COMPUTER SCIENCE 349A

Handout Number 5

Taylor's Theorem is the fundamental tool for deriving and analyzing numerical approximation formulas in this course. It states that any "smooth" function (one with a sufficient number of derivatives) can be approximated by a polynomial, and it includes an error (remainder) term that indicates how accurate the polynomial approximation is. Taylor's Theorem also provides a means to estimate the value of a function $f(x)$ at some point x_{i+1} using the values of $f(x)$ and its derivatives at some nearby point x_i.

Taylor's Theorem (page 79 of the 6th edition; page 82 of the 7th)

Let $n \ge 0$ and let a be any constant. If $f(x)$ and its first $n+1$ derivatives are continuous on some interval containing x and a, then

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$
$$+ \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n$$

where the remainder (or error) term is

$$R_n = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$$

and ξ is some value between x and a. Note that

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

is a polynomial of degree n in x, and is called the **Taylor polynomial approximation of degree *n*** for $f(x)$ expanded about a. R_n is the **truncation error** of this polynomial approximation to $f(x)$.

Example 1

Determine the Taylor polynomial approximation of order $n = 3$ for $f(x) = \ell n(x+1)$ expanded about $a = 0$.

$$f(x) = \ell n(x+1) \qquad f(0) = 0$$

$$f'(x) = \frac{1}{x+1} \qquad f'(0) = 1$$

$$f''(x) = \frac{-1}{(x+1)^2} \qquad f''(0) = -1$$

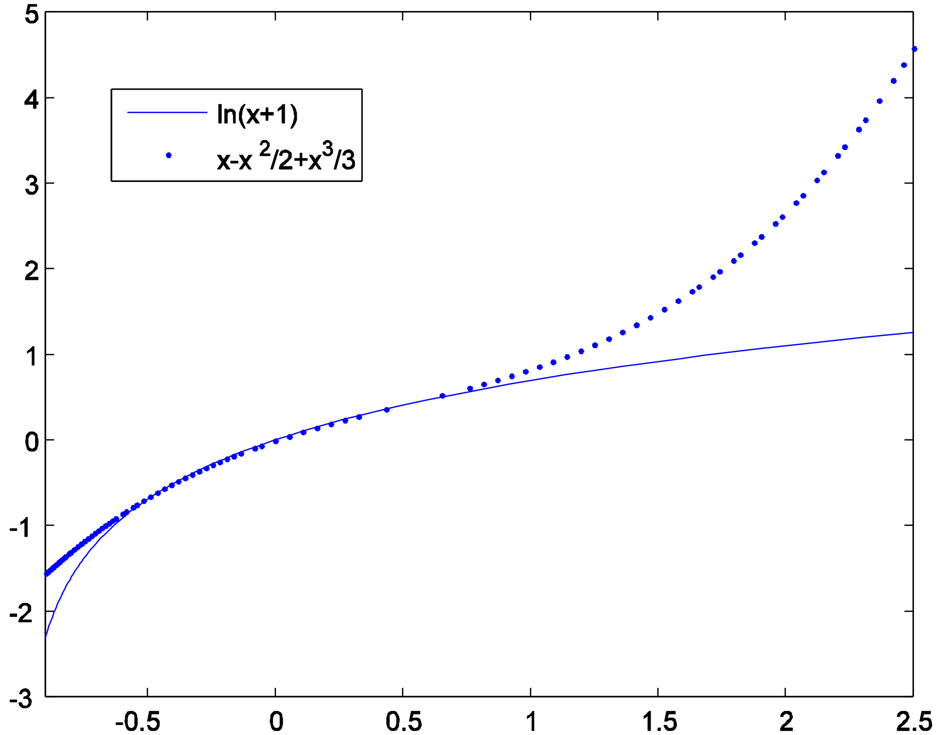
$$f'''(x) = \frac{2}{(x+1)^3} \qquad f'''(0) = 2$$

Thus

$$\ell n(x+1) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$

$$= f(0) + f'(0)\,x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3}$$



How accurate is this polynomial approximation? Let $P_3(x) = x - \frac{x^2}{2} + \frac{x^3}{3}$.

Since $f^{(4)}(x) = \frac{-6}{(x+1)^4}$, for each value of x there exists a value ξ such that

$$\ell n(x+1) - P_3(x) = R_n$$

$$= \frac{f^{(4)}(\xi)}{4!} x^4$$

$$= \frac{-x^4}{4(\xi+1)^4}$$

where ξ is some number between x and $a = 0$.

For example, if $x = 0.25$ then

$$\ell n(1.25) \approx 0.25 - \frac{(0.25)^2}{2} + \frac{(0.25)^3}{3} = 0.2239583333\cdots$$

and the truncation error of this approximation is

$$R_3 = \frac{-(0.25)^4}{4(\xi+1)^4} = \frac{-1}{1024(\xi+1)^4} \quad \text{for some value of } \xi \text{ such that } 0 \le \xi \le 0.25 \ .$$

It is not possible to determine the value of ξ that gives the exact value of R_3, but it is possible to determine an **upper bound** for this truncation error:

$$|R_3| = \left| \frac{-1}{1024(\xi+1)^4} \right| \le \frac{1}{1024(0+1)^4} = 0.0009765625$$

This gives a guaranteed upper bound for the truncation error in the above approximation to $\ell n(1.25)$. Note that the actual (absolute) error is

$$|E_t| = |0.22314355 - 0.22395833| = 0.00081478 \ .$$

Rather than just using a Taylor polynomial approximation to estimate the value of a function at one specified point, it is more common to use the polynomial approximation **for an entire interval of values of *x***. In such a case, it is also desirable to be able to determine the accuracy (that is, an upper bound for the error). For example, suppose that

$$\ell n(x+1) \approx x - \frac{x^2}{2} + \frac{x^3}{3} \quad \text{for any value of } x \in [0\ ,\ 0.5].$$

Then, for any value of $x \in [0\ ,\ 0.5]$,

$$|\ell n(x+1) - P_3(x)| = \left| \frac{-x^4}{4(\xi+1)^4} \right| \le \frac{(0.5)^4}{4(0+1)^4} = 0.015625$$

since $\max\limits_{0\le x\le 0.5}\left|x^4\right| = (0.5)^4$ and, as ξ lies between 0 and *x*, $\max\limits_{0\le \xi\le 0.5}\dfrac{1}{\left|(\xi+1)^4\right|} = 1$.

Example 2

Let $x_{i+1} = x_i + h$ so that $h = x_{i+1} - x_i$. Then Taylor's Theorem for $f(x)$ expanded about x_i and evaluated at $x = x_{i+1}$ is (see (4.7) on page 80 of the 6th ed.; page 83 of the 7th)

$$f(x_{i+1}) = f(x_i) + f'(x_i)h + \frac{f''(x_i)}{2!}h^2 + \frac{f'''(x_i)}{3!}h^3 + \cdots + \frac{f^{(n)}(x_i)}{n!}h^n + R_n.$$

Letting $n = 1$, this gives

$$f(x_{i+1}) = f(x_i) + f'(x_i)h + R_n$$

which implies that

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h} - \frac{R_1}{h}, \text{ where } \frac{R_1}{h} = \frac{1}{h}\frac{f''(\xi)}{2}h^2 = \frac{f''(\xi)}{2}h.$$

See page 86 of the 6th ed; page 89 of the 7th. This gives the first derivative approximation

$$f'(x_i) \approx \frac{f(x_{i+1}) - f(x_i)}{h}$$

that was used in Chapter 1; it also gives the truncation error of this finite difference approximation to the derivative, namely $-\dfrac{R_1}{h} = -\dfrac{f''(\xi)}{2}h$. As this is some constant times *h*, we say that this truncation error is $O(h)$.

Example 3

The Taylor polynomial approximation for $f(x) = e^x$ expanded about $a = 0$ is

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}.$$

It is clear from

$$R_n = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$$

that the truncation error of any Taylor polynomial approximation is small when *x* is close to *a* (note that $R_n = 0$ when $x = a$) and will increase as *x* gets further away from *a*. Also, as *n* increases, the Taylor polynomial approximations become better and better approximations to $f(x)$, provided of course that $f^{(n+1)}(x)$ is bounded on some interval containing x and a. These points are illustrated in the following graph.