CSC 225: Spring 2018: Lab 4
Solving Recurrence Equations

January 29, 2018

Recurrence Equation 1.

$$T(n) = \begin{cases} 1 & \text{if } n = 1 \\ 2T(\frac{n}{2}) + \log n & \text{if } n \geq 2 \end{cases}$$

Solution.

Using the repeated substitution method,

$$\begin{aligned}
T(n) &= 2T\left(\frac{n}{2}\right) + \log n \\
&= 2\left\{2T\left(\frac{n}{4}\right) + \log\frac{n}{2}\right\} + \log n \\
&= 4T\left(\frac{n}{4}\right) + 2\log\frac{n}{2} + \log n \\
&= 4\left\{2T\left(\frac{n}{8}\right) + \log\frac{n}{4}\right\} + 2\log\frac{n}{2} + \log n \\
&= 8T\left(\frac{n}{8}\right) + 4\log\frac{n}{4} + 2\log\frac{n}{2} + \log n \\
&\dots \\
&= 2^k T\left(\frac{n}{2^k}\right) + 2^{k-1}\log\frac{n}{2^{k-1}} + \cdots + 4\log\frac{n}{4} + 2\log\frac{n}{2} + \log n
\end{aligned}$$

We can re-write the above equation as a summation as follows

$$T(n) = 2^k T\left(\frac{n}{2^k}\right) + \sum_{i=0}^{k-1} 2^i \log\left(\frac{n}{2^i}\right)$$

We can visualize the recurrence with the help of a **recursion tree** as shown below.

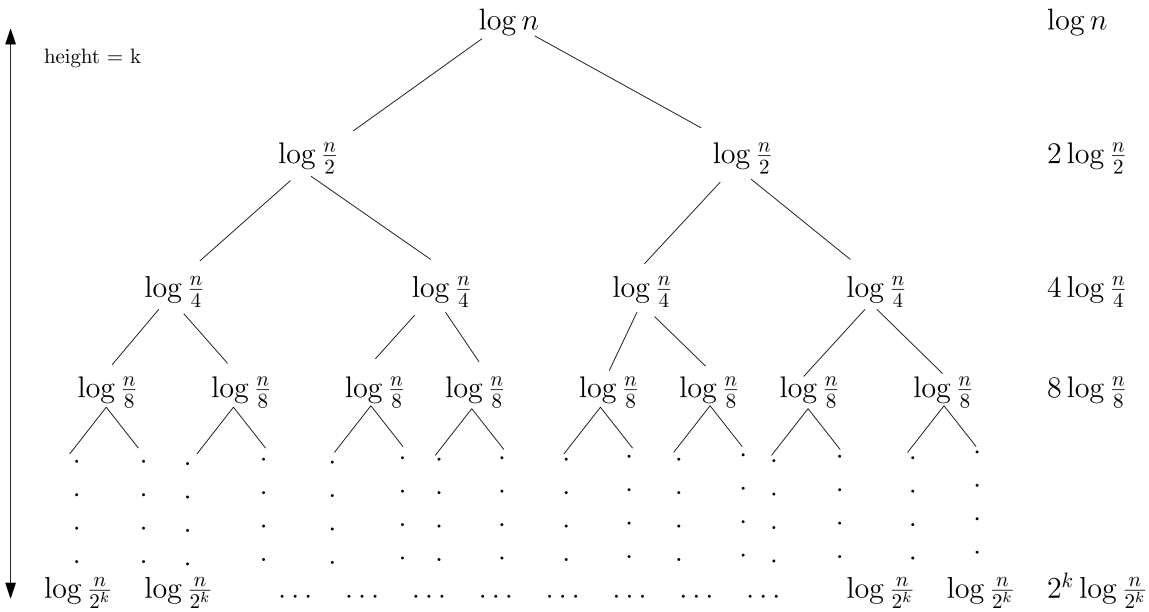

Figure 1: Recursion tree for recurrence equation 1.

In order to apply our base case we can choose $k = \log n$ such that $2^k = n$. Then,

$$\begin{aligned}
T(n) &= 2^{\log n} T\left(\frac{n}{2^{\log n}}\right) + \sum_{i=0}^{(\log n)-1} 2^i \log\left(\frac{n}{2^i}\right) \\
&= n \cdot T(1) + \sum_{i=0}^{(\log n)-1} 2^i \log\left(\frac{n}{2^i}\right) \\
&= n + \sum_{i=0}^{(\log n)-1} 2^i \log\left(\frac{n}{2^i}\right)
\end{aligned}$$

However, the resulting expression requires further work to get to a nice form. One alternative is to make an initial substitution of $n = 2^b$ in order to simplify some of the algebra. With this substitution we get

$$
\begin{aligned}
T(2^b) &= 2T(2^{b-1}) + b \\
&= 2\{2T(2^{b-2}) + (b-1)\} + b \\
&= 4T(2^{b-2}) + 2(b-1) + b \\
&= 4\{2T(2^{b-3}) + (b-2)\} + 2(b-1) + b \\
&= 8T(2^{b-3}) + 4(b-2) + 2(b-1) + b \\
&\ldots \\
&= 2^k T(2^{b-k}) + 2^{k-1}(b-(k-1)) + \cdots + 2(b-1) + b
\end{aligned}
$$

We can re-write the above equation as a summation as follows

$$
T(2^b) = 2^k T(2^{b-k}) + \sum_{i=0}^{k-1} 2^i (b-i)
$$

In order to apply our base case we can choose $k = b$. Then,

$$
\begin{aligned}
T(2^b) &= 2^b T(2^{b-b}) + \sum_{i=0}^{b-1} 2^i (b-i) \\
&= 2^b T(1) + \sum_{i=0}^{b-1} 2^i (b-i) \\
&= 2^b + 2^{b+1} - (b+2)
\end{aligned}
$$

Using the identity $\sum_{i=0}^{b-1} 2^i (b-i) = 2^{b+1} - (b+2)$. Therefore, since $b = \log n$, we get

$$
\begin{aligned}
T(n) &= n + 2n - (\log n + 2) \\
&= 3n - \log n - 2
\end{aligned}
$$

Recurrence Equation 2.

$$T(n) = \begin{cases} 1 & \text{if } n = 1 \\ T(\frac{n}{3}) + \log_3 n & \text{if } n \geq 3 \end{cases}$$

Solution 1.

$$T(n) = 1 + \sum_{i=0}^{(\log_3 n)-1} \log_3 \left(\frac{n}{3^i} \right)$$

Solution 2.
Using the substitution $n = 3^b$

$$T(3^b) = 1 + \frac{b(b+1)}{2}$$

Undoing the substitution gives,

$$T(n) = 1 + \frac{\log_3 n(1 + \log_3 n)}{2}$$