1 Fundamental Principles of Counting

Enumeration, or counting, may strike one as an obvious process that a student learns when first studying arithmetic. But then, it seems, very little attention is paid to further development in counting as the student turns to "more difficult" areas in mathematics, such as algebra, geometry, trigonometry, and calculus. Consequently, this first chapter should provide some warning about the seriousness and difficulty of "mere" counting.

Enumeration does not end with arithmetic. It also has applications in such areas as coding theory, probability and statistics, and in the analysis of algorithms. Later chapters will offer some specific examples of these applications.

As we enter this fascinating field of mathematics, we shall come upon many problems that are very simple to state but somewhat "sticky" to solve. Thus, be sure to learn and understand the basic formulas—but do *not* rely on them too heavily. For without an analysis of each problem, a mere knowledge of formulas is next to useless. Instead, welcome the challenge to solve unusual problems or those that are different from problems you have encountered in the past. Seek solutions based on your own scrutiny, regardless of whether it reproduces what the author provides. There are often several ways to solve a given problem.

1.1 The Rules of Sum and Product

Our study of discrete and combinatorial mathematics begins with two basic principles of counting: the rules of sum and product. The statements and initial applications of these rules appear quite simple. In analyzing more complicated problems, one is often able to break down such problems into parts that can be solved using these basic principles. We want to develop the ability to "decompose" such problems and piece together our partial solutions in order to arrive at the final answer. A good way to do this is to analyze and solve many diverse enumeration problems, taking note of the principles being used. This is the approach we shall follow here.

Our first principle of counting can be stated as follows:

The Rule of Sum: If a first task can be performed in m ways, while a second task can be performed in n ways, and the two tasks cannot be performed simultaneously, then performing either task can be accomplished in any one of $m + n$ ways.

Note that when we say that a particular occurrence, such as a first task, can come about in m ways, these m ways are assumed to be distinct, unless a statement is made to the contrary. This will be true throughout the entire text.

EXAMPLE 1.1

A college library has 40 textbooks on sociology and 50 textbooks dealing with anthropology. By the rule of sum, a student at this college can select among $40 + 50 = 90$ textbooks in order to learn more about one or the other of these two subjects.

EXAMPLE 1.2

The rule can be extended beyond two tasks as long as no pair of the tasks can occur simultaneously. For instance, a computer science instructor who has, say, five introductory books each on C++, FORTRAN, Java, and Pascal can recommend any one of these 20 books to a student who is interested in learning a first programming language.

EXAMPLE 1.3

The computer science instructor of Example 1.2 has two colleagues. One of these colleagues has three textbooks on the analysis of algorithms, and the other has five such textbooks. If n denotes the maximum number of different books on this topic that this instructor can borrow from them, then $5 \leq n \leq 8$, for here both colleagues *may* own copies of the same textbook(s).

The following example introduces our second principle of counting.

EXAMPLE 1.4

In trying to reach a decision on plant expansion, an administrator assigns 12 of her employees to two committees. Committee A consists of five members and is to investigate possible favorable results from such an expansion. The other seven employees, committee B, will scrutinize possible unfavorable repercussions. Should the administrator decide to speak to just one committee member before making her decision, then by the rule of sum there are 12 employees she can call upon for input. However, to be a bit more unbiased, she decides to speak with a member of committee A on Monday, and then with a member of committee B on Tuesday, before reaching a decision. Using the following principle, we find that she can select two such employees to speak with in $5 \times 7 = 35$ ways.

The Rule of Product: If a procedure can be broken down into first and second stages, and if there are m possible outcomes for the first stage and if, for each of these outcomes, there are n possible outcomes for the second stage, then the total procedure can be carried out, in the designated order, in mn ways.

At times this rule is also referred to as the *principle of choice*.

EXAMPLE 1.5

The drama club of Central University is holding tryouts for a spring play. With six men and eight women auditioning for the leading male and female roles, by the rule of product the director can cast his leading couple in $6 \times 8 = 48$ ways.

EXAMPLE 1.6

Here various extensions of the rule are illustrated by considering the manufacture of license plates consisting of two letters followed by four digits.

a) If no letter or digit can be repeated, there are $26 \times 25 \times 10 \times 9 \times 8 \times 7 =$ 3,276,000 different possible plates.

b) With repetitions of letters and digits allowed, $26 \times 26 \times 10 \times 10 \times 10 \times 10 =$ 6,760,000 different license plates are possible.

c) If repetitions are allowed, as in part (b), how many of the plates have only vowels (A, E, I, O, U) and even digits? (0 is an even integer.)

EXAMPLE 1.7

In order to store data, a computer's main memory contains a large collection of circuits, each of which is capable of storing a *bit*—that is, one of the *bi*nary digi*ts* 0 or 1. These storage circuits are arranged in units called (memory) cells. To identify the cells in a computer's main memory, each is assigned a unique name called its *address.* For some computers, such as embedded microcontrollers (as found in the ignition system of an automobile), an address is represented by an ordered list of eight bits, collectively referred to as a *byte.* Using the rule of product, there are $2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^8 = 256$ such bytes. So we have 256 addresses that may be used for cells where certain information may be stored.

A kitchen appliance, such as a microwave oven, incorporates a small computer whose main memory may contain hundreds of cells. Some older computers (such as the PDP 11 family[†]) contained thousands of memory cells and used two-byte addresses to identify these cells in their main memory. Such addresses are made up of two consecutive bytes, or 16 consecutive bits. Thus there were $256 \times 256 = 2^8 \times 2^8 = 2^{16} = 65{,}536$ available addresses that could be used to identify cells in the main memory. Other computers use addressing systems of four bytes. This 32-bit architecture is used in the Pentium[‡] and SPARC[§] processors, where there are as many as $2^8 \times 2^8 \times 2^8 \times 2^8 = 2^{32} =$ 4,294,967,296 addresses for use in identifying the cells in the main memory. When a programmer deals with the Alpha[‖] processor, he or she considers memory cells with eight-byte addresses. Each of these addresses comprises $8 \times 8 = 64$ bits, and there are $2^{64} =$ 18,446,744,073,709,551,616 possible addresses for this architecture. (Of course, not all of these possibilities are actually used.)

EXAMPLE 1.8

At times it is necessary to combine several different counting principles in the solution of one problem. Here we find that the rules of both sum and product are needed to attain the answer.

[†] The PDP 11 family of computers was a product of the Digital Equipment Corporation.
[‡] The Pentium processor is manufactured by the Intel Corporation.
[§] The SPARC processor is a product of SUN Microsystems.
[‖] The Alpha processor is manufactured by the Digital Equipment Corporation.

At the AWL corporation Mrs. Foster operates the Quick Snack Coffee Shop. The menu at her shop is limited: six kinds of muffins, eight kinds of sandwiches, and five beverages (hot coffee, hot tea, iced tea, cola, and orange juice). Ms. Dodd, an editor at AWL, sends her assistant Carl to the shop to get her lunch—either a muffin and a hot beverage or a sandwich and a cold beverage.

By the rule of product, there are $6 \times 2 = 12$ ways in which Carl can purchase a muffin and hot beverage. A second application of this rule shows that there are $8 \times 3 = 24$ possibilities for a sandwich and cold beverage. So by the rule of sum, there are $12 + 24 = 36$ ways in which Carl can purchase Ms. Dodd's lunch.

1.2 Permutations

Continuing to examine applications of the rule of product, we turn now to counting linear arrangements of objects. These arrangements are often called *permutations* when the objects are distinct. We shall develop some systematic methods for dealing with linear arrangements, starting with a typical example.

EXAMPLE 1.9

In a class of 10 students, five are to be chosen and seated in a row for a picture. How many such linear arrangements are possible?

The key word here is *arrangement*, which designates the importance of *order*. If A, B, C, . . . , I, J denote the 10 students, then BCEFI, CEFIB, and ABCFG are three such different arrangements, even though the first two involve the same five students.

To answer the question, we consider the positions and possible numbers of students we can choose from in order to fill each position. The filling of a position is a stage of our procedure.

10	×	9	×	8	×	7	×	6
1st position		2nd position		3rd position		4th position		5th position

Each of the 10 students can occupy the first position in the row. Because repetitions are not possible here, we can select only one of the nine remaining students to fill the second position. Continuing in this way, we find only six students to select from in order to fill the fifth and final position. This yields a total of 30,240 possible arrangements of five students selected from the class of 10.

Exactly the same answer is obtained if the positions are filled in the opposite order ($6 \times 7 \times 8 \times 9 \times 10$). If the order followed is 3rd, 1st, 4th, 5th, and 2nd (that is, the 3rd position is filled first, the 1st second, the 4th third, the 5th fourth, and the 2nd fifth), then the answer is $9 \times 6 \times 10 \times 8 \times 7$, still the same value, 30,240.

As in Example 1.9 the product of certain consecutive positive integers often comes into play in enumeration problems. Consequently, the following notation proves to be quite useful when we are dealing with such counting problems. It will frequently allow us to express our answers in a more convenient form.

Definition 1.1 For an integer $n \geq 0$, n *factorial* (denoted $n!$) is defined by

$$0! = 1,$$
$$n! = (n)(n-1)(n-2)\cdots(3)(2)(1), \qquad \text{for} \qquad n \geq 1.$$

One finds that $1! = 1$, $2! = 2$, $3! = 6$, $4! = 24$, and $5! = 120$. In addition, for each $n \geq 0$, $(n+1)! = (n+1)(n!)$.

We should be made aware of how fast the values of $n!$ increase. So before we proceed any further, let us try to get a somewhat better appreciation for how fast $n!$ grows. We can calculate that $10! = 3{,}628{,}800$, and it just so happens that this is exactly the number of *seconds* in six *weeks*. Consequently, $11!$ exceeds the number of seconds in one *year*, $12!$ exceeds the number in 12 years, and $13!$ surpasses the number of seconds in a *century*.

And now if we make use of the factorial notation, we find that the answer in Example 1.9 can be expressed in the following more compact form:

$$10 \times 9 \times 8 \times 7 \times 6 = 10 \times 9 \times 8 \times 7 \times 6 \times \frac{5 \times 4 \times 3 \times 2 \times 1}{5 \times 4 \times 3 \times 2 \times 1} = \frac{10!}{5!}.$$

Definition 1.2 Given a collection of n distinct objects, any (linear) arrangement of these objects is called a *permutation* of the collection.

Starting with the letters a, b, c, we find that there are six ways to arrange, or permute, all of the letters: abc, acb, bac, bca, cab, cba. If we are interested in arranging only two of the letters at a time, there are six such size-2 permutations for the collection: ab, ba, ac, ca, bc, cb.

In general, if there are n distinct objects, denoted $a_1, a_2, \ldots, a_n$, and r is an integer, with $1 \leq r \leq n$, then by the rule of product, the number of permutations of size r for the n objects is

$$\underbrace{n}_{\substack{\text{1st}\\ \text{position}}} \times \underbrace{(n-1)}_{\substack{\text{2nd}\\ \text{position}}} \times \underbrace{(n-2)}_{\substack{\text{3rd}\\ \text{position}}} \times \cdots \times \underbrace{(n-r+1)}_{\substack{r\text{th}\\ \text{position}}} =$$

$$(n)(n-1)(n-2)\cdots(n-r+1) \times \frac{(n-r)(n-r-1)\cdots(3)(2)(1)}{(n-r)(n-r-1)\cdots(3)(2)(1)} = \frac{n!}{(n-r)!}$$

We denote this number by $P(n, r)$. For $r = 0$, $P(n, 0) = 1 = n!/(n-0)!$, so $P(n, r) = n!/(n-r)!$ holds for all $0 \leq r \leq n$. A special case of this result is Example 1.9, where $n = 10$, $r = 5$, and $P(10, 5) = 30{,}240$. When permuting all of the n objects in the collection, we have $r = n$ and find that $P(n, n) = n!/0! = n!$.

Note, for example, that if $n \geq 2$, then $P(n, 2) = n!/(n-2)! = n(n-1)$. When $n > 3$ one finds that $P(n, n-3) = n!/[n-(n-3)]! = n!/3! = (n)(n-1)(n-2)\cdots(5)(4)$.

The number of permutations of size r, where $0 \leq r \leq n$, from a collection of n objects, is $P(n, r) = n!/(n-r)!$. (Remember that $P(n, r)$ counts (linear) arrangements in which the objects can*not* be repeated.) However, if repetitions are allowed, then by the rule of product there are n^r possible arrangements, with $r \geq 0$.

EXAMPLE 1.10

The number of permutations of the letters in the word COMPUTER is $8!$. If only five of the letters are used, the number of permutations (of size 5) is $P(8, 5) = 8!/(8-5)! = 8!/3! = 6720$. If repetitions of letters are allowed, the number of possible 12-letter sequences is $8^{12} \doteq 6.872 \times 10^{10}$.†

EXAMPLE 1.11

Unlike Example 1.10, the number of (linear) arrangements of the four letters in BALL is 12, not $4!$ ($= 24$). The reason is that we do not have four distinct letters to arrange. To get the 12 arrangements, we can list them as in Table 1.1(a).

Table 1.1

(a)	(b)	
A B L L	A B L_1 L_2	A B L_2 L_1
A L B L	A L_1 B L_2	A L_2 B L_1
A L L B	A L_1 L_2 B	A L_2 L_1 B
B A L L	B A L_1 L_2	B A L_2 L_1
B L A L	B L_1 A L_2	B L_2 A L_1
B L L A	B L_1 L_2 A	B L_2 L_1 A
L A B L	L_1 A B L_2	L_2 A B L_1
L A L B	L_1 A L_2 B	L_2 A L_1 B
L B A L	L_1 B A L_2	L_2 B A L_1
L B L A	L_1 B L_2 A	L_2 B L_1 A
L L A B	L_1 L_2 A B	L_2 L_1 A B
L L B A	L_1 L_2 B A	L_2 L_1 B A

If the two L's are distinguished as L_1, L_2, then we can use our previous ideas on permutations of distinct objects; with the four distinct symbols B, A, L_1, L_2, we have $4! = 24$ permutations. These are listed in Table 1.1(b). Table 1.1 reveals that for each arrangement in which the L's are indistinguishable there corresponds a *pair* of permutations with distinct L's. Consequently,

$$2 \times (\text{Number of arrangements of the letters B, A, L, L}) = (\text{Number of permutations of the symbols B, A, } L_1, L_2),$$

and the answer to the original problem of finding all the arrangements of the four letters in BALL is $4!/2 = 12$.

† The symbol "$\doteq$" is read "is approximately equal to."

EXAMPLE 1.12 Using the idea developed in Example 1.11, we now consider the arrangements of all six letters in PEPPER.

There are $3! = 6$ arrangements with the P's distinguished for each arrangement in which the P's are not distinguished. For example, $P_1EP_2P_3ER$, $P_1EP_3P_2ER$, $P_2EP_1P_3ER$, $P_2EP_3P_1ER$, $P_3EP_1P_2ER$, and $P_3EP_2P_1ER$ all correspond to PEPPER when we remove the subscripts on the P's. In addition, to the arrangement $P_1EP_2P_3ER$ there corresponds the pair of permutations $P_1E_1P_2P_3E_2R$, $P_1E_2P_2P_3E_1R$, when the E's are distinguished. Consequently,

$$\begin{aligned}&(2!)(3!)(\text{Number of arrangements of the letters in PEPPER})\\&\qquad= (\text{Number of permutations of the symbols } P_1, E_1, P_2, P_3, E_2, R),\end{aligned}$$

so the number of arrangements of the six letters in PEPPER is $6!/(2!\,3!) = 60$.

Before stating a general principle for arrangements with repeated symbols, note that in our prior two examples we solved a new type of problem by relating it to previous enumeration principles. This practice is common in mathematics in general, and often occurs in the derivations of discrete and combinatorial formulas.

> In general, if there are n objects with n_1 of a first type, n_2 of a second type, . . . , and n_r of an rth type, where $n_1 + n_2 + \cdots + n_r = n$, then there are $\dfrac{n!}{n_1!\,n_2!\cdots n_r!}$ (linear) arrangements of the given n objects. (Objects of the same type are indistinguishable.)

EXAMPLE 1.13 The MASSASAUGA is a brown and white venomous snake indigenous to North America. Arranging all of the letters in MASSASAUGA, we find that there are

$$\frac{10!}{4!\,3!\,1!\,1!\,1!} = 25{,}200$$

possible arrangements. Among these are

$$\frac{7!}{3!\,1!\,1!\,1!\,1!} = 840$$

in which all four A's are together. To get this last result, we considered all arrangements of the seven symbols AAAA (one symbol), S, S, S, M, U, G.

EXAMPLE 1.14 Determine the number of (staircase) paths in the xy-plane from (2, 1) to (7, 4), where each such path is made up of individual steps going one unit to the right (R) or one unit upward (U). The blue lines in Fig. 1.1 show two of these paths.

Beneath each path in Fig. 1.1 we have listed the individual steps. For example, in part (a) the list R, U, R, R, U, R, R, U indicates that starting at the point (2, 1), we first move one unit to the right [to (3, 1)], then one unit upward [to (3, 2)], followed by two units to the

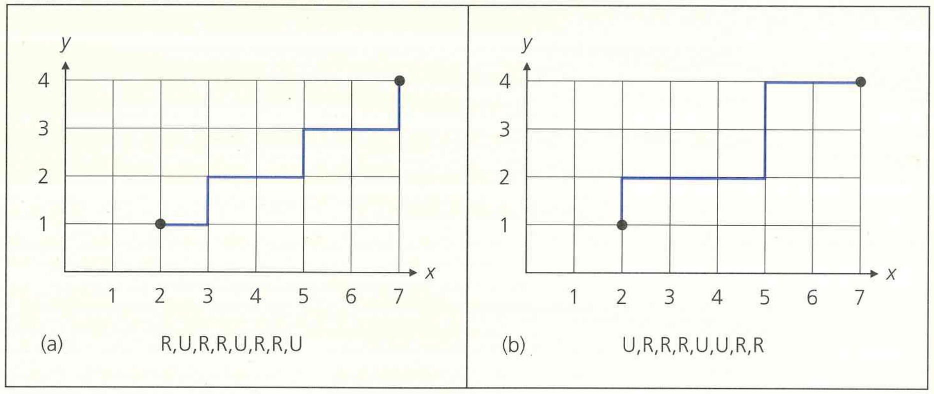

Figure 1.1

right [to (5, 2)], and so on, until we reach the point (7, 4). The path consists of five R's for moves to the right and three U's for moves upward.

The path in part (b) of the figure is also made up of five R's and three U's. In general, the overall trip from (2, 1) to (7, 4) requires $7 - 2 = 5$ horizontal moves to the right and $4 - 1 = 3$ vertical moves upward. Consequently, each path corresponds with a list of five R's and three U's, and the solution for the number of paths emerges as the number of arrangements of the five R's and three U's, which is $8!/(5!\,3!) = 56$.

EXAMPLE 1.15

We now do something a bit more abstract and prove that if n and k are positive integers with $n = 2k$, then $n!/2^k$ is an integer. Because our argument relies on counting, it is an example of a *combinatorial proof.*

Consider the n symbols $x_1, x_1, x_2, x_2, \ldots, x_k, x_k$. The number of ways in which we can arrange all of these $n = 2k$ symbols is an integer that equals

$$\frac{n!}{\underbrace{2!\,2!\cdots 2!}_{k \text{ factors of } 2!}} = \frac{n!}{2^k}.$$

Finally, we will apply what has been developed so far to a situation in which the arrangements are no longer linear.

EXAMPLE 1.16

If six people, designated as A, B, . . . , F, are seated about a round table, how many different circular arrangements are possible, if arrangements are considered the same when one can be obtained from the other by rotation? [In Fig. 1.2, arrangements (a) and (b) are considered identical, whereas (b), (c), and (d) are three distinct arrangements.]

We shall try to relate this problem to previous ones we have already encountered. Consider Figs. 1.2(a) and (b). Starting at the top of the circle and moving clockwise, we list the distinct linear arrangements ABEFCD and CDABEF, which correspond to the same circular arrangement. In addition to these two, four other linear arrangements—BEFCDA, DABEFC, EFCDAB, and FCDABE—are found to correspond to the same circular arrangement as in (a) or (b). So inasmuch as each circular arrangement corresponds to six

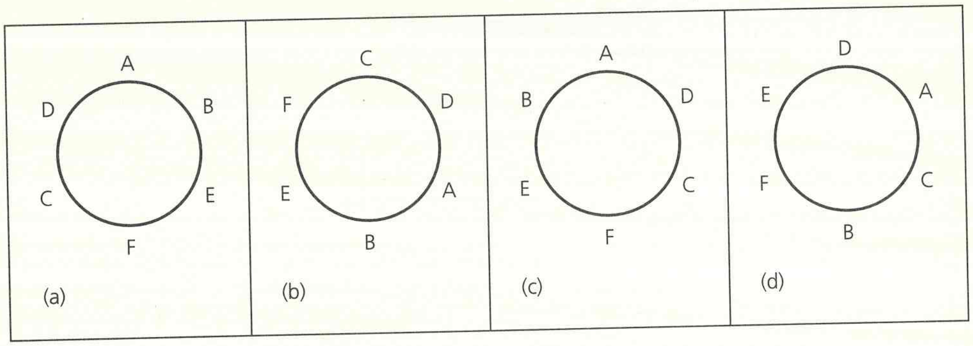

Figure 1.2

linear arrangements, we have 6 × (Number of circular arrangements of A, B, . . . , F) = (Number of linear arrangements of A, B, . . . , F) = 6!.

Consequently, there are 6!/6 = 5! = 120 arrangements of A, B, . . . , F around the circular table.

EXAMPLE 1.17

Suppose now that the six people of Example 1.16 are three married couples and that A, B, and C are the females. We want to arrange the six people around the table so that the sexes alternate. (Once again, arrangements are considered identical if one can be obtained from the other by rotation.)

Before we solve this problem, let us solve Example 1.16 by an alternative method, which will assist us in solving our present problem. If we place A at the table as shown in Fig. 1.3(a), five locations (clockwise from A) remain to be filled. Filling these locations with B, C, . . . , F is the problem of permuting B, C, . . . , F in a linear manner, and this can be done in 5! = 120 ways.

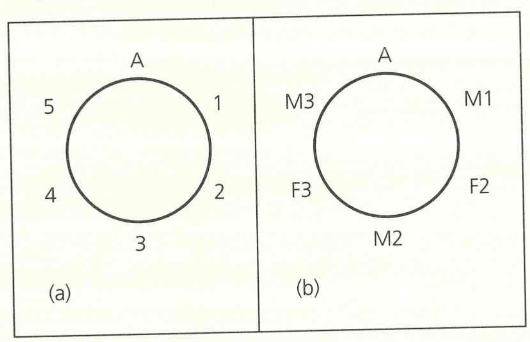

Figure 1.3

To solve the new problem of alternating the sexes, consider the method shown in Fig. 1.3(b). A (a female) is placed as before. The next position, clockwise from A, is marked M1 (Male 1) and can be filled in three ways. Continuing clockwise from A, position F2 (Female 2) can be filled in two ways. Proceeding in this manner, by the rule of product, there are 3 × 2 × 2 × 1 × 1 = 12 ways in which these six people can be arranged with no two men or women seated next to each other.

EXERCISES 1.1 AND 1.2

1. During a local campaign, eight Republican and five Democratic candidates are nominated for president of the school board.

a) If the president is to be one of these candidates, how many possibilities are there for the eventual winner?

b) How many possibilities exist for a pair of candidates (one from each party) to oppose each other for the eventual election?

c) Which counting principle is used in part (a)? in part (b)?

2. Answer part (c) of Example 1.6.

3. Buick automobiles come in four models, 12 colors, three engine sizes, and two transmission types. (a) How many distinct Buicks can be manufactured? (b) If one of the available colors is blue, how many different blue Buicks can be manufactured?

4. a) The board of directors of a pharmaceutical corporation has 10 members. An upcoming stockholders' meeting is scheduled to approve a new slate of company officers (chosen from the 10 board members). How many different slates consisting of a president, vice president, secretary, and treasurer can the board present to the stockholders for their approval?

b) Three members of the board of directors [from part (a)] are physicians. How many slates from part (a) have (i) a physician nominated for the presidency? (ii) exactly one physician appearing on the slate? (iii) at least one physician appearing on the slate?

5. While on a Saturday shopping spree Jennifer and Tiffany witnessed two men driving away from the front of a jewelry shop, just before a burglar alarm started to sound. Although everything happened rather quickly, when the two young ladies were questioned they were able to give the police the following information about the license plate (which consisted of two letters followed by four digits) on the getaway car. Tiffany was sure that the second letter on the plate was either an O or a Q and the last digit was either a 3 or an 8. Jennifer told the investigator that the first letter on the plate was either a C or a G and that the first digit was definitely a 7. How many different license plates will the police have to check out?

6. To raise money for a new municipal pool, the chamber of commerce in a certain city sponsors a race. Each participant pays a $5 entrance fee and has a chance to win one of the different-sized trophies that are to be awarded to the first eight runners who finish.

a) If 30 people enter the race, in how many ways will it be possible to award the trophies?

b) If Roberta and Candice are two participants in the race, in how many ways can the trophies be awarded with these two runners among the top three?

7. A certain "Burger Joint" advertises that a customer can have his or her hamburger with or without any or all of the following: catsup, mustard, mayonnaise, lettuce, tomato, onion, pickle, cheese, or mushrooms. How many different kinds of hamburger orders are possible?

8. Matthew works as a computer operator at a small university. One evening he finds that 12 computer programs have been submitted earlier that day for batch processing. In how many ways can Matthew order the processing of these programs if (a) there are no restrictions? (b) he considers four of the programs higher in priority than the other eight and wants to process those four first? (c) he first separates the programs into four of top priority, five of lesser priority, and three of least priority, and he wishes to process the 12 programs in such a way that the top-priority programs are processed first and the three programs of least priority are processed last?

9. Patter's Pastry Parlor offers eight different kinds of pastry and six different kinds of muffins. In addition to bakery items one can purchase small, medium, or large containers of the following beverages: coffee (black, with cream, with sugar, or with cream and sugar), tea (plain, with cream, with sugar, with cream and sugar, with lemon, or with lemon and sugar), hot cocoa, and orange juice. When Carol comes to Patter's, in how many ways can she order

a) one bakery item and one medium-sized beverage for herself?

b) one bakery item and one container of coffee for herself and one muffin and one container of tea for her boss, Ms. Didio?

c) one piece of pastry and one container of tea for herself, one muffin and a container of orange juice for Ms. Didio, and one bakery item and one container of coffee for each of her two assistants, Mr. Talbot and Mrs. Gillis?

10. Pamela has 15 different books. In how many ways can she place her books on two shelves so that there is at least one book on each shelf? (Consider the books in each arrangement to be stacked one next to the other, with the first book on each shelf at the left of the shelf.)

11. Three small towns, designated by A, B, and C, are interconnected by a system of two-way roads, as shown in Fig. 1.4.

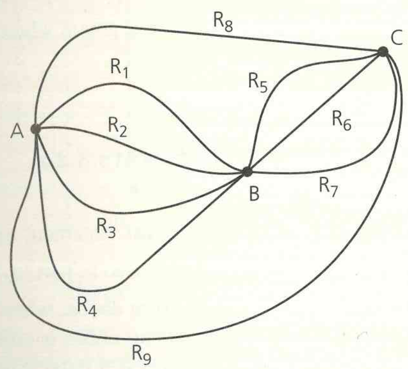

Figure 1.4

a) In how many ways can Linda travel from town A to town C?

b) How many different round trips can Linda travel from town A to town C and then back to town A?

c) How many of the round trips in part (b) are such that the return trip (from town C to town A) is at least partially different from the route Linda takes from town A to town C? (For example, if Linda travels from town A to town C along roads R_1 and R_6, then on her return she might take roads R_6 and R_3, or roads R_7 and R_2, or road R_9, among other possibilities, but she does *not* travel on roads R_6 *and* R_1.)

12. List all the permutations for the letters a, c, t.

13. a) How many permutations are there for the eight letters a, c, f, g, i, t, w, x?

b) Consider the permutations in part (a). How many start with the letter t? How many start with the letter t and end with the letter c?

14. Evaluate each of the following.

a) $P(7, 2)$ **b)** $P(8, 4)$ **c)** $P(10, 7)$ **d)** $P(12, 3)$

15. In how many ways can the symbols a, b, c, d, e, e, e, e, e be arranged so that no e is adjacent to another e?

16. An alphabet of 40 symbols is used for transmitting messages in a communication system. How many distinct messages (lists of symbols) of 25 symbols can the transmitter generate if symbols can be repeated in the message? How many if 10 of the 40 symbols can appear only as the first and/or last symbols of the message, the other 30 symbols can appear anywhere, and repetitions of all symbols are allowed?

17. In a certain implementation of the programming language Pascal, an identifier consists of a single letter or a letter followed by up to seven symbols, which may be letters or digits. (We assume that the computer does not distinguish between capital and lowercase letters; there are 26 letters and 10 digits). Certain keywords, however, are reserved for commands; consequently, these keywords may not be used as identifiers. If this implementation has 36 reserved words, how many distinct identifiers are possible in this version of Pascal?

18. Morgan is considering the purchase of a low-end computer system. After some careful investigating, she finds that there are seven basic systems (each consisting of a monitor, CPU, keyboard, and mouse) that meet her requirements. Furthermore, she also plans to buy one of four modems, one of three CD ROM drives, and one of six printers. (Here each peripheral device of a given type, such as the modem, is compatible with all seven basic systems.) In how many ways can Morgan configure her low-end computer system?

19. A computer science professor has seven different programming books on a bookshelf. Three of the books deal with C++, the other four with Java. In how many ways can the professor arrange these books on the shelf (a) if there are no restrictions? (b) if the languages should alternate? (c) if all the C++ books must be next to each other? (d) if all the C++ books must be next to each other and all the Java books must be next to each other?

20. a) In how many ways can the letters in VISITING be arranged?

b) For the arrangements of part (a), how many have all three I's together?

21. a) How many arrangements are there of all the letters in SOCIOLOGICAL?

b) In how many of the arrangements in part (a) are A and G adjacent?

c) In how many of the arrangements in part (a) are all the vowels adjacent?

22. How many positive integers n can we form using the digits 3, 4, 4, 5, 5, 6, 7 if we want n to exceed 5,000,000?

23. Twelve clay targets (identical in shape) are arranged in four hanging columns, as shown in Fig. 1.5. There are four red targets in the first column, three white ones in the second column, two green targets in the third column, and three blue ones in the fourth column. To join her college drill team, Deborah must break all 12 of these targets (using her pistol and only 12 bullets) and in so doing must always break the existing target at the bottom of a column. Under these conditions, in how many different orders can Deborah shoot down (and break) the 12 targets?

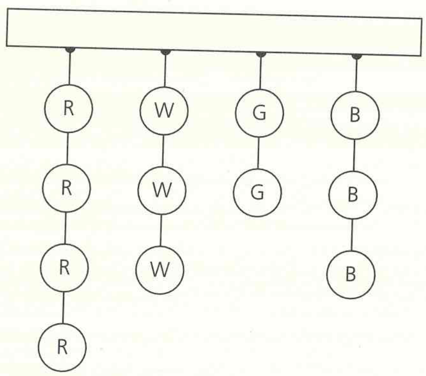

Figure 1.5

24. Show that for all integers $n, r \geq 0$, if $n + 1 > r$, then

$$P(n+1, r) = \left(\frac{n+1}{n+1-r}\right) P(n, r).$$

25. Find the value(s) of n in each of the following: (a) $P(n, 2) = 90$, (b) $P(n, 3) = 3P(n, 2)$, and (c) $2P(n, 2) + 50 = P(2n, 2)$.

26. How many different paths in the xy-plane are there from (0, 0) to (7, 7) if a path proceeds one step at a time by going either one space to the right (R) or one space upward (U)? How many such paths are there from (2, 7) to (9, 14)? Can any general statement be made that incorporates these two results?

27. **a)** How many distinct paths are there from $(-1, 2, 0)$ to $(1, 3, 7)$ in Euclidean three-space if each move is one of the following types?

(H): $(x, y, z) \rightarrow (x+1, y, z)$; (V): $(x, y, z) \rightarrow (x, y+1, z)$; (A): $(x, y, z) \rightarrow (x, y, z+1)$

b) How many such paths are there from (1, 0, 5) to (8, 1, 7)?

c) Generalize the results in parts (a) and (b).

28. **a)** Determine the value of the integer variable *counter* after execution of the following program segment. (Here i, j, and k are integer variables.)

```
counter := 0
for i := 1 to 12 do
    counter := counter + 1
for j := 5 to 10 do
    counter := counter + 2
for k := 15 downto 8 do
    counter := counter + 3
```

b) Which counting principle is at play in part (a)?

29. Consider the following program segment where i, j, and k are integer variables.

```
for i := 1 to 12 do
  for j := 5 to 10 do
    for k := 15 downto 8 do
      print (i - j)*k
```

a) How many times is the **print** statement executed?

b) Which counting principle is used in part (a)?

30. A sequence of letters of the form abcba, where the expression is unchanged upon reversing order, is an example of a *palindrome* (of five letters). (a) If a letter may appear more than twice, how many palindromes of five letters are there? of six letters? (b) Repeat part (a) under the condition that no letter appears more than twice.

31. Determine the number of six-digit integers (no leading zeros) in which (a) no digit may be repeated; (b) digits may be repeated. Answer parts (a) and (b) with the extra condition that the six-digit integer is (i) even; (ii) divisible by 5; (iii) divisible by 4.

32. **a)** Provide a combinatorial argument to show that if n and k are positive integers with $n = 3k$, then $n!/(3!)^k$ is an integer.

b) Generalize the result of part (a).

33. **a)** In how many possible ways could a student answer a 10-question true-false test?

b) In how many ways can the student answer the test in part (a) if it is possible to leave a question unanswered in order to avoid an extra penalty for a wrong answer?

34. How many distinct four-digit integers can one make from the digits 1, 3, 3, 7, 7, and 8?

35. **a)** In how many ways can seven people be arranged about a circular table?

b) If two of the people insist on sitting next to each other, how many arrangements are possible?

36. **a)** In how many ways can eight people, denoted A, B, . . . , H, be seated about the square table shown in Fig. 1.6, where Figs. 1.6(a) and 1.6(b) are considered the same but are distinct from Fig. 1.6(c)?

b) If two of the eight people, say A and B, do not get along well, how many different seatings are possible with A and B not sitting next to each other?

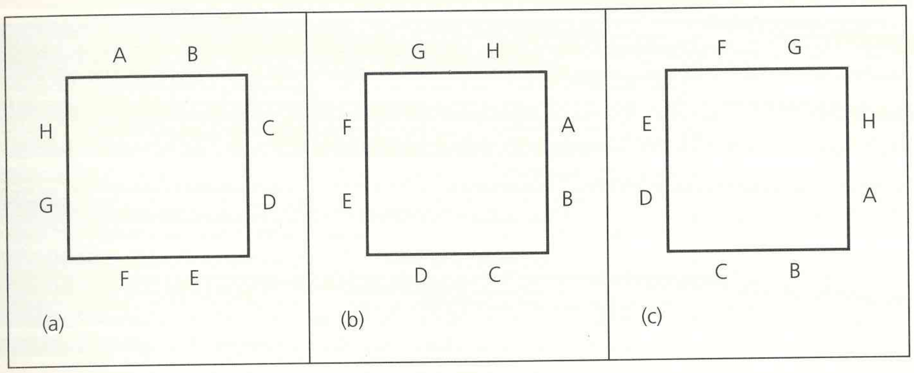

Figure 1.6

37. Write a computer program (or develop an algorithm) to determine whether there is a three-digit integer abc $(= 100a + 10b + c)$ where $abc = a! + b! + c!$.

1.3 Combinations: The Binomial Theorem

The standard deck of playing cards consists of 52 cards comprising four suits: clubs, diamonds, hearts, and spades. Each suit has 13 cards: ace, 2, 3, . . . 9, 10, jack, queen, king. If we are asked to draw three cards from a standard deck, in succession and without replacement, then by the rule of product there are

$$52 \times 51 \times 50 = \frac{52!}{49!} = P(52, 3)$$

possibilities, one of which is AH (ace of hearts), 9C (nine of clubs), KD (king of diamonds). If instead we simply select three cards at one time from the deck so that the order of selection of the cards is no longer important, then the six permutations AH–9C–KD, AH–KD–9C, 9C–AH–KD, 9C–KD–AH, KD–9C–AH, and KD–AH–9C all correspond to just one (unordered) selection. Consequently, each selection, or combination, of three cards, *with no reference to order*, corresponds to 3! permutations of three cards. In equation form this translates into

$$\begin{aligned} &(3!) \times (\text{Number of selections of size 3 from a deck of 52}) \\ &\qquad = \text{Number of permutations of size 3 for the 52 cards} \\ &\qquad = P(52, 3) = \frac{52!}{49!}. \end{aligned}$$

Consequently, three cards can be drawn, without replacement, from a standard deck in $52!/(3!\,49!) = 22{,}100$ ways.

In general, if we start with n distinct objects, each *selection*, or *combination*, of r of these objects, with no reference to order, corresponds to $r!$ permutations of size r from the n objects. Thus the number of combinations of size r from a collection of size n, denoted $C(n, r)$, where $0 \leq r \leq n$, satisfies $(r!) \times C(n, r) = P(n, r)$ and

$$C(n, r) = \frac{P(n, r)}{r!} = \frac{n!}{r!(n-r)!}, \qquad 0 \leq r \leq n.$$

In addition to $C(n, r)$ the symbol $\binom{n}{r}$ is also frequently used. Both $C(n, r)$ and $\binom{n}{r}$ are sometimes read as "n choose r." Note that $C(n, 0) = 1$, for all $n \geq 0$.

A word to the wise! When dealing with any counting problem, we should ask ourselves about the importance of order in the problem. When order is relevant, we think in terms of permutations and arrangements and the rule of product. When order is not relevant, combinations could play a key role in solving the problem.

EXAMPLE 1.18

A hostess is having a dinner party for some members of her charity committee. Because of the size of her home, she can invite only 11 of the 20 committee members. Order is not important, so she can invite "the lucky 11" in $C(20, 11) = \binom{20}{11} = 20!/(11!\,9!) = 167{,}960$ ways. However, once the 11 arrive, how she arranges them around her rectangular dining table is an arrangement problem. Unfortunately, no part of the theory of combinations and permutations can help our hostess deal with "the offended nine" who were not invited.

EXAMPLE 1.19

a) A student taking a history examination is directed to answer any seven of 10 essay questions. There is no concern about order here, so the student can answer the examination in

$$\binom{10}{7} = \frac{10!}{7!\,3!} = \frac{10 \times 9 \times 8}{3 \times 2 \times 1} = 120 \text{ ways.}$$

b) If the student must answer three questions from the first five and four questions from the last five, three questions can be selected from the first five in $\binom{5}{3} = 10$ ways, and the other four questions can be selected in $\binom{5}{4} = 5$ ways. Hence, by the rule of product, the student can complete the examination in $\binom{5}{3}\binom{5}{4} = 10 \times 5 = 50$ ways.

c) Finally, should the directions on this examination indicate that the student must answer seven of the 10 questions where at least three are selected from the first five, then there are three cases to consider:

i) The student answers three of the first five questions and four of the last five: By the rule of product this can happen in $\binom{5}{3}\binom{5}{4} = 10 \times 5 = 50$ ways, as in part (b).

ii) Four of the first five questions and three of the last five questions are selected by the student: This can come about in $\binom{5}{4}\binom{5}{3} = 5 \times 10 = 50$ ways—again by the rule of product.

iii) The student decides to answer all five of the first five questions and two of the last five: The rule of product tells us that this last case can occur in $\binom{5}{5}\binom{5}{2} = 1 \times 10 = 10$ ways.

Combining the results for cases (i), (ii), and (iii), by the rule of sum we find that the student can make $\binom{5}{3}\binom{5}{4} + \binom{5}{4}\binom{5}{3} + \binom{5}{5}\binom{5}{2} = 50 + 50 + 10 = 110$ selections of seven (out of 10) questions where each selection includes at least three of the first five questions.

EXAMPLE 1.20

a) At Rydell High School, the gym teacher must select nine girls from the junior and senior classes for a volleyball team. If there are 28 juniors and 25 seniors, she can make the selection in $\binom{53}{9} = 4{,}431{,}613{,}550$ ways.

b) If two juniors and one senior are the best spikers and must be on the team, then the rest of the team can be chosen in $\binom{50}{6} = 15{,}890{,}700$ ways.

c) For a certain tournament the team must comprise four juniors and five seniors. The teacher can select the four juniors in $\binom{28}{4}$ ways. For each of these selections she has $\binom{25}{5}$ ways to choose the five seniors. Consequently, by the rule of product, she can select her team in $\binom{28}{4}\binom{25}{5} = 1{,}087{,}836{,}750$ ways for this particular tournament.

Some problems can be treated from the viewpoint of either arrangements or combinations, depending on how one analyzes the situation. The following example demonstrates this.

EXAMPLE 1.21

The gym teacher of Example 1.20 must make up four volleyball teams of nine girls each from the 36 freshman girls in her P.E. class. In how many ways can she select these four teams? Call the teams A, B, C, and D.

a) To form team A, she can select any nine girls from the 36 enrolled in $\binom{36}{9}$ ways. For team B the selection process yields $\binom{27}{9}$ possibilities. This leaves $\binom{18}{9}$ and $\binom{9}{9}$ possible ways to select teams C and D, respectively. So by the rule of product, the four teams can be chosen in

$$\binom{36}{9}\binom{27}{9}\binom{18}{9}\binom{9}{9} = \left(\frac{36!}{9!\,27!}\right)\left(\frac{27!}{9!\,18!}\right)\left(\frac{18!}{9!\,9!}\right)\left(\frac{9!}{9!\,0!}\right)$$

$$= \frac{36!}{9!\,9!\,9!\,9!} \doteq 2.145 \times 10^{19} \text{ ways.}$$

b) For an alternative solution, consider the 36 students lined up as follows:

1st student	2nd student	3rd student	. . .	35th student	36th student

To select the four teams, we must distribute nine A's, nine B's, nine C's, and nine D's in the 36 spaces. The number of ways in which this can be done is the number of arrangements of 36 letters comprising nine each of A, B, C, and D. This is now the familiar problem of arrangements of nondistinct objects, and the answer is

$$\frac{36!}{9!\,9!\,9!\,9!}, \quad \text{as in part (a).}$$

Our next example points out how some problems require the concepts of both arrangements and combinations for their solutions.

EXAMPLE 1.22 The number of arrangements of the letters in TALLAHASSEE is

$$\frac{11!}{3!\,2!\,2!\,2!\,1!\,1!} = 831{,}600.$$

How many of these arrangements have no adjacent A's?

When we disregard the A's, there are

$$\frac{8!}{2!\,2!\,2!\,1!\,1!} = 5040$$

ways to arrange the remaining letters. One of these 5040 ways is shown in the following figure, where the arrows indicate nine possible locations for the three A's.

$$\uparrow\ \mathrm{E}\ \uparrow\ \mathrm{E}\ \uparrow\ \mathrm{S}\ \uparrow\ \mathrm{T}\ \uparrow\ \mathrm{L}\ \uparrow\ \mathrm{L}\ \uparrow\ \mathrm{S}\ \uparrow\ \mathrm{H}\ \uparrow$$

Three of these locations can be selected in $\binom{9}{3} = 84$ ways, and because this is also possible for all the other 5039 arrangements of E, E, S, T, L, L, S, H, by the rule of product there are $5040 \times 84 = 423{,}360$ arrangements of the letters in TALLAHASSEE with no consecutive A's.

Before proceeding we need to introduce a concise way of writing the sum of a list of $n + 1$ terms like $a_m, a_{m+1}, a_{m+2}, \ldots, a_{m+n}$, where m and n are integers and $n \geq 0$. This notation is called the *Sigma notation* because it involves the capital Greek letter Σ; we use it to represent a summation by writing

$$a_m + a_{m+1} + a_{m+2} + \cdots + a_{m+n} = \sum_{i=m}^{m+n} a_i.$$

Here the letter i is called the *index* of the summation, and this index accounts for all integers starting with the *lower limit* m and continuing on up to (and including) the *upper limit* $m + n$.

We may use this notation as follows.

1) $\sum_{i=3}^{7} a_i = a_3 + a_4 + a_5 + a_6 + a_7 = \sum_{j=3}^{7} a_j$, for there is nothing special about the letter i.

2) $\sum_{i=1}^{4} i^2 = 1^2 + 2^2 + 3^2 + 4^2 = 30 = \sum_{k=0}^{4} k^2$, because $0^2 = 0$.

3) $\sum_{i=11}^{100} i^3 = 11^3 + 12^3 + 13^3 + \cdots 100^3 = \sum_{j=12}^{101} (j-1)^3 = \sum_{k=10}^{99} (k+1)^3$.

4) $\sum_{i=7}^{10} 2i = 2(7) + 2(8) + 2(9) + 2(10) = 68 = 2(34) = 2(7 + 8 + 9 + 10) = 2\sum_{i=7}^{10} i$.

5) $\sum_{i=3}^{3} a_i = a_3 = \sum_{i=4}^{4} a_{i-1} = \sum_{i=2}^{2} a_{i+1}$.

6) $\sum_{i=1}^{5} a = a + a + a + a + a = 5a$.

Furthermore, using this summation notation, we see that one can express the answer to part (c) of Example 1.19 as

$$\binom{5}{3}\binom{5}{4} + \binom{5}{4}\binom{5}{3} + \binom{5}{5}\binom{5}{2} = \sum_{i=3}^{5}\binom{5}{i}\binom{5}{7-i} = \sum_{j=2}^{4}\binom{5}{7-j}\binom{5}{j}.$$

We shall find use for this new notation in the following example and in many other places throughout the remainder of this book.

EXAMPLE 1.23

In the studies of algebraic coding theory and the theory of computer languages, we consider certain arrangements, called *strings*, made up from a prescribed *alphabet* of symbols. If the prescribed alphabet consists of the symbols 0, 1, and 2, for example, then 01, 11, 21, 12, and 20 are five of the nine strings of *length* 2. Among the 27 strings of length 3 are 000, 012, 202, and 110.

In general, if n is any positive integer, then by the rule of product there are 3^n strings of length n for the alphabet 0, 1, and 2. If $x = x_1x_2x_3 \ldots x_n$ is one of these strings, we define the *weight* of x, denoted wt(x), by wt(x) $= x_1 + x_2 + x_3 + \cdots + x_n$. For example, wt(12) $= 3$ and wt(22) $= 4$ for the case where $n = 2$; wt(101) $= 2$, wt(210) $= 3$, and wt(222) $= 6$ for $n = 3$.

Among the 3^{10} strings of length 10, we wish to determine how many have even weight. Such a string has even weight precisely when the number of 1's in the string is even.

There are six different cases to consider. If the string x contains no 1's, then each of the 10 locations in x can be filled with either 0 or 2, and by the rule of product there are 2^{10} such strings. When the string contains two 1's, the locations for these two 1's can be selected in $\binom{10}{2}$ ways. Once these two locations have been specified, there are 2^8 ways to place either 0 or 2 in the other eight positions. Hence there are $\binom{10}{2}2^8$ strings of even weight that contain two 1's. The numbers of strings for the other four cases are given in Table 1.2.

Table 1.2

Number of 1's	Number of Strings	Number of 1's	Number of Strings
4	$\binom{10}{4}2^6$	8	$\binom{10}{8}2^2$
6	$\binom{10}{6}2^4$	10	$\binom{10}{10}$

Consequently, by the rule of sum, the number of strings of length 10 that have even weight is $2^{10} + \binom{10}{2}2^8 + \binom{10}{4}2^6 + \binom{10}{6}2^4 + \binom{10}{8}2^2 + \binom{10}{10} = \sum_{n=0}^{5}\binom{10}{2n}2^{10-2n}$.

Often we must be careful of *overcounting*—a situation that seems to arise in what may appear to be rather easy enumeration problems. The next example demonstrates how overcounting may come about.

EXAMPLE 1.24

a) Suppose that Ellen draws five cards from a standard deck of 52 cards. In how many ways can her selection result in a hand with no clubs? Here we are interested in counting all five-card selections such as

i) ace of hearts, three of spades, four of spades, six of diamonds, and the jack of diamonds.

ii) five of spades, seven of spades, ten of spades, seven of diamonds, and the king of diamonds.

iii) two of diamonds, three of diamonds, six of diamonds, ten of diamonds, and the jack of diamonds.

If we examine this more closely we see that Ellen is restricted to selecting her five cards from the 39 cards in the deck that are not clubs. Consequently, she can make her selection in $\binom{39}{5}$ ways.

b) Now suppose we want to count the number of Ellen's five-card selections that contain at least one club. These are precisely the selections that were *not* counted in part (a). And since there are $\binom{52}{5}$ possible five-card hands in total, we find that

$$\binom{52}{5} - \binom{39}{5} = 2{,}598{,}960 - 575{,}757 = 2{,}023{,}203$$

of all five-card hands contain at least one club.

c) Can we obtain the result in part (b) in another way? For example, since Ellen wants to have at least one club in the five-card hand, let her first select a club. This she can do in $\binom{13}{1}$ ways. And now she doesn't care what comes up for the other four cards. So after she eliminates the one club chosen from her standard deck, she can then select the other four cards in $\binom{51}{4}$ ways. Therefore, by the rule of product, we count the number of selections here as

$$\binom{13}{1}\binom{51}{4} = 13 \times 249{,}900 = 3{,}248{,}700.$$

Something here is definitely *wrong*! This answer is larger than that in part (b) by more than one million hands. Did we make a mistake in part (b)? Or is something wrong with our present reasoning?

For example, suppose that Ellen first selects

the three of clubs

and then selects

the five of clubs,
king of clubs,
seven of hearts, and
jack of spades.

If, however, she first selects

the five of clubs

and then selects

the three of clubs,
king of clubs,
seven of hearts, and
jack of spades,

is her selection here really different from the prior selection we mentioned? Unfortunately, no! And the case where she first selects

the king of clubs

and then follows this by selecting

the three of clubs,
five of clubs,
seven of hearts, and
jack of spades

is not different from the other two selections mentioned earlier.

Consequently, this approach is *wrong* because we are overcounting—by considering like selections as if they were distinct.

d) But is there any other way to arrive at the answer in part (b)? Yes! Since the five-card hands must each contain at least one club, there are five cases to consider. These are given in Table 1.3. From the results in Table 1.3 we see, for example, that there are $\binom{13}{2}\binom{39}{3}$ five-card hands that contain exactly two clubs. If we are interested in having exactly three clubs in the hand, then the results in the table indicate that there are $\binom{13}{3}\binom{39}{2}$ such hands.

Table 1.3

Number of Clubs	Number of Ways to Select This Number of Clubs	Number of Cards That Are Not Clubs	Number of Ways to Select This Number of Nonclubs
1	$\binom{13}{1}$	4	$\binom{39}{4}$
2	$\binom{13}{2}$	3	$\binom{39}{3}$
3	$\binom{13}{3}$	2	$\binom{39}{2}$
4	$\binom{13}{4}$	1	$\binom{39}{1}$
5	$\binom{13}{5}$	0	$\binom{39}{0}$

Since no two of the cases in Table 1.3 have any five-card hand in common, the number of hands that Ellen can select with at least one club is

$$\binom{13}{1}\binom{39}{4} + \binom{13}{2}\binom{39}{3} + \binom{13}{3}\binom{39}{2} + \binom{13}{4}\binom{39}{1} + \binom{13}{5}\binom{39}{0} = \sum_{i=1}^{5}\binom{13}{i}\binom{39}{5-i}$$
$$= (13)(82{,}251) + (78)(9139) + (286)(741) + (715)(39) + (1287)(1)$$
$$= 2{,}023{,}203.$$

We shall close this section with three results related to the concept of combinations.

First we note that for integers n, r, with $n \geq r \geq 0$, $\binom{n}{r} = \binom{n}{n-r}$. This can be established algebraically from the formula for $\binom{n}{r}$, but we prefer to observe that when dealing with a selection of size r from a collection of n distinct objects, the selection process leaves behind $n - r$ objects. Consequently, $\binom{n}{r} = \binom{n}{n-r}$ affirms the existence of a correspondence between the selections of size r (objects chosen) and the selections of size $n - r$ (objects left behind). An example of this correspondence is shown in Table 1.4, where $n = 5$, $r = 2$, and the distinct objects are 1, 2, 3, 4, and 5. This type of correspondence will be more formally defined in Chapter 5 and used in other counting situations.

Table 1.4

Selections of Size $r = 2$ (Objects Chosen)		Selections of Size $n - r = 3$ (Objects Left Behind)	
1. 1, 2	6. 2, 4	1. 3, 4, 5	6. 1, 3, 5
2. 1, 3	7. 2, 5	2. 2, 4, 5	7. 1, 3, 4
3. 1, 4	8. 3, 4	3. 2, 3, 5	8. 1, 2, 5
4. 1, 5	9. 3, 5	4. 2, 3, 4	9. 1, 2, 4
5. 2, 3	10. 4, 5	5. 1, 4, 5	10. 1, 2, 3

Our second result is a theorem from our past experience in algebra.

THEOREM 1.1 *The Binomial Theorem.* If x and y are variables and n is a positive integer, then

$$(x+y)^n = \binom{n}{0}x^0y^n + \binom{n}{1}x^1y^{n-1} + \binom{n}{2}x^2y^{n-2} + \cdots$$
$$+ \binom{n}{n-1}x^{n-1}y^1 + \binom{n}{n}x^ny^0 = \sum_{k=0}^{n}\binom{n}{k}x^ky^{n-k}.$$

Before considering the general proof, we examine a special case. In $n = 4$, the coefficient of x^2y^2 in the expansion of the product

$$\underset{\text{1st factor}}{(x+y)}\ \underset{\text{2nd factor}}{(x+y)}\ \underset{\text{3rd factor}}{(x+y)}\ \underset{\text{4th factor}}{(x+y)}$$

is the number of ways in which we can select two x's from the four x's, one of which is available in each factor. (Although the x's are the same in appearance, we distinguish them as the x in the first factor, the x in the second factor, . . . , and the x in the fourth factor.

Also, we note that when we select two x's, we use two factors, leaving us with two other factors from which we can select the two y's that are needed.) For example, among the possibilities, we can select (1) x from the first two factors and y from the last two or (2) x from the first and third factors and y from the second and fourth. Table 1.5 summarizes the six possible selections.

Table 1.5

Factors Selected for x		Factors Selected for y	
(1)	1, 2	(1)	3, 4
(2)	1, 3	(2)	2, 4
(3)	1, 4	(3)	2, 3
(4)	2, 3	(4)	1, 4
(5)	2, 4	(5)	1, 3
(6)	3, 4	(6)	1, 2

Consequently, the coefficient of x^2y^2 in the expansion of $(x+y)^4$ is $\binom{4}{2} = 6$, the number of ways to select two distinct objects from a collection of four distinct objects.

Now we turn to the proof of the general case.

Proof: In the expansion of the product

$$\underset{\substack{\text{1st}\\ \text{factor}}}{(x+y)}\underset{\substack{\text{2nd}\\ \text{factor}}}{(x+y)}\underset{\substack{\text{3rd}\\ \text{factor}}}{(x+y)}\cdots\underset{\substack{n\text{th}\\ \text{factor}}}{(x+y)}$$

the coefficient of $x^k y^{n-k}$, where $0 \le k \le n$, is the number of different ways in which we can select k x's (and consequently $(n-k)$ y's) from the n available factors. (One way, for example, is to choose x from the first k factors and y from the last $n-k$ factors.) The total number of such selections of size k from a collection of size n is $C(n, k) = \binom{n}{k}$, and from this the binomial theorem follows.

In view of this theorem, $\binom{n}{k}$ is often referred to as a *binomial coefficient.* Notice that it is also possible to express the result of Theorem 1.1 as

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{n-k} x^k y^{n-k}.$$

EXAMPLE 1.25

a) From the binomial theorem it follows that the coefficient of x^5y^2 in the expansion of $(x+y)^7$ is $\binom{7}{5} = \binom{7}{2} = 21$.

b) To obtain the coefficient of a^5b^2 in the expansion of $(2a-3b)^7$, replace $2a$ by x and $-3b$ by y. From the binomial theorem the coefficient of x^5y^2 in $(x+y)^7$ is $\binom{7}{5}$, and $\binom{7}{5}x^5y^2 = \binom{7}{5}(2a)^5(-3b)^2 = \binom{7}{5}(2)^5(-3)^2a^5b^2 = 6048a^5b^2$.

COROLLARY 1.1 For each integer $n > 0$,

a) $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$, and

b) $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n\binom{n}{n} = 0$.

Proof: Part (a) follows from the binomial theorem when we set $x = y = 1$. When $x = -1$ and $y = 1$, part (b) results.

Our third and final result generalizes the binomial theorem and is called the *multinomial theorem.*

THEOREM 1.2 For positive integers n, t, the coefficient of $x_1^{n_1}x_2^{n_2}x_3^{n_3}\cdots x_t^{n_t}$ in the expansion of $(x_1 + x_2 + x_3 + \cdots + x_t)^n$ is

$$\frac{n!}{n_1!\,n_2!\,n_3!\cdots n_t!},$$

where each n_i is an integer with $0 \le n_i \le n$, for all $1 \le i \le t$, and $n_1 + n_2 + n_3 + \cdots + n_t = n$.

Proof: As in the proof of the binomial theorem, the coefficient of $x_1^{n_1}x_2^{n_2}x_3^{n_3}\cdots x_t^{n_t}$ is the number of ways we can select x_1 from n_1 of the n factors, x_2 from n_2 of the $n - n_1$ remaining factors, x_3 from n_3 of the $n - n_1 - n_2$ now remaining factors, . . . , and x_t from n_t of the last $n - n_1 - n_2 - n_3 - \cdots - n_{t-1} = n_t$ remaining factors. This can be carried out, as in part (a) of Example 1.21, in

$$\binom{n}{n_1}\binom{n - n_1}{n_2}\binom{n - n_1 - n_2}{n_3}\cdots\binom{n - n_1 - n_2 - n_3 - \cdots - n_{t-1}}{n_t}$$

ways. We leave to the reader the details of showing that this is equal to

$$\frac{n!}{n_1!\,n_2!\,n_3!\cdots n_t!},$$

which is also written as

$$\binom{n}{n_1, n_2, n_3, \ldots, n_t}$$

and is called a *multinomial coefficient.*

EXAMPLE 1.26

a) In the expansion of $(x + y + z)^7$ it follows from the multinomial theorem that the coefficient of $x^2y^2z^3$ is $\binom{7}{2,2,3} = \frac{7!}{2!\,2!\,3!} = 210$, while the coefficient of xyz^5 is $\binom{7}{1,1,5} = 42$ and that of x^3z^4 is $\binom{7}{3,0,4} = \frac{7!}{3!\,0!\,4!} = 35$.

b) Suppose we need to know the coefficient of $a^2b^3c^2d^5$ in the expansion of $(a + 2b - 3c + 2d + 5)^{16}$. If we replace a by v, $2b$ by w, $-3c$ by x, $2d$ by y, and 5 by z, then we can apply the multinomial theorem to $(v + w + x + y + z)^{16}$ and determine the coefficient of $v^2w^3x^2y^5z^4$ as $\binom{16}{2,3,2,5,4} = 302{,}702{,}400$. But $\binom{16}{2,3,2,5,4}(a)^2(2b)^3(-3c)^2(2d)^5(5)^4 = \binom{16}{2,3,2,5,4}(1)^2(2)^3(-3)^2(2)^5(5)^4(a^2b^3c^2d^5)$ $= 435{,}891{,}456{,}000{,}000\, a^2b^3c^2d^5$.

EXERCISES 1.3

1. Calculate $\binom{6}{2}$ and check your answer by listing all the selections of size 2 that can be made from the letters a, b, c, d, e, and f.

2. Facing a four-hour bus trip back to college, Diane decides to take along five magazines from the 12 that her sister Ann Marie has recently acquired. In how many ways can Diane make her selection?

3. Evaluate each of the following.

a) $C(10, 4)$ **b)** $\binom{12}{7}$ **c)** $C(14, 12)$ **d)** $\binom{15}{10}$

4. In the Braille system a symbol, such as a lowercase letter, punctuation mark, suffix, and so on, is given by raising at least one of the dots in the six-dot arrangement shown in part (a) of Fig. 1.7. (The six Braille positions are labeled in this part of the figure.) For example, in part (b) of the figure the dots in positions 1 and 4 are raised and this six-dot arrangement represents the letter c. In parts (c) and (d) of the figure we have the representations for the letters m and t, respectively. The definite article "the" is shown in part (e) of the figure, while part (f) contains the form for the suffix "ow." Finally, the semicolon, ; , is given by the six-dot arrangement in part (g), where the dots at positions 2 and 3 are raised.

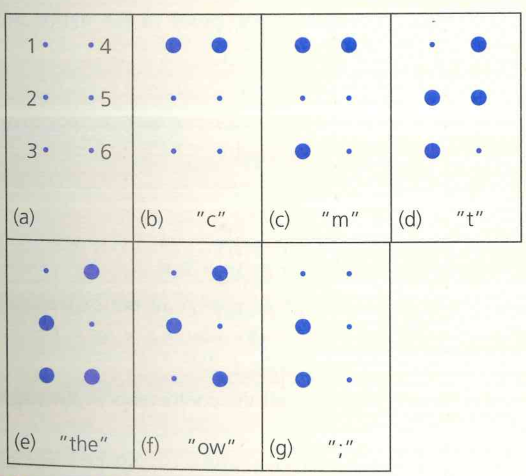

Figure 1.7

a) How many different symbols can we represent in the Braille system?

b) How many symbols have exactly three raised dots?

c) How many symbols have an even number of raised dots?

5. a) How many *permutations* of size 3 can one produce with the letters m, r, a, f, and t?

b) List all the *combinations* of size 3 that result for the letters m, r, a, f, and t.

6. If n is a positive integer and $n > 1$, prove that $\binom{n}{2} + \binom{n-1}{2}$ is a perfect square.

7. A committee of 12 is to be selected from 10 men and 10 women. In how many ways can the selection be carried out if (a) there are no restrictions? (b) there must be six men and six women? (c) there must be an even number of women? (d) there must be more women than men? (e) there must be at least eight men?

8. In how many ways can a gambler draw five cards from a standard deck and get (a) a flush (five cards of the same suit)? (b) four aces? (c) four of a kind? (d) three aces and two jacks? (e) three aces and a pair? (f) a full house (three of a kind and a pair)? (g) three of a kind? (h) two pairs?

9. How many bytes contain (a) exactly two 1's; (b) exactly four 1's; (c) exactly six 1's; (d) at least six 1's?

10. How many ways are there to pick a five-person basketball team from 12 possible players? How many selections include the weakest and the strongest players?

11. A student is to answer seven out of 10 questions on an examination. In how many ways can he make his selection if (a) there are no restrictions? (b) he must answer the first two questions? (c) he must answer at least four of the first six questions?

12. In how many ways can 12 different books be distributed among four children so that (a) each child gets three books? (b) the two oldest children get four books each and the two youngest get two books each?

13. How many arrangements of the letters in MISSISSIPPI have no consecutive S's?

14. A gym coach must select 11 seniors to play on a football team. If he can make his selection in 12,376 ways, how many seniors are eligible to play?

15. a) Fifteen points, no three of which are collinear, are given on a plane. How many lines do they determine?

b) Twenty-five points, no four of which are coplanar, are given in space. How many triangles do they determine? How many planes? How many tetrahedra (pyramidlike solids with four triangular faces)?

16. Determine the value of each of the following summations.

a) $\sum_{i=1}^{6} (i^2 + 1)$ **b)** $\sum_{j=-2}^{2} (j^3 - 1)$ **c)** $\sum_{i=0}^{10} [1 + (-1)^i]$

d) $\sum_{k=n}^{2n} (-1)^k$, where n is an odd positive integer

e) $\sum_{i=1}^{6} i(-1)^i$

17. Express each of the following using the summation (or Sigma) notation. In parts (a), (d), and (e), n denotes a positive integer.

a) $\frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{n!}, n \geq 2$

b) $1 + 4 + 9 + 16 + 25 + 36 + 49$

c) $1^3 - 2^3 + 3^3 - 4^3 + 5^3 - 6^3 + 7^3$

d) $\frac{1}{n} + \frac{2}{n+1} + \frac{3}{n+2} + \cdots + \frac{n+1}{2n}$

e) $n - \left(\frac{n+1}{2!}\right) + \left(\frac{n+2}{4!}\right) - \left(\frac{n+3}{6!}\right) + \cdots + (-1)^n\left(\frac{2n}{(2n)!}\right)$

18. For the strings of length 10 in Example 1.23, how many have (a) four 0's, three 1's, and three 2's; (b) at least eight 1's; (c) weight 4?

19. Consider the collection of all strings of length 10 made up from the alphabet 0, 1, 2, and 3. How many of these strings have weight 3? How many have weight 4? How many have even weight?

20. In the three parts of Fig. 1.8, eight points are equally spaced and marked on the circumference of a given circle.

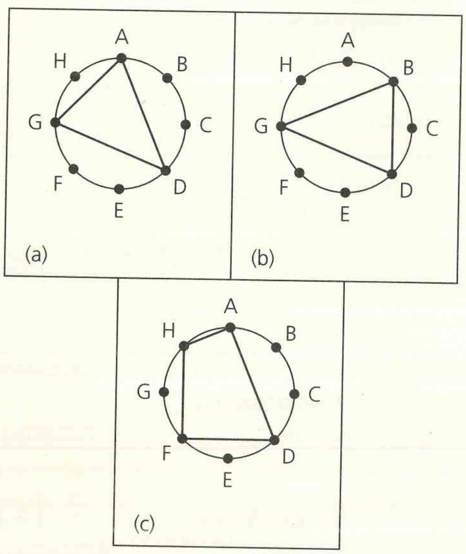

Figure 1.8

a) For parts (a) and (b) of Fig. 1.8 we have two different (though congruent) triangles. These two triangles (distinguished by their vertices) result from two selections of size 3 from the vertices A, B, C, D, E, F, G, H. How many different (whether congruent or not) triangles can we inscribe in the circle in this way?

b) How many different quadrilaterals can we inscribe in the circle, using the marked vertices? [One such quadrilateral appears in part (c) of Fig. 1.8.]

c) How many different polygons of three or more sides can we inscribe in the given circle by using three or more of the marked vertices?

21. How many triangles are determined by the vertices of a regular polygon of n sides? How many if no side of the polygon is to be a side of any triangle?

22. a) In the complete expansion of $(a + b + c + d) \cdot (e + f + g + h)(u + v + w + x + y + z)$ one obtains the sum of terms such as agw, cfx, and dgv. How many such terms appear in this complete expansion?

b) Which of the following terms do *not* appear in the complete expansion from part (a)?

i) afx **ii)** bvx **iii)** chz **iv)** cgw **v)** egu **vi)** dfz

23. Determine the coefficient of x^9y^3 in the expansions of (a) $(x + y)^{12}$, (b) $(x + 2y)^{12}$, and (c) $(2x - 3y)^{12}$.

24. Complete the details in the proof of the multinomial theorem.

25. Determine the coefficient of

a) xyz^2 in $(x + y + z)^4$

b) xyz^2 in $(w + x + y + z)^4$

c) xyz^2 in $(2x - y - z)^4$

d) xyz^{-2} in $(x - 2y + 3z^{-1})^4$

e) $w^3x^2yz^2$ in $(2w - x + 3y - 2z)^8$

26. Find the coefficient of $w^2x^2y^2z^2$ in the expansion of (a) $(w + x + y + z + 1)^{10}$, (b) $(2w - x + 3y + z - 2)^{12}$, and (c) $(v + w - 2x + y + 5z + 3)^{12}$.

27. Determine the sum of all the coefficients in the expansions of

a) $(x + y)^3$ **b)** $(x + y)^{10}$ **c)** $(x + y + z)^{10}$

d) $(w + x + y + z)^5$

e) $(2s - 3t + 5u + 6v - 11w + 3x + 2y)^{10}$

28. For any positive integer n determine

a) $\sum_{i=0}^{n} \frac{1}{i!(n-i)!}$ **b)** $\sum_{i=0}^{n} \frac{(-1)^i}{i!(n-i)!}$

29. Show that for all positive integers m and n, $n\binom{m+n}{m} = (m+1)\binom{m+n}{m+1}$.

30. With n a positive integer, evaluate the sum

$$\binom{n}{0} + 2\binom{n}{1} + 2^2\binom{n}{2} + \cdots + 2^k\binom{n}{k} + \cdots + 2^n\binom{n}{n}.$$

31. For x a real number and n a positive integer, show that

a) $1 = (1+x)^n - \binom{n}{1}x^1(1+x)^{n-1}$
$+ \binom{n}{2}x^2(1+x)^{n-2} - \cdots + (-1)^n\binom{n}{n}x^n$

b) $1 = (2+x)^n - \binom{n}{1}(x+1)(2+x)^{n-1} +$
$\binom{n}{2}(x+1)^2(2+x)^{n-2} - \cdots + (-1)^n\binom{n}{n}(x+1)^n$

c) $2^n = (2+x)^n - \binom{n}{1}x^1(2+x)^{n-1}$
$+ \binom{n}{2}x^2(2+x)^{n-2} - \cdots + (-1)^n\binom{n}{n}x^n$

32. Determine x if $\sum_{i=0}^{50}\binom{50}{i}8^i = x^{100}$.

33. a) If a_0, a_1, a_2, a_3 is a list of four real numbers, what is $\sum_{i=1}^{3}(a_i - a_{i-1})$?

b) Given a list—$a_0, a_1, a_2, \ldots, a_n$—of $n+1$ real numbers, where n is a positive integer, determine $\sum_{i=1}^{n}(a_i - a_{i-1})$.

c) Determine the value of $\sum_{i=1}^{100}\left(\frac{1}{i+2} - \frac{1}{i+1}\right)$.

34. a) Write a computer program (or develop an algorithm) that lists all selections of size 2 from the objects 1, 2, 3, 4, 5, 6.

b) Repeat part (a) for selections of size 3.

1.4 Combinations with Repetition

When repetitions are allowed, we have seen that for n distinct objects an arrangement of size r of these objects can be obtained in n^r ways, for an integer $r \geq 0$. We now turn to the comparable problem for combinations and once again obtain a related problem whose solution follows from our previous enumeration principles.

EXAMPLE 1.27 On their way home from track practice, seven high school freshmen stop at a restaurant, where each of them has one of the following: a cheeseburger, a hot dog, a taco, or a fish sandwich. How many different purchases are possible?

Let c, h, t, and f represent cheeseburger, hot dog, taco, and fish sandwich, respectively. Here we are concerned with how many of each item are purchased, not with the order in which they are purchased, so the problem is one of selections, or combinations, with repetition.

In Table 1.6 we list some possible purchases in column (a) and another means of representing each purchase in column (b).

Table 1.6

<table>
<tr><td>1. c, c, h, h, t, t, f</td><td>1. x x | x x | x x | x</td></tr>
<tr><td>2. c, c, c, c, h, t, f</td><td>2. x x x x | x | x | x</td></tr>
<tr><td>3. c, c, c, c, c, c, f</td><td>3. x x x x x x | | | x</td></tr>
<tr><td>4. h, t, t, f, f, f, f</td><td>4. | x | x x | x x x x</td></tr>
<tr><td>5. t, t, t, t, t, f, f</td><td>5. | | x x x x x | x x</td></tr>
<tr><td>6. t, t, t, t, t, t, t</td><td>6. | | x x x x x x x |</td></tr>
<tr><td>7. f, f, f, f, f, f, f</td><td>7. | | | x x x x x x x</td></tr>
<tr><td>(a)</td><td>(b)</td></tr>
</table>

For a purchase in column (b) of Table 1.6 we realize that each x to the left of the first bar (|) represents a c, each x between the first and second bars represents an h, the x's between the second and third bars stand for t's, and each x to the right of the third bar stands for an f. The third purchase, for example, has three consecutive bars because no one bought a hot dog or taco; the bar at the start of the fourth purchase indicates that there were no cheeseburgers in that purchase.

Once again a correspondence has been established between two collections of objects, where we know how to count the number in one collection. For the representations in column (b) of Table 1.6, we are enumerating all arrangements of 10 symbols consisting of seven x's and three |'s, so by our correspondence the number of different orders for column (a) is

$$\frac{10!}{7!\,3!} = \binom{10}{7}.$$

In this example we note that the seven x's (one for each freshman) correspond to the size of the selection and that the three bars are needed to separate the $3 + 1 = 4$ possible food items that can be chosen.

In general, when we wish to select, *with repetition*, r of n distinct objects, we find (as in Table 1.6) that we are considering all arrangements of r x's and $n - 1$ |'s and that their number is

$$\frac{(n+r-1)!}{r!(n-1)!} = \binom{n+r-1}{r}.$$

Consequently, the number of combinations of n objects taken r at a time, *with repetition*, is $C(n + r - 1, r)$.

(In Example 1.27, $n = 4$, $r = 7$, so it is possible for r to exceed n when repetitions are allowed.)

EXAMPLE 1.28

A donut shop offers 20 kinds of donuts. Assuming that there are at least a dozen of each kind when we enter the shop, we can select a dozen donuts in $C(20 + 12 - 1, 12) = C(31, 12) = 141{,}120{,}525$ ways. (Here $n = 20$, $r = 12$.)

EXAMPLE 1.29

President Helen has four vice presidents: (1) Betty, (2) Goldie, (3) Mary Lou, and (4) Mona. She wishes to distribute among them \$1000 in Christmas bonus checks, where each check will be written for a multiple of \$100.

a) Allowing the situation in which one or more of the vice presidents get nothing, President Helen is making a selection of size 10 (one for each unit of \$100) from a collection of size 4 (four vice presidents), with repetition. This can be done in $C(4 + 10 - 1, 10) = C(13, 10) = 286$ ways.

b) If there are to be no hard feelings, each vice president should receive at least \$100. With this restriction, President Helen is now faced with making a selection of size 6 (the remaining six units of \$100) from the same collection of size 4, and the choices now number $C(4 + 6 - 1, 6) = C(9, 6) = 84$. [For example, here the selection 2, 3, 3, 4, 4, 4 is interpreted as follows: Betty does not get anything extra—for there is no 1 in the selection. The one 2 in the selection indicates that Goldie gets an additional \$100. Mary Lou receives an additional \$200 (\$100 for each of the two 3's in the selection). Due to the three 4's, Mona's bonus check will total \$100 +3(\$100) = \$400.]

c) If each vice president must get at least \$100 and Mona, as executive vice president, gets at least \$500, then the number of ways President Helen can distribute the bonus checks is

$$\underbrace{C(3+2-1,2)}_{\text{Mona gets exactly \$500}} + \underbrace{C(3+1-1,1)}_{\text{Mona gets exactly \$600}} + \underbrace{C(3+0-1,0)}_{\text{Mona gets exactly \$700}} = 10 = \underbrace{C(4+2-1,1)}_{\text{Using the technique in part (b)}}$$

2?

Having worked examples utilizing combinations with repetition, we now consider two examples involving other counting principles as well.

EXAMPLE 1.30 In how many ways can we distribute seven apples and six oranges among four children so that each child receives at least one apple?

Giving each child an apple, we have $C(4 + 3 - 1, 3) = 20$ ways to distribute the other three apples and $C(4 + 6 - 1, 6) = 84$ ways to distribute the six oranges. So by the rule of product, there are $20 \times 84 = 1680$ ways to distribute the fruit under the stated conditions.

EXAMPLE 1.31 A message is made up of 12 different symbols and is to be transmitted through a communication channel. In addition to the 12 symbols, the transmitter will also send a total of 45 (blank) spaces between the symbols, with at least three spaces between each pair of consecutive symbols. In how many ways can the transmitter send such a message?

There are 12! ways to arrange the 12 different symbols, and for each of these arrangements there are 11 positions between the 12 symbols. Because there must be at least three spaces between successive symbols, we use up 33 of the 45 spaces and must now locate the remaining 12 spaces. This is now a selection, with repetition, of size 12 (the spaces) from a collection of size 11 (the locations), and this can be accomplished in $C(11 + 12 - 1, 12) = 646{,}646$ ways.

Consequently, by the rule of product the transmitter can send such messages with the required spacing in $(12!)\binom{22}{12} \doteq 3.097 \times 10^{14}$ ways.

In the next example an idea is introduced that appears to have more to do with number theory than with combinations or arrangements. Nonetheless, the solution of this example will turn out to be equivalent to counting combinations with repetitions.

EXAMPLE 1.32 Determine all integer solutions to the equation

$$x_1 + x_2 + x_3 + x_4 = 7, \qquad \text{where } x_i \geq 0 \quad \text{for all } 1 \leq i \leq 4.$$

One solution of the equation is $x_1 = 3, x_2 = 3, x_3 = 0, x_4 = 1$. (This is different from a solution such as $x_1 = 1, x_2 = 0, x_3 = 3, x_4 = 3$, even though the same four integers are being used.) A possible interpretation for the solution $x_1 = 3, x_2 = 3, x_3 = 0, x_4 = 1$ is that we are distributing seven pennies (identical objects) among four children (distinct containers), and here we have given three pennies to each of the first two children, nothing to the third child, and the last penny to the fourth child. Continuing with this interpretation, we see that each nonnegative integer solution of the equation corresponds to a selection, with repetition, of size 7 (the *identical* pennies) from a collection of size 4 (the *distinct* children), so there are $C(4 + 7 - 1, 7) = 120$ solutions.

At this point it is crucial that we recognize the equivalence of the following:

a) The number of integer solutions of the equation

$$x_1 + x_2 + \cdots + x_n = r, \qquad x_i \geq 0, \qquad 1 \leq i \leq n.$$

b) The number of selections, with repetition, of size r from a collection of size n.

c) The number of ways r identical objects can be distributed among n distinct containers.

In terms of distributions, part (c) is valid only when the r objects being distributed are identical and the n containers are distinct. When both the r objects and the n containers are distinct, we can select any of the n containers for each one of the objects and get n^r distributions by the rule of product.

When the objects are distinct but the containers are identical, we shall solve the problem using the Stirling numbers of the second kind (Chapter 5). For the final case, in which both objects and containers are identical, the theory of partitions of integers (Chapter 9) will provide some necessary results.

EXAMPLE 1.33

In how many ways can one distribute 10 (identical) white marbles among six distinct containers?

Solving this problem is equivalent to finding the number of nonnegative integer solutions to the equation $x_1 + x_2 + \cdots + x_6 = 10$. That number is the number of selections of size 10, with repetition, from a collection of size 6. Hence the answer is $C(6 + 10 - 1, 10) = 3003$.

We now examine two other examples related to the theme of the section.

EXAMPLE 1.34

From Example 1.33 we know that there are 3003 nonnegative integer solutions to the equation $x_1 + x_2 + \cdots + x_6 = 10$. How many such solutions are there to the inequality $x_1 + x_2 + \cdots + x_6 < 10$?

One approach that may seem feasible in dealing with this inequality is to determine the number of such solutions to $x_1 + x_2 + \cdots + x_6 = k$, where k is an integer and $0 \leq k \leq 9$. Although feasible now, the technique becomes unrealistic if 10 is replaced by a somewhat larger number, say 100. In Example 3.11 of Chapter 3, however, we shall establish a combinatorial identity that will help us obtain an alternative solution to the problem by using this approach.

For the present we transform the problem by noting the correspondence between the nonnegative integer solutions of

$$x_1 + x_2 + \cdots + x_6 < 10 \tag{1}$$

and the integer solutions of

$$x_1 + x_2 + \cdots + x_6 + x_7 = 10, \qquad 0 \le x_i, \qquad 1 \le i \le 6, \qquad 0 < x_7. \tag{2}$$

The number of solutions of Eq. (2) is the same as the number of nonnegative integer solutions of $y_1 + y_2 + \cdots + y_6 + y_7 = 9$, where $y_i = x_i$ for $1 \le i \le 6$, and $y_7 = x_7 - 1$. This is $C(7 + 9 - 1, 9) = 5005$.

Our next result takes us back to the binomial and multinomial expansions.

EXAMPLE 1.35

In the binomial expansion for $(x + y)^n$, each term is of the form $\binom{n}{k}x^k y^{n-k}$, so the total number of terms in the expansion is the number of nonnegative integer solutions of $n_1 + n_2 = n$ (n_1 is the exponent for x, n_2 the exponent for y). This number is $C(2 + n - 1, n) = n + 1$.

Perhaps it seems that we have used a rather long-winded argument to get this result. Many of us would probably be willing to believe the result on the basis of our experiences in expanding $(x + y)^n$ for various small values of n.

Although experience is worthwhile in pattern recognition, it is not always enough to find a general principle. Here it would prove of little value if we wanted to know how many terms there are in the expansion of $(w + x + y + z)^{10}$.

Each distinct term here is of the form $\binom{10}{n_1, n_2, n_3, n_4} w^{n_1} x^{n_2} y^{n_3} z^{n_4}$, where $0 \le n_i$ for $1 \le i \le 4$, and $n_1 + n_2 + n_3 + n_4 = 10$. This last equation can be solved in $C(4 + 10 - 1, 10) = 286$ ways, so there are 286 terms in the expansion of $(w + x + y + z)^{10}$.

And now once again the binomial expansion will come into play, as we find ourselves using part (a) of Corollary 1.1.

EXAMPLE 1.36

a) Let us determine all the different ways in which we can write the number 4 as a sum of positive integers, where the order of the summands is considered relevant. These representations are called the *compositions* of 4 and may be listed as follows:

1) 4	**5)** 2 + 1 + 1
2) 3 + 1	**6)** 1 + 2 + 1
3) 1 + 3	**7)** 1 + 1 + 2
4) 2 + 2	**8)** 1 + 1 + 1 + 1

Here we include the sum consisting of only one summand—namely, 4. We find that for the number 4 there are eight compositions in total. (If we do *not* care about the order of the summands, then the representations in (2) and (3) are no longer considered to be different—nor are the representations in (5), (6), and (7). Under these circumstances we find that there are five *partitions* for the number 4—namely, 4; 3 + 1; 2 + 2; 2 + 1 + 1; and 1 + 1 + 1 + 1. We shall learn more about partitions of positive integers in Section 9.3.)

b) Now suppose that we wish to *count* the number of compositions for the number 7. Here we do *not* want to list all of the possibilities—which include 7; 6 + 1; 1 + 6; 5 + 2; 1 + 2 + 4; 2 + 4 + 1; and 3 + 1 + 2 + 1. To count all of these compositions, let us consider the number of possible summands.

i) For one summand there is only one composition—namely, 7.

ii) If there are two (positive) summands, we want to count the number of integer solutions for

$$w_1 + w_2 = 7, \qquad \text{where } w_1, w_2 > 0.$$

This is equal to the number of integer solutions for

$$x_1 + x_2 = 5, \qquad \text{where } x_1, x_2 \geq 0.$$

The number of such solutions is $\binom{2+5-1}{5} = \binom{6}{5}$.

iii) Continuing with our next case, we examine the compositions with three (positive) summands. So now we want to count the number of *positive* integer solutions for

$$y_1 + y_2 + y_3 = 7.$$

This is equal to the number of *nonnegative* integer solutions for

$$z_1 + z_2 + z_3 = 4,$$

and that number is $\binom{3+4-1}{4} = \binom{6}{4}$.

We summarize cases (i), (ii), and (iii), and the other four cases in Table 1.7, where we recall for case (i) that $1 = \binom{6}{6}$.

Table 1.7

n = The Number of Summands in a Composition of 7		The Number of Compositions of 7 with n Summands	
(i)	$n = 1$	(i)	$\binom{6}{6}$
(ii)	$n = 2$	(ii)	$\binom{6}{5}$
(iii)	$n = 3$	(iii)	$\binom{6}{4}$
(iv)	$n = 4$	(iv)	$\binom{6}{3}$
(v)	$n = 5$	(v)	$\binom{6}{2}$
(vi)	$n = 6$	(vi)	$\binom{6}{1}$
(vii)	$n = 7$	(vii)	$\binom{6}{0}$

Consequently, the results from the right-hand side of our table tell us that the (total) number of compositions of 7 is

$$\binom{6}{6}+\binom{6}{5}+\binom{6}{4}+\binom{6}{3}+\binom{6}{2}+\binom{6}{1}+\binom{6}{0}=\sum_{k=0}^{6}\binom{6}{k}.$$

From part (a) of Corollary 1.1 this reduces to 2^6.

In general, one finds that for each positive integer m, there are $\sum_{k=0}^{m-1}\binom{m-1}{k}=2^{m-1}$ compositions.

Our last two examples for this section provide applications from the area of computer science. Furthermore, the final example will lead to an important summation formula that we shall use in many later chapters.

EXAMPLE 1.37 Consider the following program segment, where i, j, and k are integer variables.

```
for i := 1 to 20 do
  for j := 1 to i do
    for k := 1 to j do
      print (i * j + k)
```

How many times is the **print** statement executed in this program segment?

Among the possible choices for i, j, and k (in the order i–first, j–second, k–third) that will lead to execution of the **print** statement, we list (1) 1, 1, 1; (2) 2, 1, 1; (3) 15, 10, 1; and (4) 15, 10, 7. We note that $i=10, j=12, k=5$ is not one of the selections to be considered, because $j=12>10=i$; this violates the condition set forth in the second **for** loop. Each of the above four selections where the **print** statement is executed satisfies the condition $1\leq k\leq j\leq i\leq 20$. In fact, any selection a, b, c $(a\leq b\leq c)$ of size 3, with repetitions allowed, from the list 1, 2, 3, . . . , 20 results in one of the correct selections: here, $k=a, j=b, i=c$. Consequently the **print** statement is executed

$$\binom{20+3-1}{3}=\binom{22}{3}=1540 \text{ times.}$$

If there had been r (≥ 1) **for** loops instead of three, the **print** statement would have been executed $\binom{20+r-1}{r}$ times.

EXAMPLE 1.38 Here we use a program segment to derive a summation formula. In this program segment, the variables i, j, n, and *counter* are integer variables. Furthermore, we assume that the value of n has been set prior to this segment.

```
counter := 0
for i := 1 to n do
  for j := 1 to i do
    counter := counter + 1
```

From the results in Example 1.37, after this segment is executed the value of (the variable) *counter* will be $\binom{n+2-1}{2} = \binom{n+1}{2}$. (This is also the number of times that the statement

(*) `counter := counter + 1`

is executed.)

This result can also be obtained as follows: When $i := 1$, then j varies from 1 to 1 and (*) is executed once; when i is assigned the value 2, then j varies from 1 to 2 and (*) is executed twice; j varies from 1 to 3 when i is assigned the value 3, and (*) is executed three times; in general, for $1 \leq k \leq n$, when $i := k$, then j varies from 1 to k and (*) is executed k times. In total, the variable *counter* is incremented [and the statement (*) is executed] $1 + 2 + 3 + \cdots + n$ times.

Consequently,

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n = \binom{n+1}{2} = \frac{n(n+1)}{2}.$$

The derivation of this summation formula, obtained by counting the same result in two different ways, constitutes a combinatorial proof.

EXERCISES 1.4

1. In how many ways can 10 (identical) dimes be distributed among five children if (a) there are no restrictions? (b) each child gets at least one dime? (c) the oldest child gets at least two dimes?

2. In how many ways can 15 (identical) candy bars be distributed among five children so that the youngest gets only one or two of them?

3. Determine how many ways 20 coins can be selected from four large containers filled with pennies, nickels, dimes, and quarters. (Each container is filled with only one type of coin.)

4. A certain ice cream store has 31 flavors of ice cream available. In how many ways can we order a dozen ice cream cones if (a) we do not want the same flavor more than once? (b) a flavor may be ordered as many as 12 times? (c) a flavor may be ordered no more than 11 times?

5. a) In how many ways can we select five coins from a collection of 10 consisting of one penny, one nickel, one dime, one quarter, one half-dollar, and five (identical) Susan B. Anthony dollars?

b) In how many ways can we select n objects from a collection of size $2n$ that consists of n distinct and n identical objects?

6. Answer Example 1.31 where the 12 symbols being transmitted are four A's, four B's, and four C's.

7. Determine the number of integer solutions of $x_1 + x_2 + x_3 + x_4 = 32$, where

a) $x_i \geq 0, \quad 1 \leq i \leq 4$

b) $x_i > 0, \quad 1 \leq i \leq 4$

c) $x_1, x_2 \geq 5, \quad x_3, x_4 \geq 7$

d) $x_i \geq 8, \quad 1 \leq i \leq 4$

e) $x_i \geq -2, \quad 1 \leq i \leq 4$

f) $x_1, x_2, x_3 > 0, \quad 0 < x_4 \leq 25$

8. In how many ways can a teacher distribute eight chocolate donuts and seven jelly donuts among three student helpers if each helper wants at least one donut of each kind?

9. Columba has two dozen each of n different colored beads. If she can select 20 beads (with repetitions of colors allowed) in 230,230 ways, what is the value of n?

10. In how many ways can Lisa toss 100 (identical) dice so that at least three of each type of face will be showing?

11. Two n-digit integers (leading zeros allowed) are considered equivalent if one is a rearrangement of the other. (For example, 12033, 20331, and 01332 are considered equivalent five-digit integers.) (a) How many five-digit integers are not equivalent? (b) If the digits 1, 3, and 7 can appear at most once, how many nonequivalent five-digit integers are there?

12. Determine the number of integer solutions for $x_1 + x_2 + x_3 + x_4 + x_5 < 40$, where

a) $x_i \geq 0, \quad 1 \leq i \leq 5$

b) $x_i \geq -3, \quad 1 \leq i \leq 5$

13. In how many ways can we distribute eight identical white balls into four distinct containers so that (a) no container is left empty? (b) the fourth container has an odd number of balls in it?

14. a) Find the coefficient of v^2w^4xz in the expansion of $(3v + 2w + x + y + z)^8$.

b) How many distinct terms arise in the expansion in part (a)?

15. In how many ways can Beth place 24 different books on four shelves so that there is at least one book on each shelf? (For any of these arrangements consider the books on each shelf to be placed one next to the other, with the first book at the left of the shelf.)

16. For which positive integer n will the equations

(1) $x_1 + x_2 + x_3 + \cdots + x_{19} = n$, and

(2) $y_1 + y_2 + y_3 + \cdots + y_{64} = n$

have the same number of positive integer solutions?

17. How many ways are there to place 12 marbles of the same size in five distinct jars if (a) the marbles are all black? (b) each marble is a different color?

18. a) How many nonnegative integer solutions are there to the pair of equations $x_1 + x_2 + x_3 + \cdots + x_7 = 37$, $x_1 + x_2 + x_3 = 6$?

b) How many solutions in part (a) have $x_1, x_2, x_3 > 0$?

19. How many times is the **print** statement executed for the following program segment? (Here i, j, k, and m are integer variables.)

```
for i := 1 to 20 do
  for j := 1 to i do
    for k := 1 to j do
      for m := 1 to k do
        print (i * j) + (k * m)
```

20. In the following program segment, i, j, k, and *counter* are integer variables. Determine the value that the variable *counter* will have after the segment is executed.

```
counter := 10
for i := 1 to 15 do
  for j := i to 15 do
    for k := j to 15 do
      counter := counter + 1
```

21. Find the value of *sum* after the given program segment is executed. (Here i, j, k, *increment*, and *sum* are integer variables.)

```
increment := 0
sum := 0
for i := 1 to 10 do
  for j := 1 to i do
    for k := 1 to j do
      begin
        increment := increment + 1
        sum := sum + increment
      end
```

22. Consider the following program segment, where i, j, k, n, and *counter* are integer variables and the value of n (a positive integer) is set prior to this segment.

```
counter := 0
for i := 1 to n do
  for j := 1 to i do
    for k := 1 to j do
      counter := counter + 1
```

We shall determine, in two different ways, the number of times the statement

```
counter := counter + 1
```

is executed. (This is also the value of *counter* after execution of the program segment.) From the result in Example 1.37, we know that the statement is executed $\binom{n+3-1}{3} = \binom{n+2}{3}$ times. For a fixed value of i, the **for** loops involving j and k result in $\binom{i+1}{2}$ executions of the counter increment statement. Consequently, $\binom{n+2}{3} = \sum_{i=1}^{n} \binom{i+1}{2}$. Use this result to obtain a summation formula for

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \sum_{i=1}^{n} i^2.$$

23. a) Given positive integers m, n with $m \geq n$, show that the number of ways to distribute m identical objects into n distinct containers with no container left empty is

$$C(m-1, m-n) = C(m-1, n-1).$$

b) Show that the number of distributions in part (a) where each container holds at least r objects ($m \geq nr$) is

$$C(m-1+(1-r)n, n-1).$$

24. Write a computer program (or develop an algorithm) to list the integer solutions for

a) $x_1 + x_2 + x_3 = 10, \quad 0 \leq x_i, \quad 1 \leq i \leq 3$

b) $x_1 + x_2 + x_3 + x_4 = 4, \quad -2 \leq x_i, \quad 1 \leq i \leq 4$