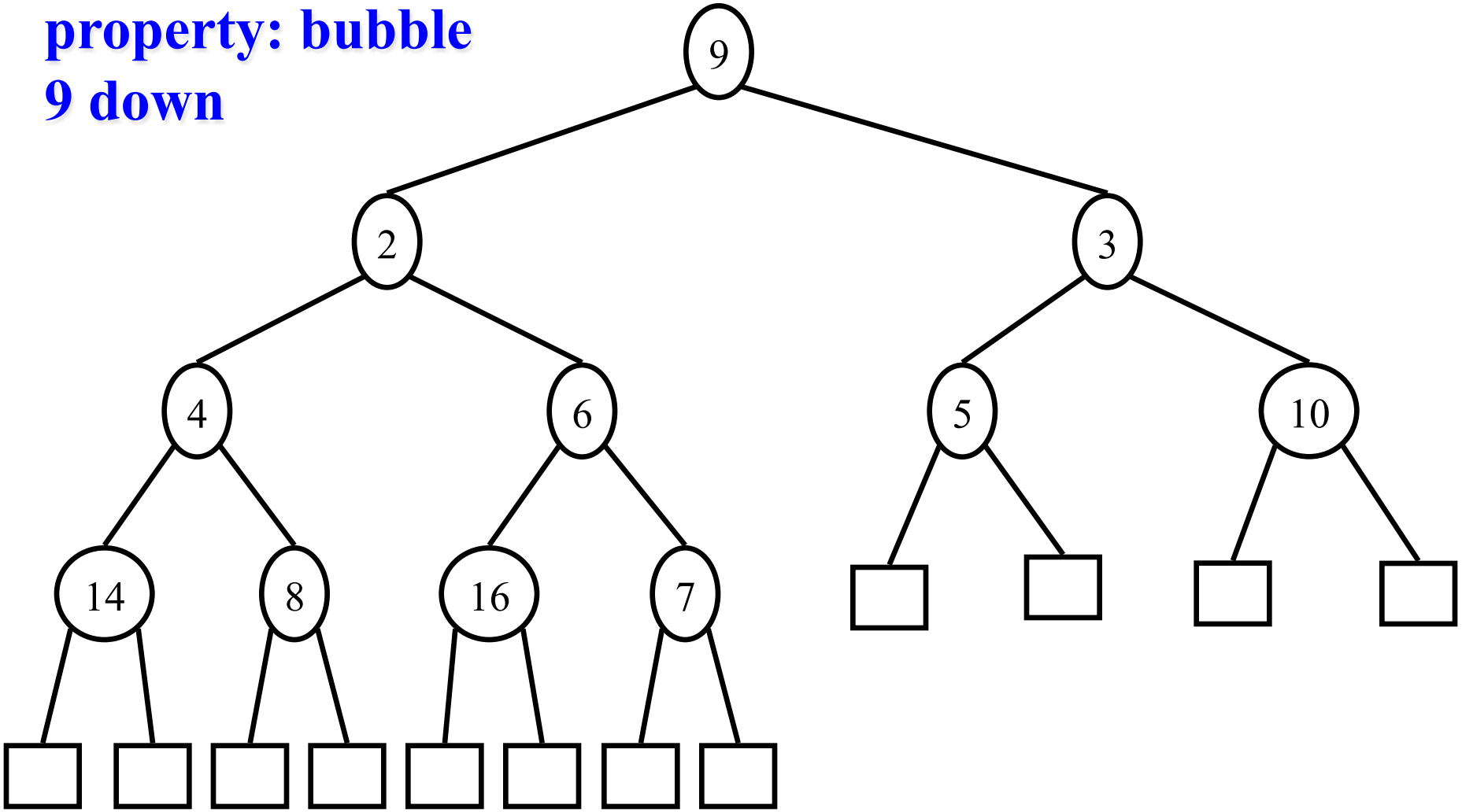
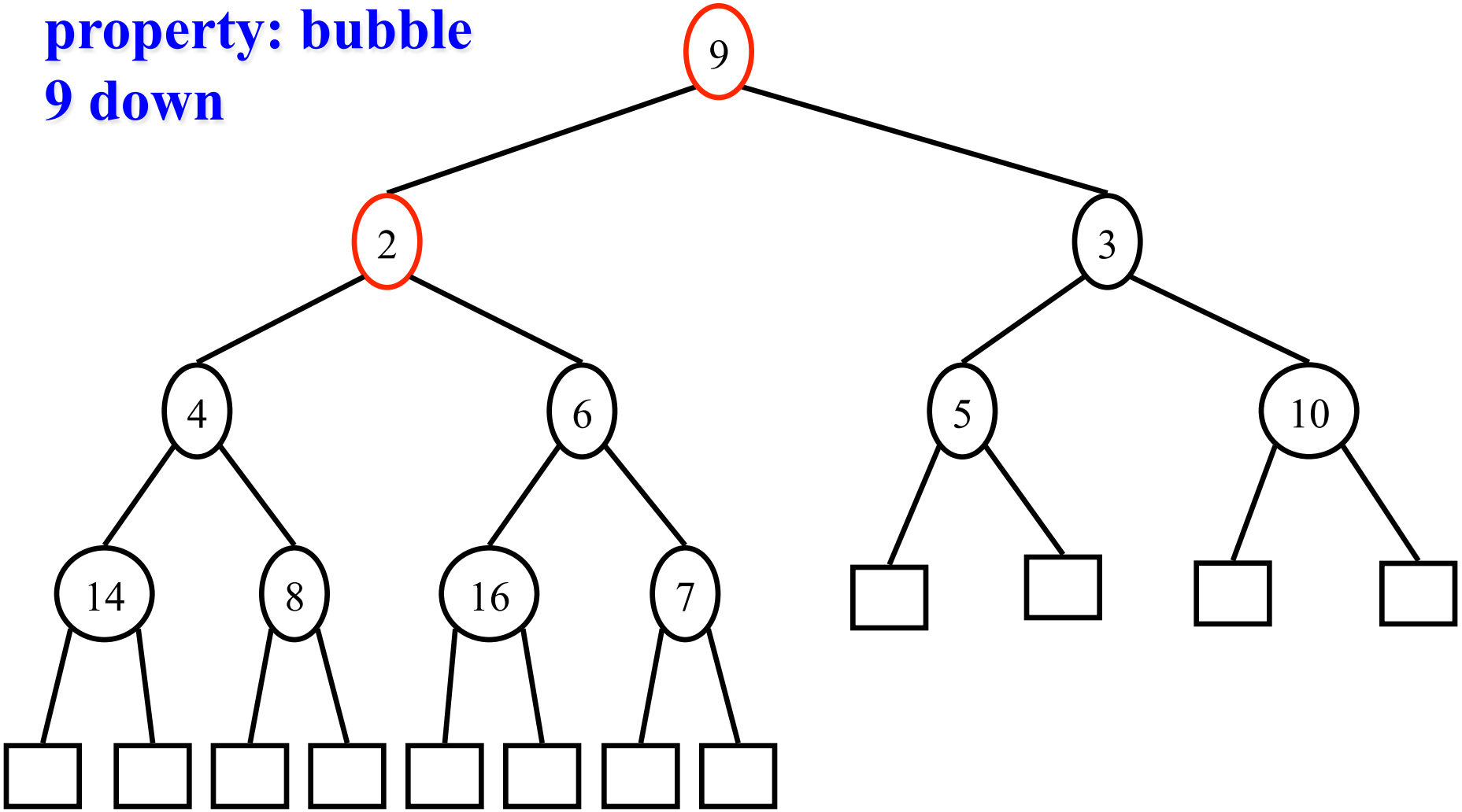
# Reestablish order property: bubble 9 down



# Reestablish order property: bubble 9 down



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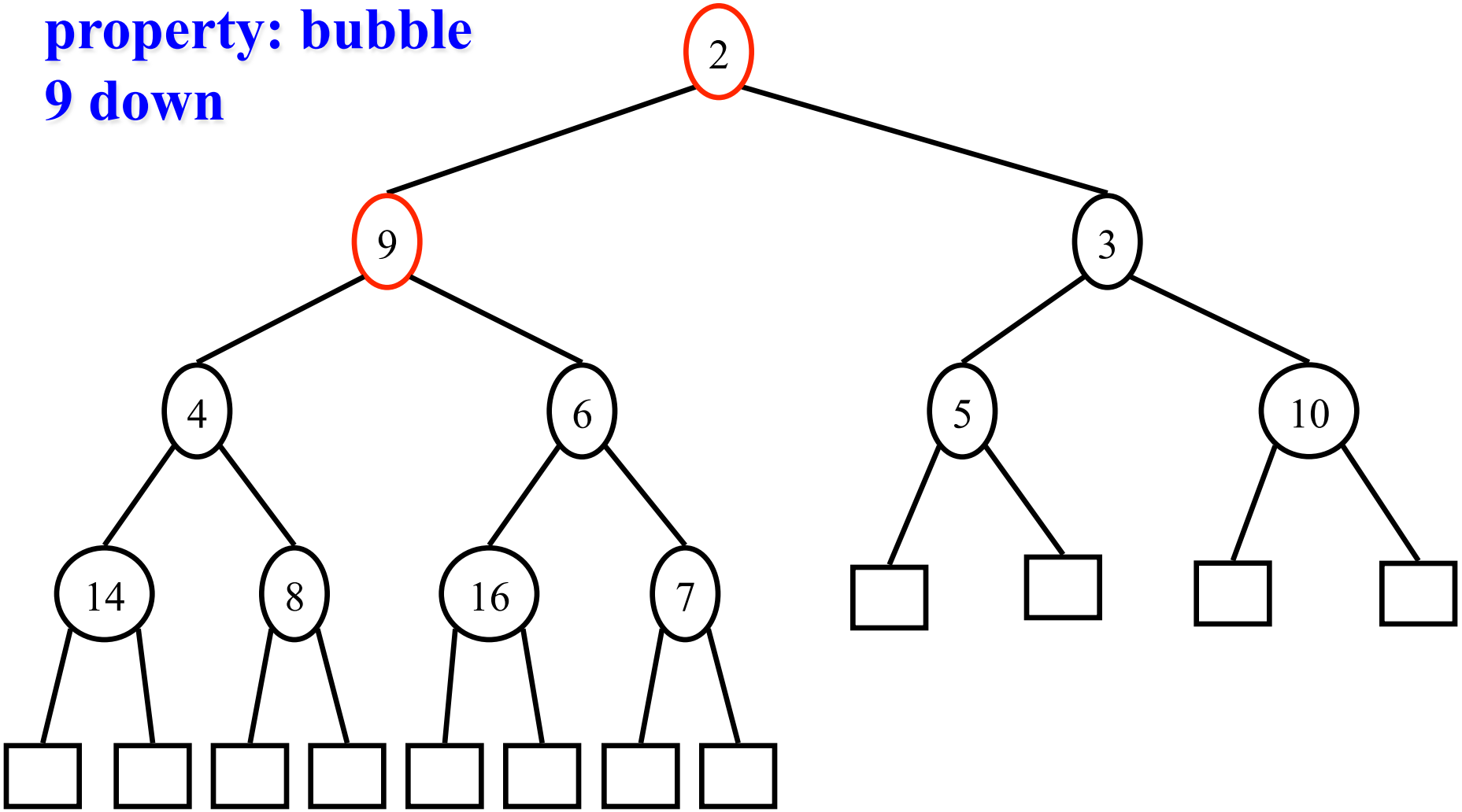

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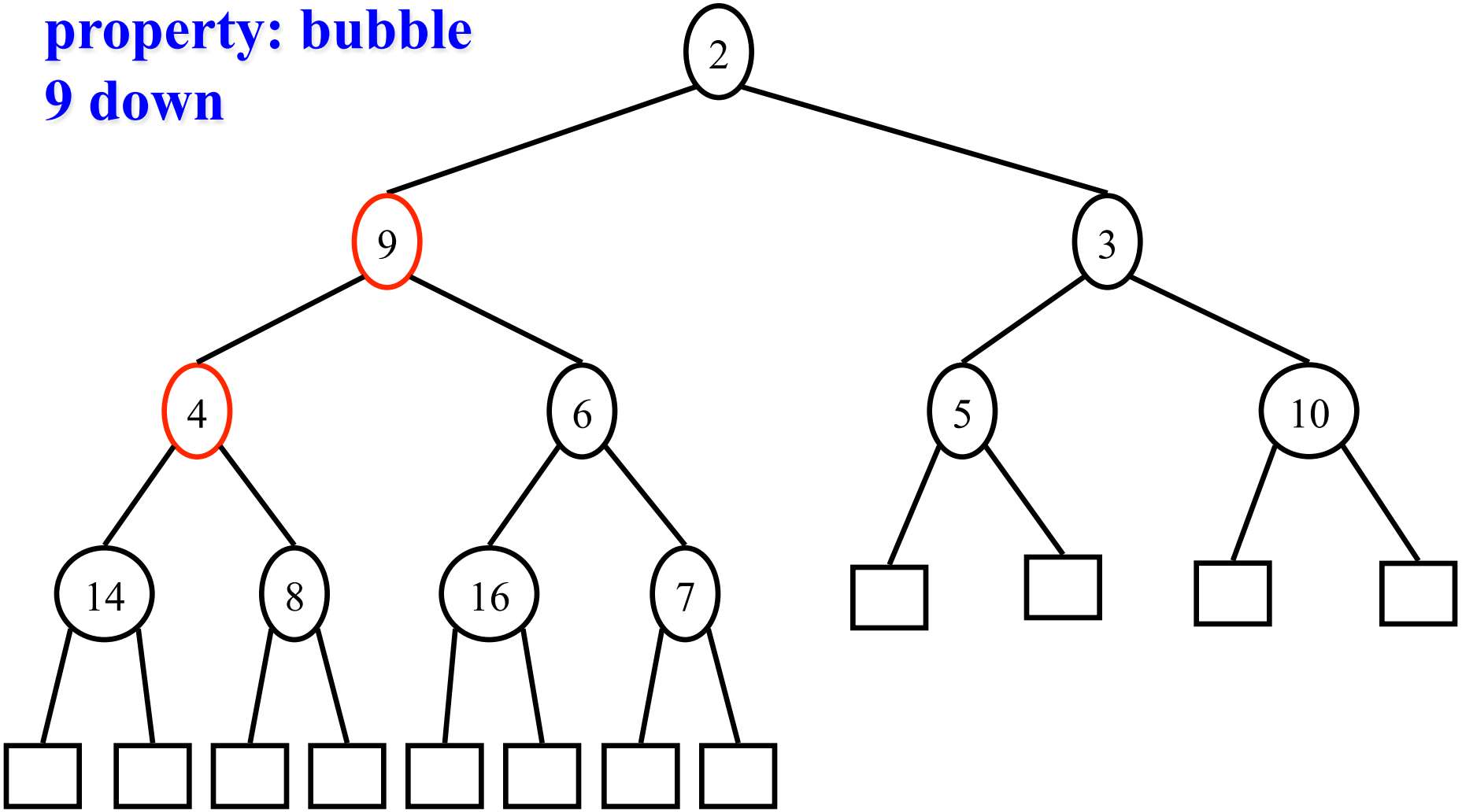
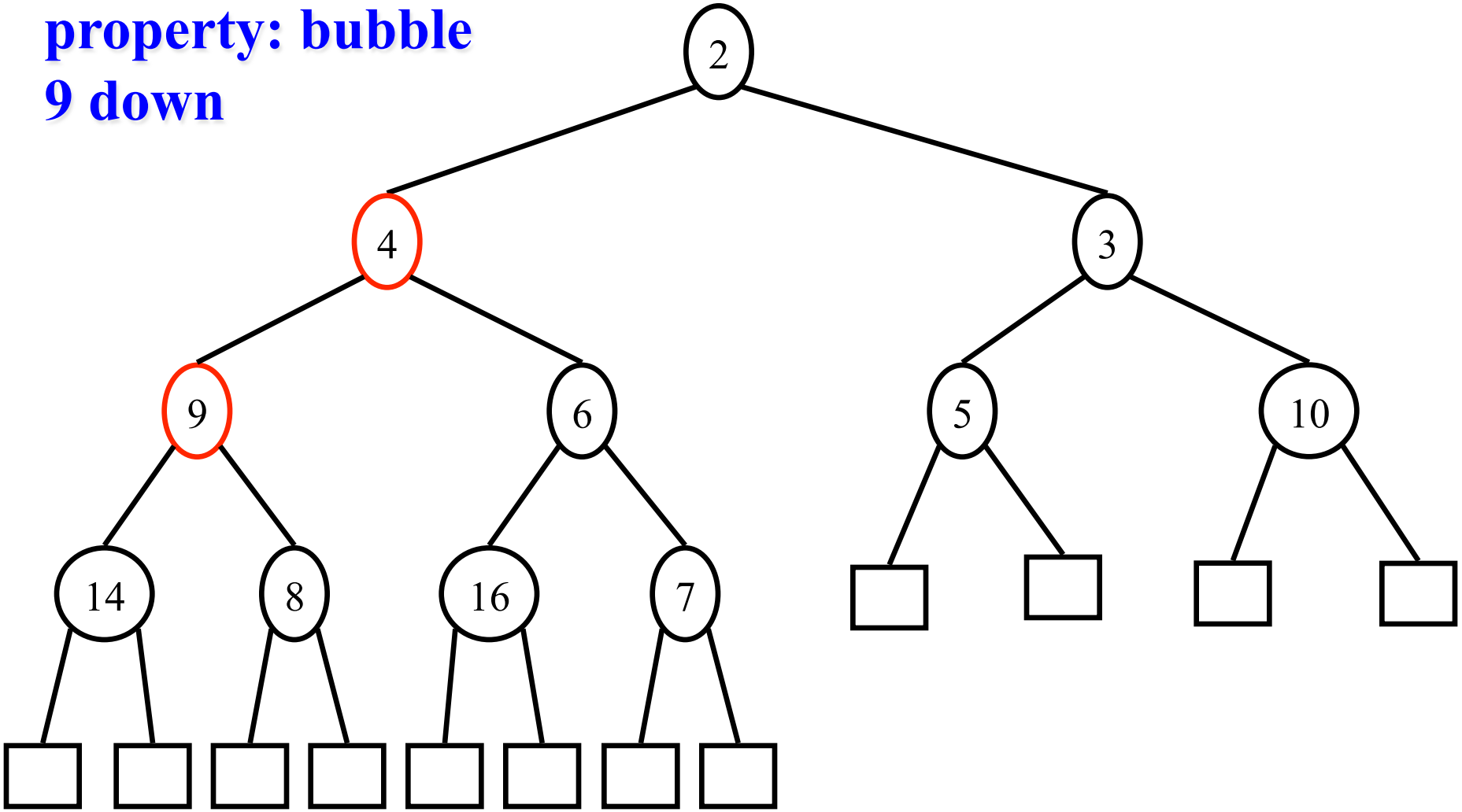
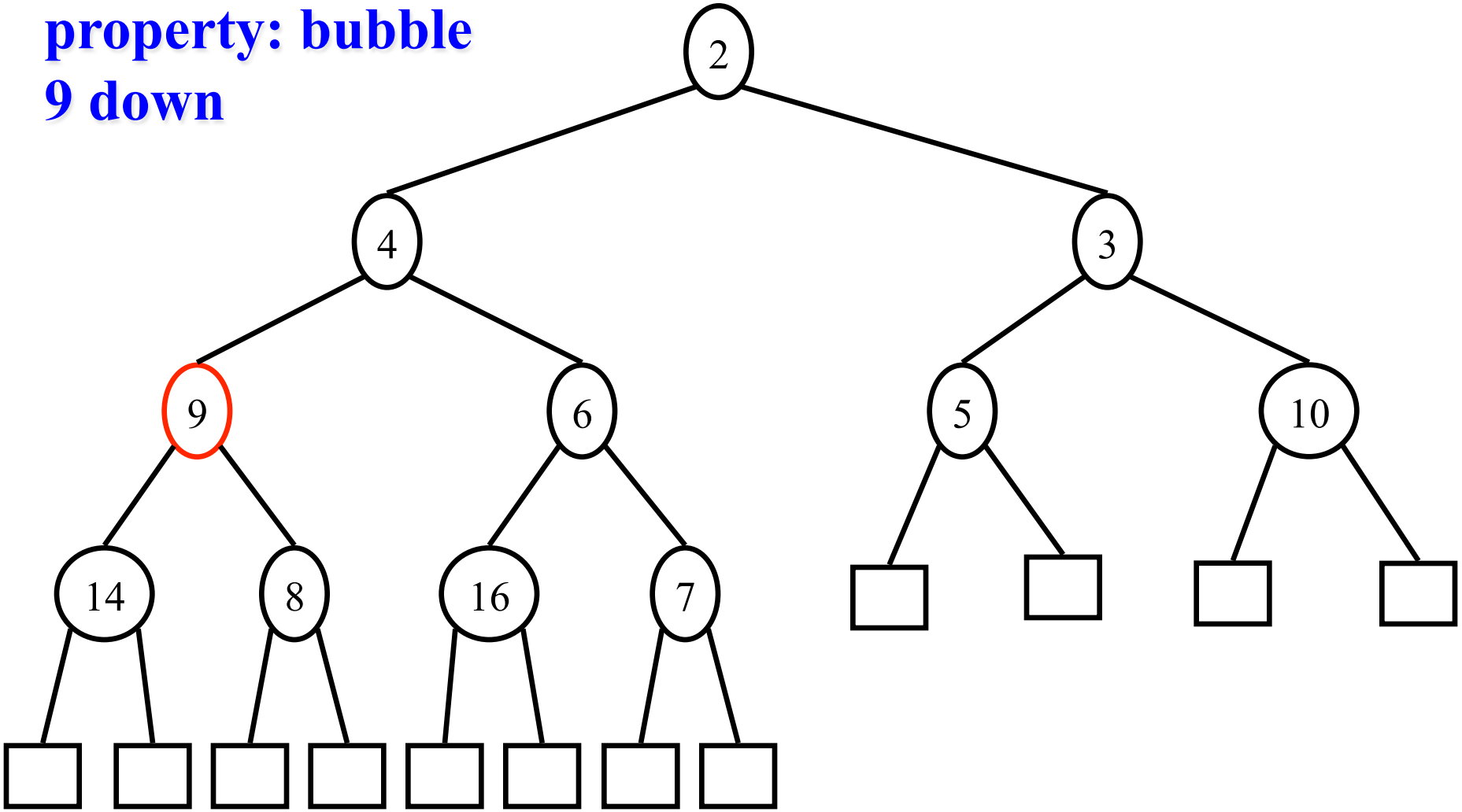


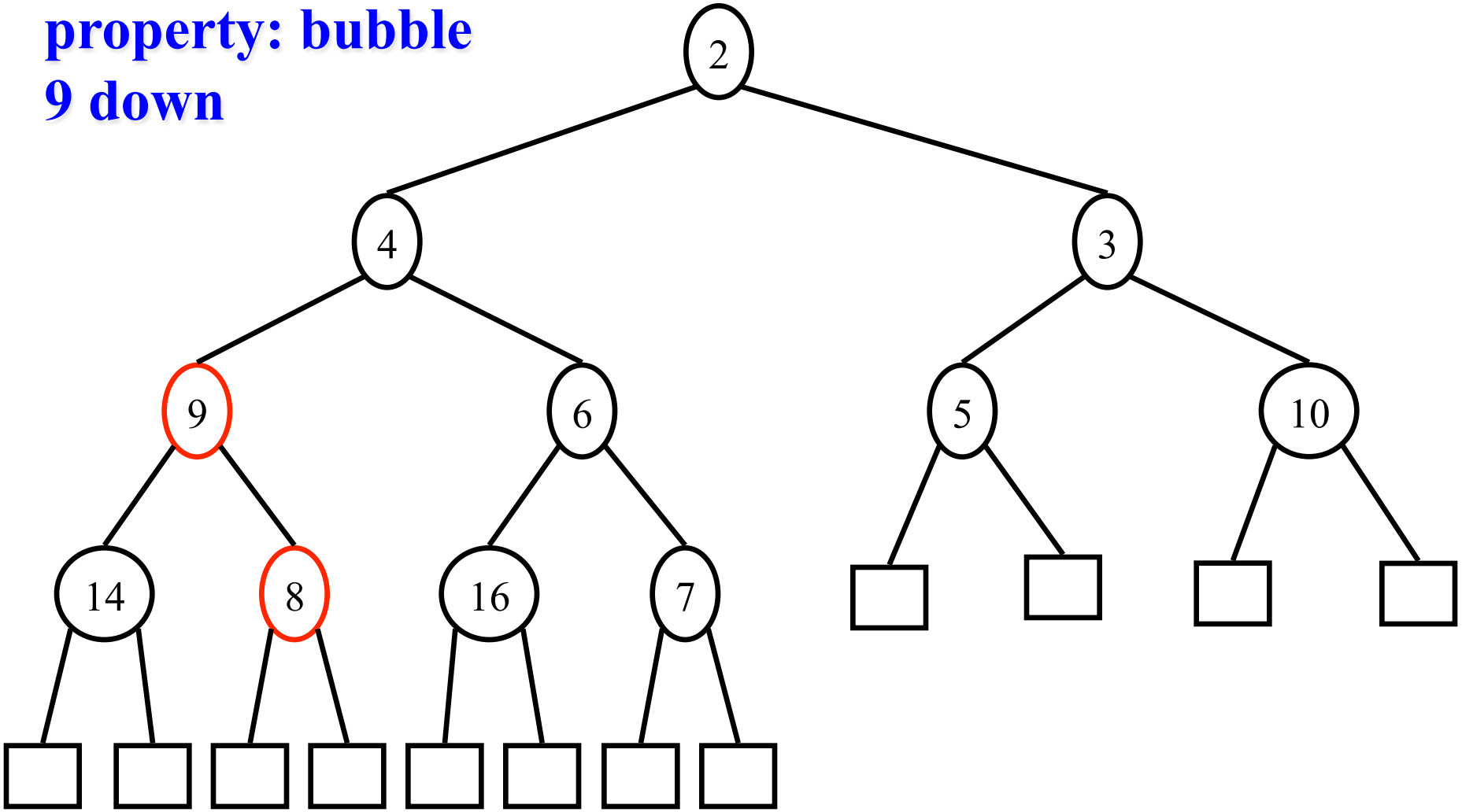
# Reestablish order property: bubble 9 down



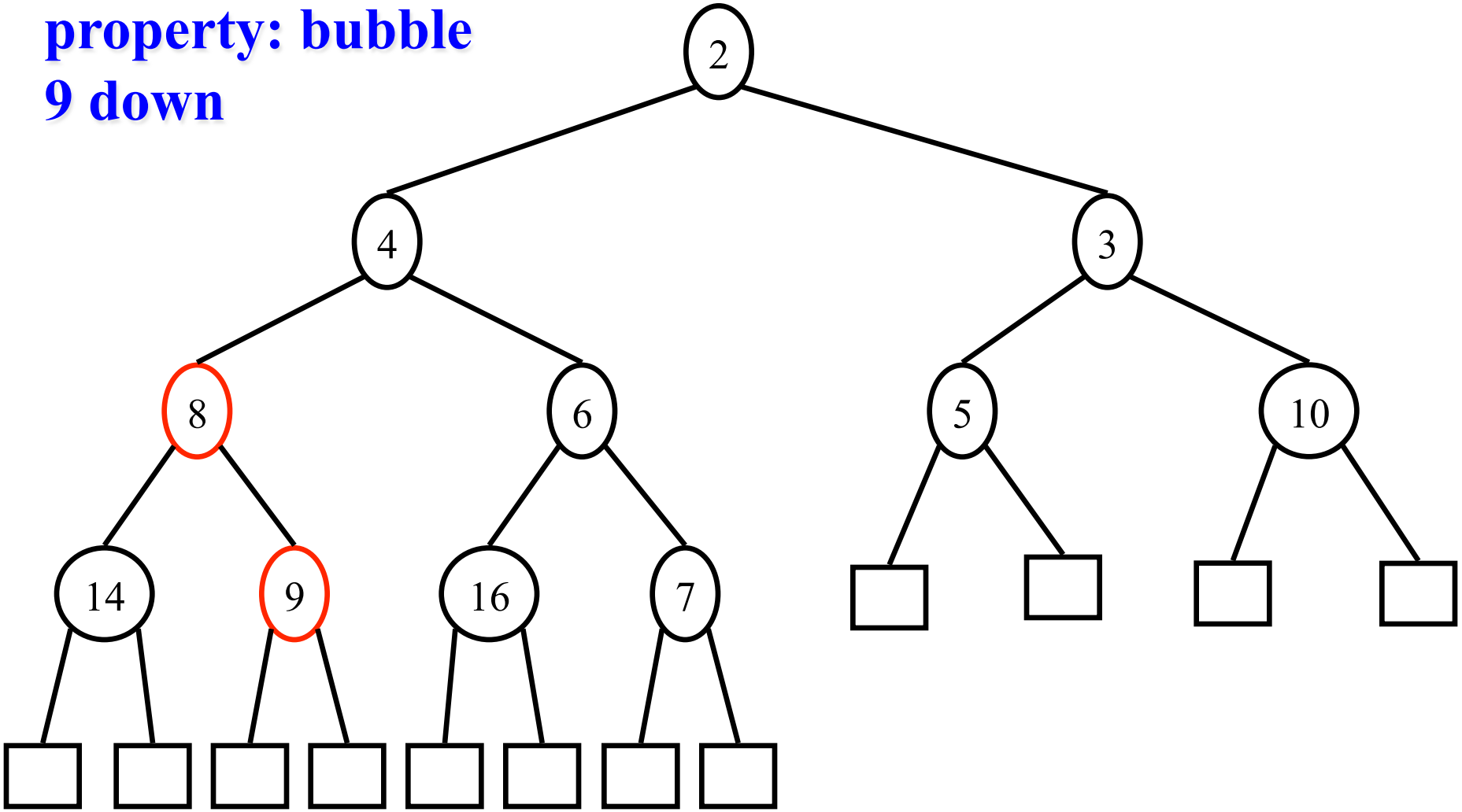
# Reestablish order property: bubble 9 down



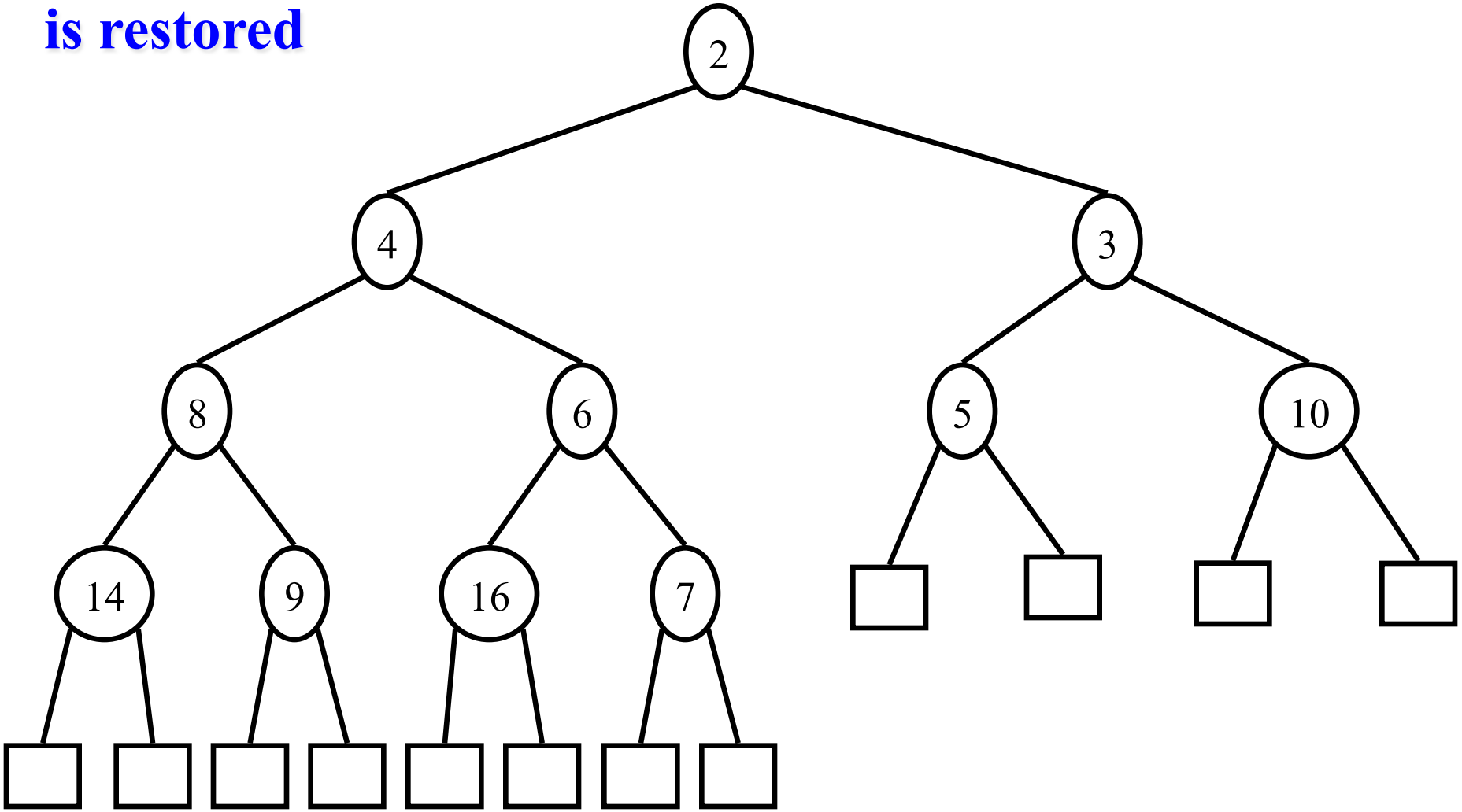
# Reestablish order property: bubble 9 down



# Reestablish order property: bubble 9 down



# Order property is restored



# Time Complexity of Heap Operations

- **Theorem**
  - The height of a balanced binary tree is $\log n$ ( i.e., $\log_2 n$)
- **Theorem**
  - The running times of the **Bubble up** and **Bubble down** operations in a Heap data structure are $O(\log n)$
- **Theorem**
  - The running times of the **insert** and **deleteMin** operations in a Heap data structure are $O(\log n)$

# Heapsort

1. Insert all elements from a given sequence $S$ into a Priority Queue implemented as a Heap
2. Remove minimum element and insert element into $S$
3. Repeat Step 2 until every element is inserted back in $S$

# Heapsort

```
Algorithm Heapsort(Sequence s)
// assertion: s contains a sequence of unsorted items
PriorityQueue p
// assertion: p is empty
while not s.empty() do
    p.insert(s.delete())
end
// assertion: s is empty
while not p.empty() do
    s.insert(p.deleteMin())
end
// assertion: s contains a sequence of sorted items
```

# Building up a heap

- Both loops are executed $n$ times
- **p.deleteMin()** and **p.insert()** take $O(\log n)$ time each
- Thus, each loop takes $O(n \log n)$ time

- **Theorem**
  - The worst-case time complexity of Heapsort is O(n log n)