**COMPUTER SCIENCE 349A**
**Handout Number 24**

# CUBIC SPLINE INTERPOLANTS

The following definition is the same as given in points 1-5 on pages 515-516 of the 6th ed. or pages 517-518 of the 7th ed., but is more precise.

**Definition**

$$\text{Given data} \quad \begin{cases} x_0,\ x_1,\ \ldots,\ x_n \ \text{ with } \ x_i < x_{i+1}, \ \text{ and} \\ f(x_0),\ f(x_1),\ \ldots,\ f(x_n), \end{cases}$$

$S(x)$ is a **cubic spline interpolant** for $f(x)$ if

(a) $S(x)$ is a cubic polynomial, denoted by $S_j(x)$, on each subinterval $[x_j, x_{j+1}], \quad 0 \le j \le n-1$

(b) $S_j(x_j) = f(x_j)$, for $0 \le j \le n-1$ and $S_{n-1}(x_n) = f(x_n)$

(c) $S_{j+1}(x_{j+1}) = S_j(x_{j+1})$, for $0 \le j \le n-2$

(d) $S'_{j+1}(x_{j+1}) = S'_j(x_{j+1})$, for $0 \le j \le n-2$

(e) $S''_{j+1}(x_{j+1}) = S''_j(x_{j+1})$, for $0 \le j \le n-2$

and (f) either one of the following hold:

(i) $S''(x_0) = S''(x_n) = 0$
--- the free or natural boundary conditions

or

(ii) $S'(x_0) = f'(x_0)$ and $S'(x_n) = f'(x_n)$
--- the clamped boundary conditions

**Note**: for any $f(x)$, there exist an <u>infinite number</u> of cubic splines satisfying conditions (a) - (e). The reason:

There are $n$ cubic polynomials $S_j(x)$ to specify, each one is defined by 4 coefficients, giving a total of $4n$ unknowns to be specified.

However, condition (b) gives $n+1$ conditions to be satisfied, and (c), (d) and (e) each give $n-1$ conditions to be satisfied.

Thus, there are $(n+1)+3(n-1) = 4n-2$ conditions (equations) to be satisfied in $4n$ unknowns.

But if either (i) or (ii) is also required to be satisfied, then there are $4n$ conditions in $4n$ unknowns and there exists a <u>unique</u> cubic spline interpolant satisfying (a) – (f ).

If $S(x)$ satisfies conditions (a), (c), (d) and (e), then (by the definition given in class) it is a spline function with $q = 3$; that is, $S(x)$ is a cubic polynomial on each of the subintervals $[x_i, x_{i+1}]$, and each of $S(x)$, $S'(x)$ and $S''(x)$ is continuous on $[x_0, x_n]$. Condition (b) implies that $S(x)$ interpolates the data $f(x_i)$, and condition (f) gives a unique spline.

**The boundary conditions (i) and (ii)**

If $S(x)$ satisfies (i), it is called a natural cubic spline. Physically, such a spline corresponds to the shape of a long flexible rod that is constrained to pass through the $n+1$ points $(x_i, f(x_i))$. See Figure 18.15 on page 511 of the 6th ed. or page 513 of the 7th ed. Mathematically, a natural cubic spline is the unique function possessing minimum curvature (the rate of change of the tangent vector to a curve) of all functions that interpolate the data $(x_i, f(x_i))$ and have a square integrable second derivative.

If $S(x)$ satisfies the clamped boundary conditions (ii), then these specified first derivative values at the end points usually make this spline a more accurate approximation to the data $(x_i, f(x_i))$. But, of course, it requires this additional information about $f(x)$.

Other boundary conditions are also possible. MATLAB allows two different types of boundary conditions: the clamped boundary conditions and the not-a-knot boundary conditions.