CSC 226

Algorithms and Data Structures: II

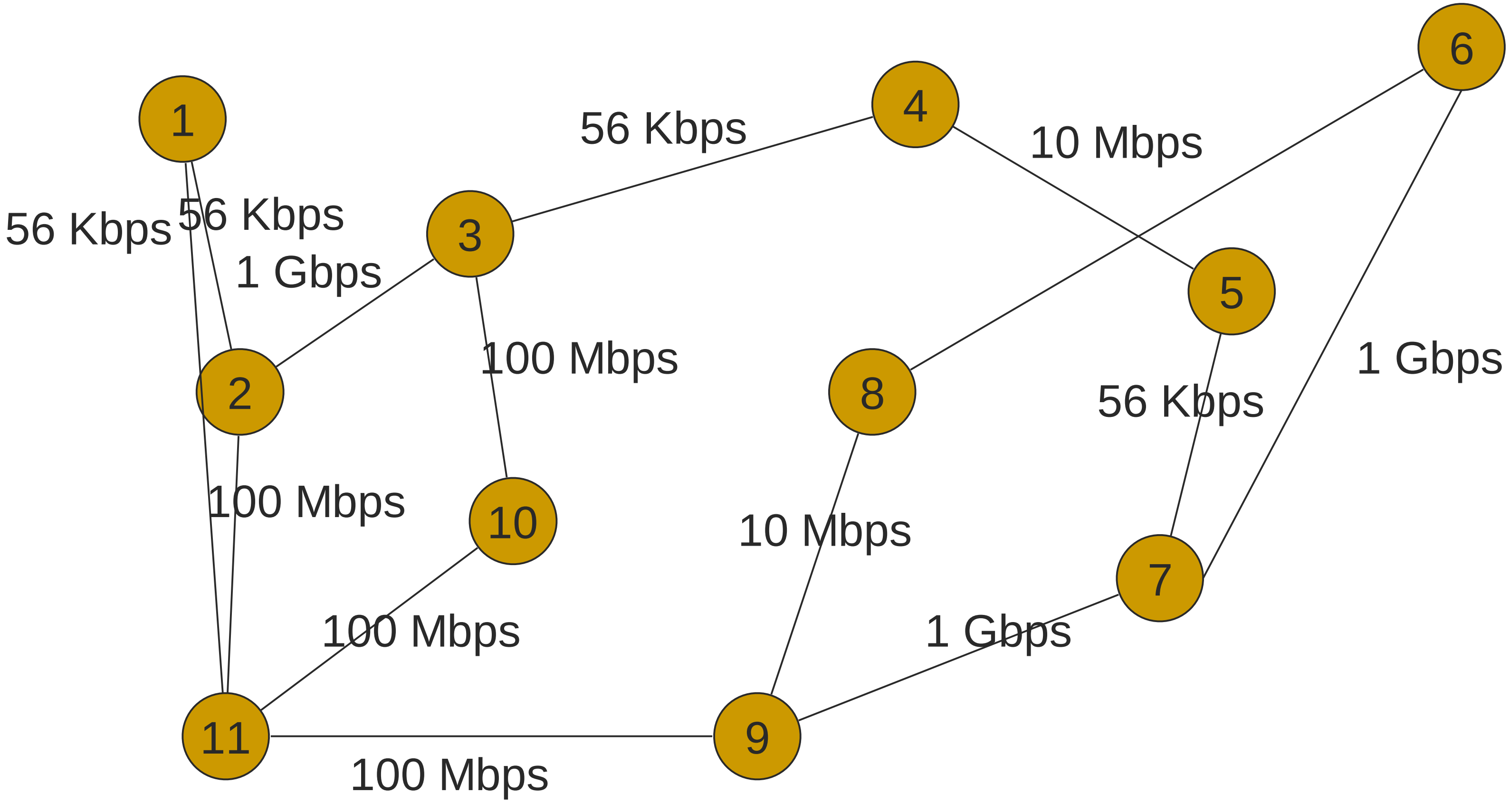
Rich Little
rlittle@uvic.ca
ECS 516

Shortest Paths

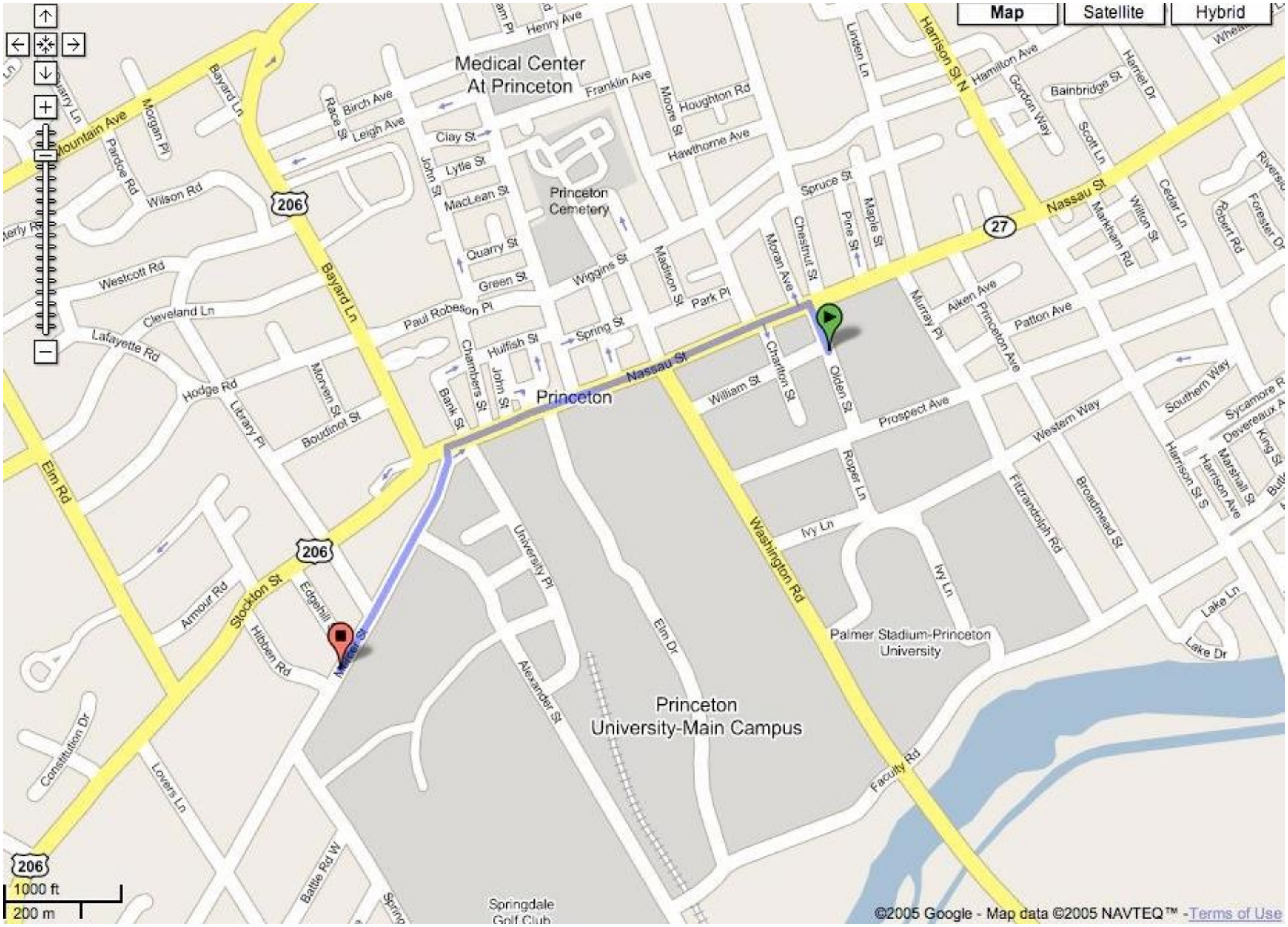
in edge-weighted graphs

Communication Speeds in a Computer Network

Find fastest way to route a data packet between two computers



Google maps


Map
Satellite
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Mercer St
Olden St
Prospect Ave
Harrison St N
Palmer Stadium-Princeton University
Princeton University-Main Campus
Springdale Golf Club
206
27
1000 ft
200 m
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Shortest path applications

- PERT/CPM.
- Map routing.
- Seam carving.
- Texture mapping.
- Robot navigation.
- Typesetting in TeX.
- Urban traffic planning.
- Optimal pipelining of VLSI chip.
- Telemarketer operator scheduling.
- Routing of telecommunications messages.
- Network routing protocols (OSPF, BGP, RIP).
- Exploiting arbitrage opportunities in currency exchange.
- Optimal truck routing through given traffic congestion pattern.

http://en.wikipedia.org/wiki/Seam_carving

Reference: Network Flows: Theory, Algorithms, and Applications, R. K. Ahuja, T. L. Magnanti, and J. B. Orlin, Prentice Hall, 1993.

Shortest Path problems

- Find a shortest path between two given vertices
- Single source shortest paths
- Single sink shortest paths
- All pair shortest paths

Single Source Shortest Path problems

- Undirected graphs with non-negative edge weights
- Directed graphs with non-negative edge weights
- Directed graphs with arbitrary weights

Single Source Shortest Paths

- If graph is not weighted (all edge-weights are unit-weight): BFS works
- Now assume: graph is edge-weighted
  - Every edge is associated with a positive number
    - Possible weights: integers, real numbers, rational numbers
    - Edge-weights can represent: distance, connection cost, affinity

Single Source Shortest Paths

- Input: An edge-weighted undirected graph and a source node v with: for every edge e edge-weight w(e) > 0
- Output: All single-source shortest paths (and their weight) for v in G: for every node $w \neq v$ in G a shortest path from v to w.
  - Here, a *path* p from v to w consisting of edges e_0, e_1, … , e_{k-1} is shortest in G, if its length

$$w(p) = \sum_{i=0}^{k-1} w(e_i)$$

is minimum (i.e., there is no path from v to w in G that is shorter).

Algorithm
`DijkstraShortestPaths(G,v)`

Input: A simple undirected graph G with nonnegative edge-weights, a distinguished vertex v in G

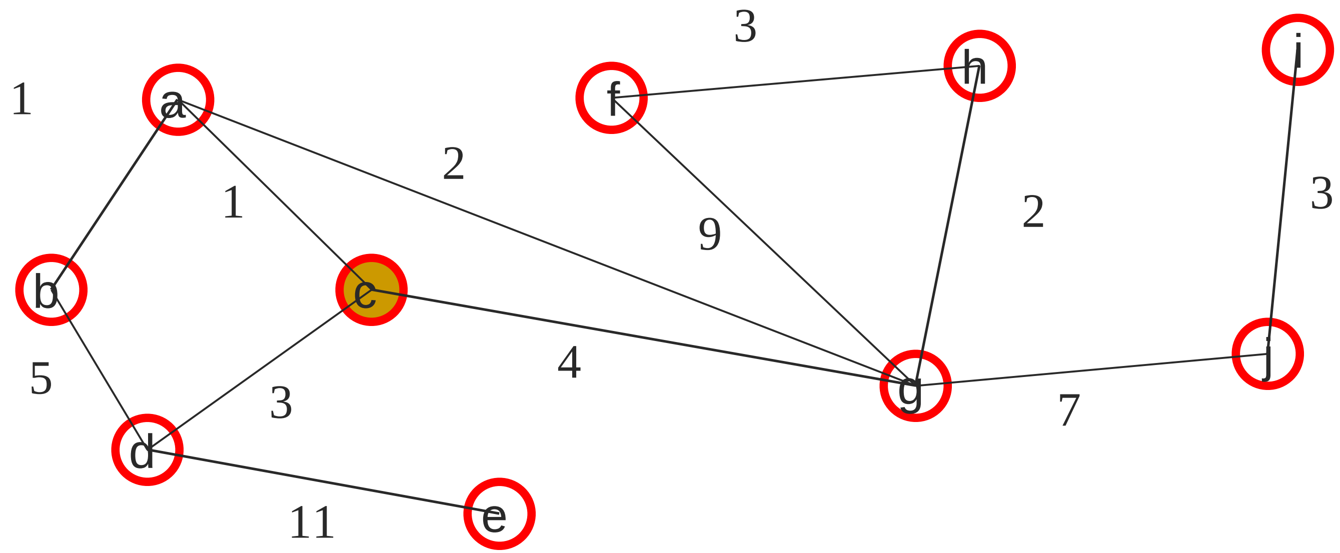
Output: A label D[u] for each vertex u in G such that D[u] is the shortest distance from v to u in G.

Algorithm
DijkstraShortestPaths(G,v)

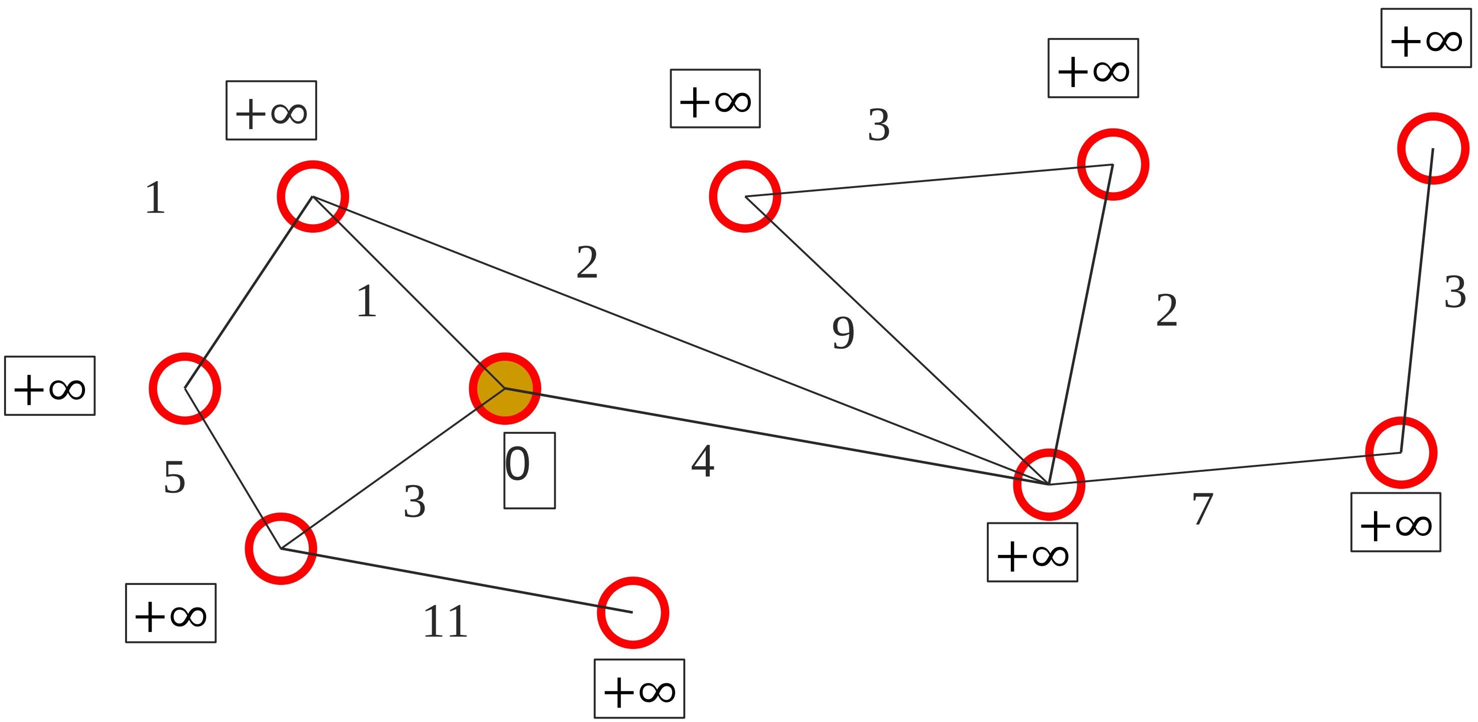
```
D[v]←0
for each vertex u≠v of G do
   D[u]← +∞
Let Q be a priority queue containing all
   vertices of G using D[.] as keys
while Q is not empty do
   u←Q.removeMin() //u is added to cloud
   for each vertex z∈N(u) with z∈Q do
     if D[u]+w((u,z)) < D[z] then
        D[z]←D[u]+w((u,z))
        update z's key in Q to D[z]
return D
```

Relaxation

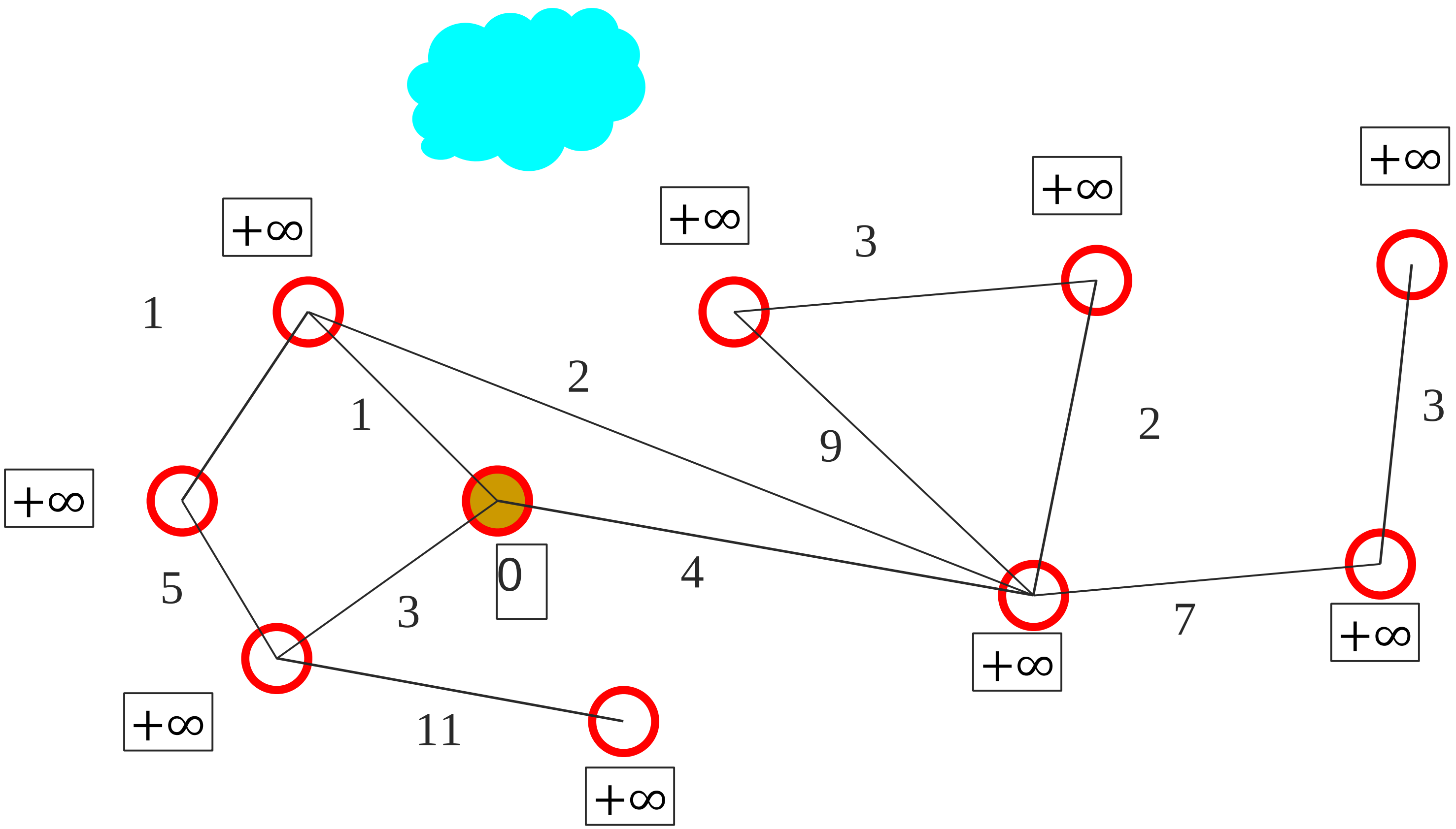
Dijkstra’s algorithm: a greedy algorithm



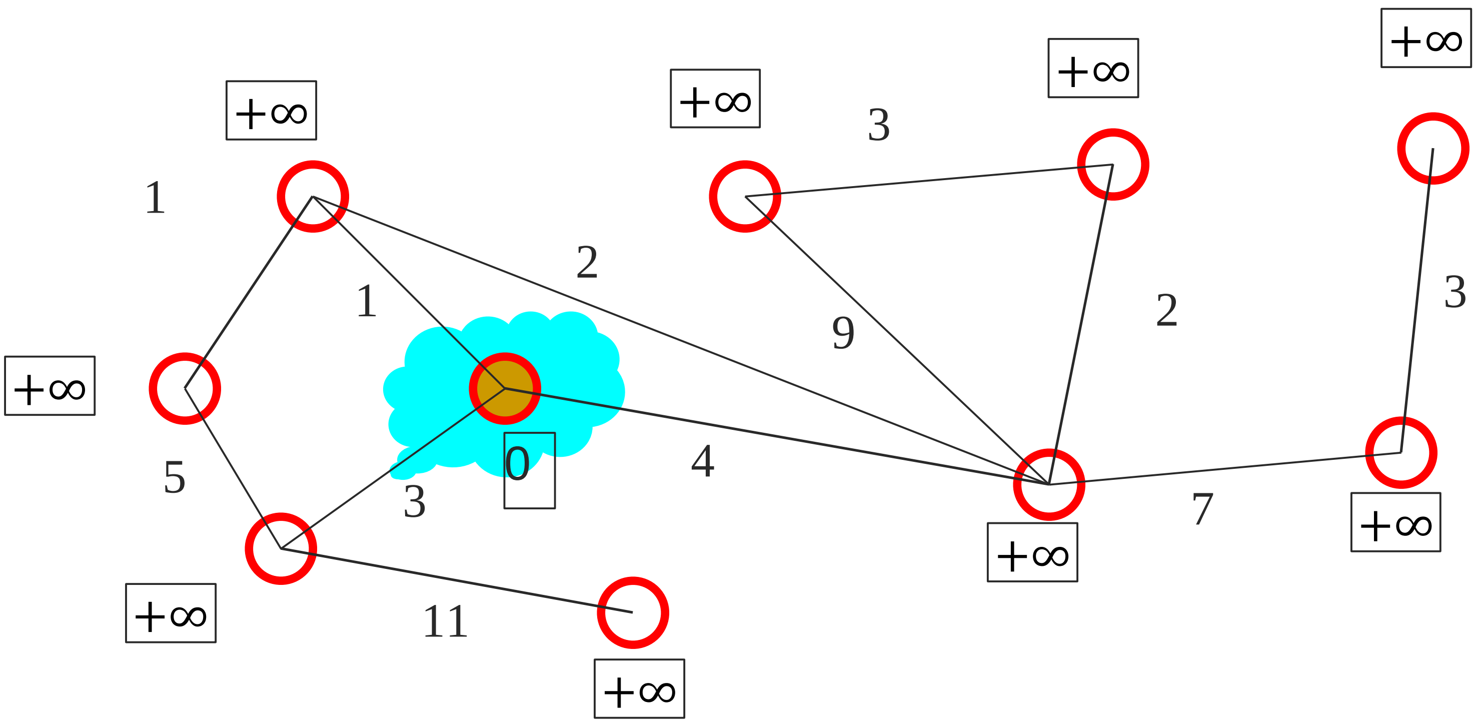
Dijkstra's algorithm: Initializing



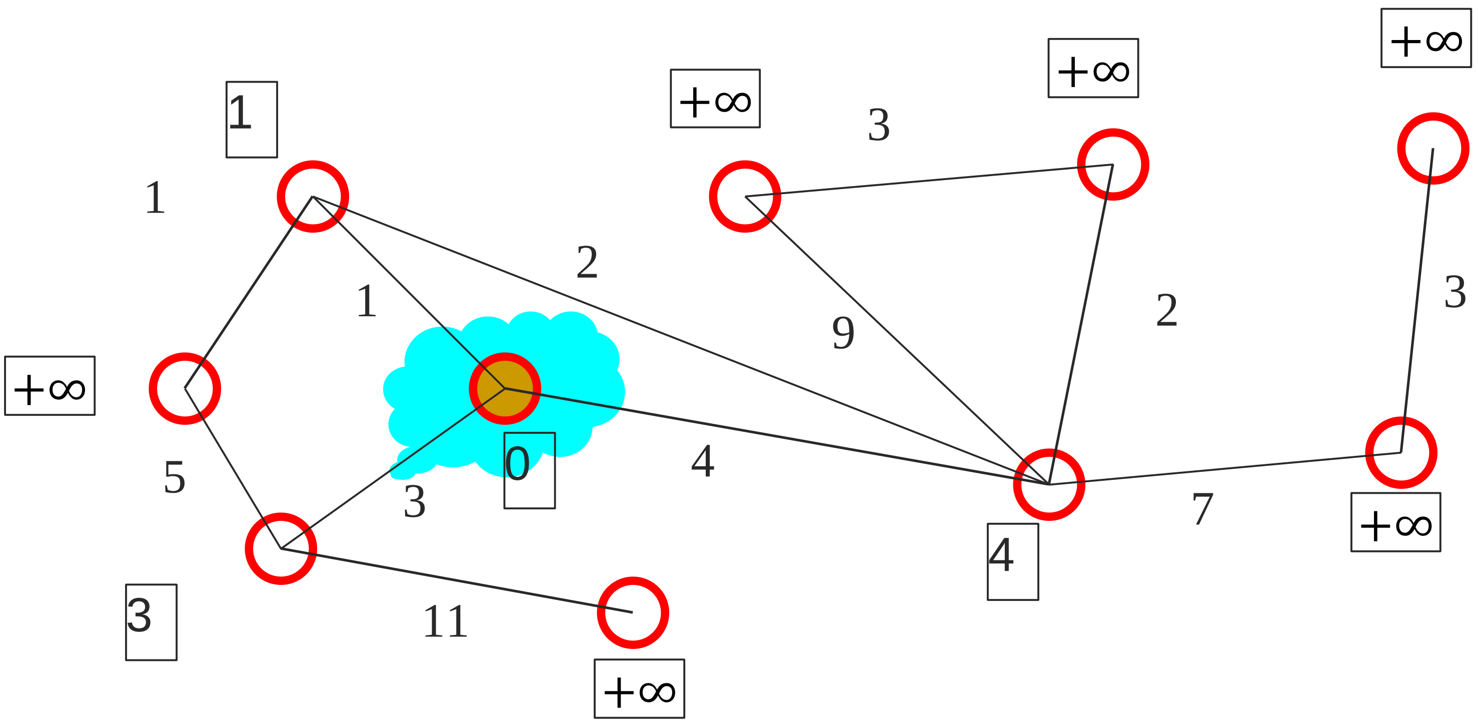
Dijkstra's algorithm: Initializing Cloud C (consisting of "solved" subgraph)



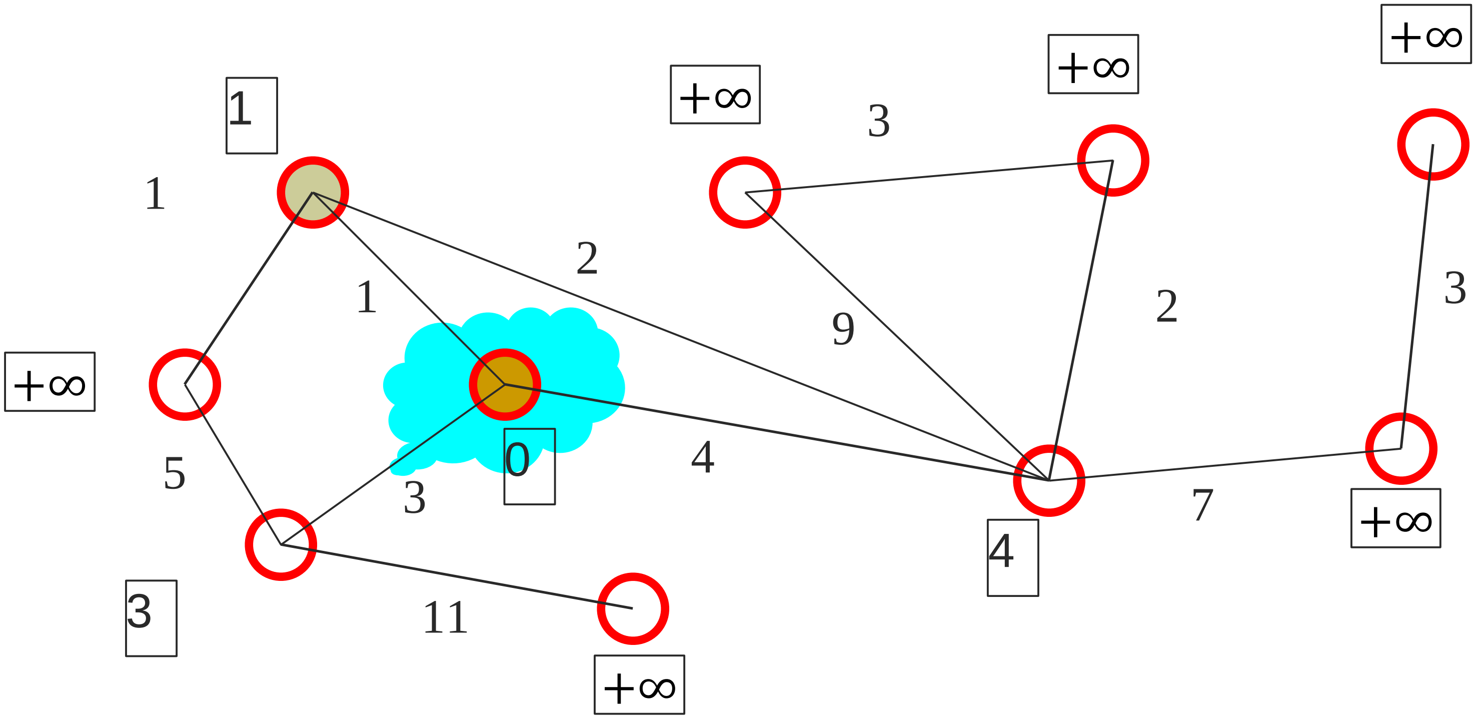
Dijkstra's algorithm: pull v into C



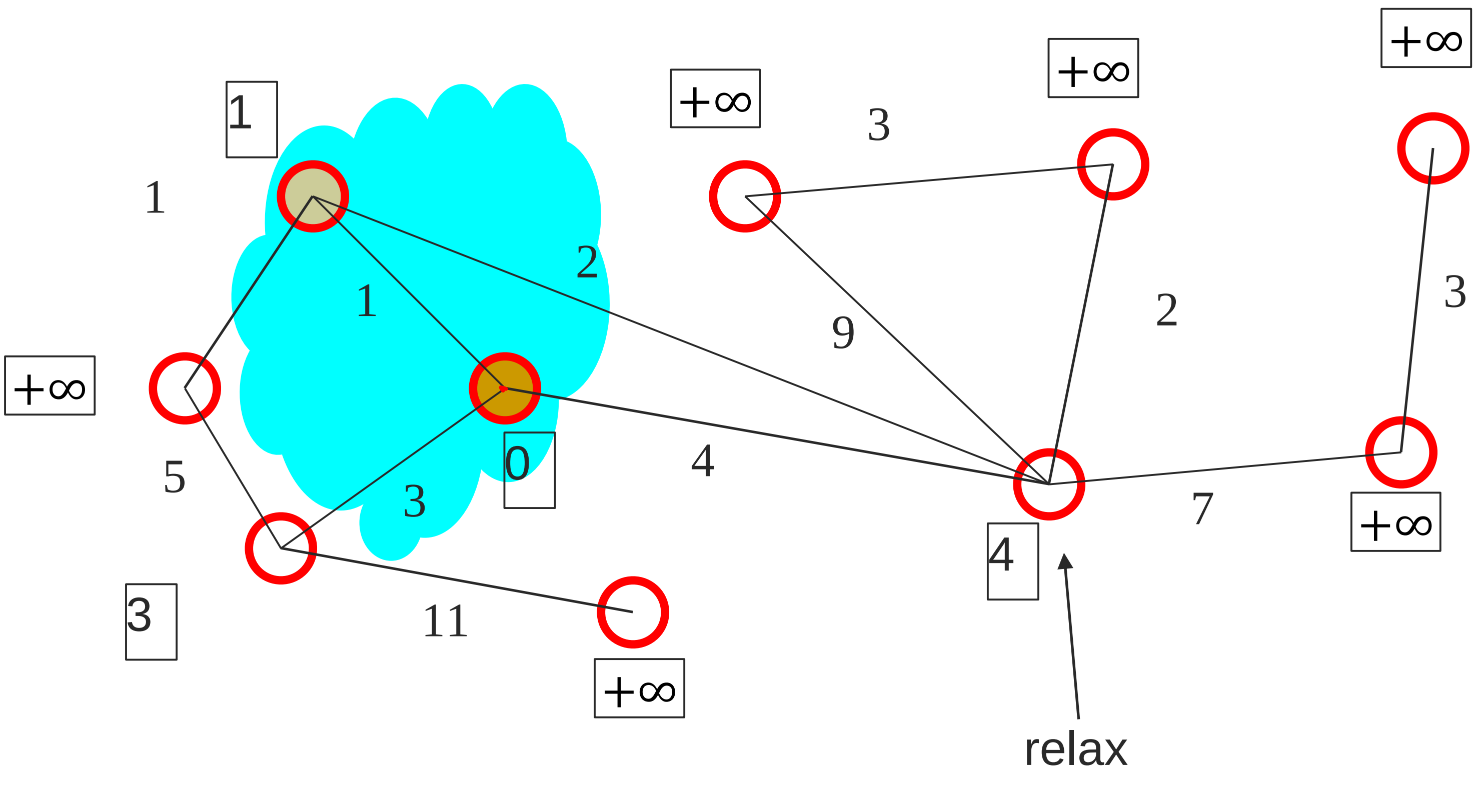
Dijkstra's algorithm: update *C's* neighborhood



Dijkstra's algorithm: pick closest vertex u outside C



Dijkstra's algorithm: pull u into C



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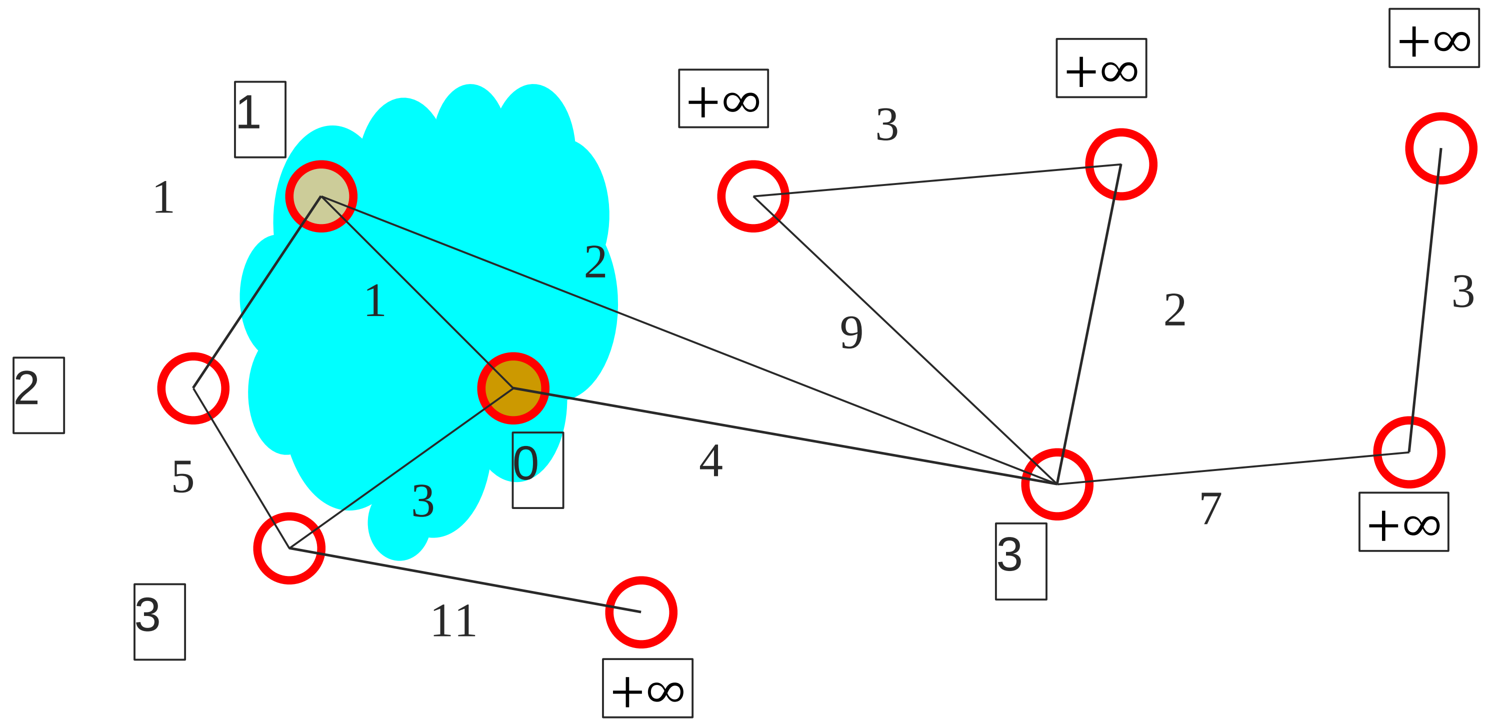

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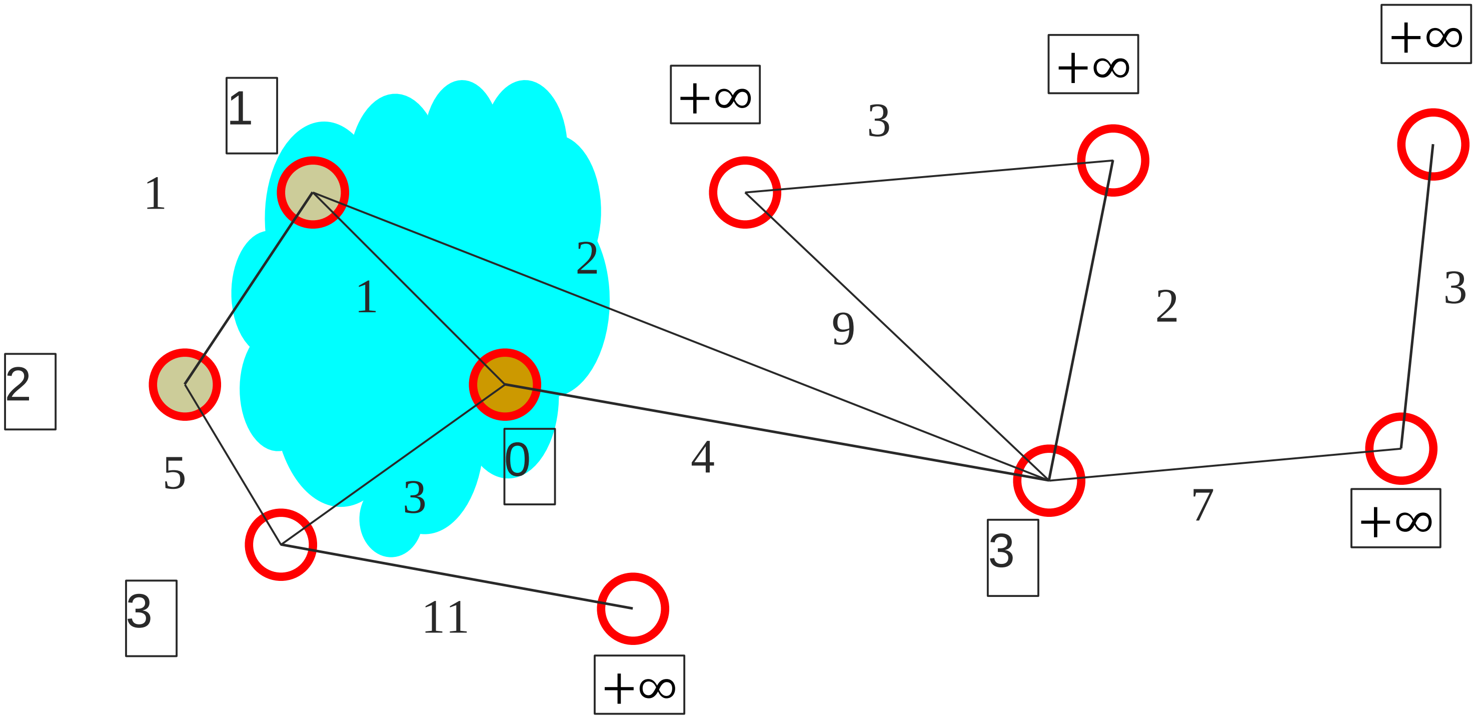

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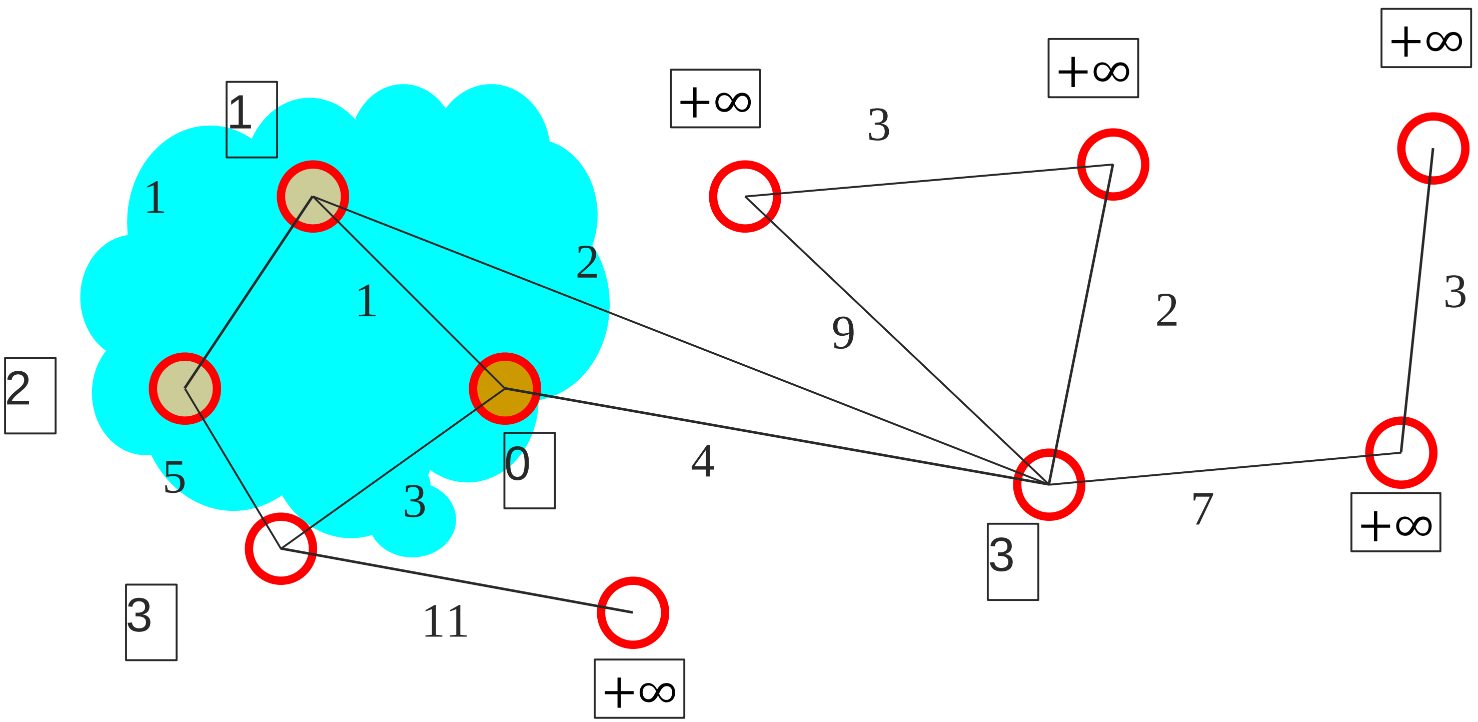

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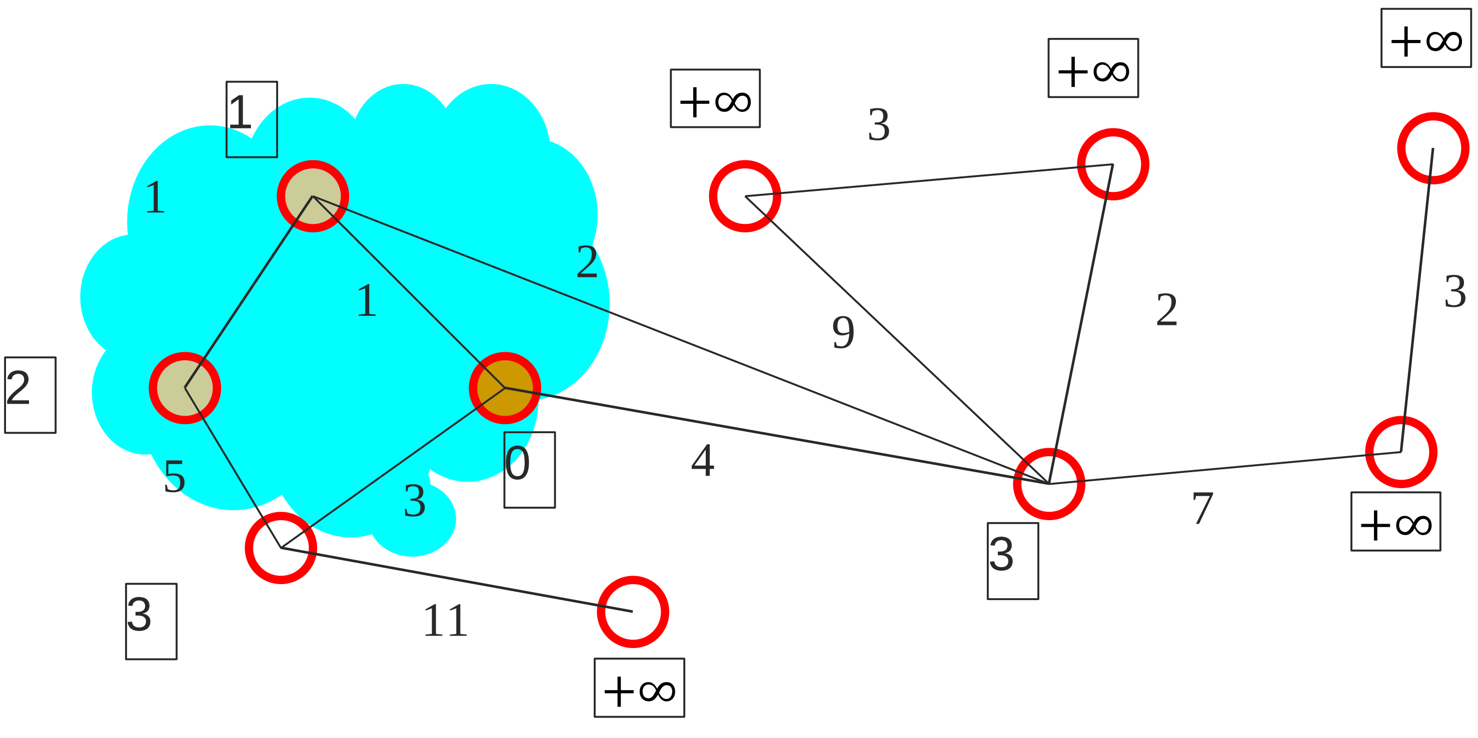

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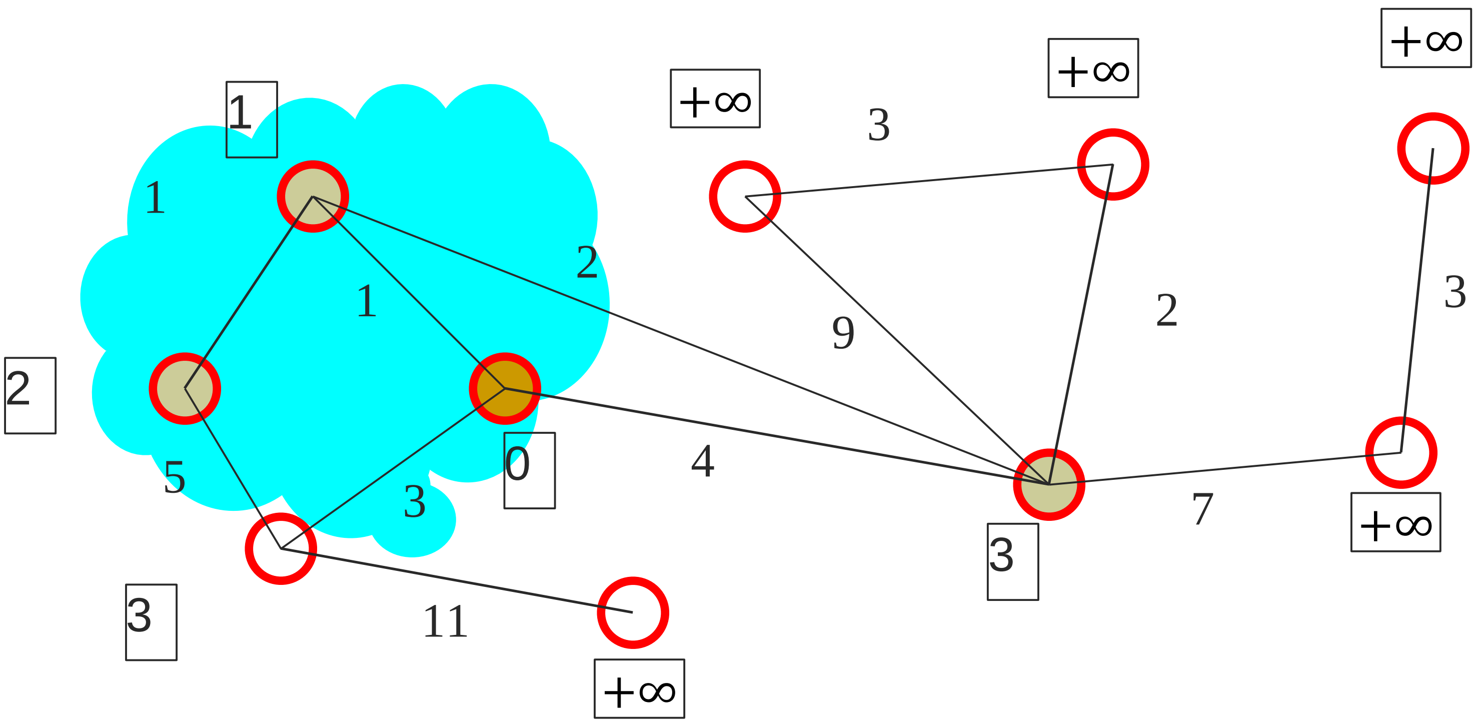

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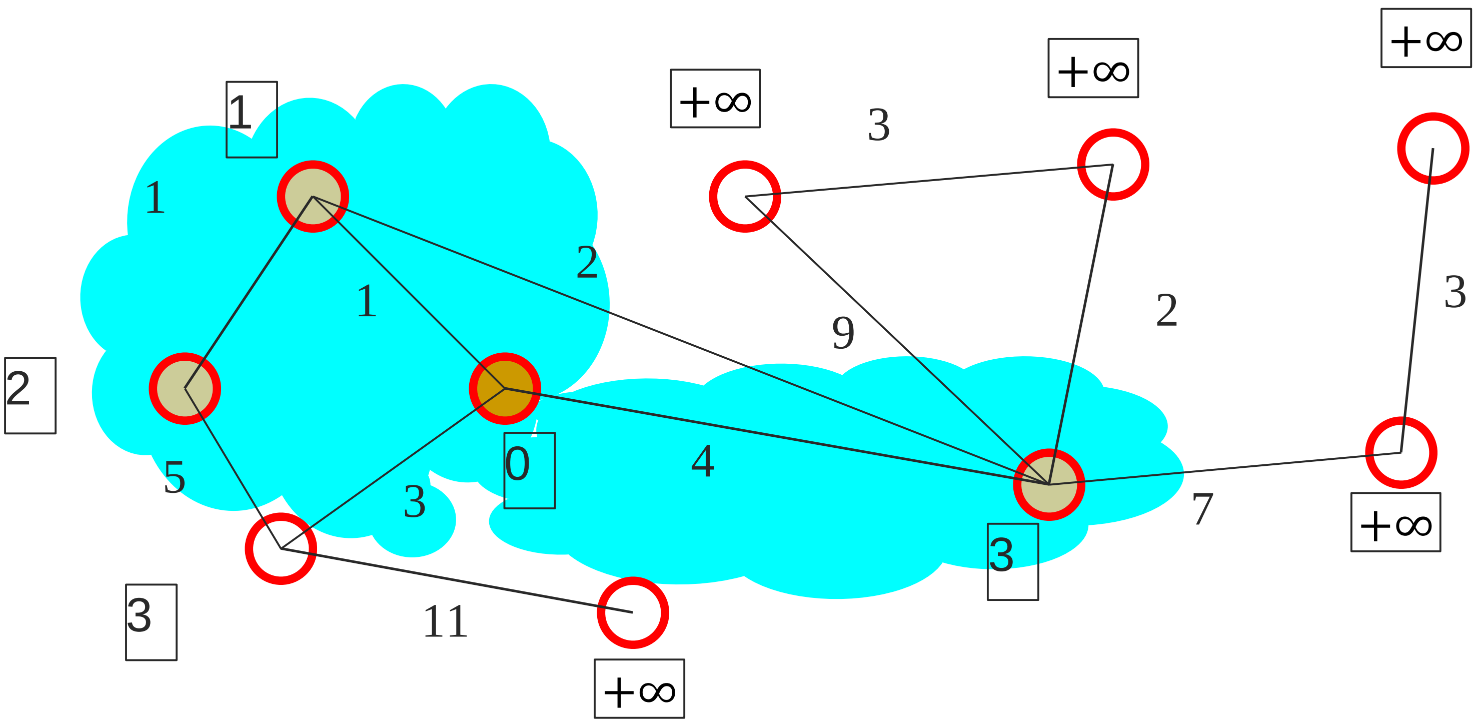

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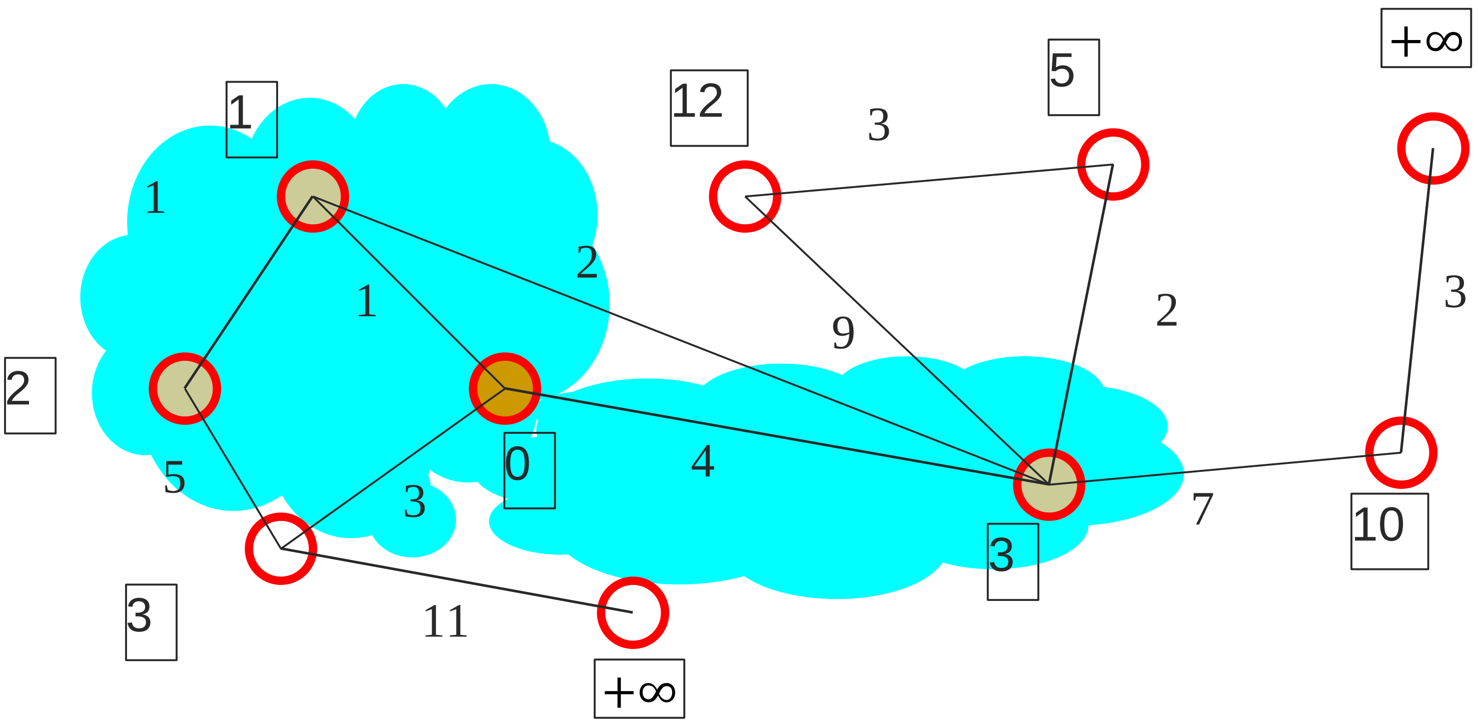
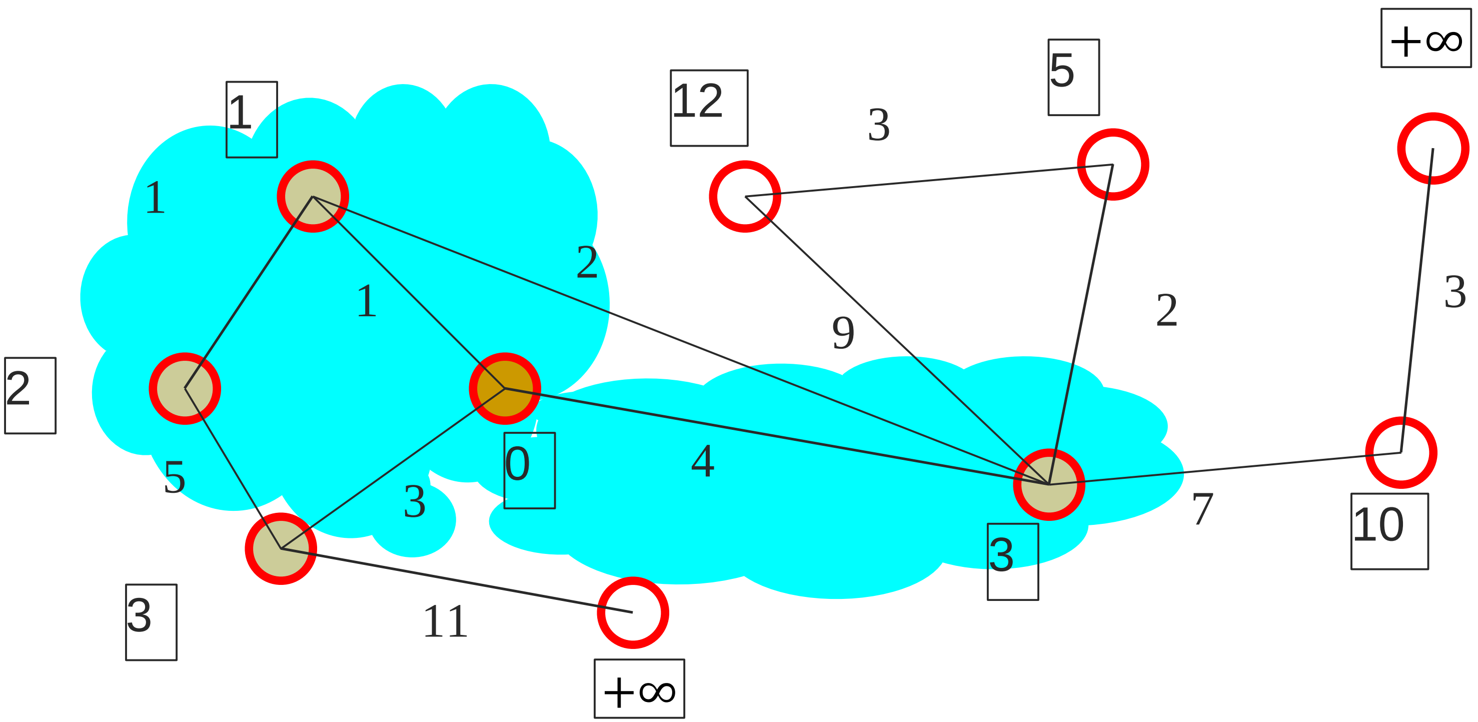


Dijkstra's algorithm: pick closest vertex u outside C



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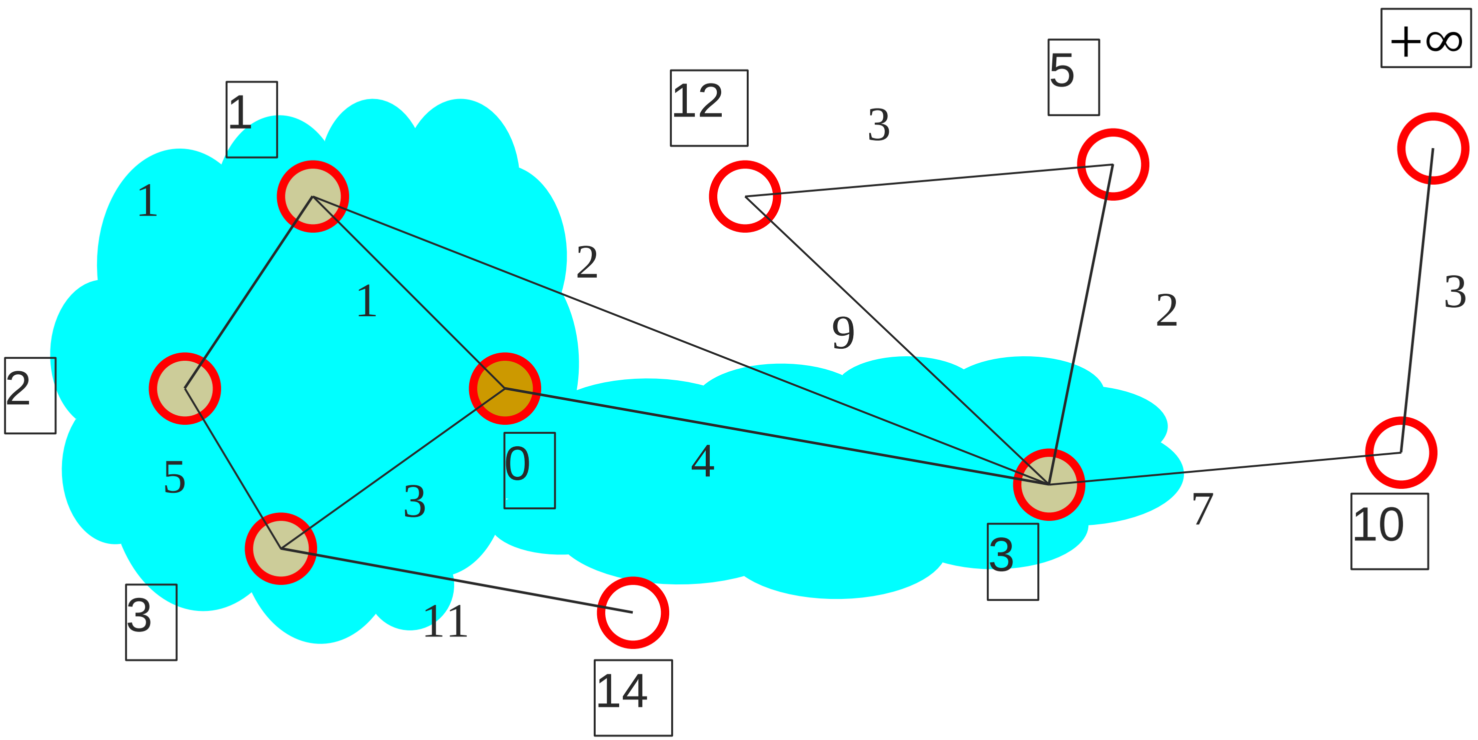

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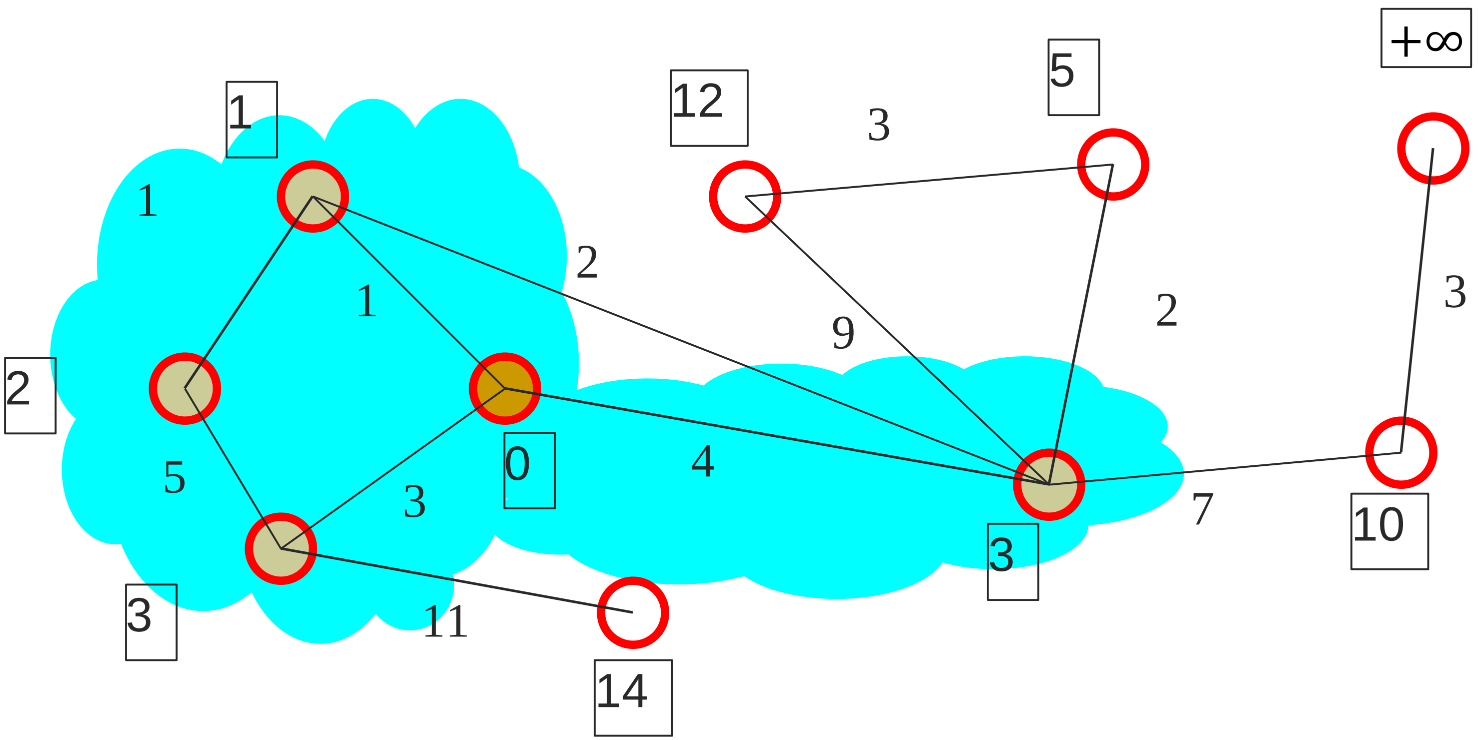
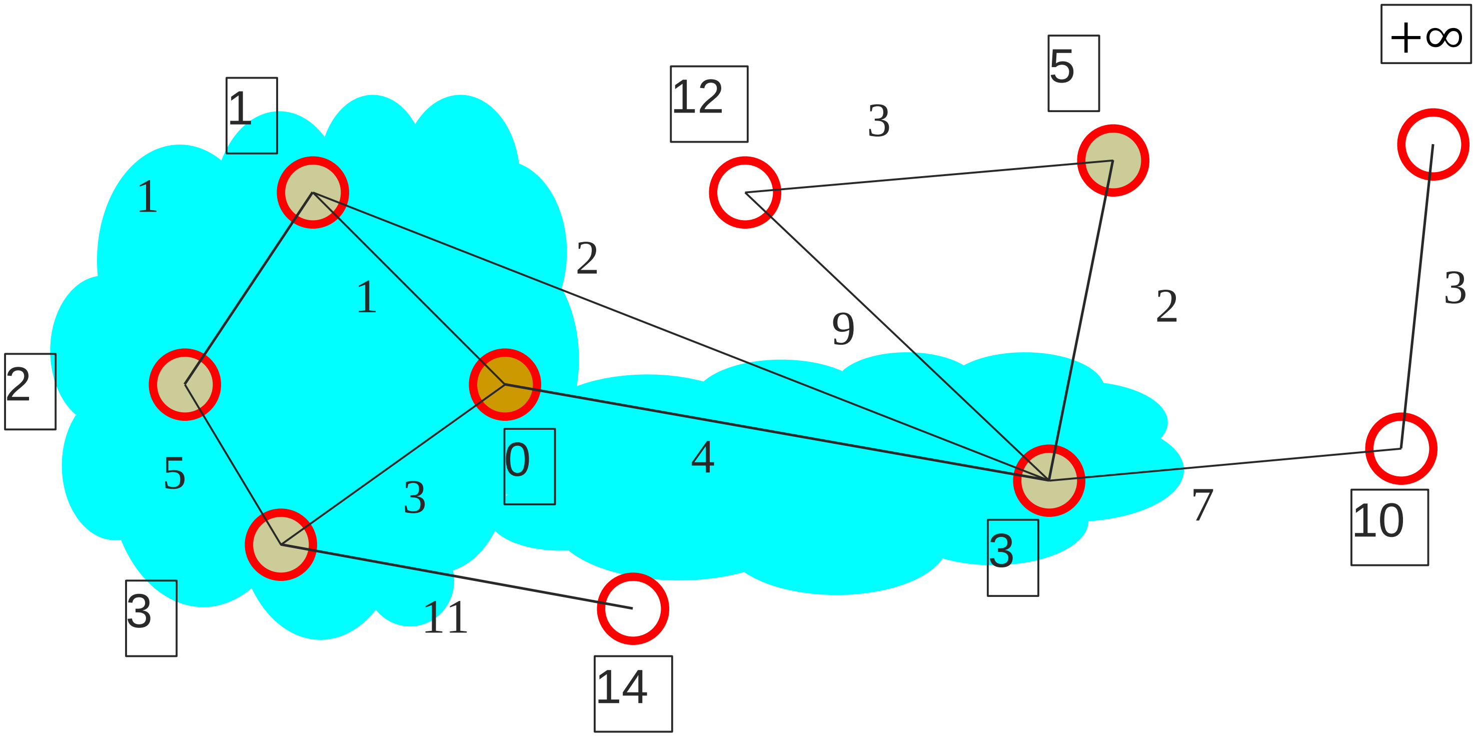
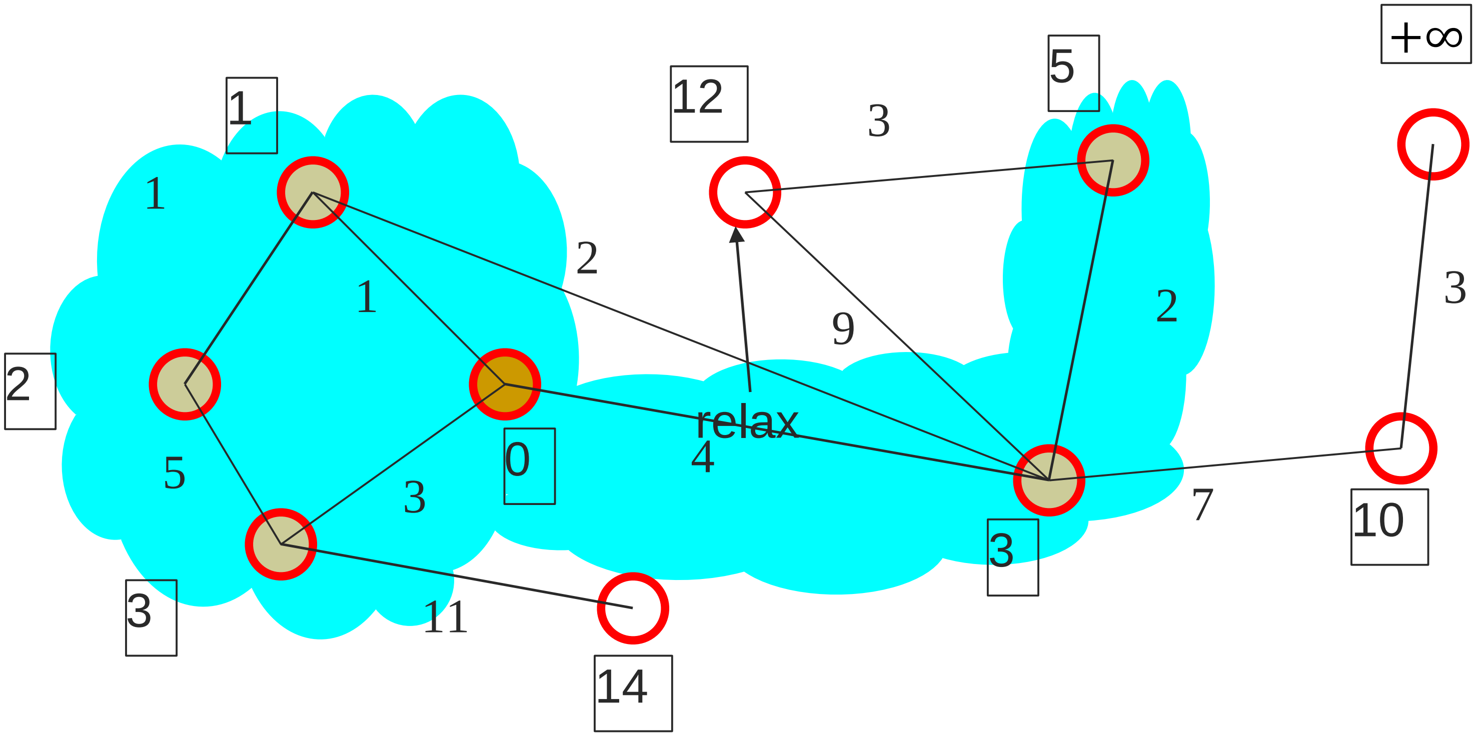


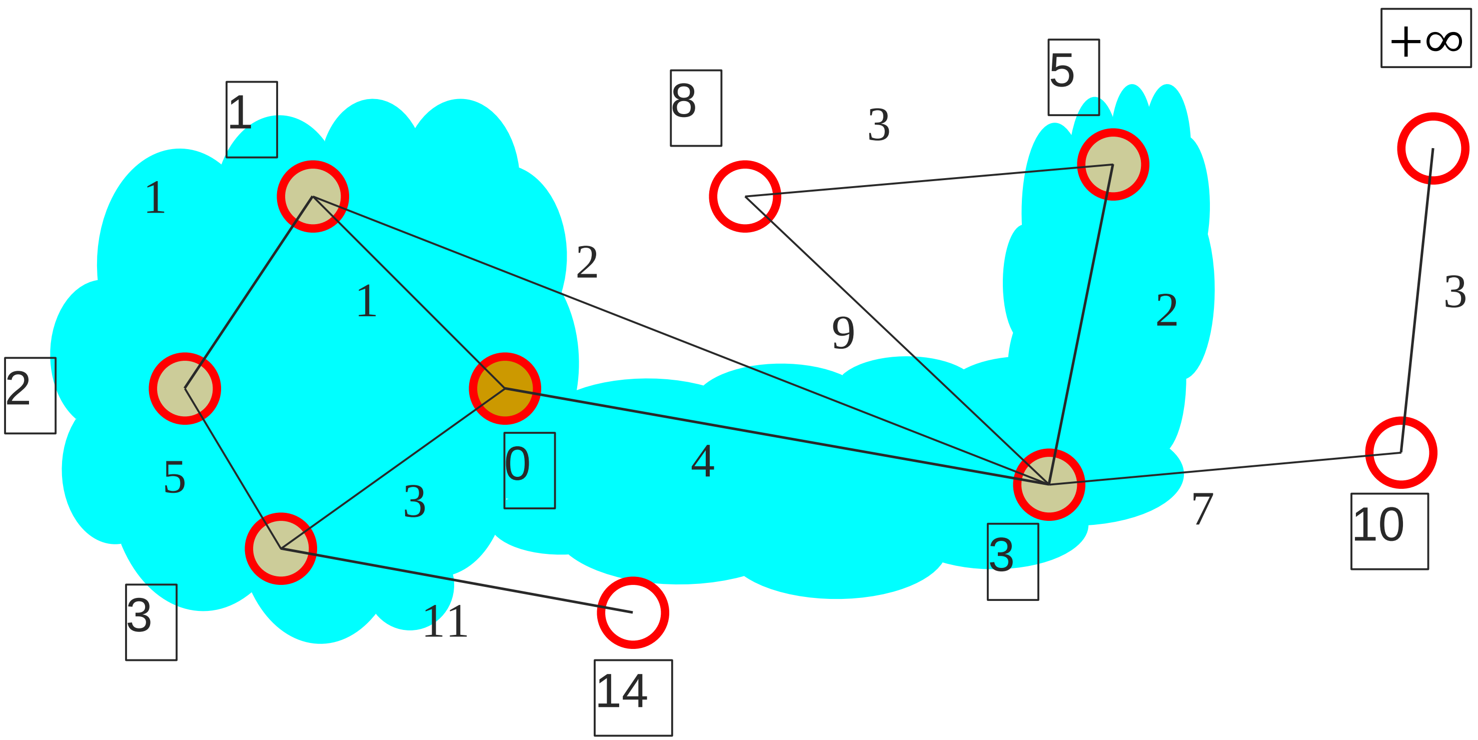
Dijkstra's algorithm: pick closest vertex u outside C



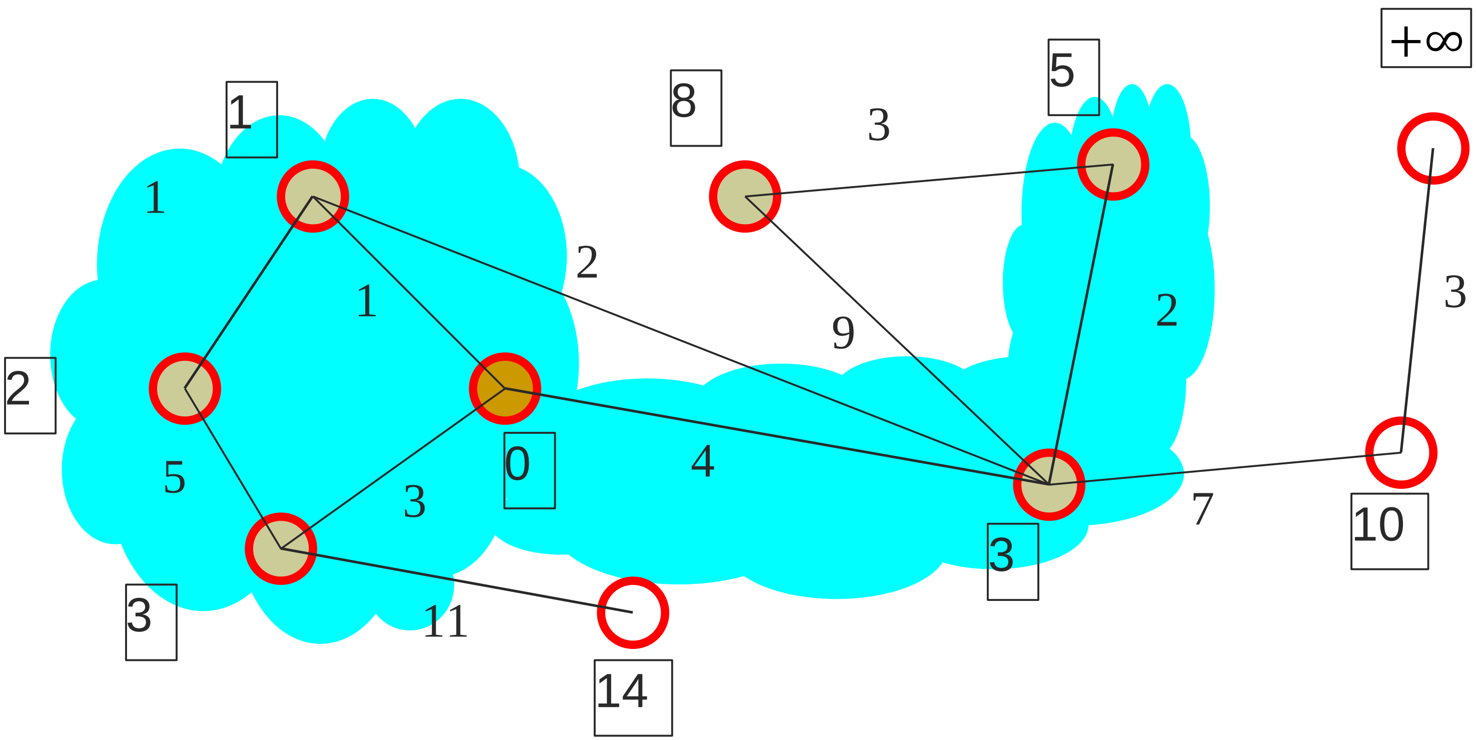
Dijkstra's algorithm: pull u into C



Dijkstra's algorithm: update *C*'s neighborhood



Dijkstra's algorithm: pick closest vertex u outside C



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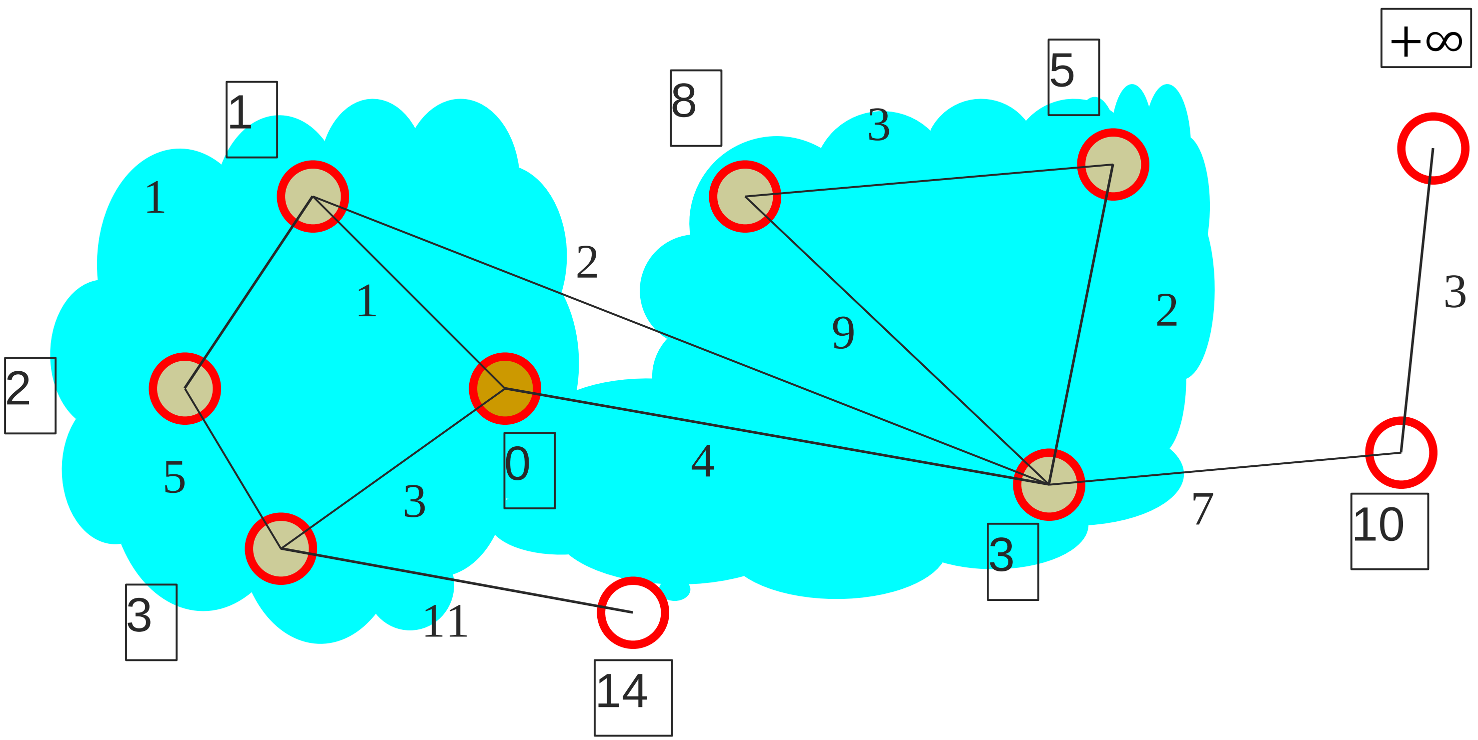
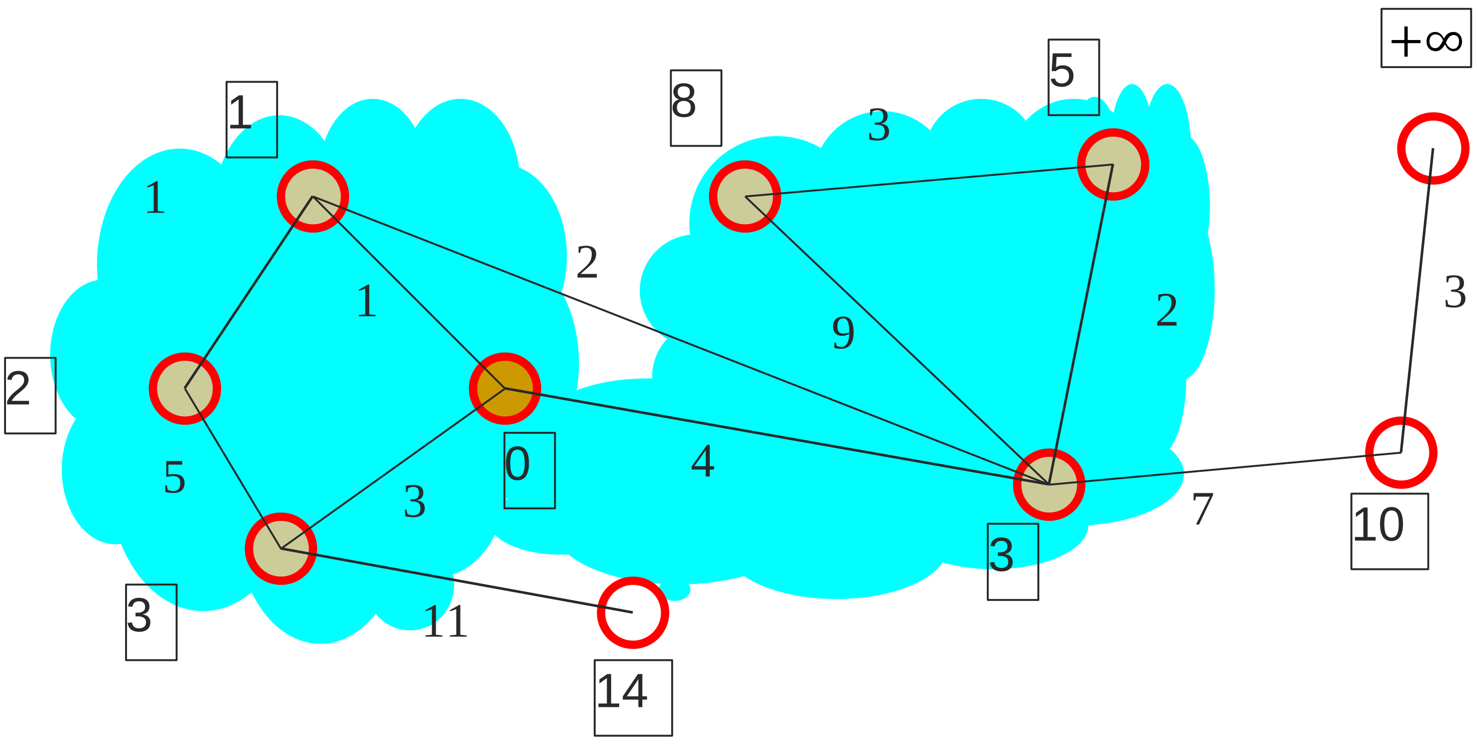


Dijkstra's algorithm: update *C's* neighborhood



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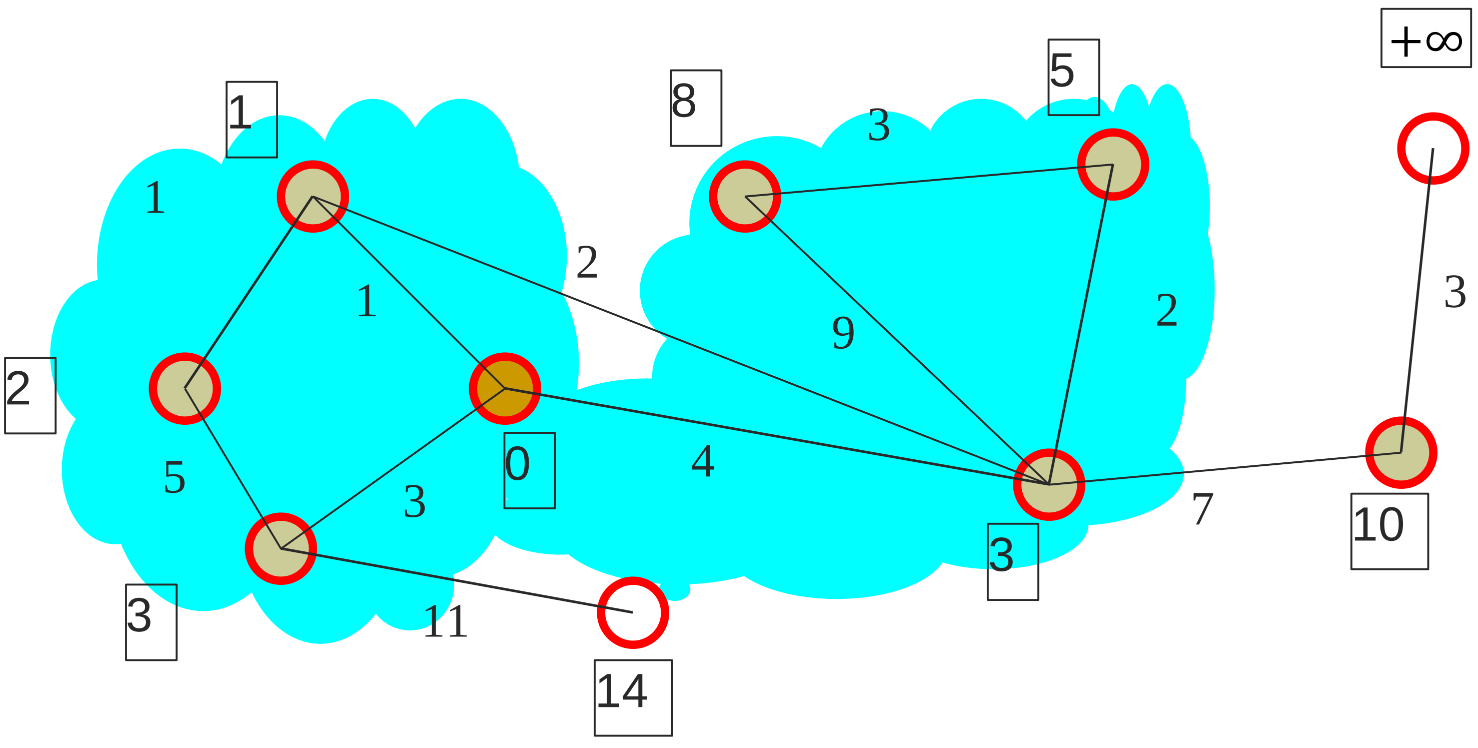

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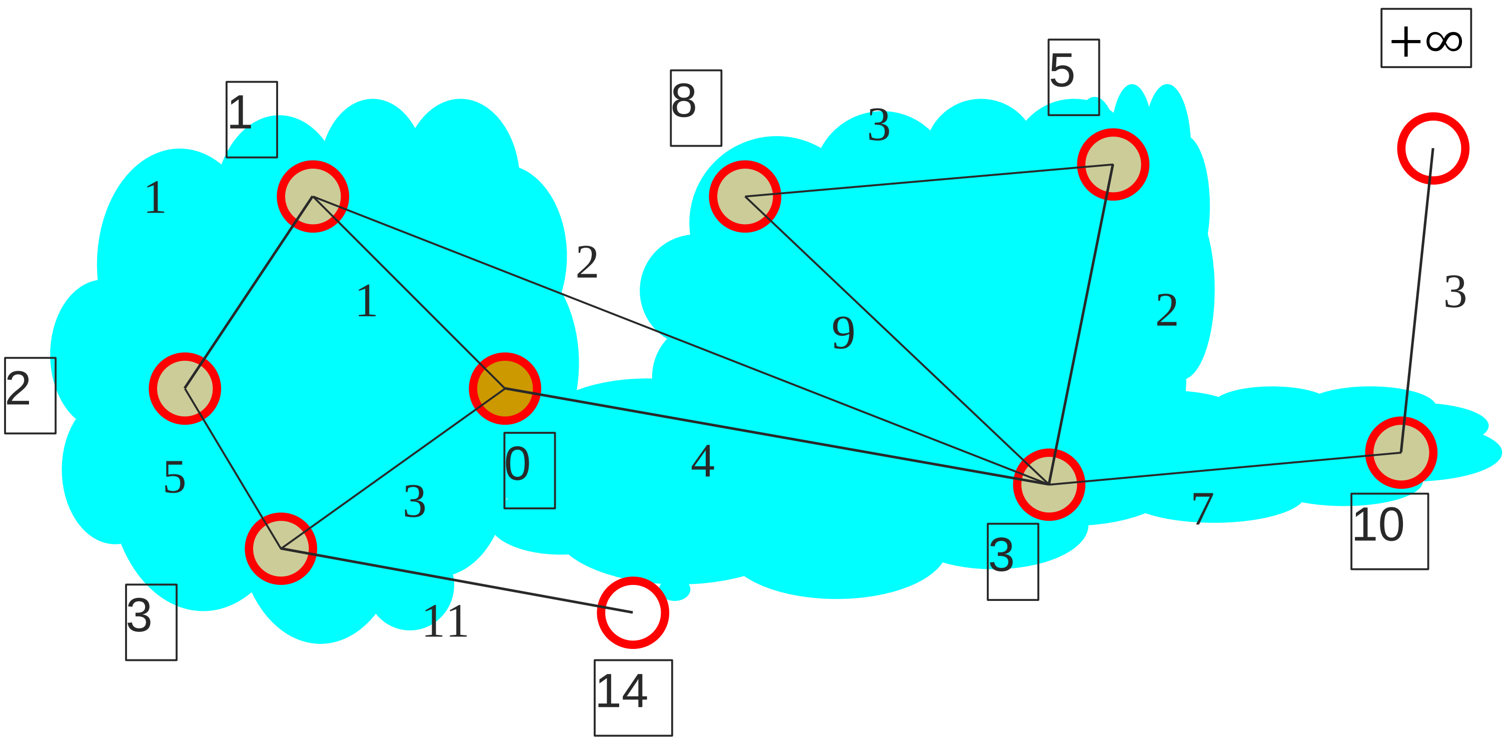
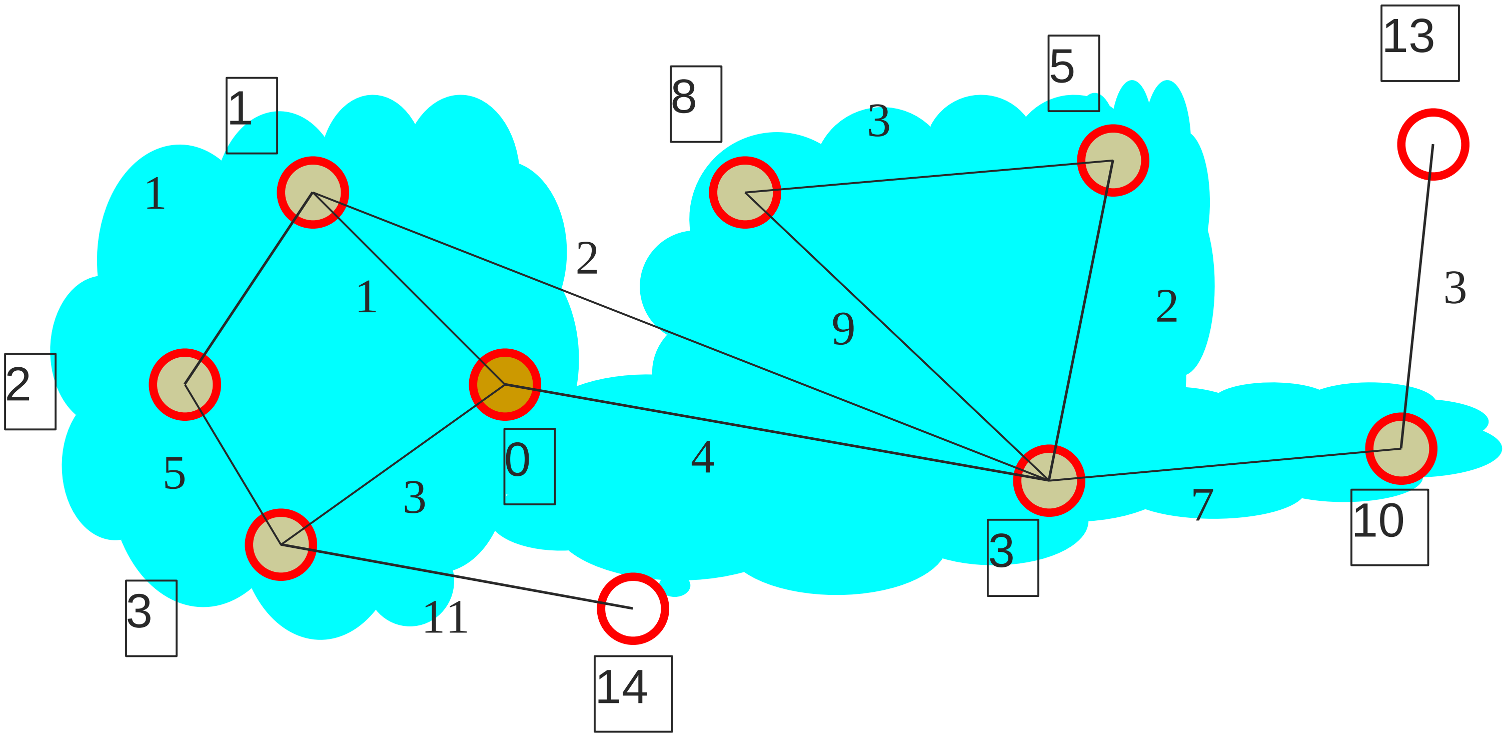


Dijkstra’s algorithm: update *C’s* neighborhood



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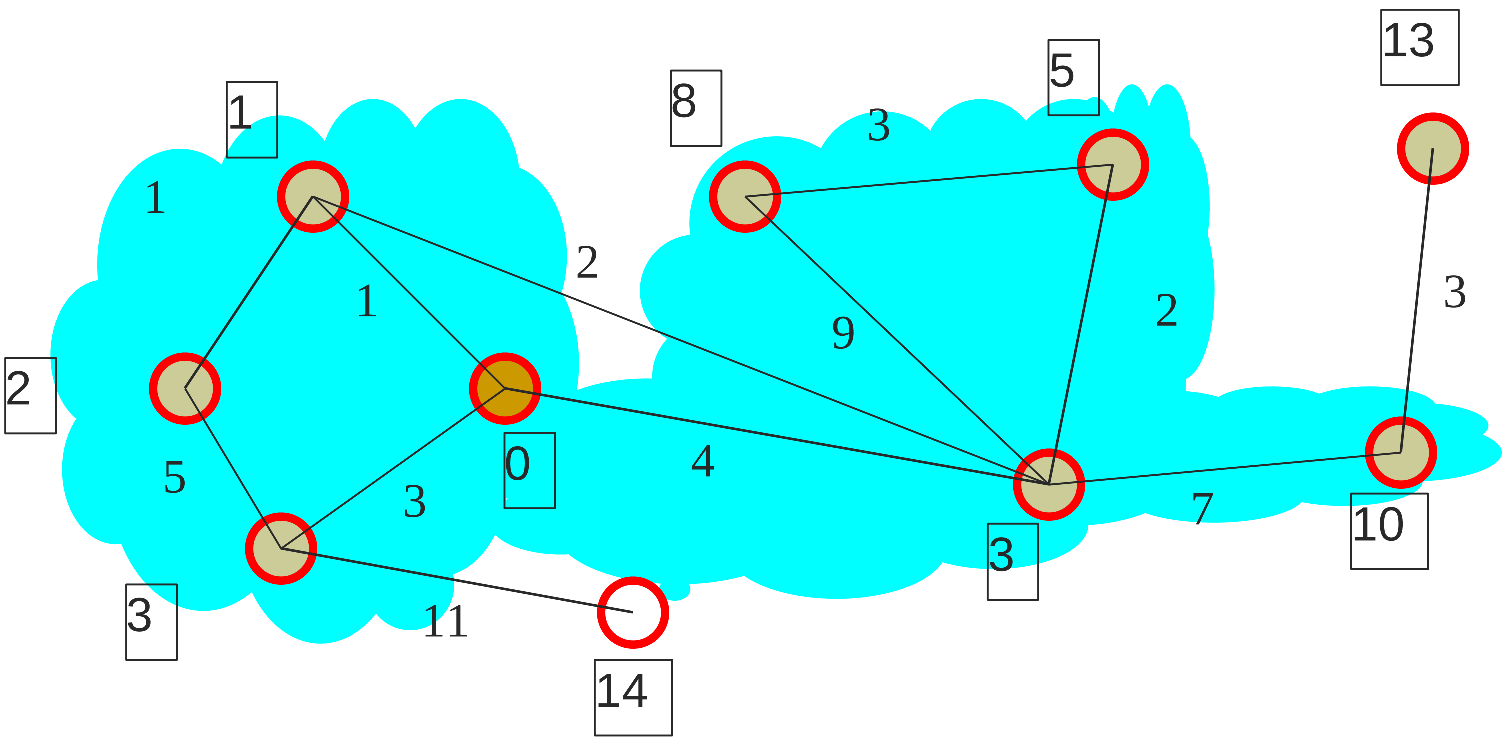
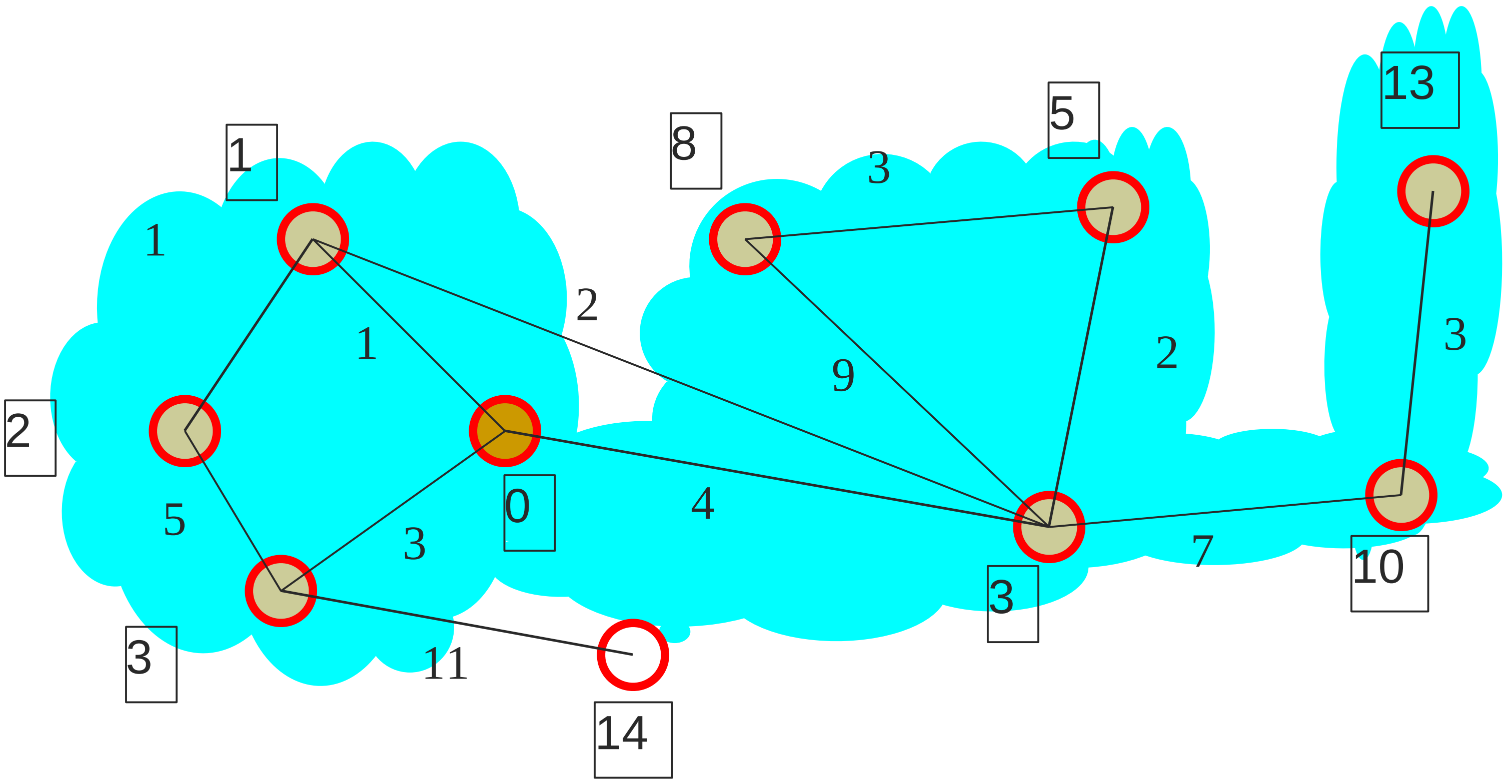
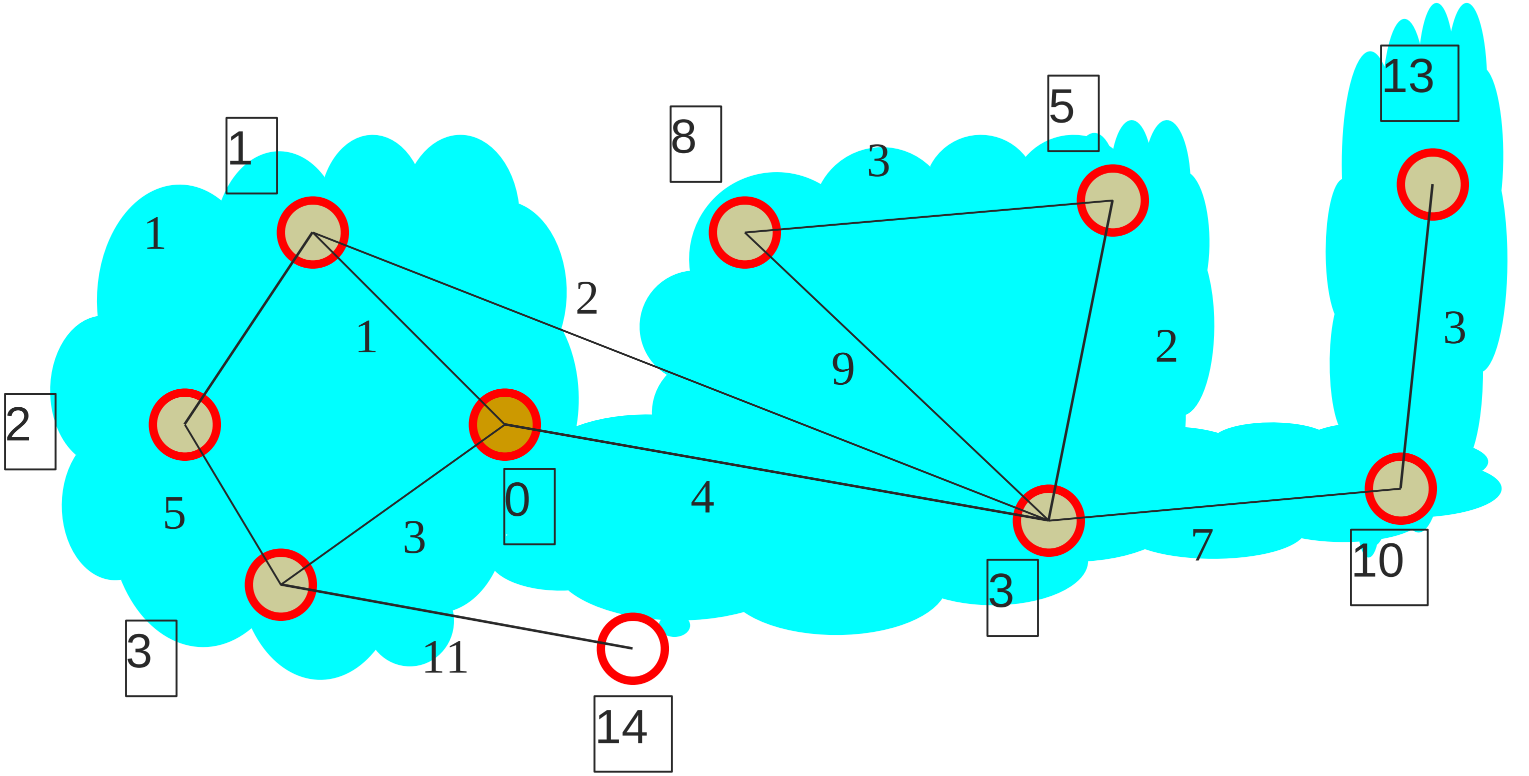


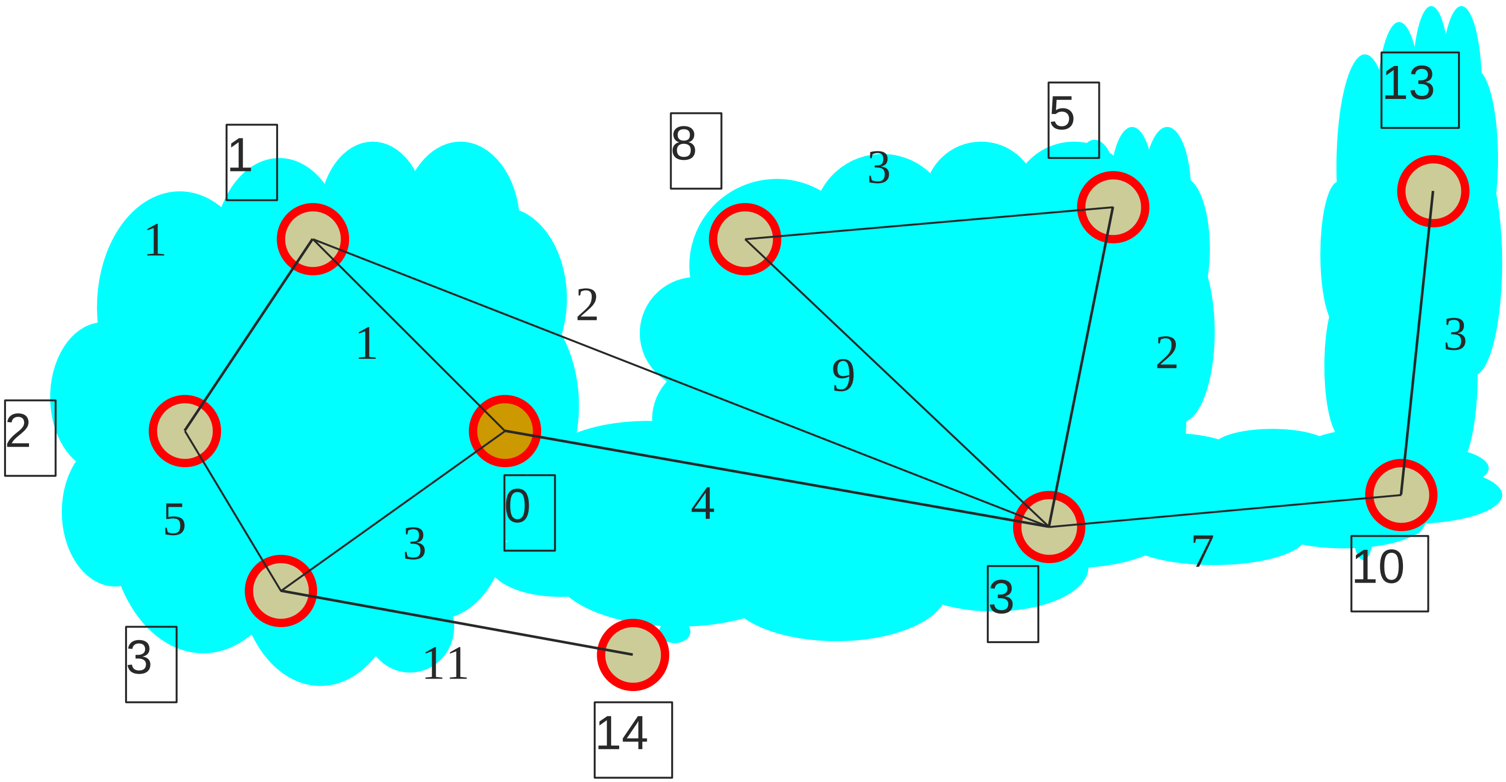
Dijkstra's algorithm: pull u into C



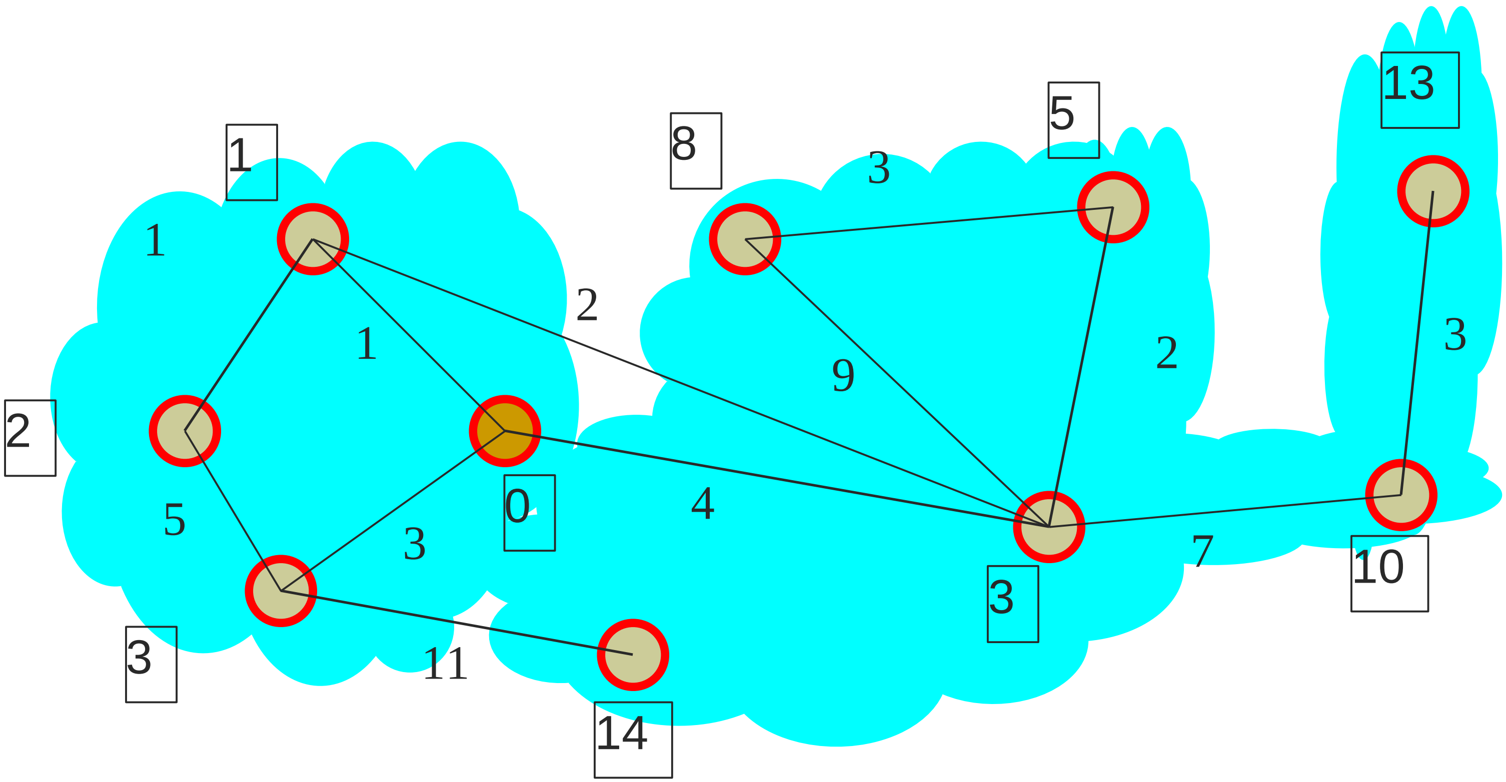
Dijkstra's algorithm: update *C's* neighborhood



Dijkstra's algorithm: pick closest vertex u outside C



Dijkstra's algorithm: pull u into C



When pulling a neighbour u of C into C

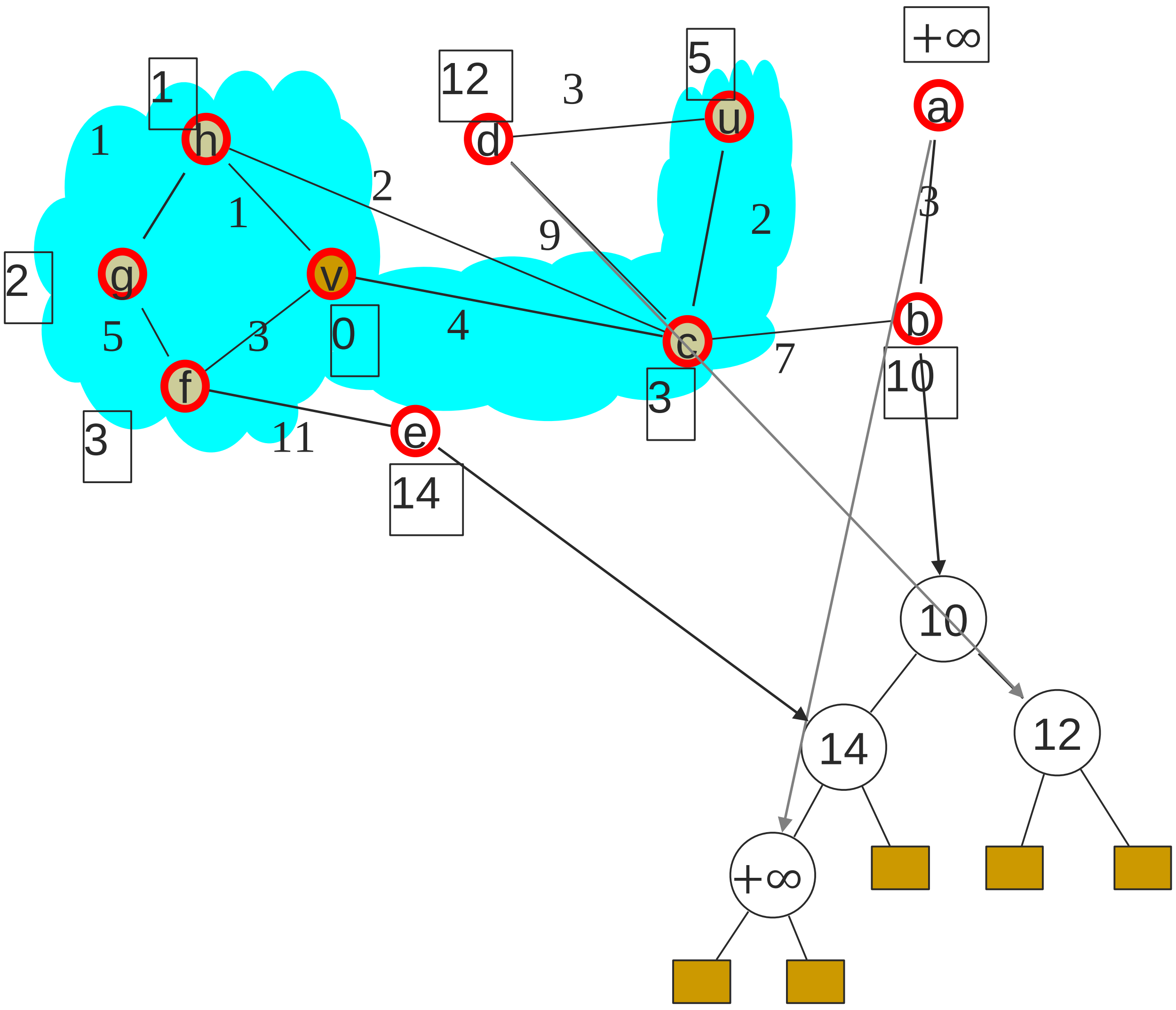
- The value associated with u denotes the length of a shortest path from v to u
- For any vertex x not in the cloud
  - the value associated with x denotes a shortest path from v to x without the use of other vertices outside of the cloud
  - $+\infty$ denotes that the vertex cannot be reached yet from v via cloud vertices only

Algorithm DijkstraShortestPaths(G,v)

```
D[v]←0
for each vertex u≠v of G do
   D[u]← +∞
Let Q be a priority queue containing all
   vertices of G using D[.] as keys
while Q is not empty do
   u←Q.removeMin() //u is added to cloud
   for each vertex z∈N(u) with z∈Q do
     if D[u]+w((u,z)) < D[z] then
        D[z]←D[u]+w((u,z))
        update z's key in Q to D[z]
return D
```

Relaxation

Dijkstra’s algorithm using heaps



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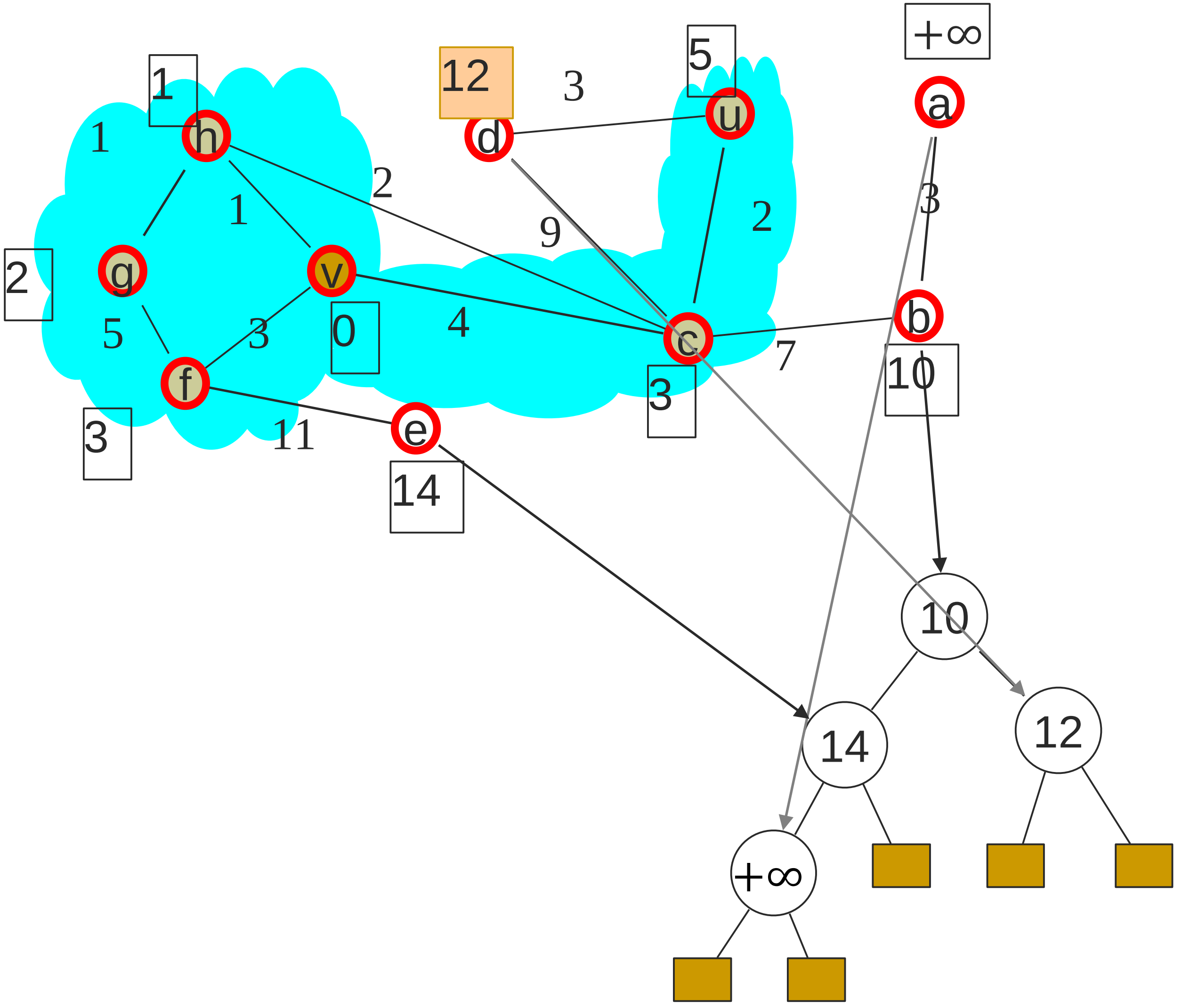

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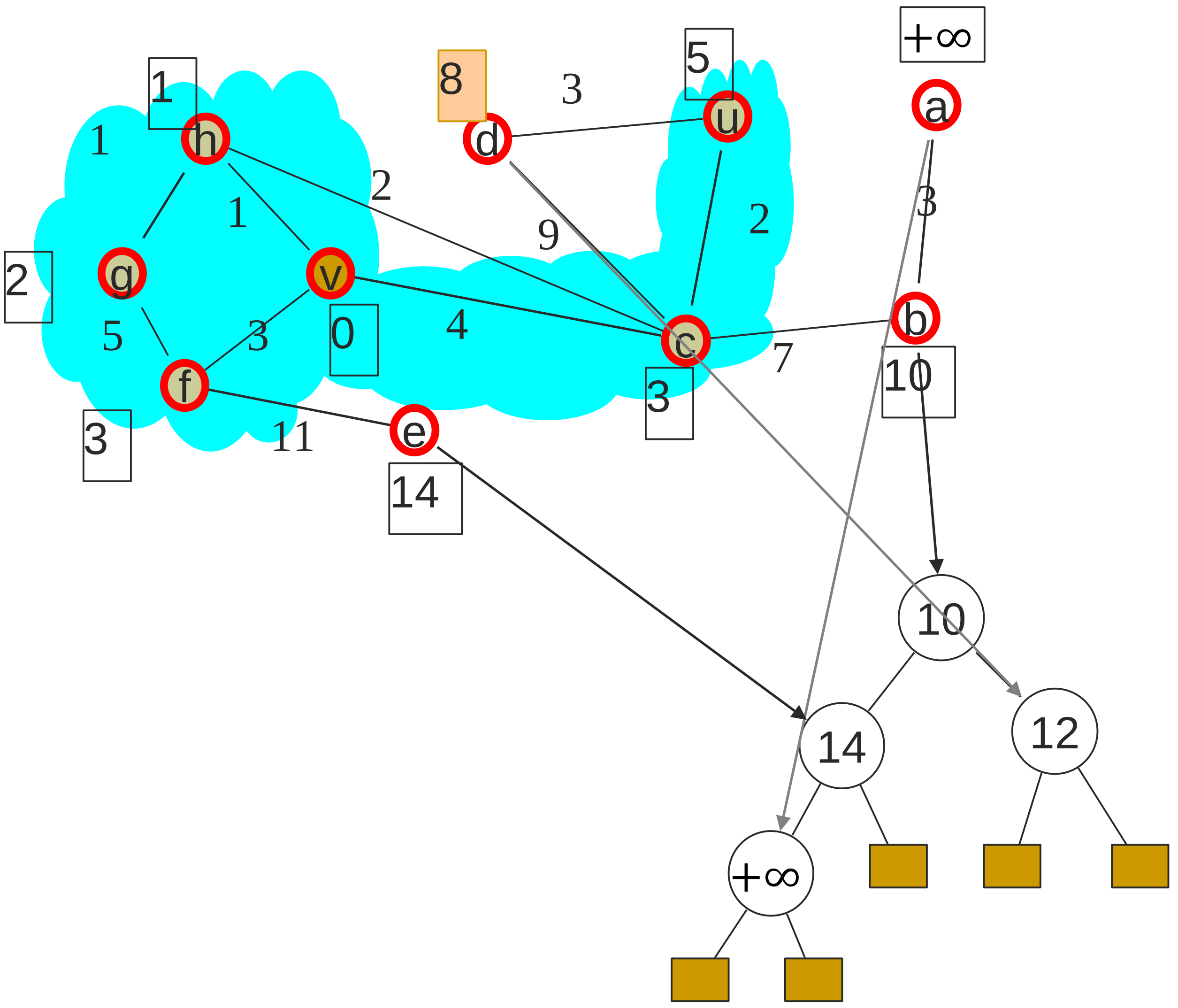
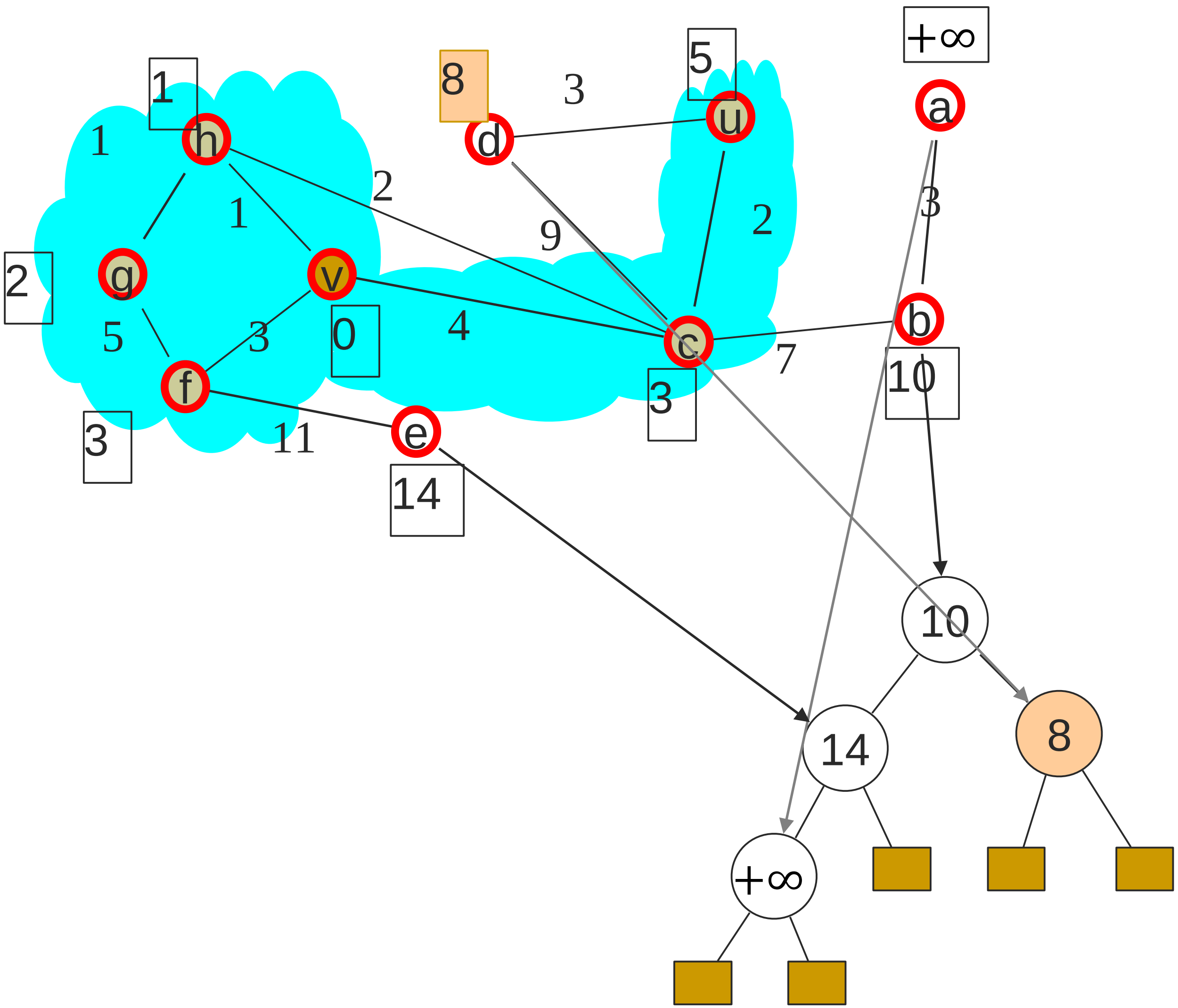
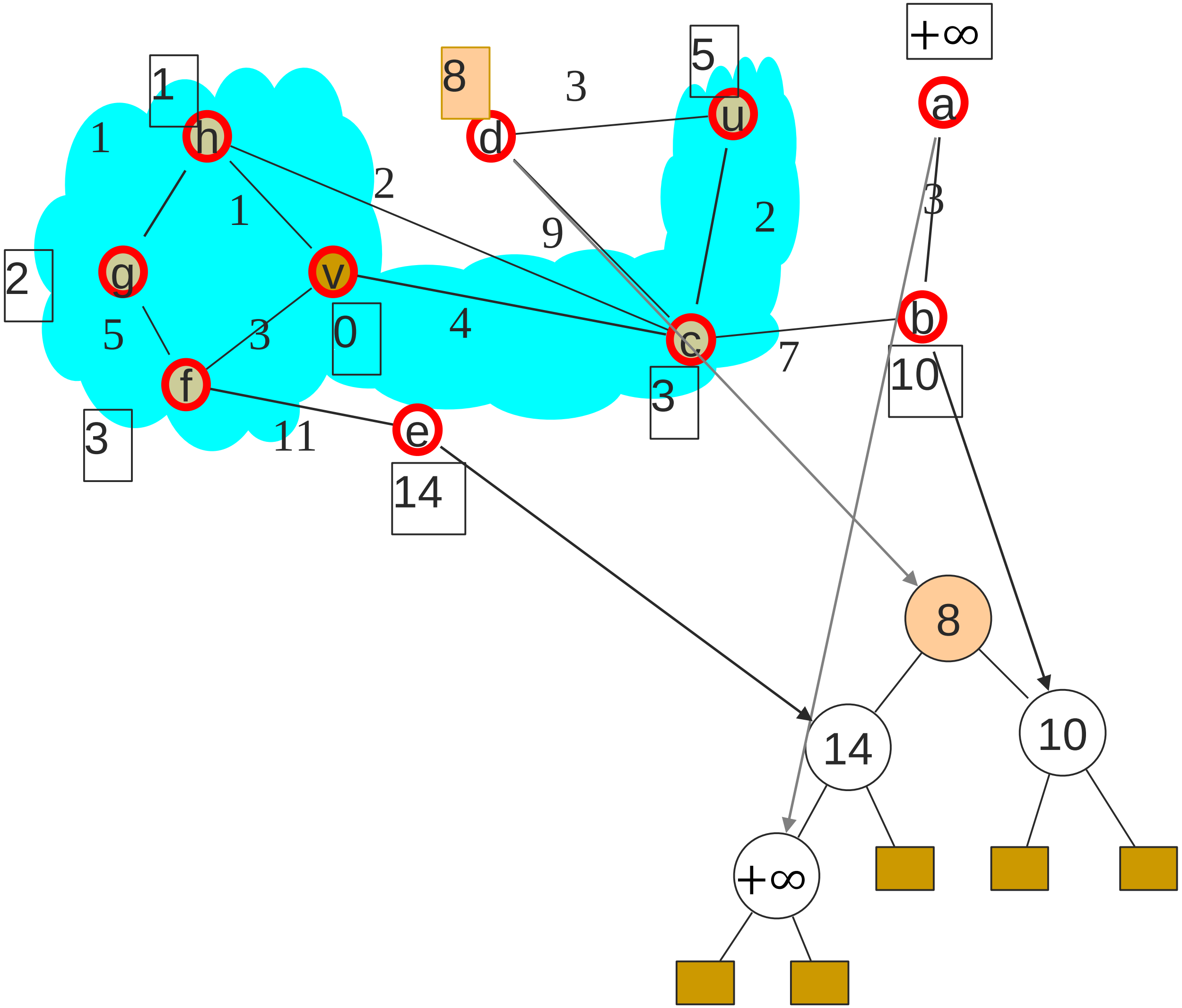


Dijkstra’s algorithm using heaps



Dijkstra's algorithm using heaps



Running time

```
D[v]←0
for each vertex u≠v of G do
   D[u]← +∞
Let Q be a priority queue containing all
   vertices of G using D[.] as keys
while Q is not empty do
   u←Q.removeMin() //u is added to cloud
   for each vertex z∈N(u) with z∈Q do
     if D[u]+w((u,z)) < D[z] then
         D[z]←D[u]+w((u,z))
         update z's key in Q to D[z]
return D
```

Relaxation

Running time for G=(V,E) with $|V|$=n and $|E|$=m

- Insertion of vertices in priority queue Q
  - $O(n)$ when using bottom-up heap construction
- While loop:
  - Per iteration:
    - Remove vertex from Q $\quad O(\log n)$
    - Relaxation $\quad O(\deg(u)\log(n))$
  - $\sum_{u \in G}\big(1 + \deg(u)\big)\log n$ is $O((n+m)\log n)$

- Overall running time: $O(m \log n)$

In real life applications

- Often the graphs are sparse
- Then $O(m \log n)$ may be $O(n \log n)$

Algorithm DijkstraShortestPaths(G,v)

Correctness

```
D[v]←0
for each vertex u≠v of G do
   D[u]← +∞
Let Q be a priority queue containing all
   vertices of G using D[.] as keys
while Q is not empty do
   u←Q.removeMin() //u is added to cloud
   for each vertex z∈N(u) with z∈Q do
     if D[u]+w((u,z)) < D[z] then
       D[z]←D[u]+w((u,z))
       update z's key in Q to D[z]
return D
```

Relaxation

Correctness of Dijkstra's algorithm

- **To show:** whenever u is pulled into cloud C, D[u] stores the length of a shortest path from v (the starting vertex) to u

- **Definition:** For vertices u and v in G, we denote with $d(v,u)$ the length of a shortest path from v to u.

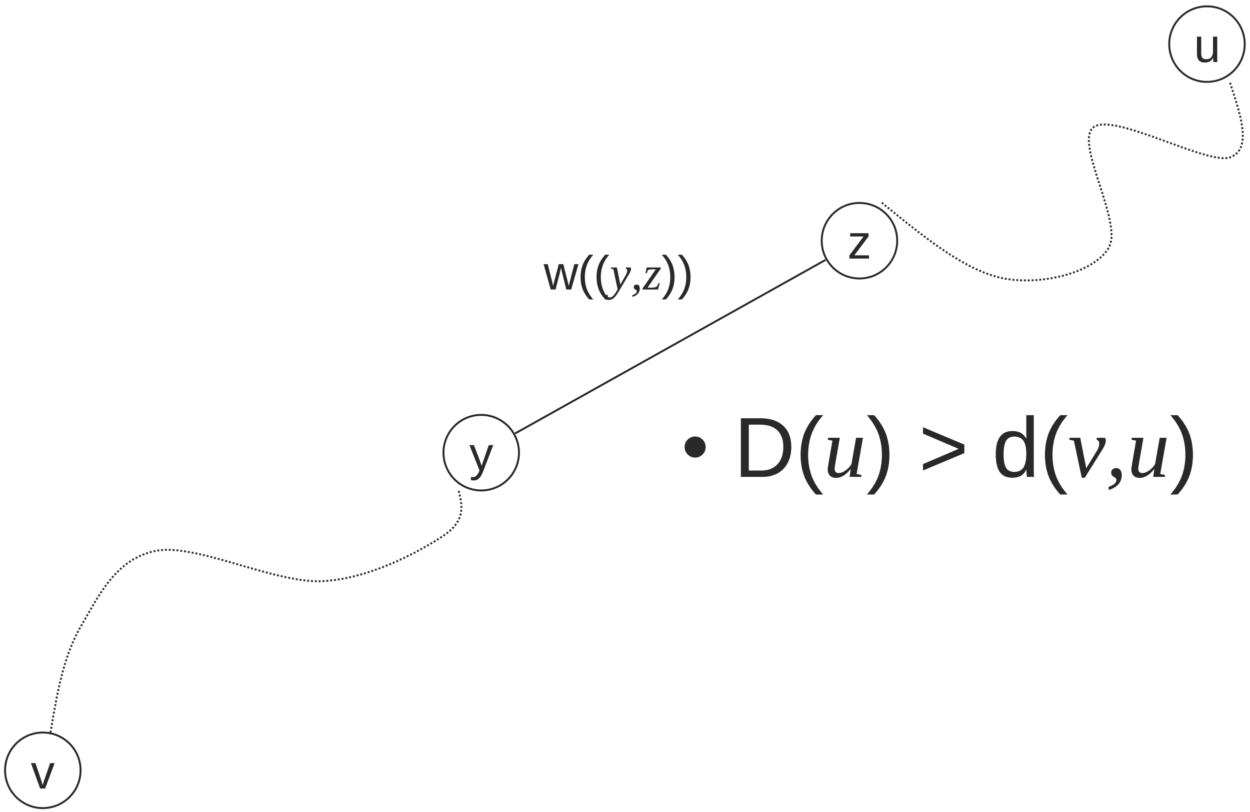
Whenever u is pulled into cloud C, D[u] stores the length of a shortest path from v to u

Proof. Assume: claim is wrong. Then:

there exists a vertex t that is pulled into cloud C and D[t] > d(v,t)

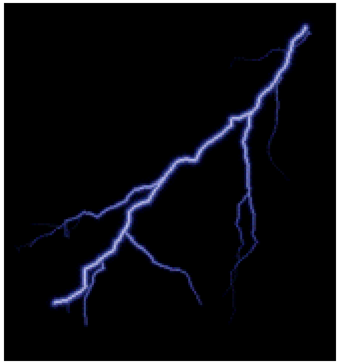
We define:

- u the first such vertex (currently) pulled into C
- P a shortest path in G from source v to vertex u
- y the last vertex that lies on P and is pulled “correctly” into C
- z the vertex closest to y that lies on P and is not in C



- $D(u) > d(v,u)$

- $D(u) > d(v,u)$
- $y \in C$, $D[y] = d(v,y)$
- $D(u) \le D(z)$
- $D(z) = d(v,z)$



- $D(u) \le D(z) = d(v,z) \le d(v,z) + d(z,u)$
  $= d(v,u)$

Prim's algorithm: eager implementation

```
public class PrimMST {
  private Edge[] edgeTo;                  // shortest edge from tree to vertex
  private double[] distTo;                // distTo[w] = edgeTo[w].weight()
  private boolean[] marked;               // true if v in mst
  private IndexMinPQ<Double> pq;          // eligible crossing edges

   public PrimMST(WeightedGraph G) {
     edgeTo = new Edge[G.V()];
     distTo = new double[G.V()];
     marked = new boolean[G.V()];
     for(int v = 0; v < G.V(); v++)
         distTo[v] = Double.POSITIVE_INFINITY;
     pq = new IndexMinPQ<Double>(G.V());
     distTo[0] = 0.0;
     pq.insert(0, 0.0);
     while(!pq.isEmpty())
          visit(G, pq.delMin());
  }
```

assume G is connected

repeatedly delete the min weight edge e = v–w from PQ

Prim's algorithm: eager implementation

```
private void visit(WeightedGraph G, int v) {
   marked[v] = true;
   for (Edge e : G.adj(v)) {
      int w = e.other(v);
      if (marked[w]) continue;
      if (e.weight() < distTo[w]) {
           edgeTo[w] = e;
           distTo[w] = e.weight();
           if (pq.contains(w)) pq.changeKey(w, distTo[w]);
           else pq.insert(w, distTo[w]);
       }
   }
}
public Iterable<Edge> edges(){
   Queue<Edge> mst = new Queue<Edge>();
   for (int v = 0; v < edgeTo.length; v++)
      Edge e = edgeTo[v];
      if (e != null) {
         mst.enqueue(e);
      }
   }
   return mst; }
```

add v to T

for each edge e = v–w, add to PQ if w not already in T

add edge e to tree

Update distance to w or Insert distance to w

Create the mst

Djikstra's algorithm: eager implementation

```
public class DijkstraUndirectedSP {
  private Edge[] edgeTo;                  // last edge from on path to v
  private double[] distTo;                // distance to v from s
  private IndexMinPQ<Double> pq;     // eligible crossing edges

   public DijkstraUndirectedSP(WeightedGraph G, int s) {
      edgeTo = new Edge[G.V()];
      distTo = new double[G.V()];

      for(int v = 0; v < G.V(); v++)
           distTo[v] = Double.POSITIVE_INFINITY;
      pq = new IndexMinPQ<Double>(G.V());
      distTo[s] = 0.0;
      pq.insert(s, distTo[s]);
      while(!pq.isEmpty())
             relax(G, pq.delMin());
  }
```

assume G is connected

repeatedly delete the edge e = v–w from PQ that is closest to s.

Dijkstra's algorithm: eager implementation

```
private void relax(WeightedGraph G, int v) {

  for (Edge e : G.adj(v)) {
    int w = e.other(v);

    if (distTo[v] + e.weight() < distTo[w]) {
        edgeTo[w] = e;
        distTo[w] = distTo[v] + e.weight();
        if (pq.contains(w)) pq.changeKey(w, distTo[w]);
        else pq.insert(w, distTo[w]);
     }
  }
}
public Iterable<Edge> edges(){
  Queue<Edge> spt = new Queue<Edge>();
  for (int v = 0; v < edgeTo.length; v++)
    Edge e = edgeTo[v];
    if (e != null) {
      spt.enqueue(e);
    }
  }
  return spt; }
```

for each edge e = v–w, add to PQ if w not already in T

add edge e to tree

Update distance to w or Insert distance to w

Create the spt