COMPUTER SCIENCE 349A
ASSIGNMENT #7 - SOLUTION

1. Here we have the two Taylor expansions,

$$f(x_0+h) = f(x_0) + hf'(x_0) + \frac{h^2}{2}f''(x_0) + \frac{h^3}{6}f'''(x_0) + O(h^4)$$
$$f(x_0+2h) = f(x_0) + 2hf'(x_0) + \frac{4h^2}{2}f''(x_0) + \frac{8h^3}{6}f'''(x_0) + O(h^4)$$

Thus,

$$\begin{aligned}
&-3f(x_0) + 4f(x_0+h) - f(x_0+2h)\\
&= -3f(x_0) + 4\left[f(x_0) + hf'(x_0) + \frac{h^2}{2}f''(x_0) + \frac{h^3}{6}f'''(x_0) + O(h^4)\right]\\
&\qquad - \left[f(x_0) + 2hf'(x_0) + \frac{4h^2}{2}f''(x_0) + \frac{8h^3}{6}f'''(x_0) + O(h^4)\right]\\
&= -3f(x_0) + 4f(x_0) + 4hf'(x_0) + \frac{4h^2}{2}f''(x_0) + \frac{4h^3}{6}f'''(x_0)\\
&\qquad - f(x_0) - 2hf'(x_0) - \frac{4h^2}{2}f''(x_0) - \frac{8h^3}{6}f'''(x_0) + O(h^4)\\
&\qquad = 2hf'(x_0) - \frac{4h^3}{6}f'''(x_0) + O(h^4)
\end{aligned}$$

Solving for $f'(x_0)$ we obtain

$$f'(x_0) = \frac{-3f(x_0) + 4f(x_0+h) - f(x_0+2h)}{2h} + \frac{h^2}{3}f'''(x_0) + O(h^3).$$

Here, the error term is $\frac{h^2}{3}f'''(x_0)+O(h^3)$ making the overall error of this approximation $O(h^2)$.

2. (a) Since $y'(x) = 2 + (x - y(x))^2$, we have

$$\begin{aligned}
y''(x) &= f'(x, y(x))\\
&= 2(x - y(x))(1 - y'(x))\\
&= 2(x - y(x))(-1 - (x - y(x))^2)\\
&= -2(x - y(x))(1 + (x - y(x))^2)
\end{aligned}$$

Thus the iterative formula is

$$\begin{aligned}
y_{i+1} &= y_i + hf(x_i, y_i) + \frac{h}{2}f'(x, y_i)\\
&= y_i + h\left(2 + (x_i - y_i)^2\right) - h^2(x_i - y_i)(1 + (x_i - y_i)^2)
\end{aligned}$$

(b)
```
function  taylor2
fprintf('values of x       approximation y\n')
x = 1 ;
y = 1 ;
h = 0.01 ;
fprintf ( '%8.2f ',x), fprintf ('%19.8f \n',y)
for  i = 1:25
     y = y + h*(2+(x-y)^2) - h^2*(x-y)*(1+(x-y)^2);
     x = x + h;
     fprintf ( '%8.2f ',x), fprintf ('%19.8f \n',y)
end
```
```
>> taylor2
values of x       approximation y
    1.00          1.00000000
    1.01          1.02000000
    1.02          1.04000200
    1.03          1.06000800
    1.04          1.08002001
    1.05          1.10004003
    1.06          1.12007009
    1.07          1.14011220
    1.08          1.16016841
    1.09          1.18024075
    1.10          1.20033128
    1.11          1.22044207
    1.12          1.24057523
    1.13          1.26073284
    1.14          1.28091705
    1.15          1.30113000
    1.16          1.32137386
    1.17          1.34165083
    1.18          1.36196314
    1.19          1.38231305
    1.20          1.40270283
    1.21          1.42313482
    1.22          1.44361137
    1.23          1.46413487
    1.24          1.48470776
    1.25          1.50533251
```

(c) At $x_i = 1.25$ the exact value is $y(x_i) = 1.505341921221036$. Thus the global truncation error is $|1.50533251 - 1.50534192| = 0.00000941$.

3. Consider the initial-value problem $y'(x) = 2 + y(x)/x$, with $y(1) = 3$.

(a) Here $f(x, y) = 2 + y/x$, $x_0 = 1.00$ and $y_0 = 3.00$. Thus $y_0/x_0 = 3.00$ and $f(x_0, y_0) =$

5.00.

$$
\begin{aligned}
y_1 &= y_0 + \frac{h}{2}\left(f(x_0,y_0) + f(x_0+h, y_0 + h\cdot f(x_0,y_0))\right)\\
&= y_0 + 0.005\left(5 + f(x_0+0.01,\ y_0+0.01(5))\right)\\
&= 3 + 0.005(5 + f(1.01,\ 3+0.05))\\
&= 3 + 0.005(5 + 2 + 3.05/1.01)\\
&= 3 + 0.005(7 + 3.01980198)\\
&= 3 + 0.005(10.01980198)\\
&= 3.0500990
\end{aligned}
$$

(b) The global truncation error is

$$|3.2015765 - 3.2015791| = 0.0000026.$$

For the local truncation error note that

$$f(1.03, y(1.03)) = f(1.03, 3.1508911) = 2{+}3.1508911/1.03 = 5.059117573(5.059117604).$$

Thus

$$
\begin{aligned}
v &= y(1.03) + \frac{h}{2}\left(f(1.03,y(1.03)) + f(1.03+h, y(1.03) + h\cdot f(1.03,y(1.03)))\right)\\
&= 3.1508911 + 0.005(\ f(1.03, 3.1508911)\\
&\quad + f(1.03+0.01, 3.1508911 + 0.01\cdot f(1.03, 3.1508911))\ )\\
&= 3.1508911 + 0.005\left(5.059117573 + f(1.04, 3.1508911 + 0.01\cdot 5.059117573)\right))\\
&= 3.1508911 + 0.005(5.059117573 + f(1.04, 3.1508911 + 0.05059117573))\\
&= 3.1508911 + 0.005(5.059117573 + 5.078348342)\\
&= 3.201578430
\end{aligned}
$$

We can now calculate the local truncation error as being about

$$|3.2015791 - 3.2015784| = 0.0000008.$$

4. (a)

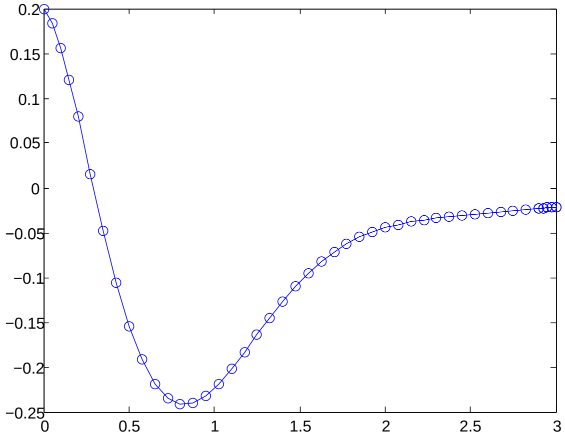
```
function z = ass7q4(x,y)
z = -y-2*exp(-2*x)*sin(3*x)
end

>> [x,y] = ode45('ass7q4',[0,3],0.2)
x =                 y =
         0             0.2000
```

```
   0.0502          0.1832
   0.1005          0.1554
   0.1507          0.1199
   0.2010          0.0794
   0.2760          0.0150
   0.3510         -0.0482
   0.4260         -0.1057
   0.5010         -0.1542
   0.5760         -0.1921
   0.6510         -0.2190
   0.7260         -0.2352
   0.8010         -0.2418
   0.8760         -0.2402
   0.9510         -0.2319
   1.0260         -0.2187
   1.1010         -0.2021
   1.1760         -0.1835
   1.2510         -0.1642
   1.3260         -0.1450
   1.4010         -0.1269
   1.4760         -0.1103
   1.5510         -0.0954
   1.6260         -0.0825
   1.7010         -0.0716
   1.7760         -0.0625
   1.8510         -0.0551
   1.9260         -0.0492
   2.0010         -0.0446
   2.0760         -0.0409
   2.1510         -0.0381
   2.2260         -0.0358
   2.3010         -0.0340
   2.3760         -0.0324
   2.4510         -0.0310
   2.5260         -0.0297
   2.6010         -0.0284
   2.6760         -0.0271
   2.7510         -0.0257
   2.8260         -0.0244
   2.9010         -0.0229
   2.9257         -0.0225
   2.9505         -0.0220
   2.9752         -0.0215
   3.0000         -0.0211
```

(b)

```
>> ode45('ass7q4',[0,3],0.2)
```



0.2
0.15
0.1
0.05
0
−0.05
−0.1
−0.15
−0.2
−0.25
0
0.5
1
1.5
2
2.5
3