CSC 226: Summer 2018: Lab 1

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1 Asymptotic Notation

Let f and g be two functions that take integers as input and outputs real numbers.
Big-Oh: $f(n)$ is $O(g(n))$ if and only if there exists a *real* constant $c > 0$ and an integer $n_0 > 0$ such that $f(n) \leq c.g(n)\ \forall n \geq n_0$.
Big-Omega: $f(n)$ is $\Omega(g(n))$ if and only if there exists a *real* constant $c > 0$ and an integer $n_0 > 0$ such that $f(n) \geq c.g(n)\ \forall n \geq n_0$.
Big-Theta: $f(n)$ is $\Theta(g(n))$ if and only if $f(n)$ is $O(g(n))$ and $f(n)$ is $\Omega(g(n))$.

Based on the definitions above, prove the followings.

1. $5n^2 + 6n + 12$ is $O(n^3)$. `Ans:` Here, $f(n) = 5n^2 + 6n + 12$ and $g(n) = n^3$. $5n^2 + 6n + 12 \leq 5n^3 + 6n^3 + 12n^3 = 23n^3$. For $c = 23$ and $n_0 = 1$, $f(n) \leq c.g(n)$, and therefore, $f(n)$ is $O(n^3)$.

2. $5n^2 + 6n + 12$ is $\Omega(n^2)$. `Ans:` Here, $f(n) = 5n^2 + 6n + 12$ and $g(n) = n^2$. $5n^2 + 6n + 12 \geq 5n^2$. For $c = 5$ and $n_0 = 1$, $f(n) \geq c.g(n)$, and therefore, $f(n)$ is $\Omega(n^2)$.

3. $5n^2 + 6n + 12$ is $\Theta(n^2)$. `Ans:` Here, $f(n) = 5n^2 + 6n + 12$ and $g(n) = n^2$. We have to prove that $f(n)$ is $O(n^2)$ and $f(n)$ is $\Omega(n^2)$. We have proved that $f(n)$ is $\Omega(n^2)$. We now prove that $f(n)$ is $O(n^2)$. $5n^2 + 6n + 12 \leq 5n^2 + 6n^2 + 12n^2 = 23n^2$. For $c = 23$ and $n_0 = 1$, $f(n) \leq c.g(n)$, and therefore, $f(n)$ is $O(n^2)$. By definition of Big-Theta, $f(n)$ is $\Theta(n^2)$.

2 Rules of Big-Oh

Prove the following theorems using the definition of Big-Oh from above.

1. **R1 (Scaling):** If $f(n)$ is $O(g(n))$ then $af(n)$ is $O(g(n))$, $a > 0$. `Ans:` If $f(n)$ is $O(g(n))$, then there exists $c > 0$ and $n_0 > 0$ such that $f(n) \leq c.g(n)$ for all $n \geq n_0$. Then $af(n) \leq ac.g(n) = c'.g(n)$, where $c' = ac$. For $c' > 0$, since $a > 0$ and $c > 0$, and $n_0 > 0$, $af(n) \leq c'g(n)$. Therefore, $af(n)$ is $O(g(n))$.

2. **R4 (Transitivity):** If $d(n)$ is $O(f(n))$ and $f(n)$ is $O(g(n))$, then $d(n)$ is $O(g(n))$. `Ans:` If $d(n)$ is $O(f(n))$, then there exists $c > 0$ and $n_0 > 0$ such that $d(n) \leq c.f(n)$ for all $n \geq n_0$. Since $f(n)$ is $O(g(n))$, then there exists $a > 0$ and $n_1 > 0$ such that $f(n) \leq a.g(n)$ for all $n \geq n_1$. Then $d(n) \leq ac.g(n) = c'.g(n)$, where $c' = ac$, for $c' > 0$ and $\max\{n_0, n_1\} > 0$. Therefore, $d(n)$ is $O(g(n))$.

3. **R7:** $\log(n^x)$ is $O(\log n)$ for any fixed $x > 0$. `Ans:` Since $\log(n^x) = x \log n$, for $c = x > 0$ and $n_0 = 1$, $\log(n^x) \leq c \log n$. Therefore, $\log(n^x)$ is $O(\log n)$.

4. **R6:** n^x is $O(a^n)$ for any fixed $x > 0$ and $a > 1$. `Ans:` We have to show that a^n grows faster than n^x.
   The Limit Rule [3]: Suppose $\lim_{n\to\infty} \frac{f(n)}{g(n)} = L$ exists. Then,

$$f(n) = \begin{cases} O(g(n)) & \text{if } L = 0 \\ \Theta(g(n)) & \text{if } 0 < L < \infty \\ \Omega(g(n)) & \text{if } L = \infty \end{cases}$$

   In our case, $f(n) = n^x$ and $g(n) = a^n$. To find L, we use L'Hopital rule.
   L'Hopital Rule [3]: $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f'(n)}{g'(n)}$.
   Now, $f' = xn^{x-1}$, $f'' = x(x-1)n^{x-2}$. In this way, $f^k = xx - 1(x - 2) \ldots 2.1 = x!$ and $f^{k+1} = 0$.
   $g' = a^n \ln a$, $g'' = a^n(\ln a)^2$, $\ldots$, $g^{k+1} = a^n(\ln a)^{k+1}$.
   Therefore, $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f^{k+1}(n)}{g^{k+1}(n)} = \lim_{n\to\infty} \frac{0}{a^n(\ln a)^{k+1}} = 0$ and $f(n)$ is $O(g(n))$.

3 Permutations and Combinations

3.1 Poker Hands

If you have played poker, you probably know some or all the hands below [1]. You can choose 5 cards from 52 in $\binom{52}{5}$ ways. But how many of them would be a *Royal Flush* or a *Four-of-a-Kind*? Let's try to calculate the numbers for all the following hands. The green ones have already been covered in the class.

`Ans:` We are posting solutions to some of the poker hands, all of them will be posted later.

1. **Royal Flush:** All five cards are of the same suit and are of the sequence 10 J Q K A. `Ans:`There are 4 suits and each suit can have exactly 1 royal flush suit. So, the number of possible royal flush is 4.

2. **Four-of-a-Kind:** Four cards are all of the same rank. `Ans:`To be of the same rank, the 4 cards have to come from 4 suits. There are 13 such sets of four and we can choose 1 in $\binom{13}{1}$ ways. The fifth card in the hand can be any of the remaining 48 cards. So, number of possible hands is $13 \times 48 = 624$.

3. **Full House:** A hand consisting of one pair and a three-of-a-kind of a different rank than the pair. `Ans:` We choose a rank for the pair in 13 ways and a rank for the three-of-a-kind in 12 ways. We did not divide by 2! to remove permutation, because pair of 2s and three 4s is different than a pair of 4s and three 2s. So we do not need to remove permutations. For the pair, we can choose two cards in $\binom{4}{2}$ ways and for the three-of-a-kind we can choose in $\binom{4}{3}$ ways. The total number is $13 \times 12 \times \binom{4}{2} \times \binom{4}{3}$.

4. **Straight:** All five cards are sequential in rank but are not all of the same suit. `Ans:` There are 9 possible sequences as in Straight Flush. For each of the 5 cards, we can choose one from four suit. Then we have to deduct the 32 Straight Flushes and 4 Royal Flushes. So, the total number is $\binom{9}{1}4^5 - 32 - 4 = 9180$.

5. **Three-of-a-Kind:** Three cards are all of the same rank and the other two are each of different ranks from the first three and each other. `Ans:` Choosing a rank has $\binom{13}{1} = 13$ ways. For a chosen rank, we can choose 3 cards out of 4 in $\binom{4}{3} = 4$ ways. Then we choose the forth card from the remaining 48, and then one from remaining 44. Since the order of the last two cards do not matter, the number is $13 \times 4 \times 48 \times 44/2$

6. **One Pair:** Only two cards of the five are of the same rank with the other three cards all having different ranks from each other and from that of the pair. `Ans:`Choosing a pair can be done in $\binom{13}{1}\binom{4}{2}$ ways. The remaining three cards can be chosen in $(48 \times 44 \times 40)/3!$ ways. So the total number is $13 \times \binom{4}{2} \times (48 \times 44 \times 40)/6$.

7. **Two Pair:** Two pairs of two cards of the same rank (the ranks of each pair are different in rank, obviously, to avoid a Four-of-a-Kind). `Ans:`Choosing two pairs can be done in $\binom{13}{2}$ ways and choosing two cards for each rank can be done in $\binom{4}{2}$ ways. The remaining card can be chosen in 44 ways. So the total number is $\binom{13}{2} \times \binom{4}{2} \times 44$.

3.2 Some other problems

1. Six friends want to play enough games of chess and every one wants to play everyone else. How many games will they have to play? `Ans:` Two players can be chosen in $\binom{6}{2}$ ways.

2. There are five flavors of ice cream: banana, chocolate, lemon, strawberry and vanilla. We can have three scoops. How many variations will there be? [2] `Ans: Check out the link.`

References

[1] Jeff Duda, *Probabilities of Poker Hands with Variations.* http://www.meteor.iastate.edu/~jdduda/portfolio/492.pdf

[2] https://www.mathsisfun.com/combinatorics/combinations-permutations.html

[3] https://www.cs.auckland.ac.nz/courses/compsci220s1c/lectures/2016S1C/CS220-Lecture04.pdf