CSC 226

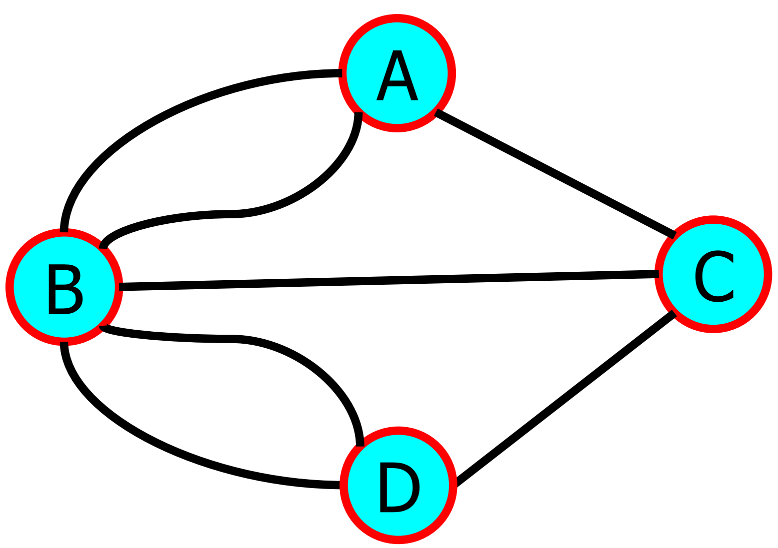
Algorithms and Data Structures: II

Rich Little
rlittle@uvic.ca
ECS 516

Abstract Meaning of the Term Graph

- A ***graph*** G = (V, E) is a set V of ***vertices*** (*nodes*) and a collection E of pairs from V, called ***edges*** (*arcs*).

- **Graph Example:**



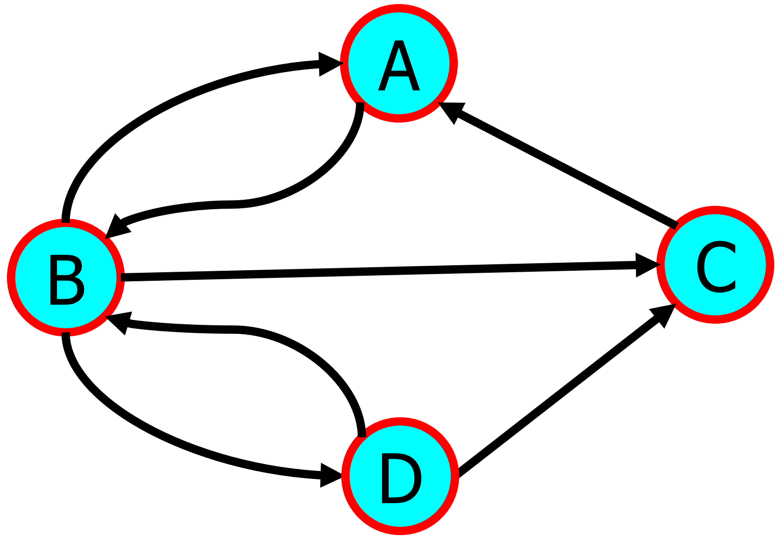
$$V = \{A, B, C, D\}$$

$$E = \left\{ \begin{array}{l} \{A, B\}, \{A, B\}, \{A, C\}, \\ \{B, C\}, \{B, D\}, \{B, D\}, \{C, D\} \end{array} \right\}$$

Abstract Meaning of the Term Graph

- A ***digraph*** $G = (V, E)$ is a set V of ***vertices*** (*nodes*) and a collection E of ordered pairs from V, called ***edges*** (*arcs*).

- **Digraph Example:**

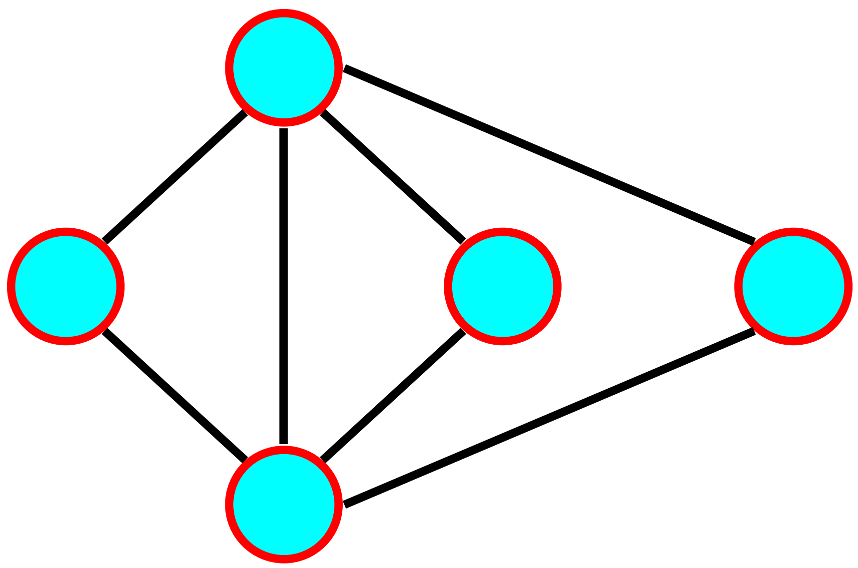


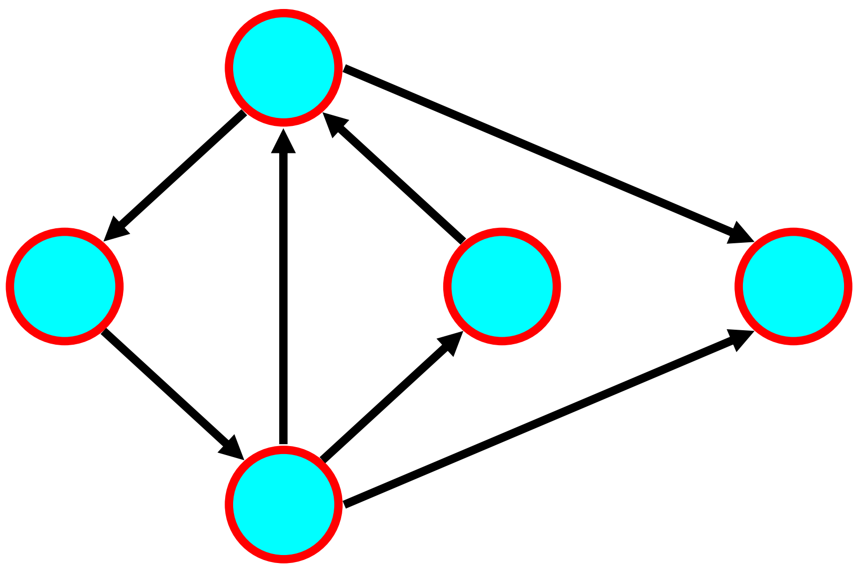
$$V = \{A, B, C, D\}$$

$$E = \left\{ \begin{array}{l} (A, B), (B, A), (B, D), \\ (D, B), (B, C), (D, C), (C, A) \end{array} \right\}$$

Graph Terminology

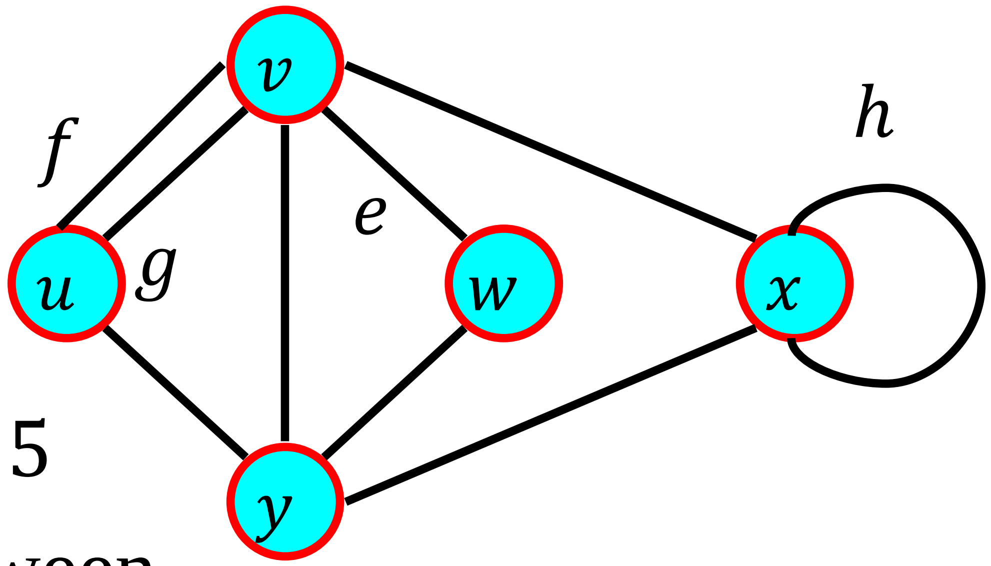
- Much of the terminology for graphs is applicable to undirected graphs and directed graphs





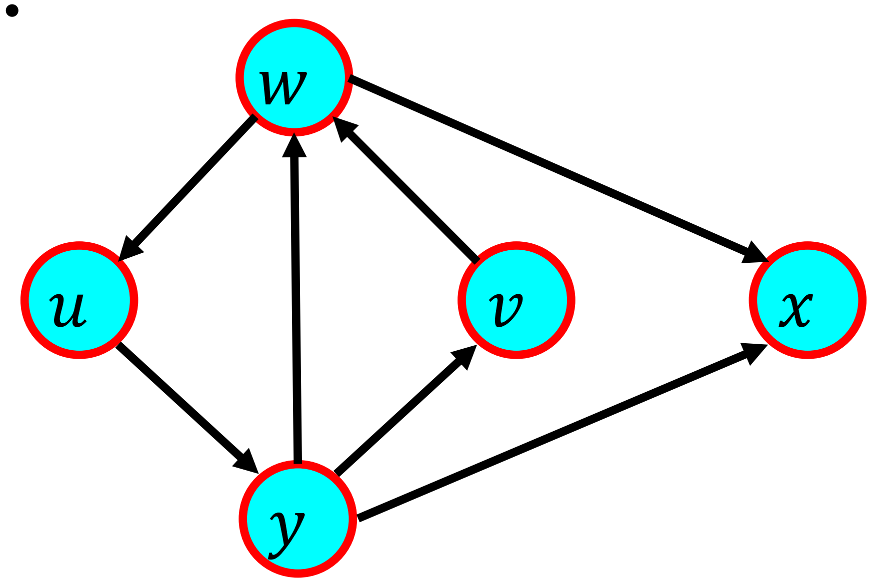
Undirected Edges

- An *undirected edge e* represents a *symmetric* relation between two vertices *v* and *w* represented by the vertices.
  - We usually write $e = \{v, w\}$, where $\{v, w\}$ is an unordered pair.
  - *v*, *w* are the *endpoints* of the edge
  - *v* is *adjacent* to *w*
  - *e* is *incident* upon *v* and *w*
  - The *degree* of a vertex is the number of incident edges, eg. $\deg(v) = 5$
  - *parallel* edges – more than one edge between a pair of vertices, eg. f and g
  - *self-loop* – edge that connects a vertex to itself, eg. *h*
  - Typically, the number of vertices is denoted by *n* and the number of edges by *m*.



Directed Edges or Arcs

- A *directed edge* (*or arc*) *e* represents an *asymmetric* relation between two vertices *v* and *w*.
  e = (*v*, w) denotes an ordered pair.
  - *v*, *w* are the endpoints of the edge
  - *v* is *adjacent* to *w*
  - *e* is *incident* upon *v* and *w*
  - The arc goes from the *source* vertex *v* to the *destination* vertex *w*



- The *indegree* of a vertex is the number of incoming arcs
- The *outdegree* of a vertex is the number of outgoing arcs

Walks

- A ***walk*** in a graph is a sequence of vertices $v_1, v_2, \dots, v_n$ such that there exist edges $\{v_1, v_2\}$, $\{v_2, v_3\}$, …, $\{v_{n-1}, v_n\}$
- If $v_1 = v_n$ it's ***closed***, otherwise it's ***open***.
- The ***length*** of a walk is the number of edges.
- If no edge is repeated it's a ***trail***. If closed a ***circuit***.
- If no vertex is repeated it's a ***path***. If closed a ***cycle***.

Graphs

- A graph is ***connected*** if every pair of vertices is connected by a path.
- A ***simple graph*** is a graph with no self-loops and no parallel or multi-edges
- A ***complete graph*** is a simple graph where an edge connects every pair of vertices

Connected Digraphs

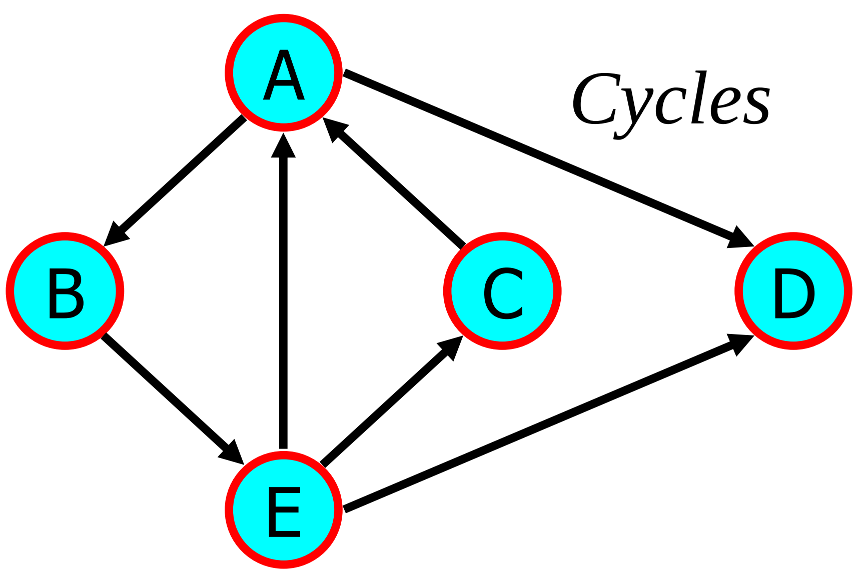
- Given vertices u and v of a digraph G, we say v is ***reachable*** from u if G has a directed path from u to v.

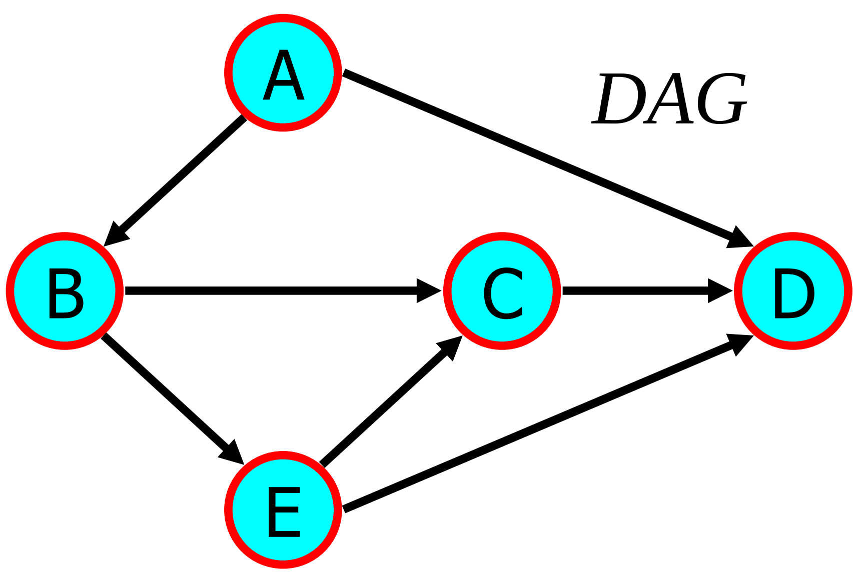
- A digraph G is ***connected*** if every pair of vertices is connected by an undirected path.

- A digraph G is ***strongly connected*** if for every pair of vertices u and v of G, u is reachable from v and v is reachable from u.

Directed Acyclic Graphs (DAGs)

- A ***directed acyclic graph*** (DAG) is a directed graph with no cycles.





Subgraphs

- A ***subgraph*** of $G = (V, E)$ *is a graph* $G' = (V', E')$ where
  - ➢ V' is a subset of V
  - ➢ E' consists of edges $\{v, w\}$ in E such that both v and w are in V'
- A ***spanning subgraph*** of G contains all the vertices of G

Theorem

- **Theorem:** If $G = (V, E)$ is an undirected graph, then
$$\sum_{v \in V} \deg(v) = 2|E|.$$
- *Proof:*
  - ➢Every edge contributes 2 to the total degree.
- **Corollary:** For any undirected graph, the number of vertices of odd degree must be even.

Euler Circuits

- Let $G = (V, E)$ be an undirected graph with no isolated vertices. Then G is said to have an ***Euler circuit*** if there is a circuit in G that traverses every edge exactly once.
  - If there is a trail from vertex a to b which traverses every edge exactly once, it is an ***Euler trail***.

Theorem

- **Theorem:** Let $G = (V, E)$ be an undirected graph with no isolated vertices. Then, G has an Euler circuit if and only if G is connected and every vertex has an even degree.
  - This is the 7 bridges of Konigsberg.

- ***Corollary:*** There exists an Euler trail in G if and only if G is connected and has exactly two vertices of odd degree.

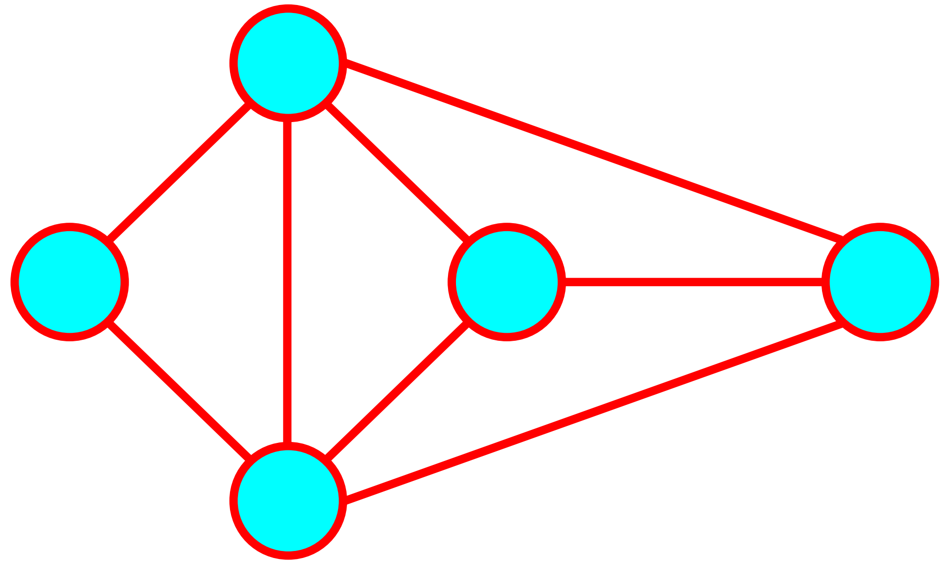
Trees and Forests

- A ***(free) tree*** is an undirected graph T such that
  - T is connected
  - T has no cycles

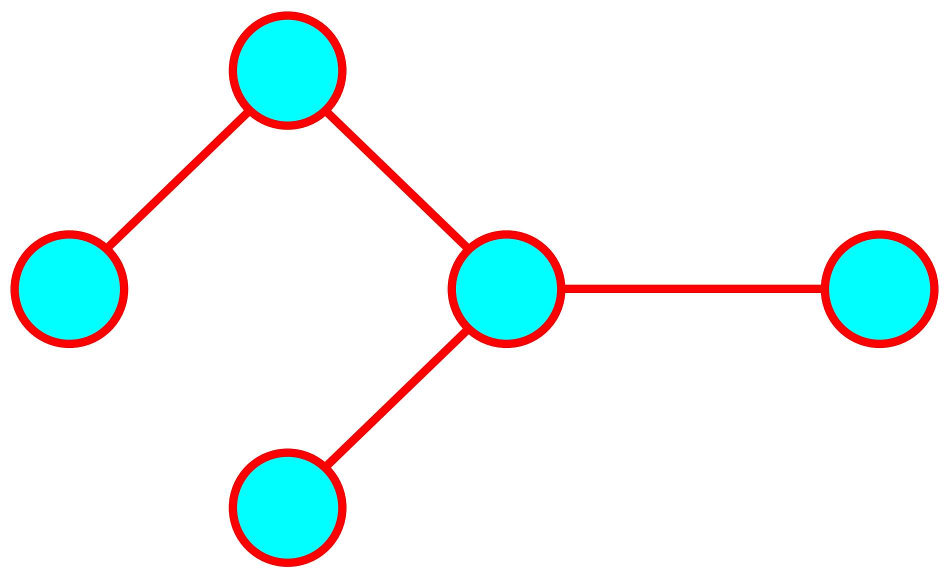
  This definition of tree is different from the one of a rooted tree
- A ***forest*** is an undirected graph without cycles
  - The connected components of a forest are trees

Spanning Trees and Forests

- A ***spanning tree*** of a connected graph is a spanning subgraph that is a tree
- A spanning tree is not unique unless the graph is a tree
- Spanning trees have applications to the design of communication networks
- A ***spanning forest*** of a graph is a spanning subgraph that is a forest



Graph



Spanning tree