CSC 226

Algorithms and Data Structures: II

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ECS 516

2-3 tree: implementation?

Direct implementation is complicated, because:

- Maintaining multiple node types is cumbersome.
- Need multiple compares to move down tree.
- Need to move back up the tree to split 4-nodes.
- Large number of cases for splitting.

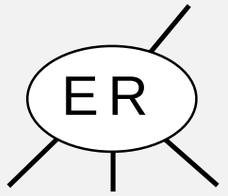
fantasy code

```
public void put(Key key, Value val)
{
  Node x = root;
  while (x.getTheCorrectChild(key) != null)
  {
    x = x.getTheCorrectChildKey();
    if (x.is4Node()) x.split();
  }
  if      (x.is2Node()) x.make3Node(key, val);
  else if (x.is3Node()) x.make4Node(key, val);
}
```

Bottom line. Could do it, but there's a better way.

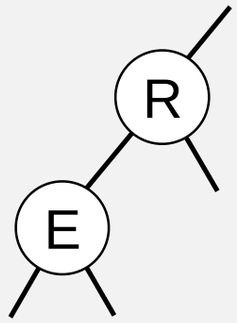
How to implement 2-3 trees with binary trees?

Challenge. How to represent a 3 node?



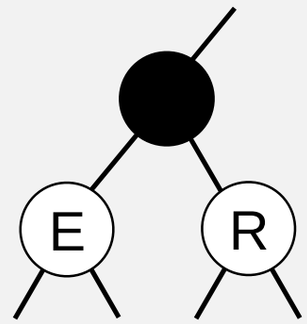
Approach 1: regular BST.

- No way to tell a 3-node from a 2-node.
- Cannot map from BST back to 2-3 tree.



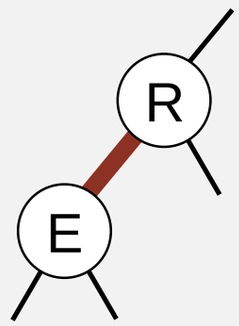
Approach 2: regular BST with "glue" nodes.

- Wastes space, wasted link.
- Code probably messy.



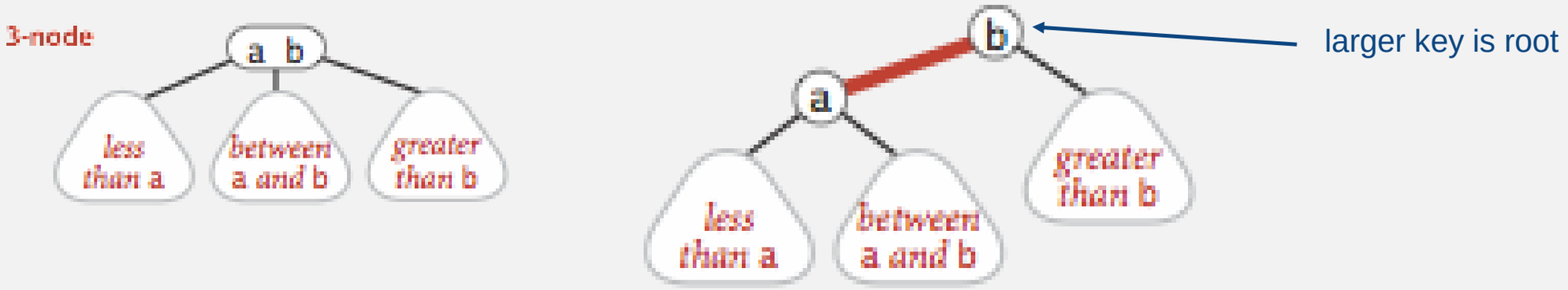
Approach 3: regular BST with red "glue" links.

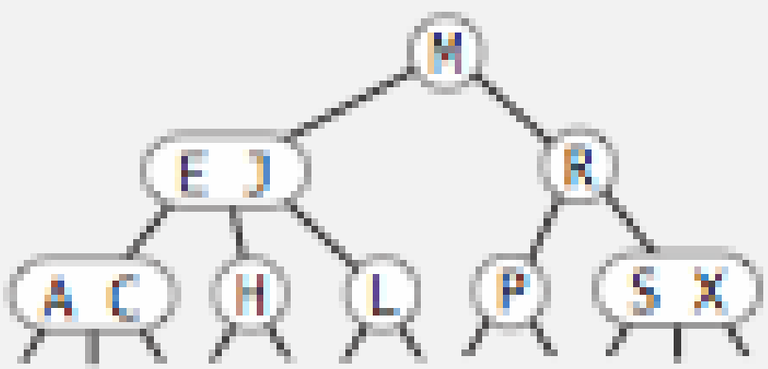
- Widely used in practice.
- Arbitrary restriction: red links lean left.



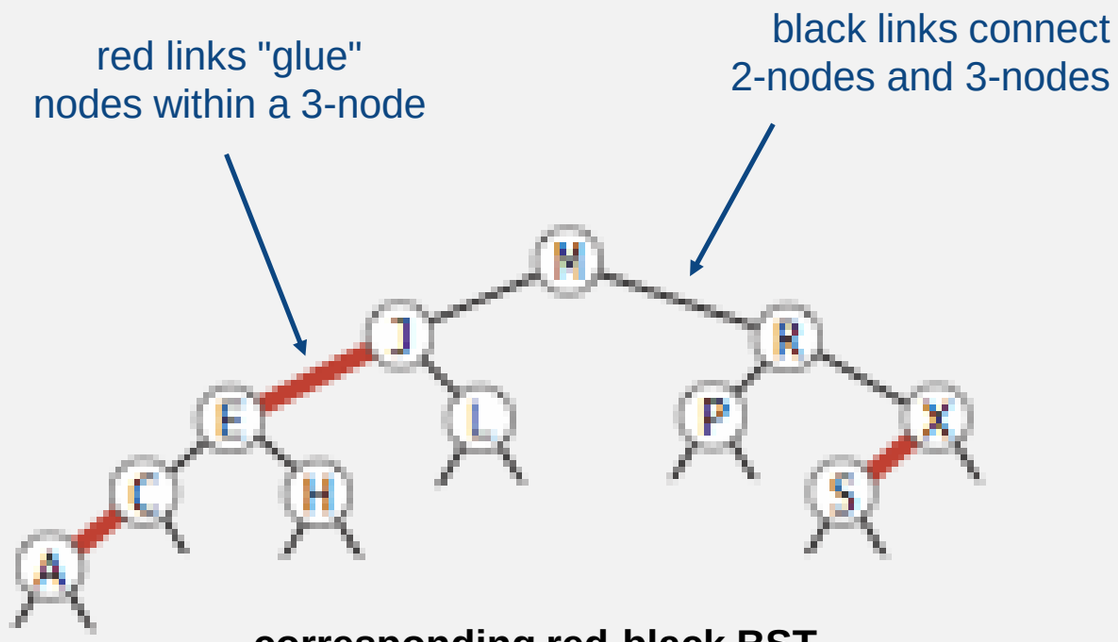
Left-leaning red-black BSTs (Guibas-Sedgewick 1979 and Sedgewick 2007)

1. Represent 2-3 tree as a BST.
2. Use "internal" left-leaning links as "glue" for 3-nodes.





2-3 tree

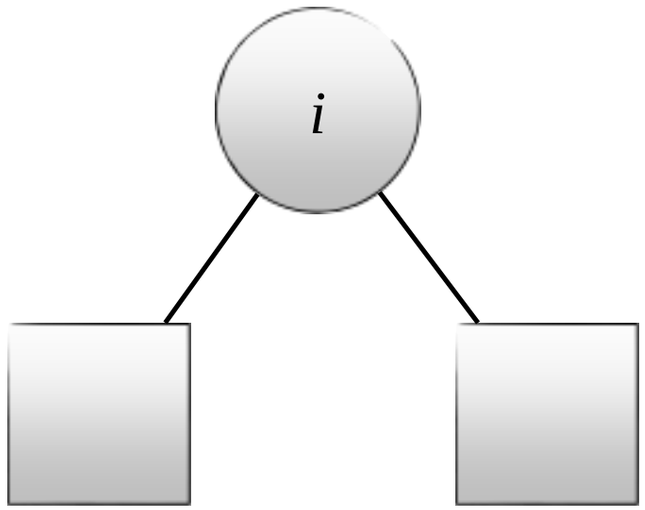


corresponding red-black BST

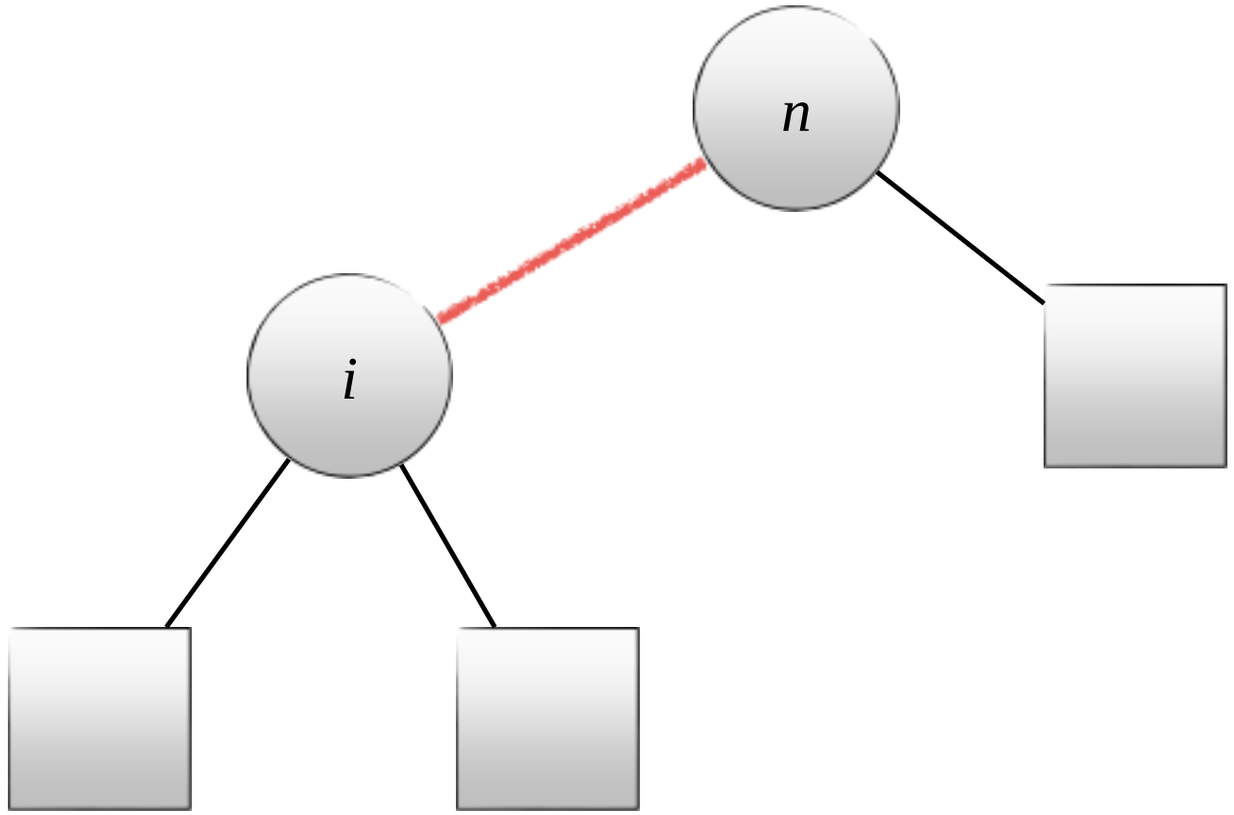
Definition: Red-Black Tree

- A red-black tree is a binary search tree where each link/edge is either red or black. Further
  - All red links lean left
  - No node has two red links connected to it
  - The tree has a balance: every path from the root to a leaf has the same number of black links
  - Links to leaves are black

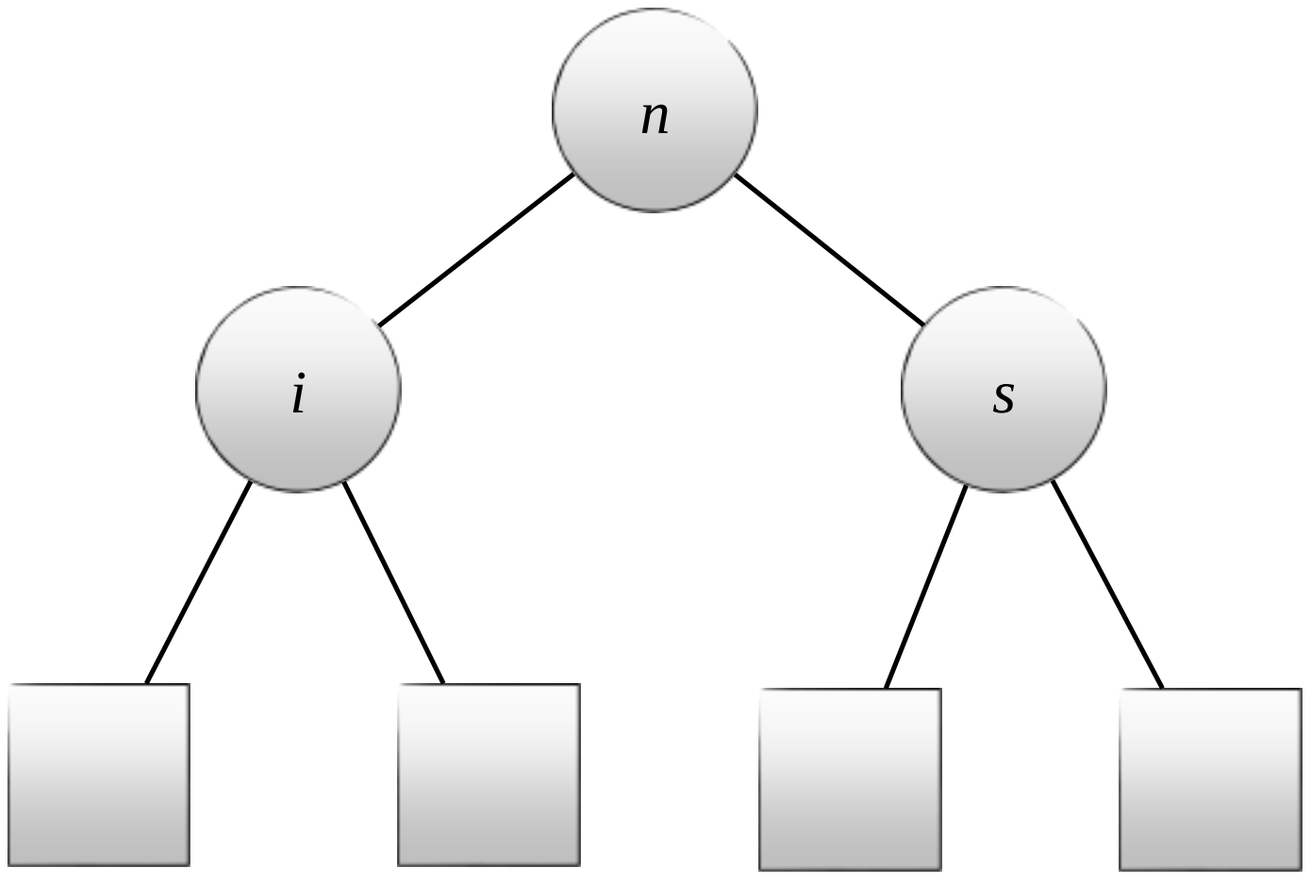
Example 1: red-black tree?



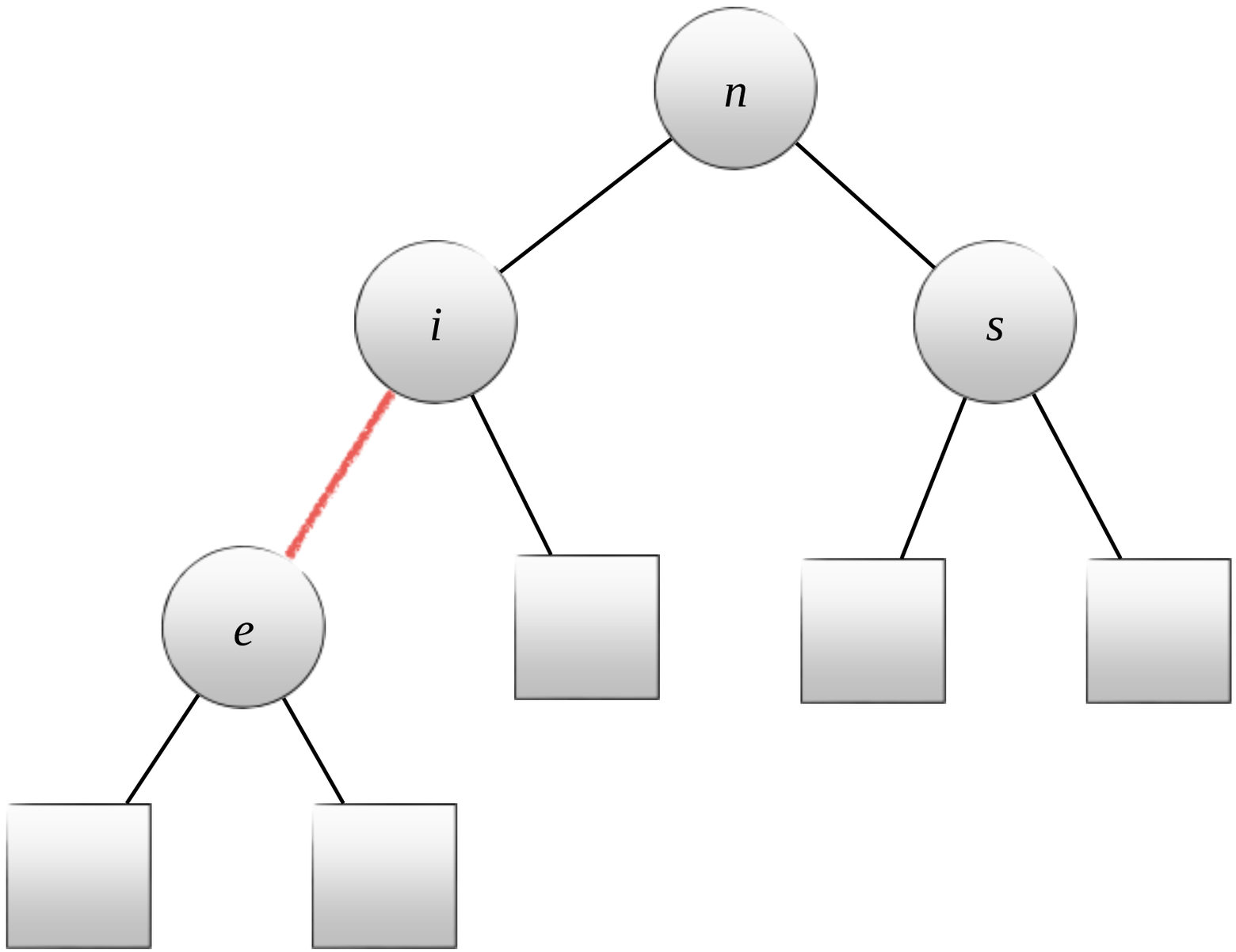
Example 2: red-black tree?



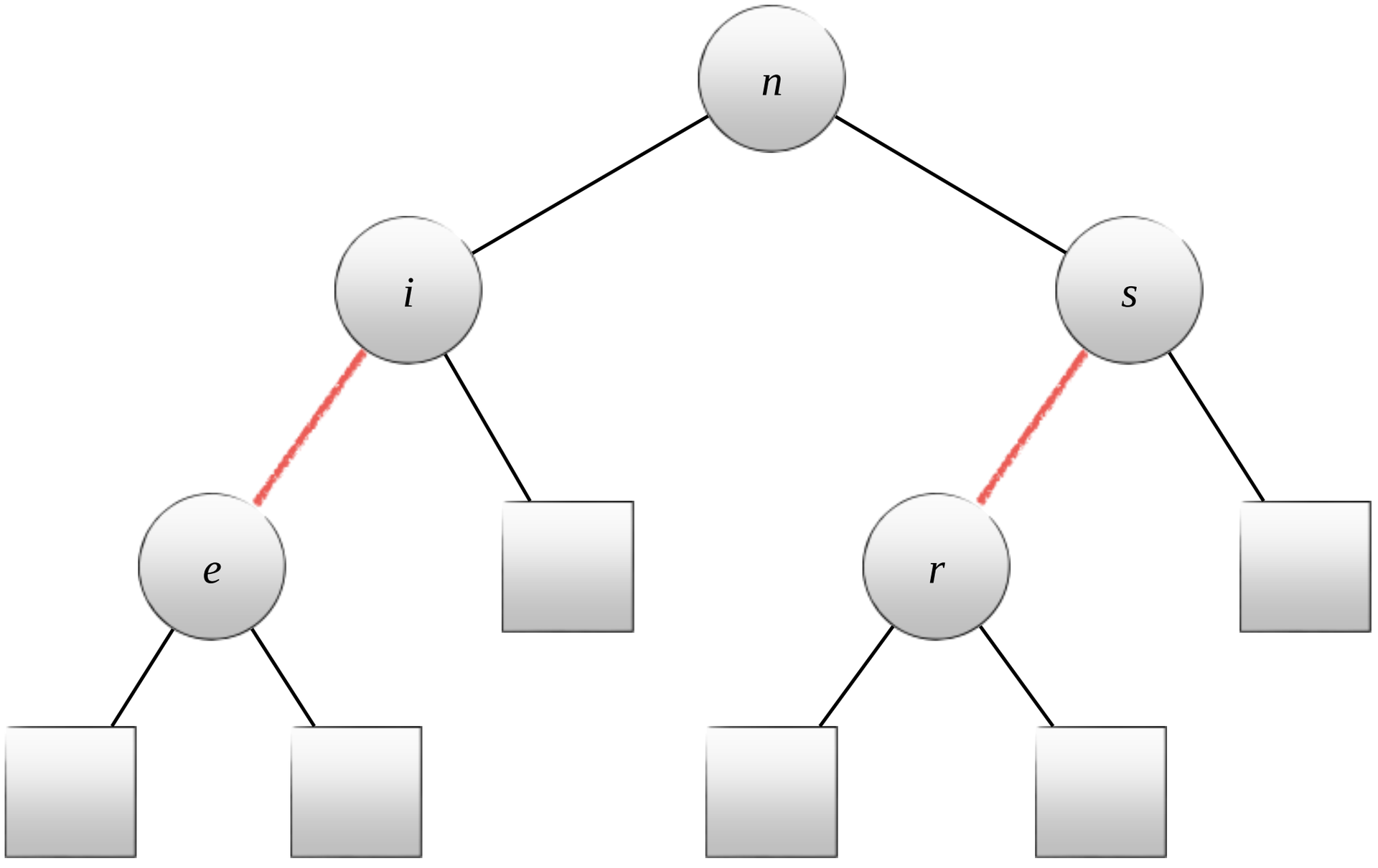
Example 3: red-black tree?



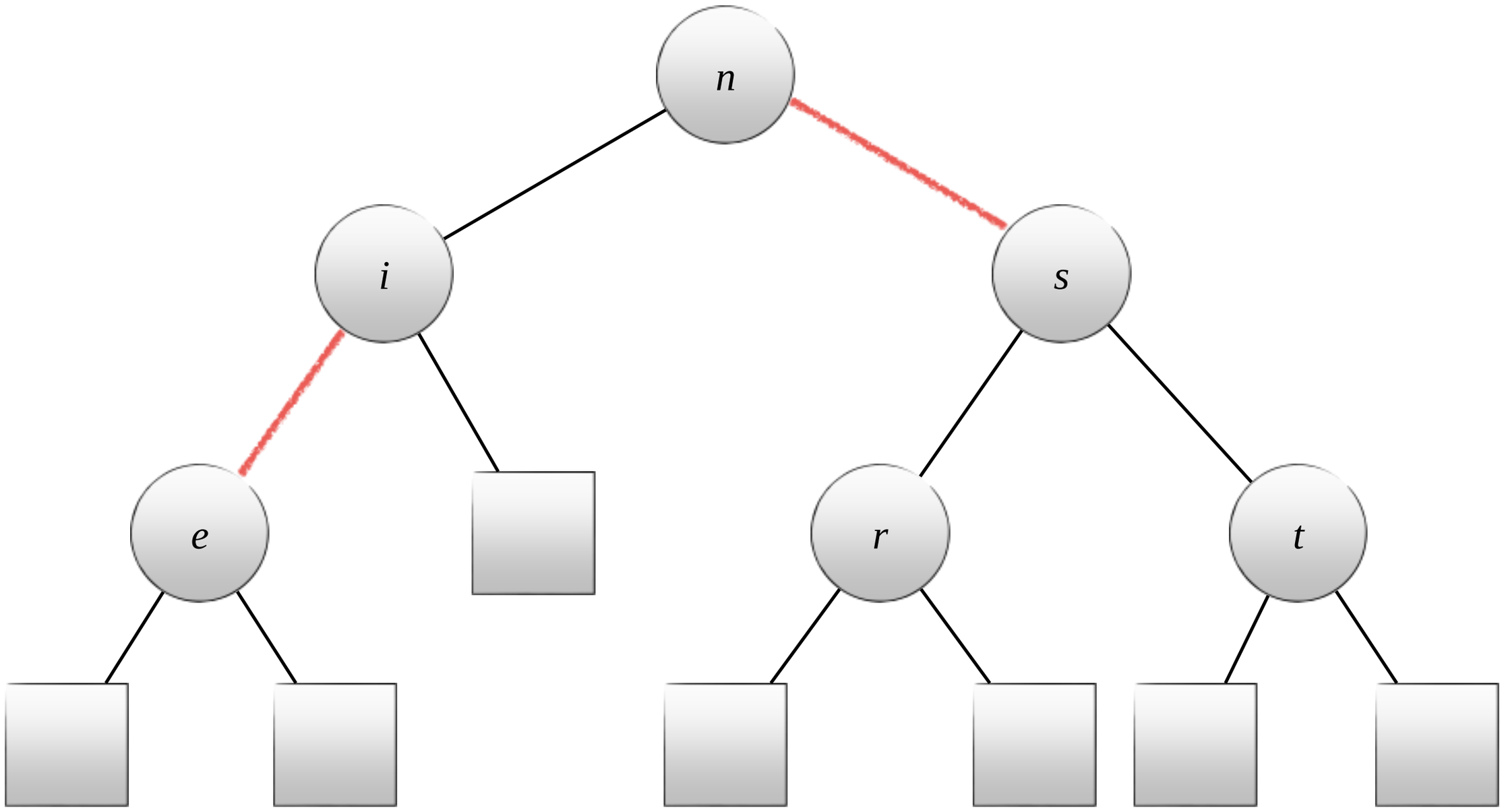
Example 4: red-black tree?



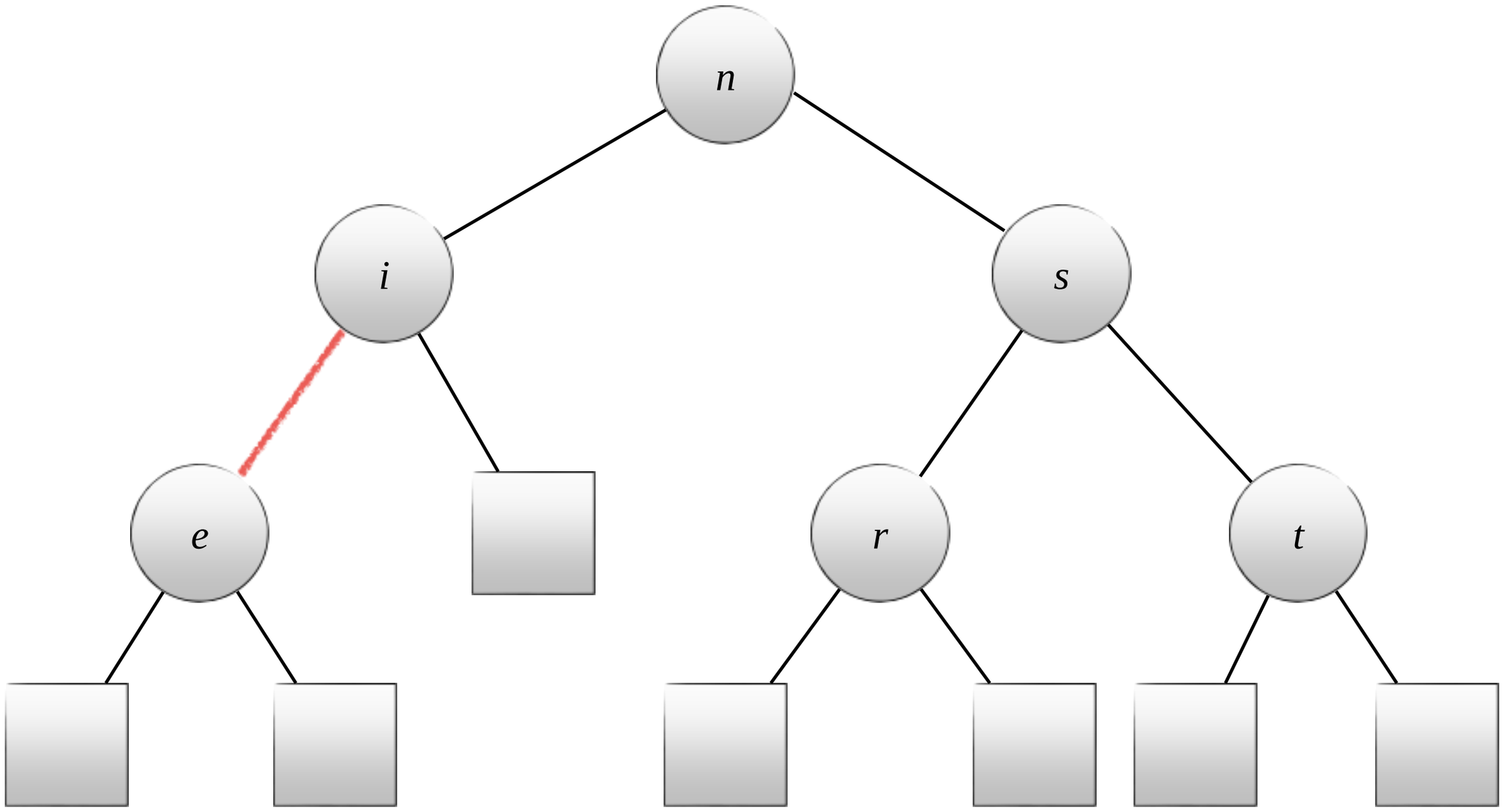
Example 5: red-black tree?



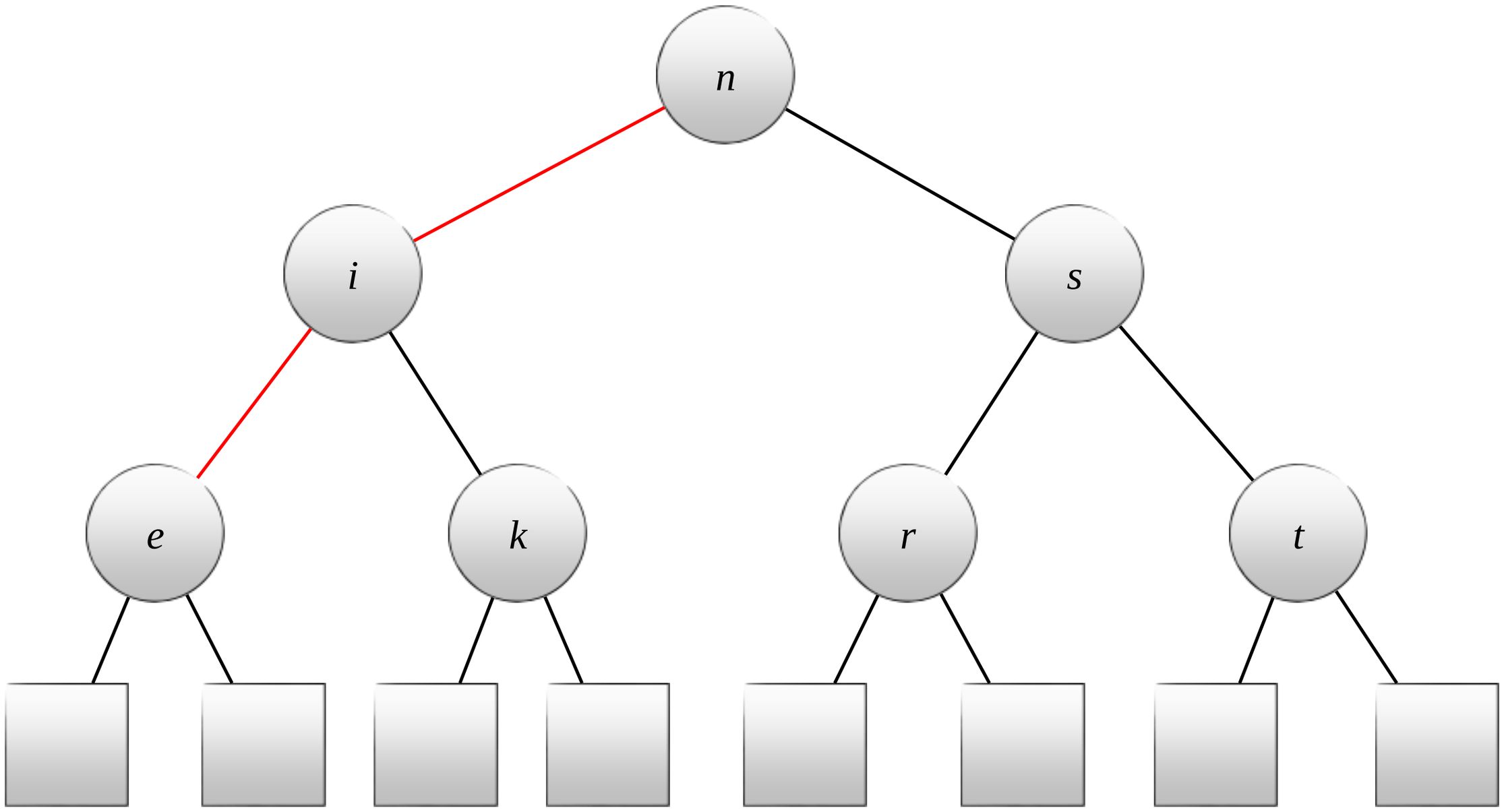
Example 6: red-black tree?



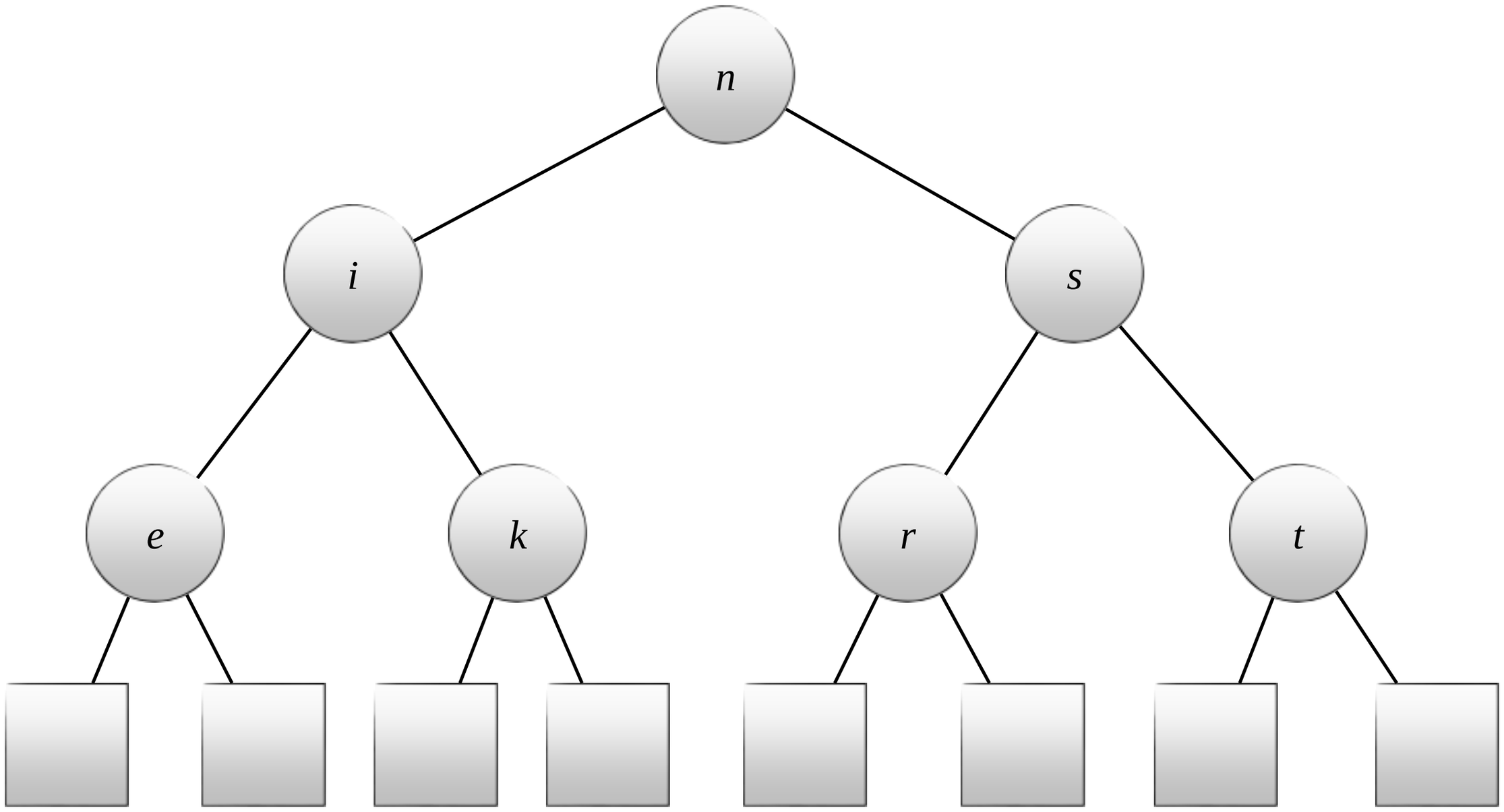
Example 7: red-black tree?



Example 8: red-black tree?



Example 9: red-black tree?



Red-black BST representation

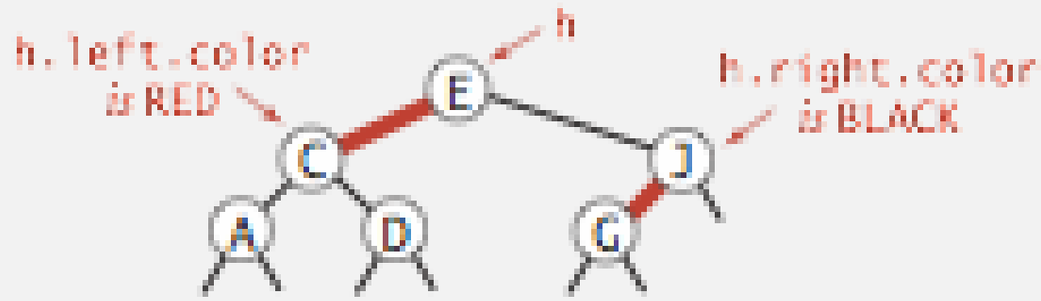
Each node is pointed to by precisely one link (from its parent)
can encode color of links in nodes.

```
private static final boolean RED   = true;
private static final boolean BLACK = false;

private class Node
{
   Key key;
   Value val;
   Node left, right;
   boolean color;   // color of parent link
}

private boolean isRed(Node x)
{
   if (x == null) return false;
   return x.color == RED;
}
```

null links are black



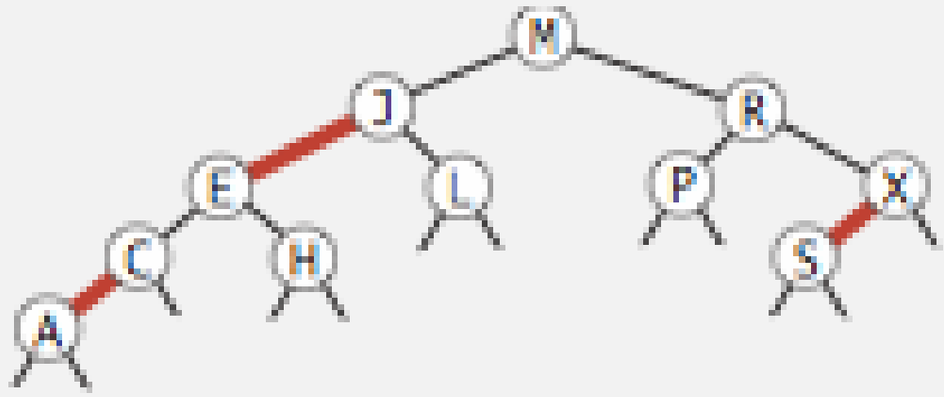
Search implementation for red-black BSTs

Observation. Search is the same as for elementary BST (ignore color).

but runs faster

because of better balance

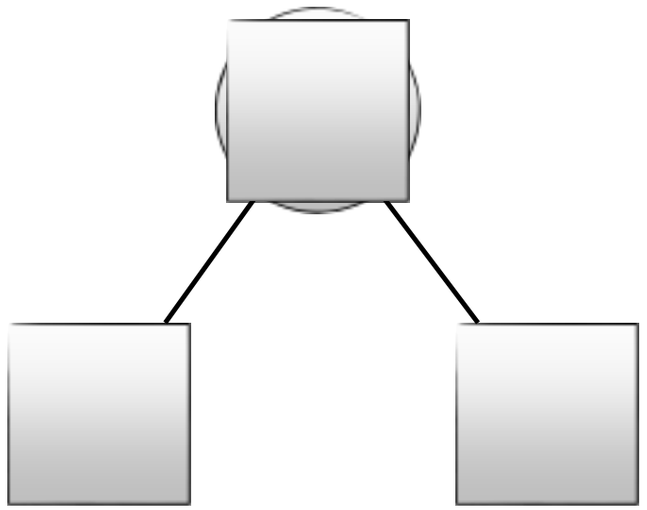
```
public Val get(Key key)
{
   Node x = root;
   while (x != null)
   {
      int cmp = key.compareTo(x.key);
      if      (cmp  < 0)
         x = x.left;
      else if (cmp  > 0)
         x = x.right;
      else
         return x.val;
   }
   return null;
}
```



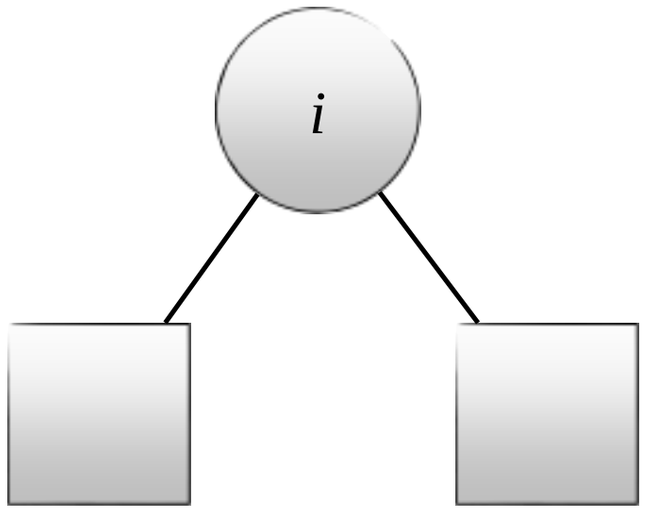
Inserting into a red-black tree

- Insert just as in BST
- but: link/edge to new node is red
- rotations and color flipping (depending on case)

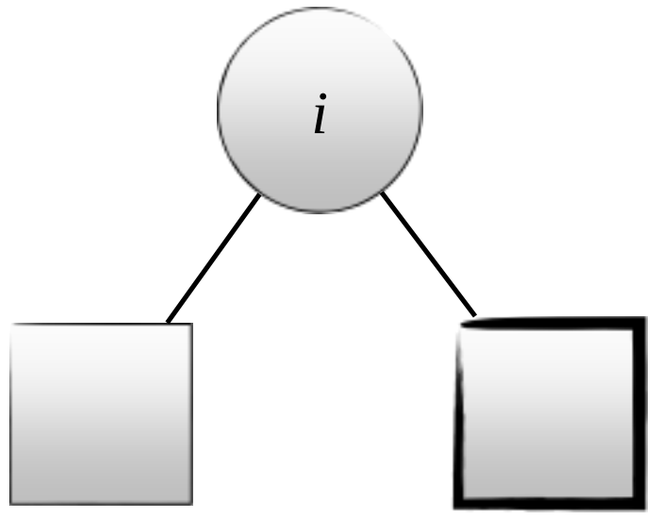
Insert key *i* into empty tree



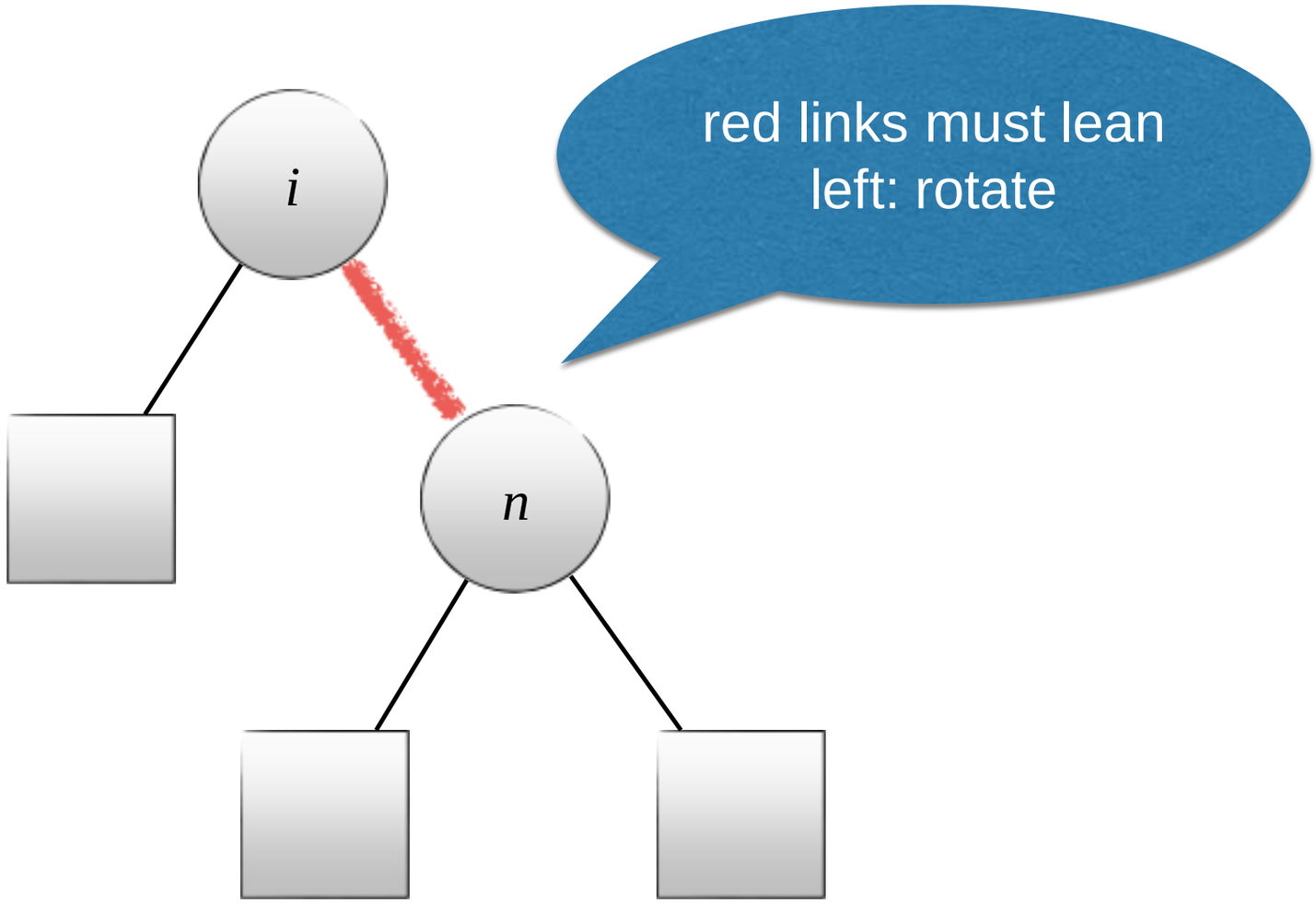
Insert key n



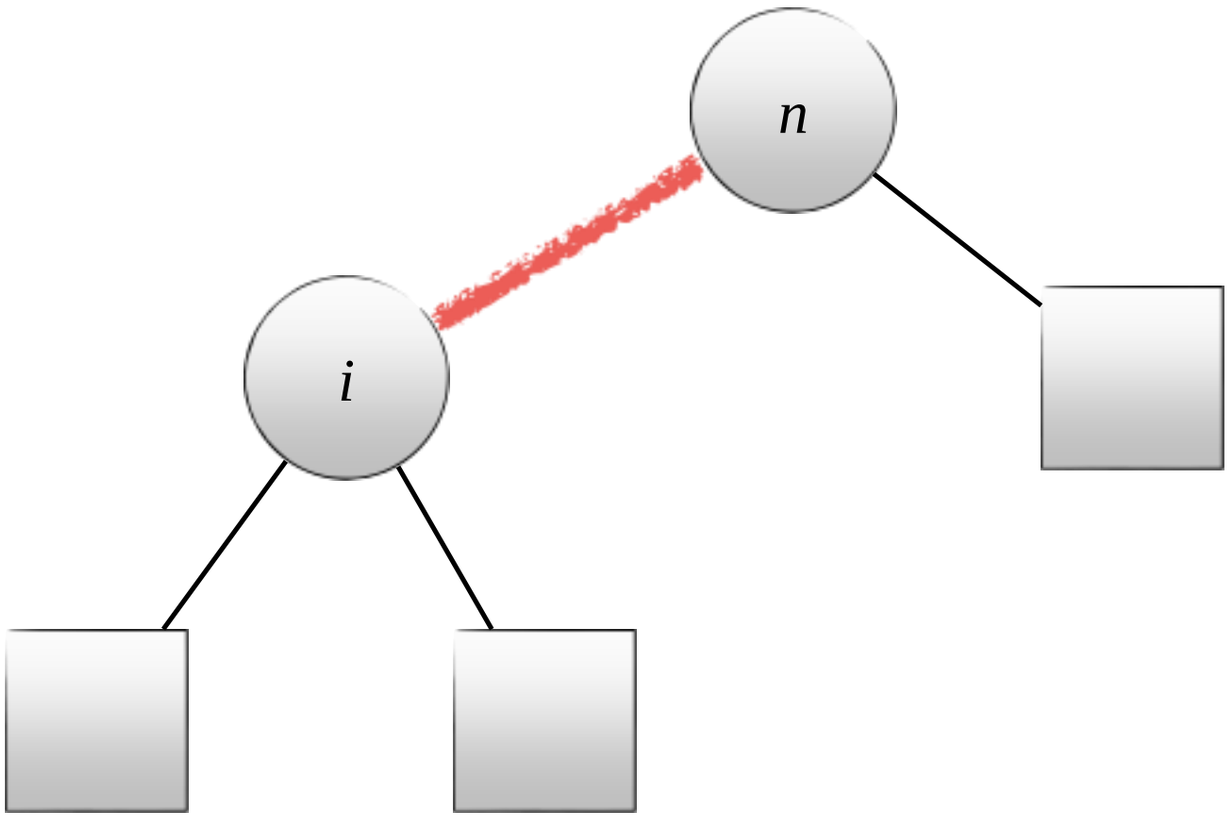
Insert key n



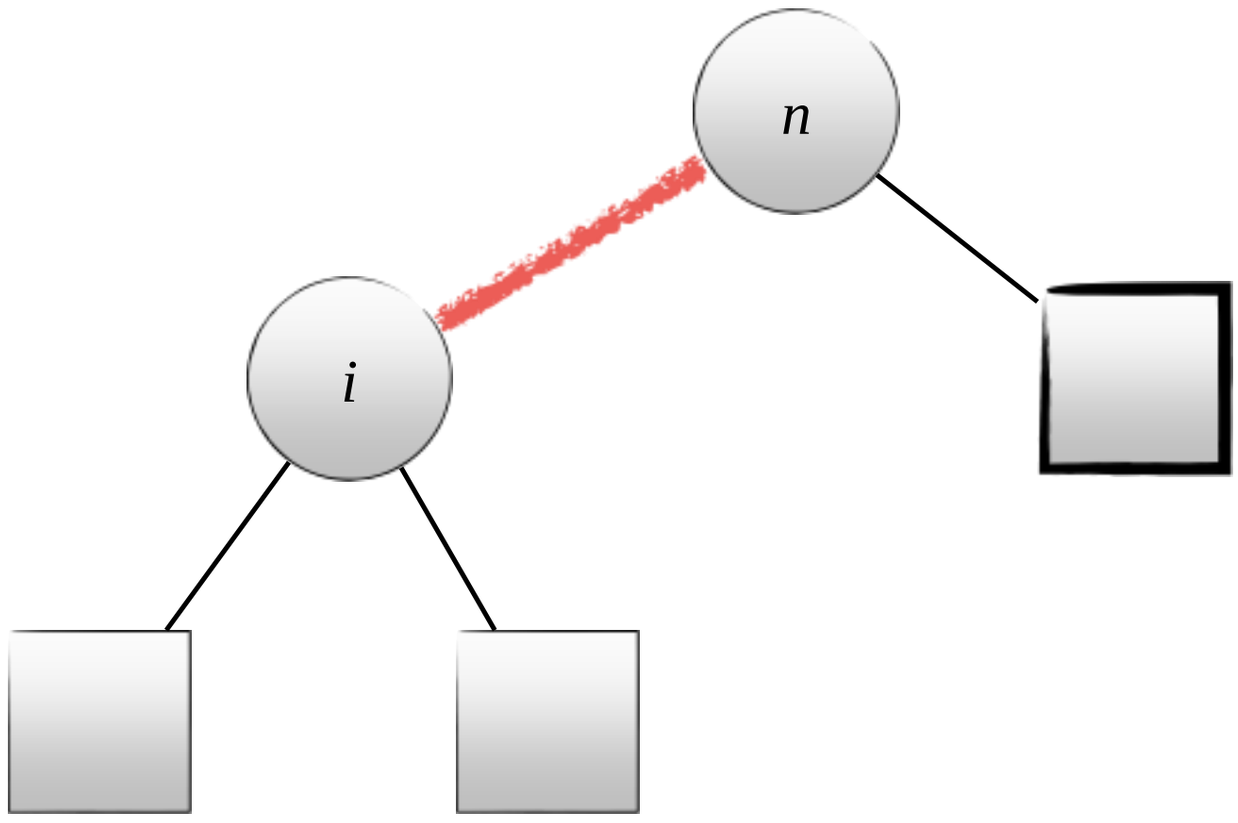
Link to new node is red



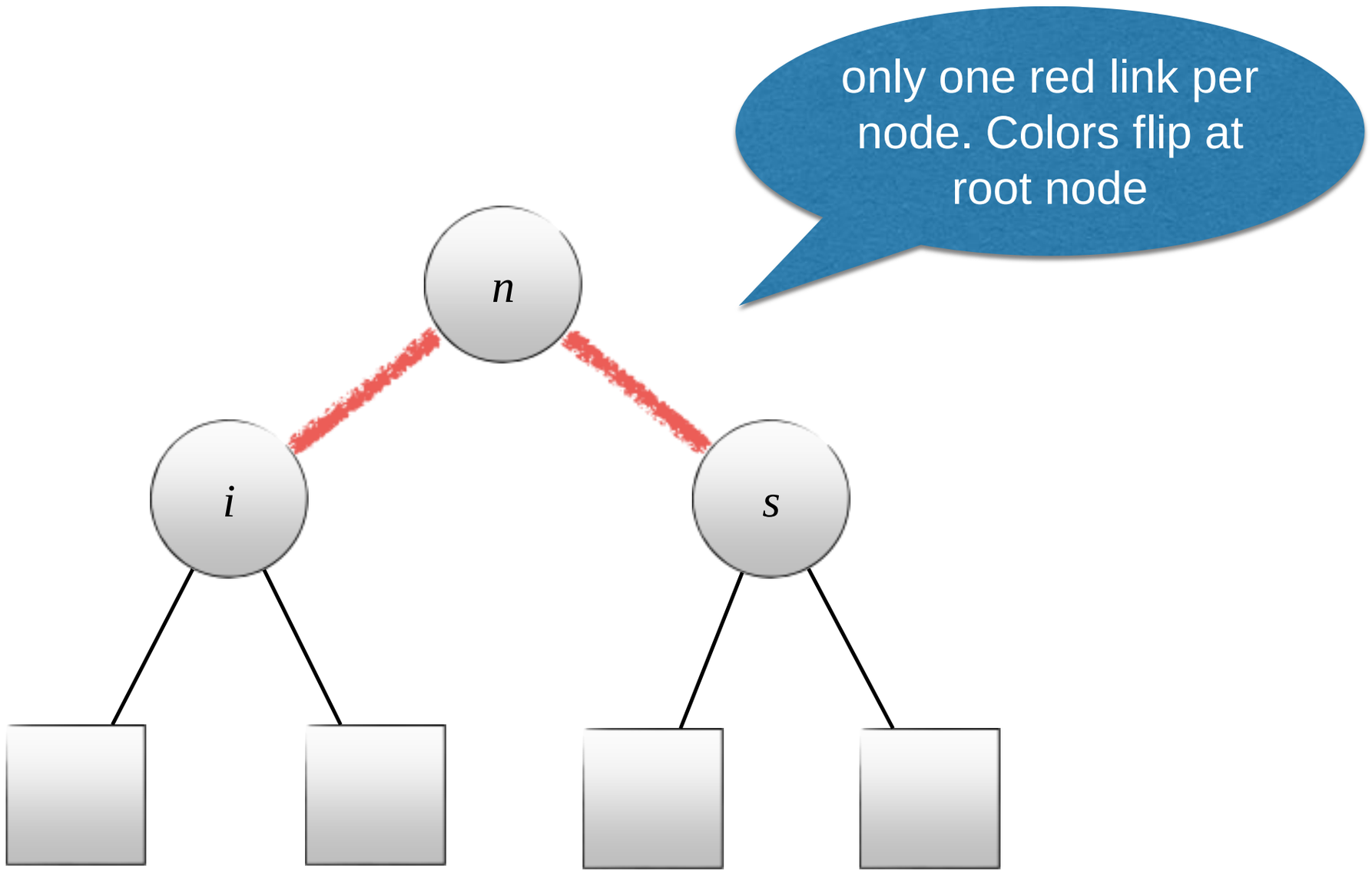
After rotation



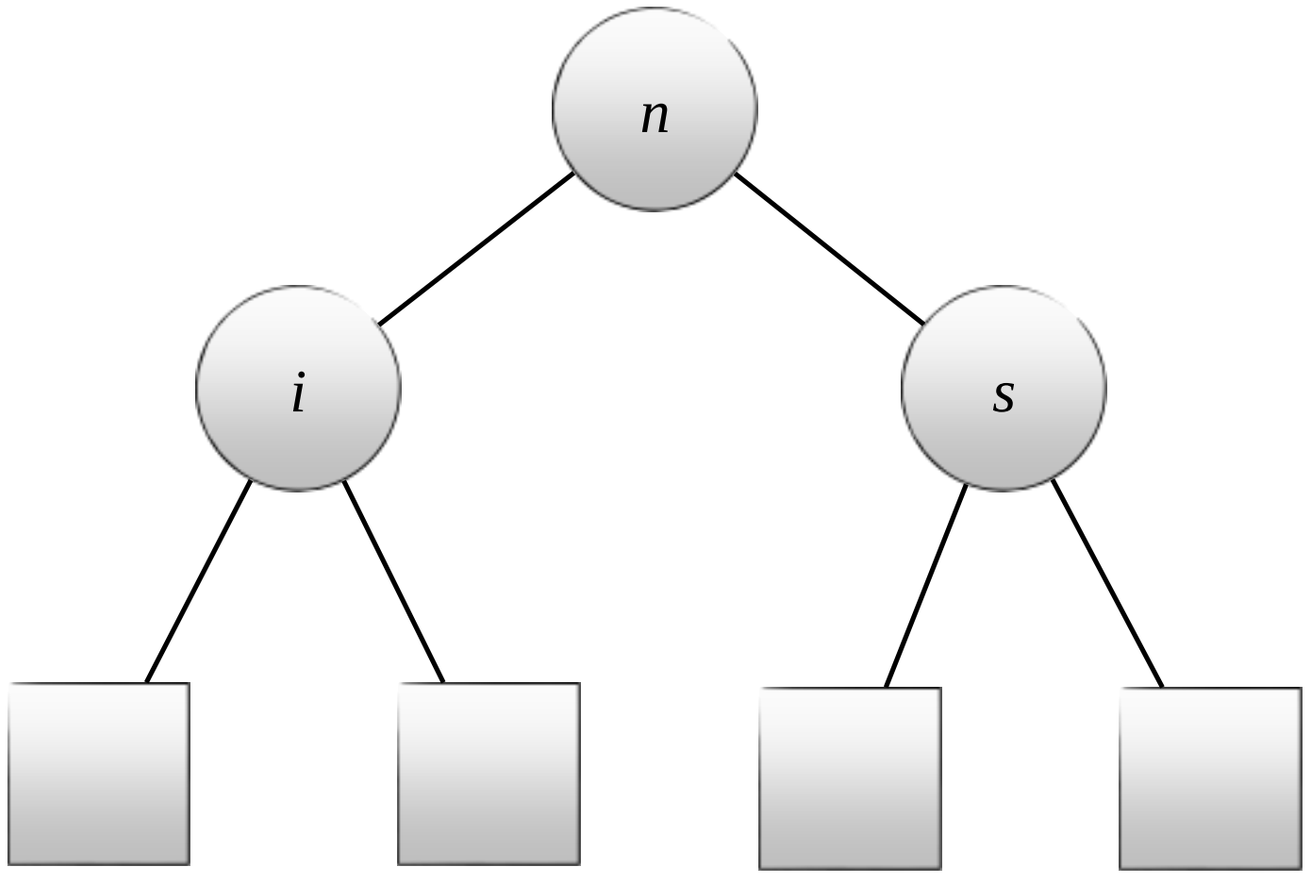
Insert key s



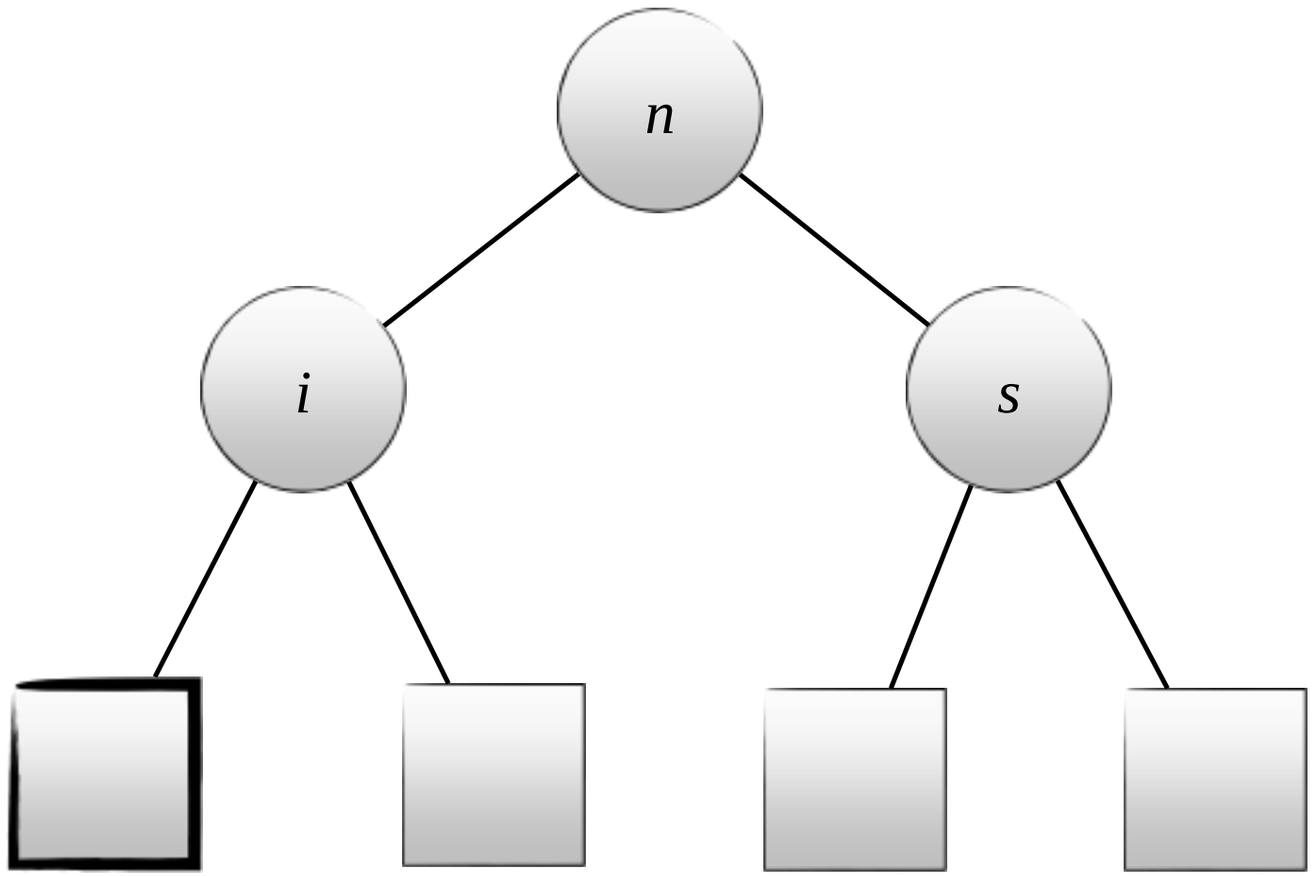
Insert key s, new link is red



Colors flipped to black

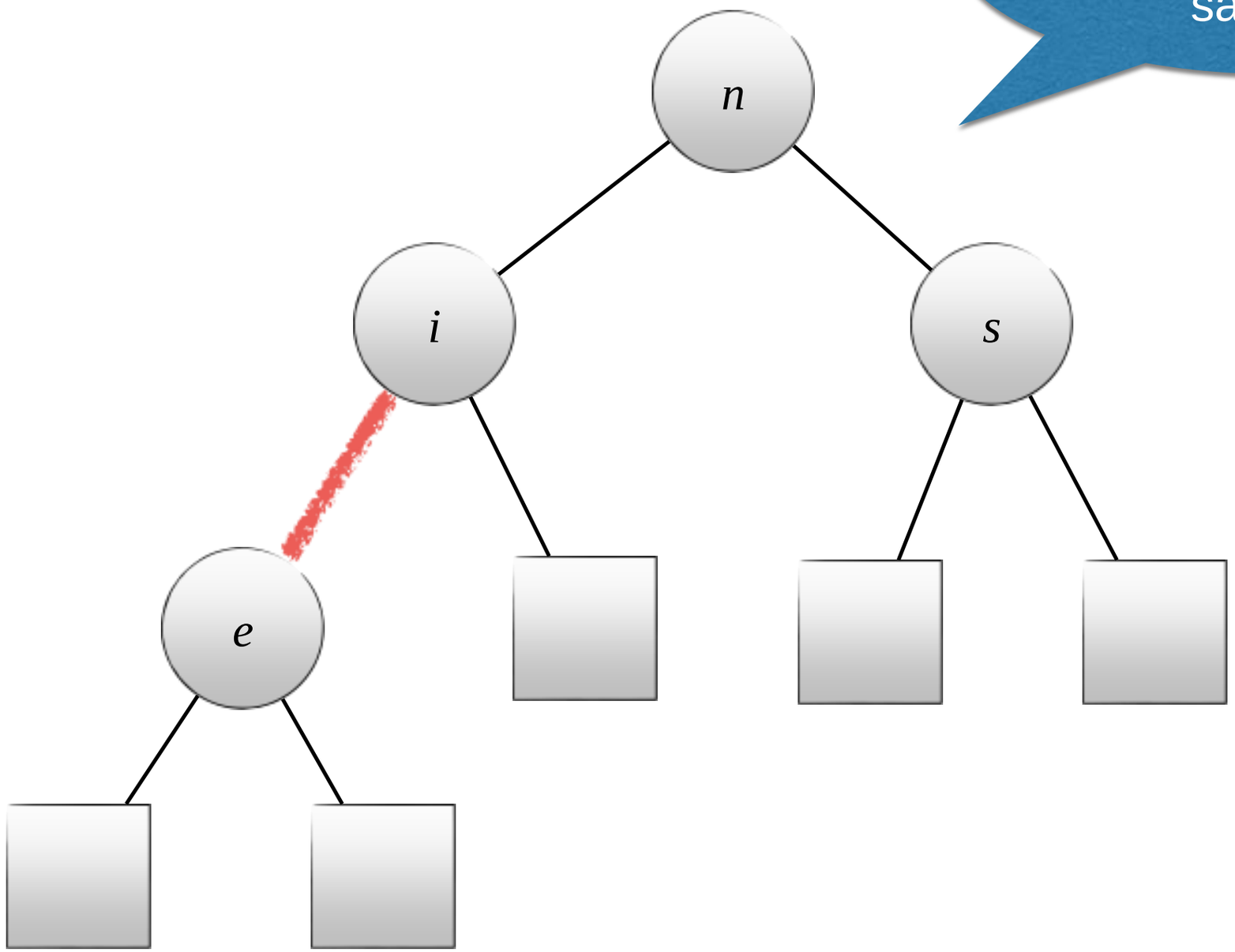


Insert key e

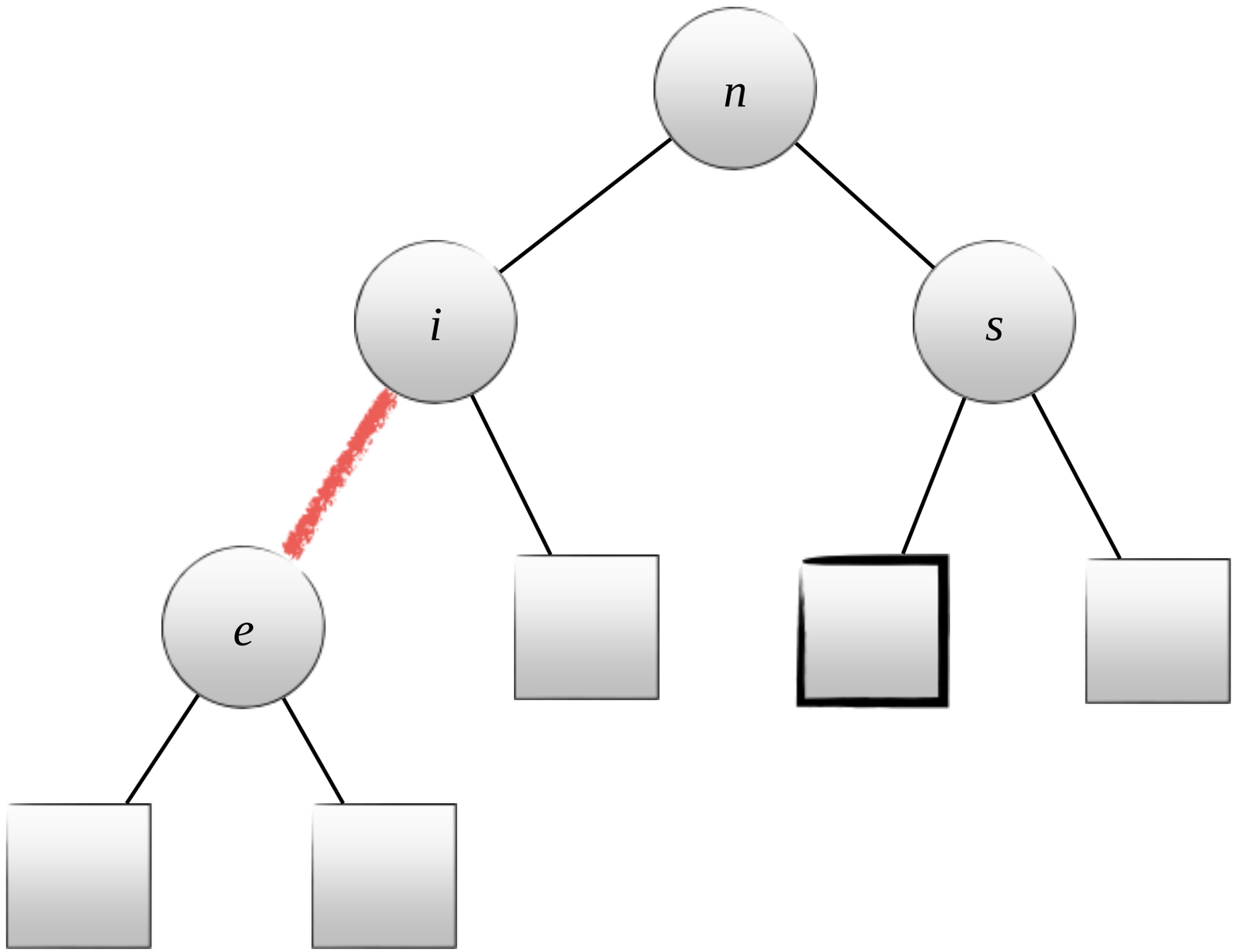


Insert key e, new link is red

All properties of red-black tree are satisfied

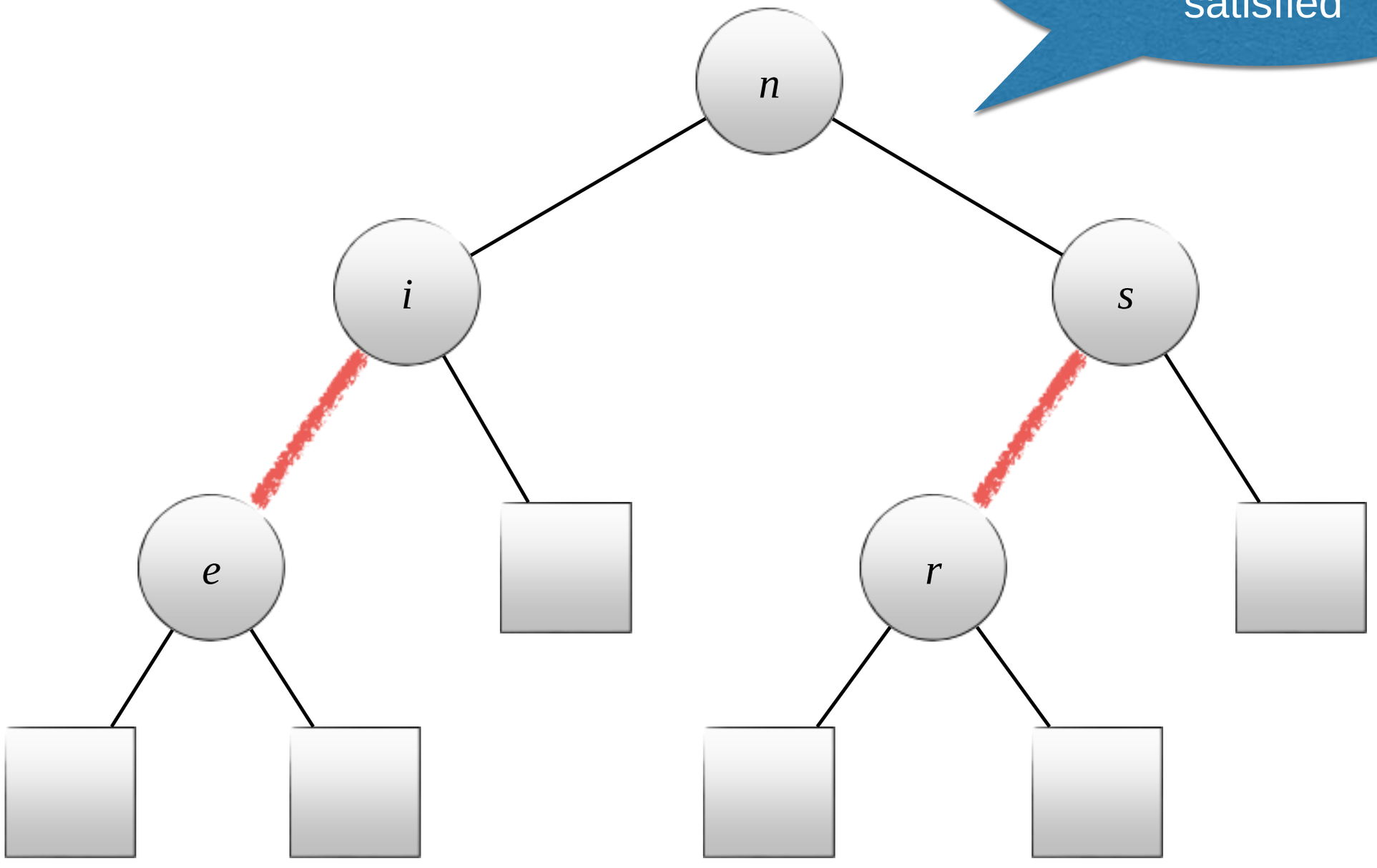


Insert key r

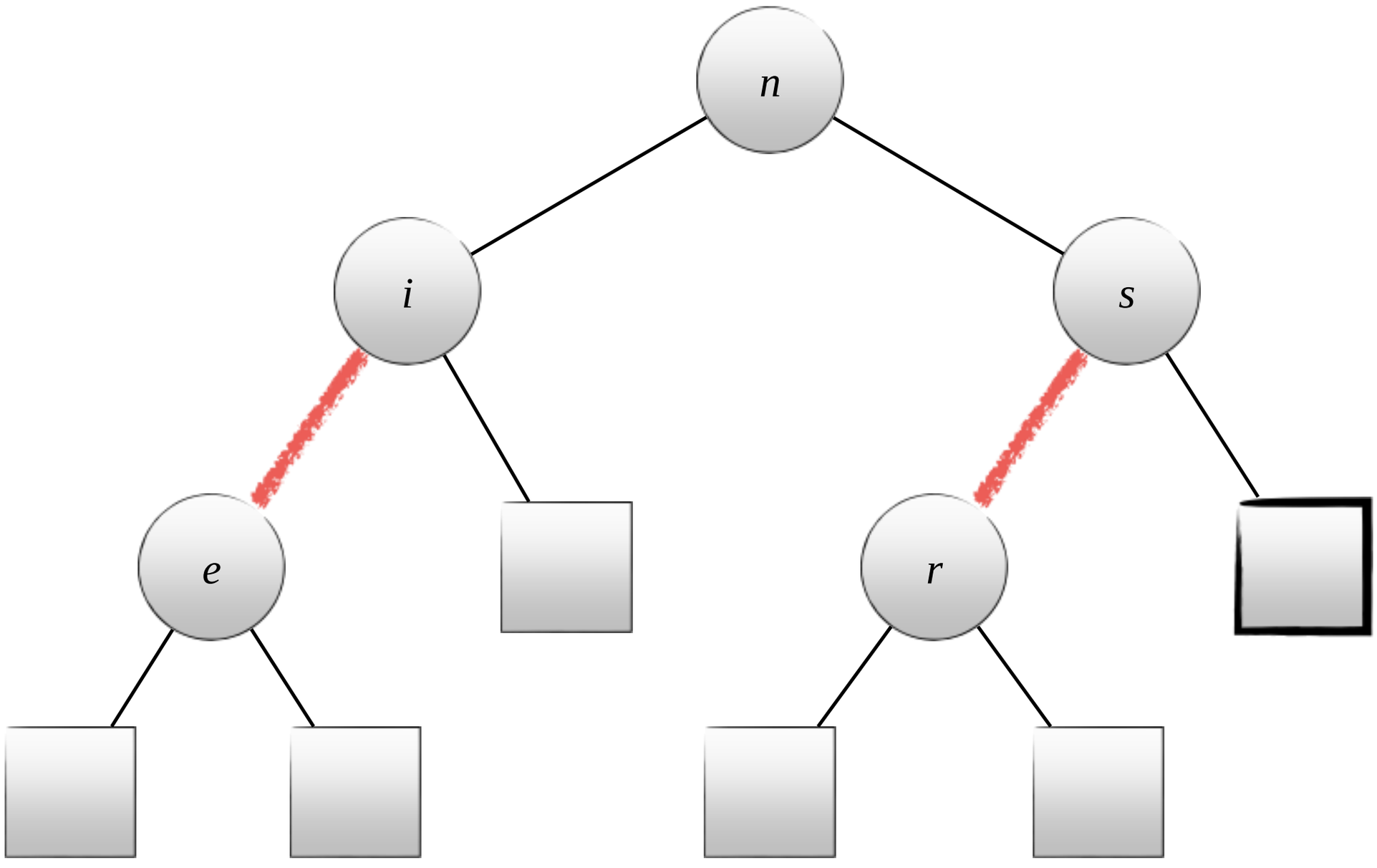


Insert key r, new link is red

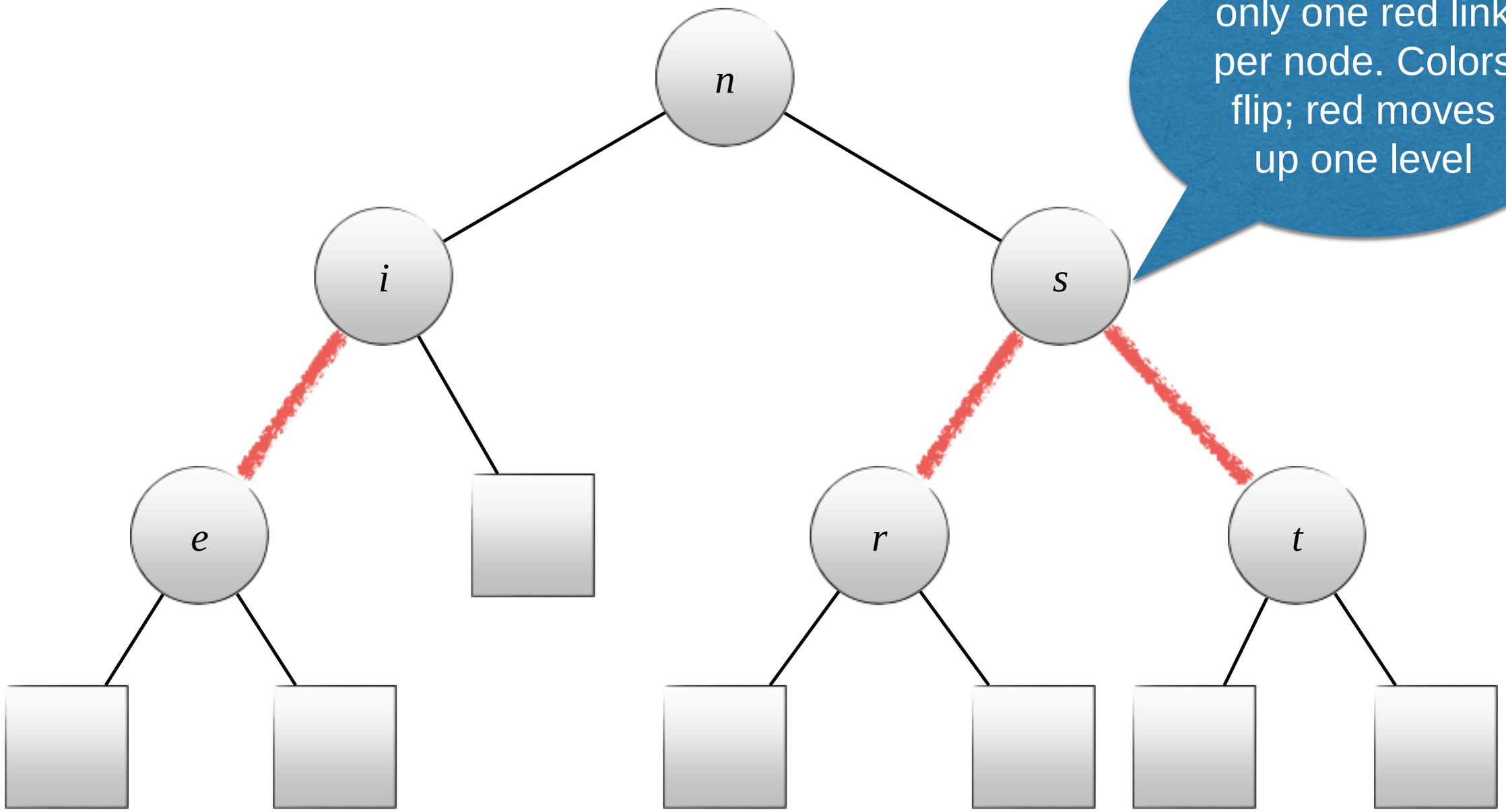
All properties of red-black tree are satisfied



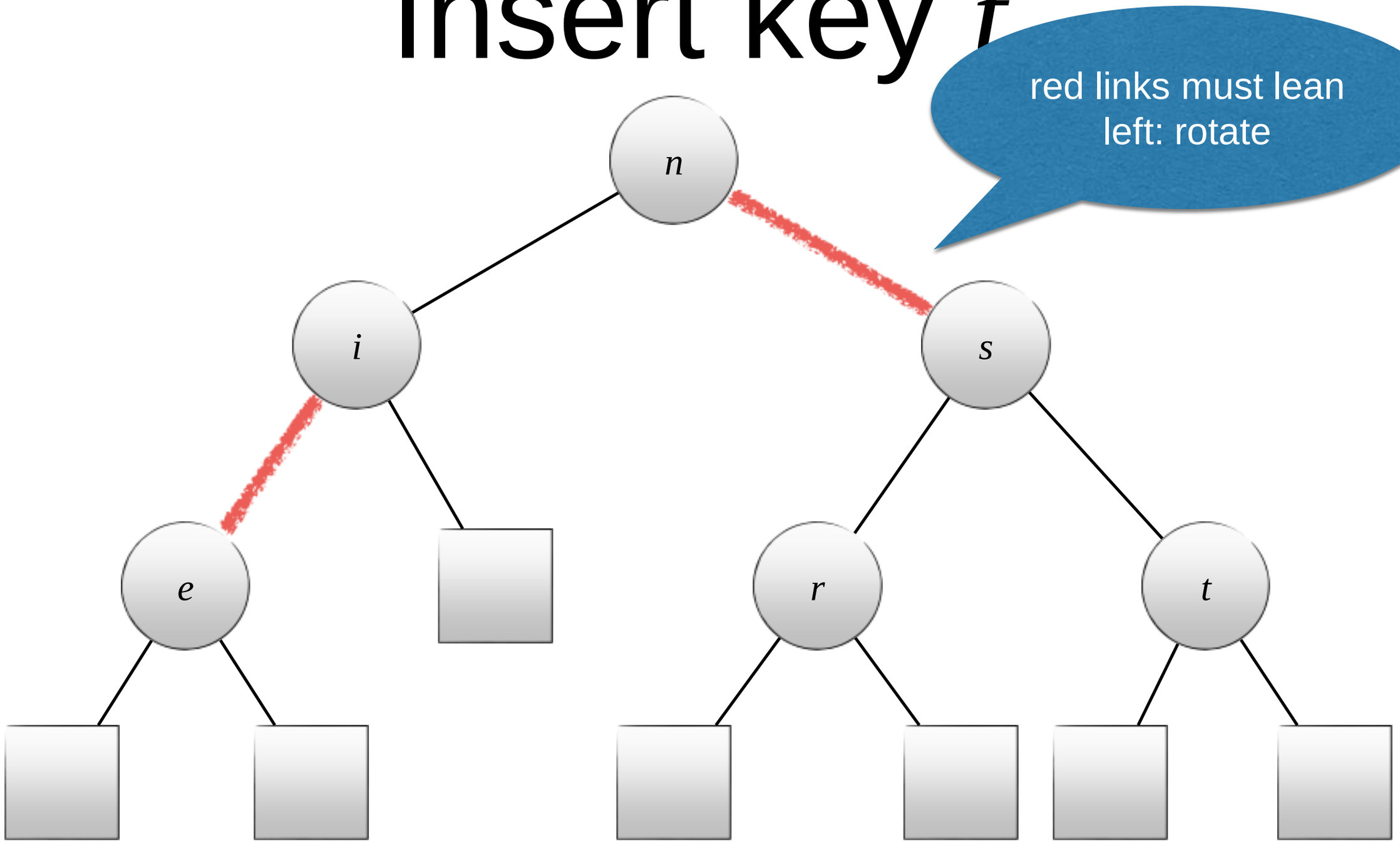
Insert key *t*



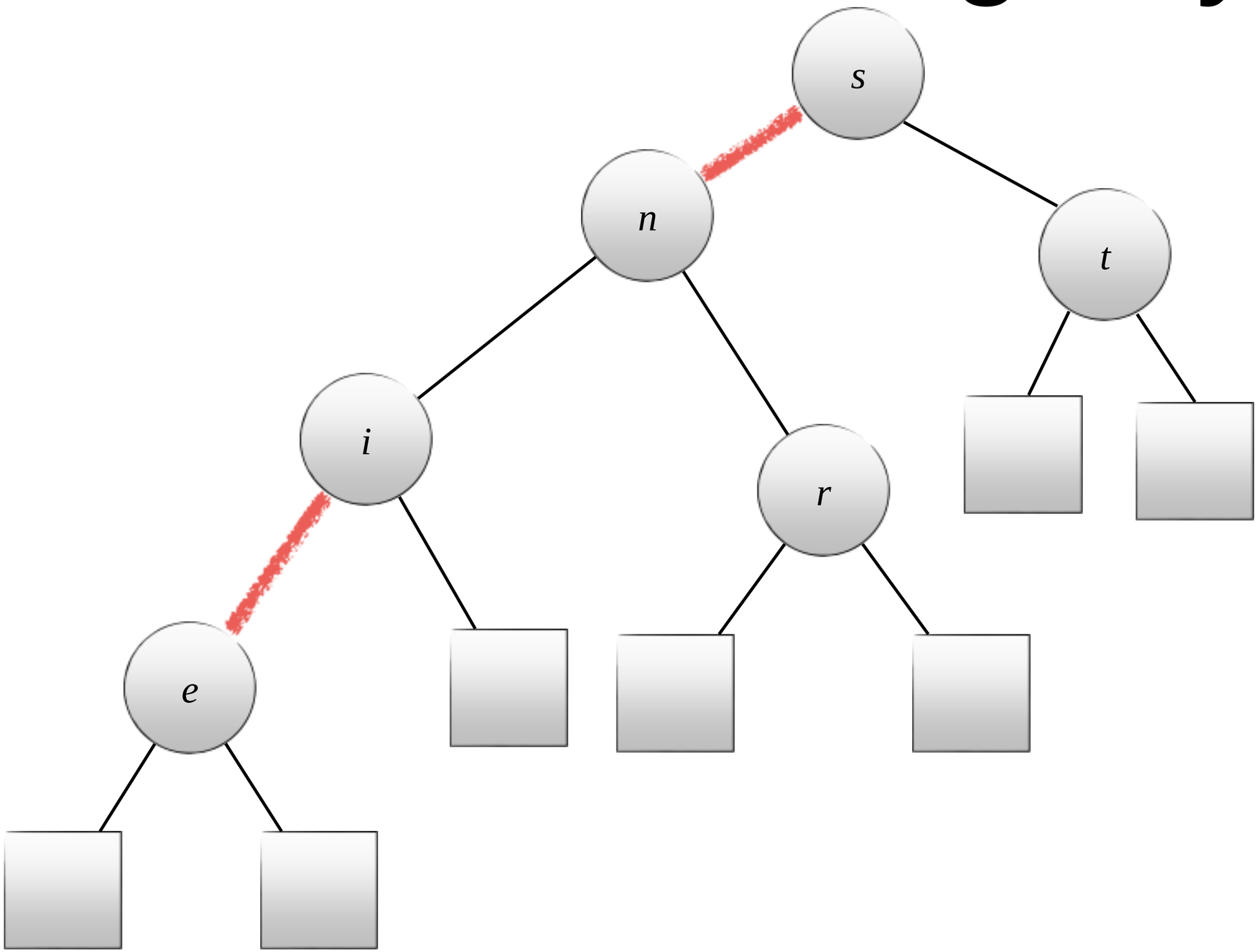
Insert key *t*, new link is red



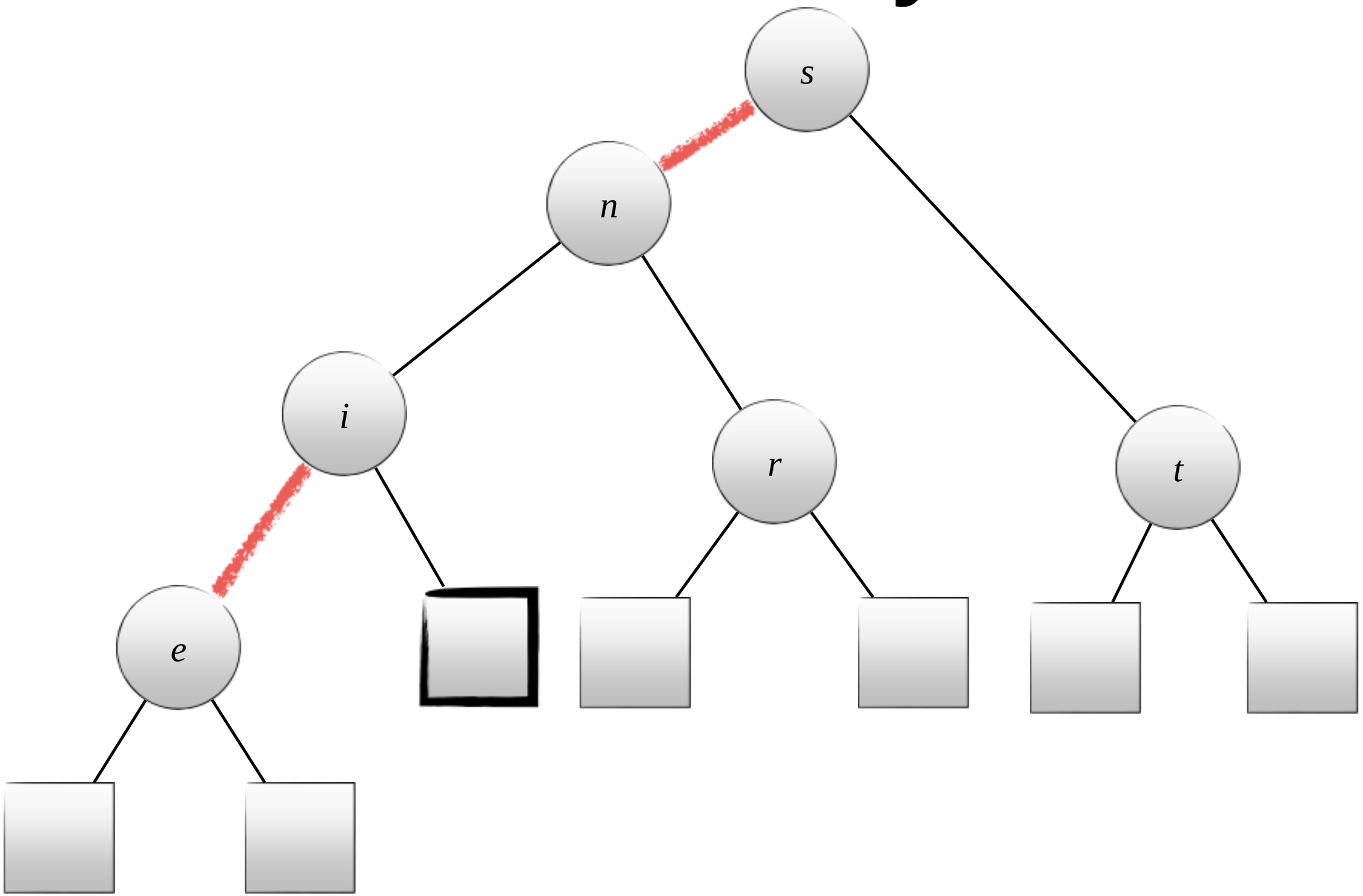
Insert key *t*



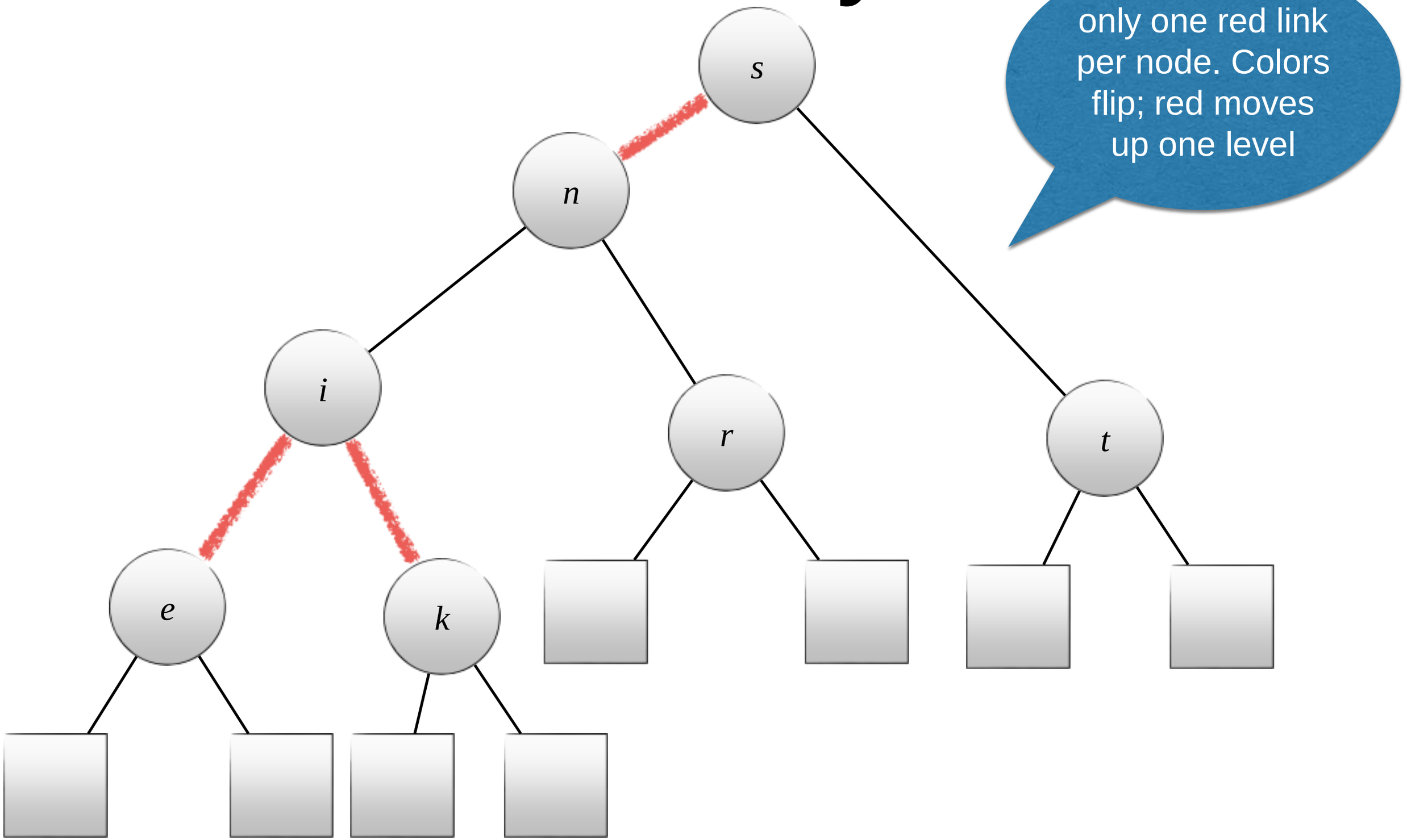
After inserting key t



Insert key k

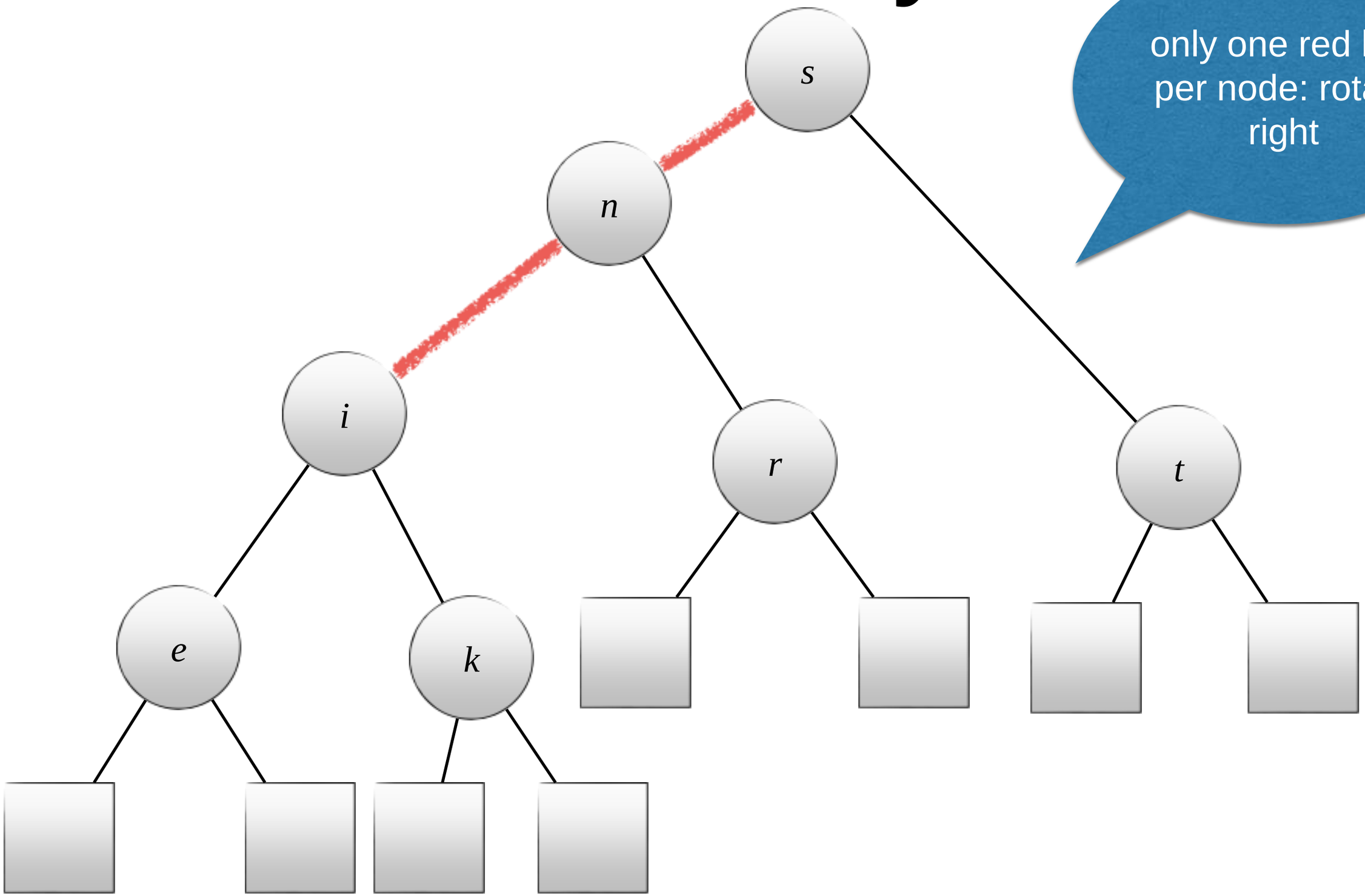


Insert key k



only one red link per node. Colors flip; red moves up one level

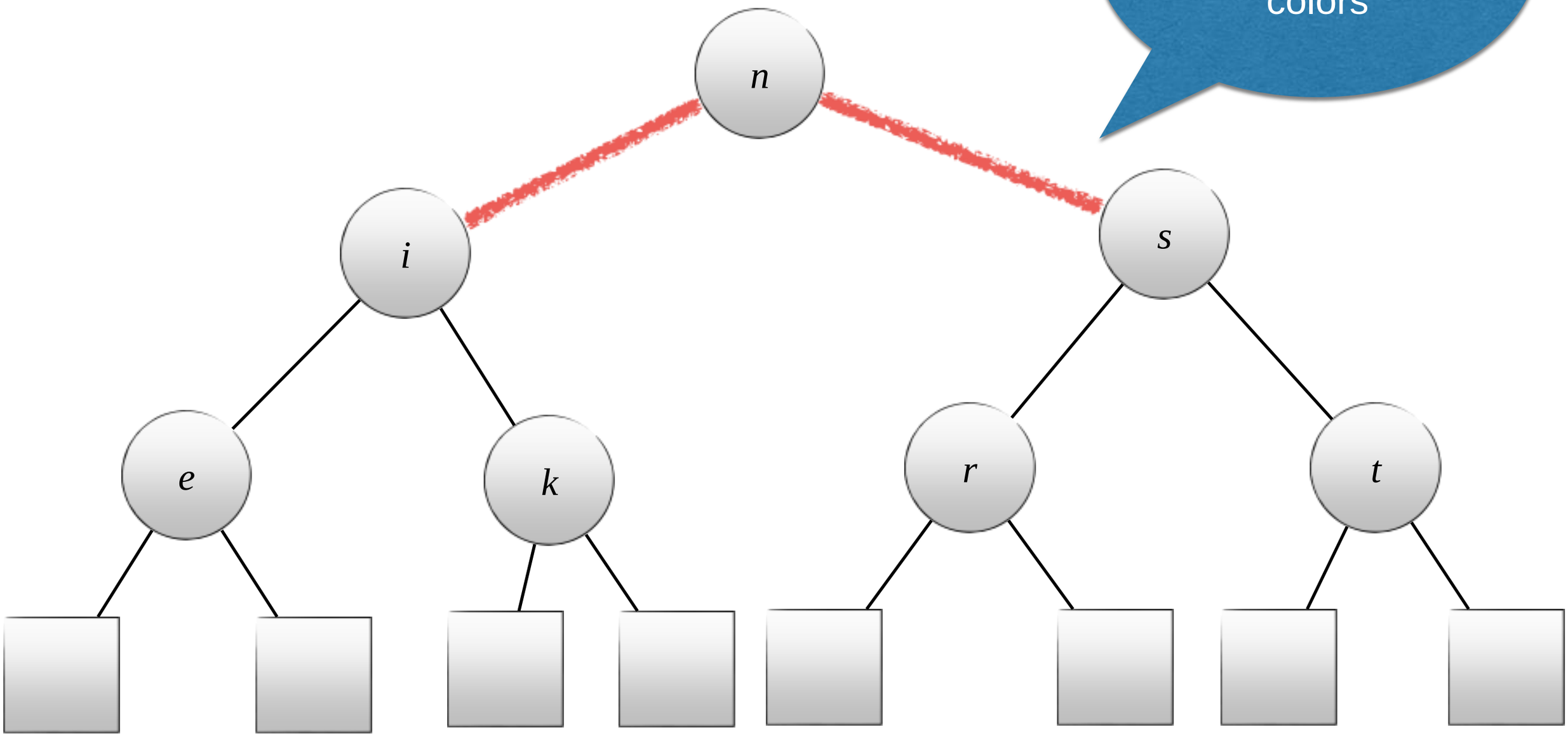
Insert key k



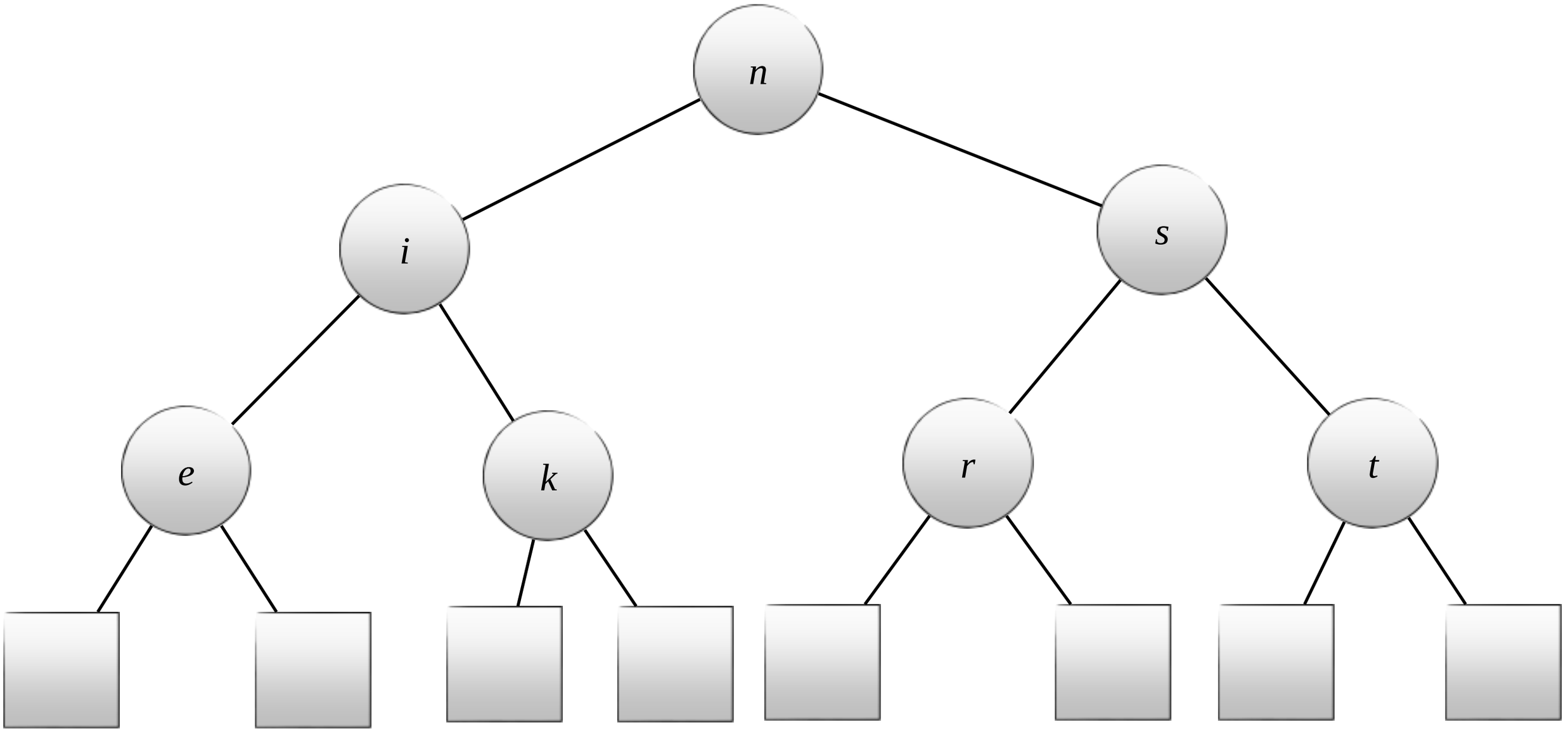
only one red link per node: rotate right

Insert key k

only one red link per node: flip colors



After inserting key k



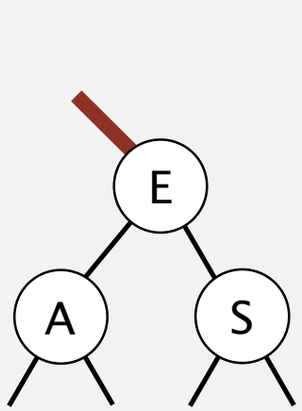
Insertion in a LLRB tree: overview

Basic strategy. Maintain 1-1 correspondence with 2-3 trees.

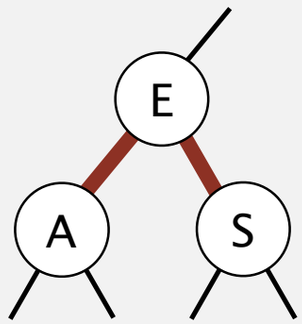
During internal operations, maintain:

- Symmetric order.
- Perfect black balance.

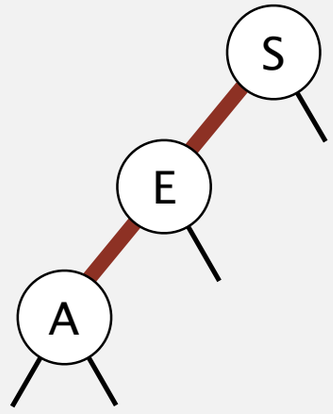
  [but not necessarily color invariants]



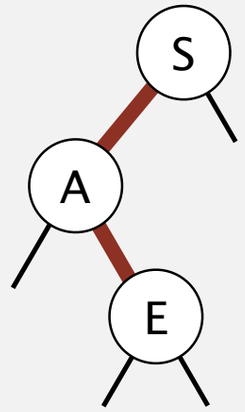
right-leaning red link



two red children (a temporary 4-node)



left-left red (a temporary 4-node)

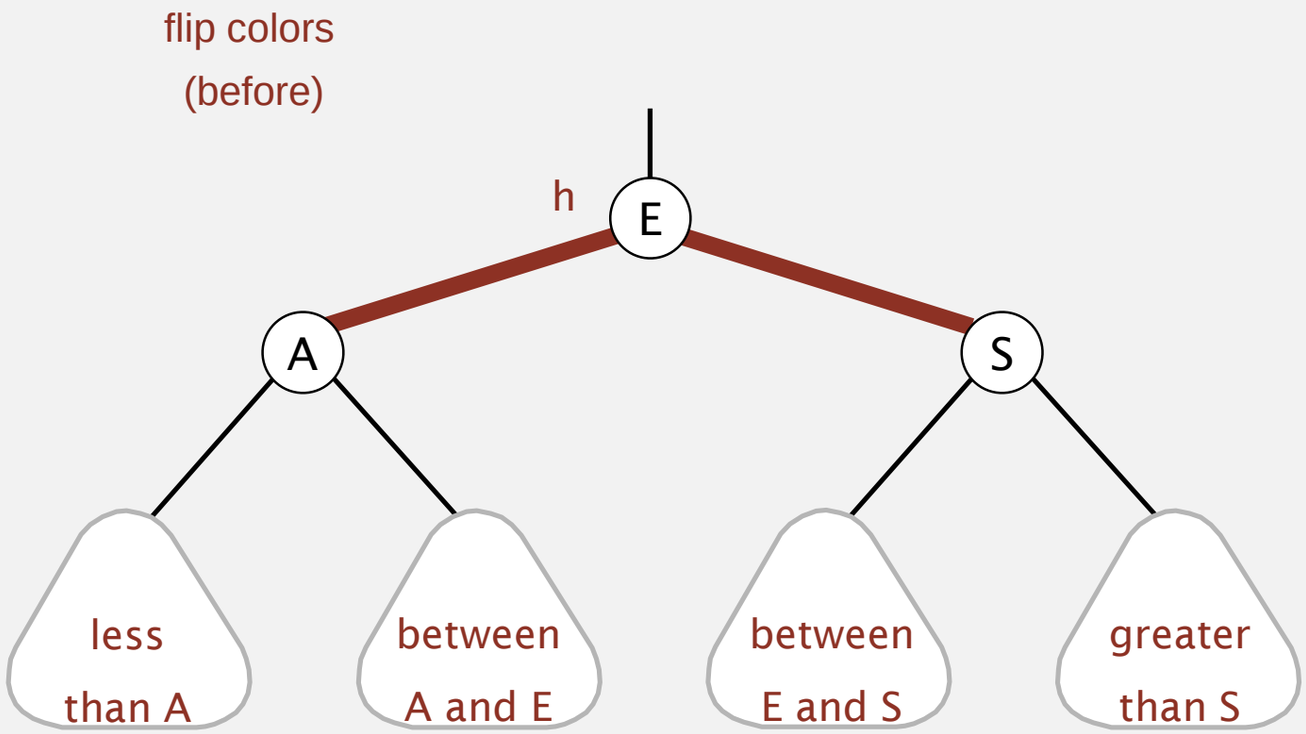


left-right red (a temporary 4-node)

How? Apply elementary red-black BST operations: rotation and color flip.

Elementary red-black BST operations

Color flip. Recolor to split a (temporary) 4-node.

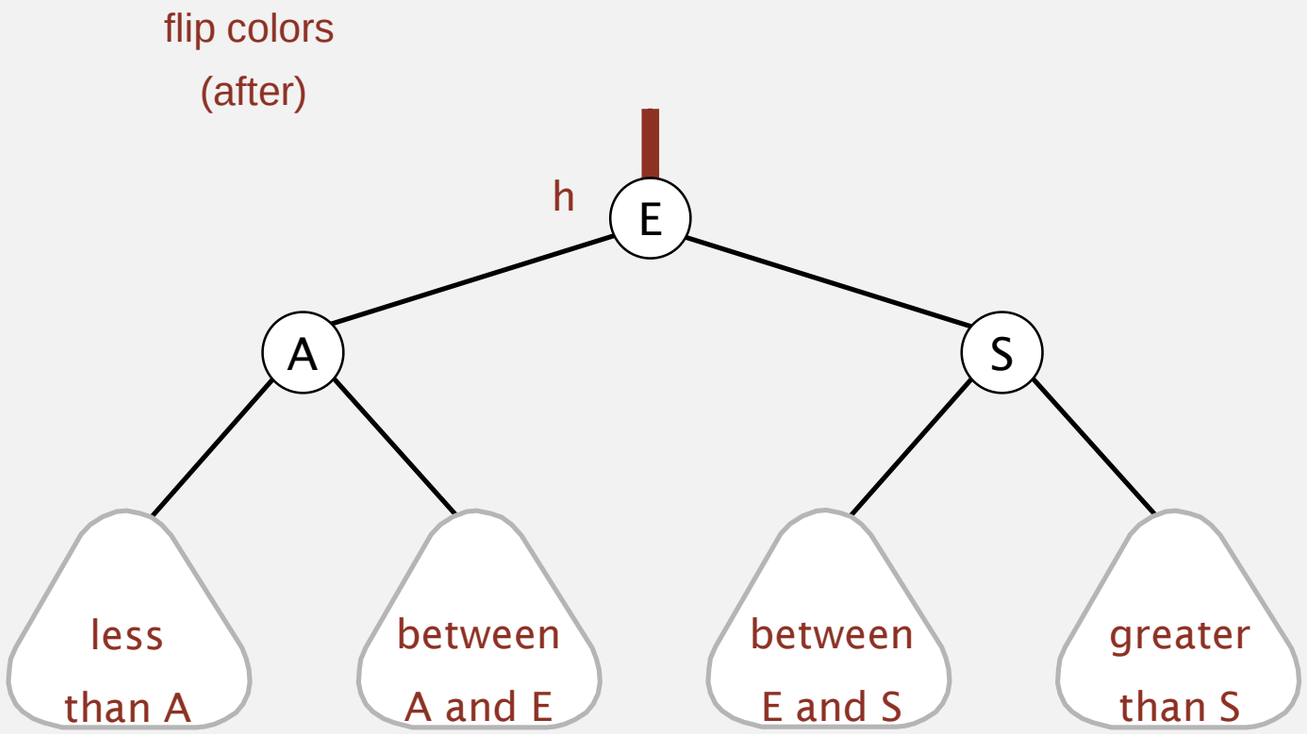



```
private void flipColors(Node h)
{
   h.color = RED;
   h.left.color = BLACK;
   h.right.color = BLACK;
}
```

Invariants. Maintains symmetric order and perfect black balance.

Elementary red-black BST operations

Color flip. Recolor to split a (temporary) 4-node.




```
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}
```
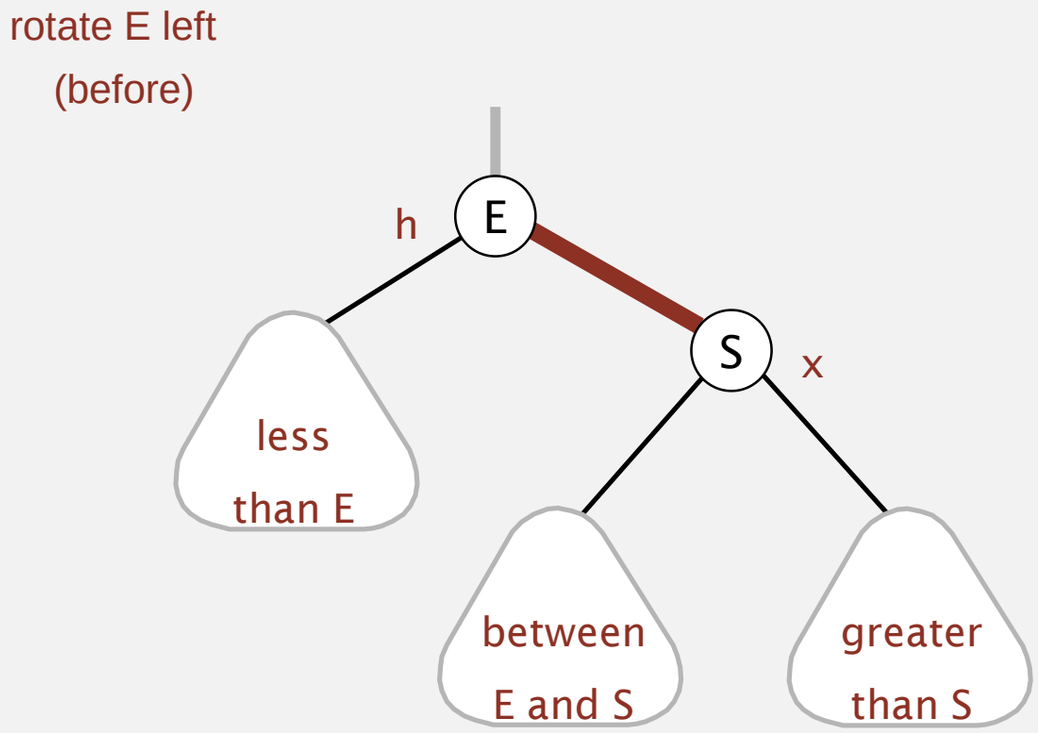

Invariants. Maintains symmetric order and perfect black balance.

Elementary red-black BST operations

Left rotation. Orient a (temporarily) right-leaning red link to lean left.

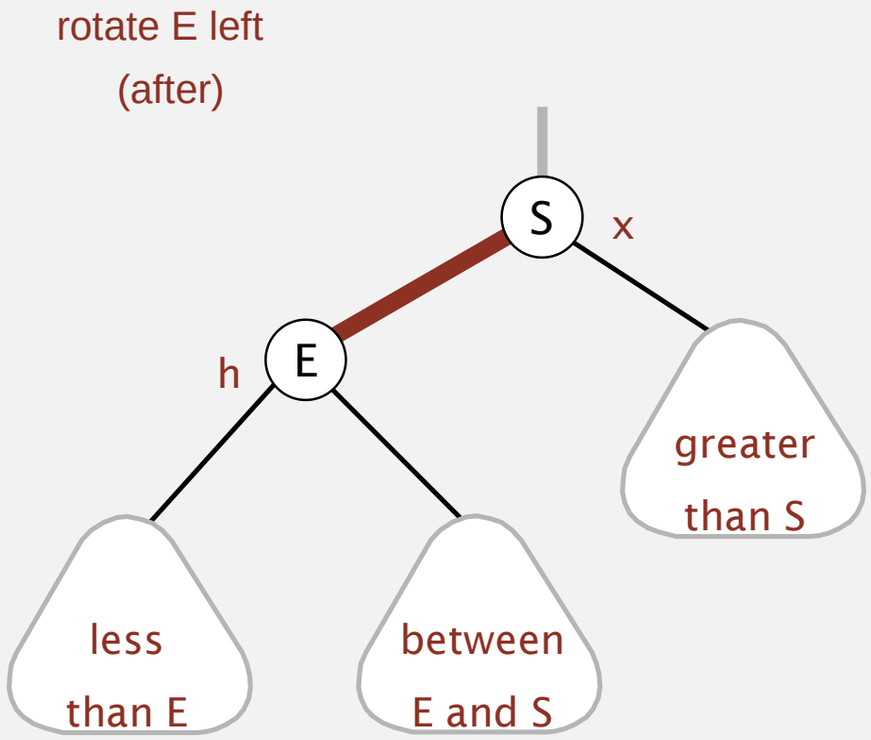



```
private Node rotateLeft(Node h)
{
   Node x = h.right;
   h.right = x.left;
   x.left = h;
   x.color = h.color;
   h.color = RED;
   return x;
}
```

Invariants. Maintains symmetric order and perfect black balance.

Elementary red-black BST operations

Left rotation. Orient a (temporarily) right-leaning red link to lean left.

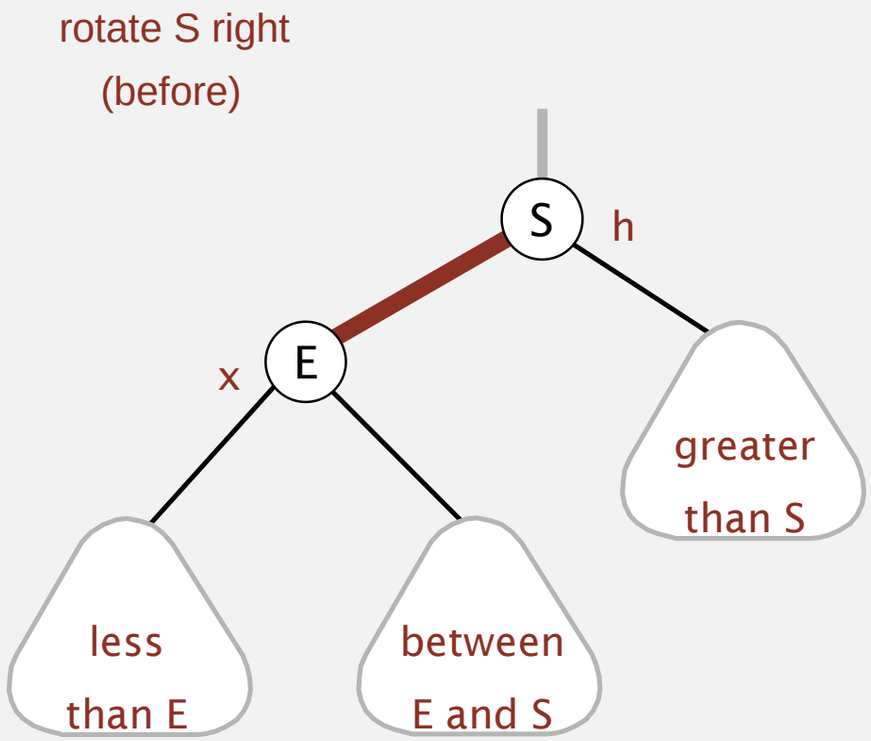



```
private Node rotateLeft(Node h)
{
   Node x = h.right;
   h.right = x.left;
   x.left = h;
   x.color = h.color;
   h.color = RED;
   return x;
}
```

Invariants. Maintains symmetric order and perfect black balance.

Elementary red-black BST operations

Right rotation. Orient a left-leaning red link to (temporarily) lean right.

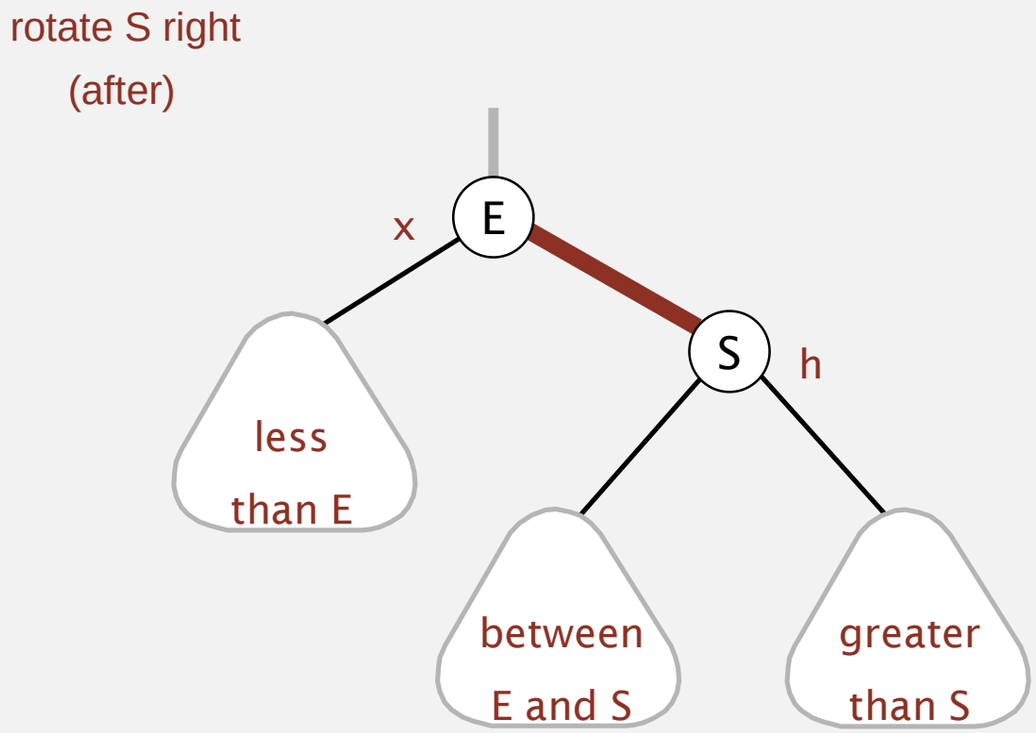



```
private Node rotateRight(Node h)
{
   Node x = h.left;
   h.left = x.right;
   x.right = h;
   x.color = h.color;
   h.color = RED;
   return x;
}
```

Invariants. Maintains symmetric order and perfect black balance.

Elementary red-black BST operations

Right rotation. Orient a left-leaning red link to (temporarily) lean right.



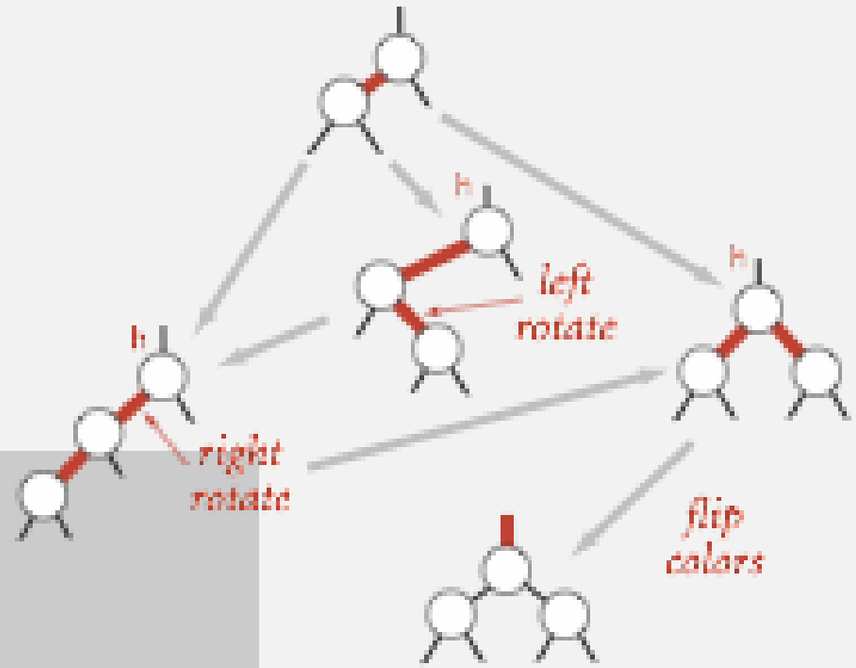

```
private Node rotateRight(Node h)
{
   Node x = h.left;
   h.left = x.right;
   x.right = h;
   x.color = h.color;
   h.color = RED;
   return x;
}
```

Invariants. Maintains symmetric order and perfect black balance.

Insertion in a LLRB tree: Java implementation

Same code for all cases.

- Right child red, left child black: rotate left.
- Left child, left-left grandchild red: rotate right.
- Both children red: flip colors.




```
private void put(Key key, Value val) {
   root = put(root, key, val);
   root.color = BLACK; }
```

```
private Node put(Node h, Key key, Value val) {
   if (h == null) return new Node(key, val, RED);
   int cmp = key.compareTo(h.key);
   if      (cmp  < 0) h.left  = put(h.left,  key, val);
   else if (cmp  > 0) h.right = put(h.right, key, val);
   else h.val = val;

   if (isRed(h.right) && !isRed(h.left))     h = rotateLeft(h);
   if (isRed(h.left)  && isRed(h.left.left)) h = rotateRight(h);
   if (isRed(h.left)  && isRed(h.right))     flipColors(h);

   return h; }
```

insert at bottom (and color it red)

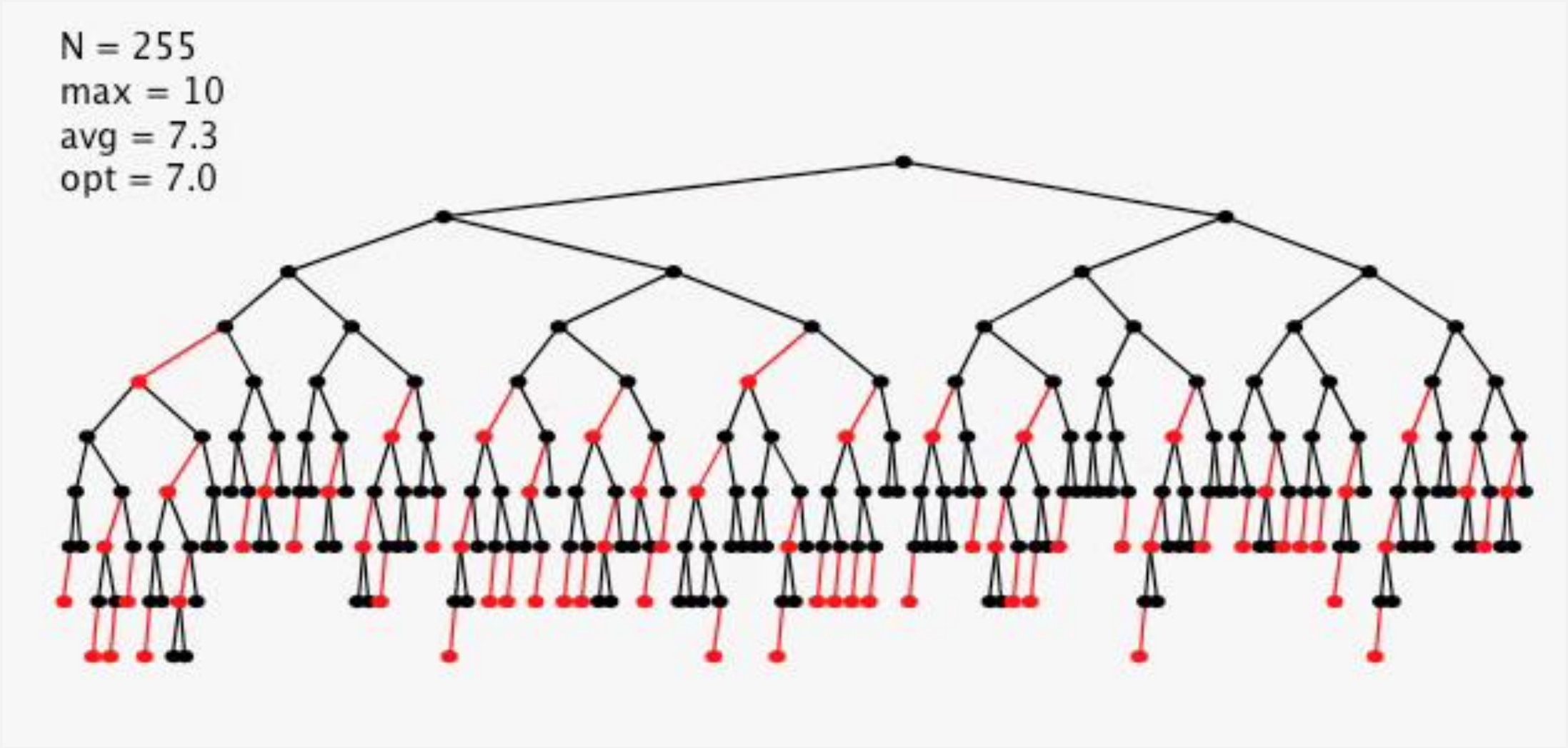
lean left

balance 4-node

split 4-node

only a few extra lines of code provides near-perfect balance

Insertion in a LLRB tree: visualization



255 random insertions