COMPUTER SCIENCE 349A
Handout Number 16

GAUSSIAN ELIMINATION WITH PARTIAL PIVOTING
(Section 9.4)

The Naïve Gaussian elimination algorithm will fail if any of the pivots $a_{11}, a_{22}^{(1)}, a_{33}^{(2)}, \ldots$ is equal to 0. Mathematically speaking, the algorithm only requires that the pivots be nonzero. However, as the algorithm fails when a pivot is exactly equal to 0, it often gives very poor numerical results (using floating-point arithmetic) when a pivot is close to 0.

Example

Consider the $n = 2$ linear system with augmented matrix

$$\left[\begin{array}{cc|c} 0.003 & 59.14 & 59.17 \\ 5.291 & -6.13 & 46.78 \end{array}\right].$$

The exact solution is

$$x = \begin{bmatrix} 10 \\ 1 \end{bmatrix}.$$

However, using 4 decimal-digit floating-point rounding arithmetic and Naïve Gaussian elimination, a very inaccurate solution is computed:

$$\begin{aligned}
\text{mult} &\leftarrow f\ell(5.291/0.003) = f\ell(1763.666\ldots) = 1764. \\
a_{22}^{(1)} &\leftarrow f\ell(-6.13 - f\ell(1764 \times 59.14)) \\
&= f\ell(-6.13 - (1043 \times 10^2)) \\
&= -1043 \times 10^2 \quad \text{or} \quad -0.1043 \times 10^6 \\
b_2^{(1)} &\leftarrow f\ell(46.78 - f\ell(1764 \times 59.17)) \\
&= f\ell(46.78 - (1044 \times 10^2)) \\
&= -1044 \times 10^2 \quad \text{or} \quad -0.1044 \times 10^6
\end{aligned}$$

That is, the reduced linear system (in which the coefficient matrix has been reduced to upper triangular form) is

$$\left[\begin{array}{cc|c} 0.003 & 59.14 & 59.17 \\ 0.0 & -1043 \times 10^2 & -1044 \times 10^2 \end{array}\right].$$

From this, back substitution gives a computed solution of

$$\hat{x} = \begin{bmatrix} -10.00 \\ 1.001 \end{bmatrix}.$$

Analysis of the above example. The source of the extremely inaccurate computed solution $\hat{x}$ is the large magnitude of the multiplier (1764), which is much larger than the numbers in the given linear system. This multiplier is large because the pivot $a_{11} = 0.003$ is very small (relative to other numbers in the linear system).

Consequently, in the floating-point computation of $a_{22}^{(1)}$ and $b_2^{(1)}$, the numbers -6.13 and 46.78 are so small that they are lost (or discarded) in the 4-digit floating-point computation.

The **partial pivoting strategy** is designed to avoid the selection of a small pivot, and thus to avoid the computation of unnecessarily large multipliers (relative to other numbers in the linear system):

at step k of the forward elimination (where $1 \le k \le n-1$), choose the pivot to be the largest entry in absolute value from

$$\begin{bmatrix} a_{kk} \\ a_{k+1,k} \\ a_{k+2,k} \\ \vdots \\ a_{nk} \end{bmatrix}.$$

If this largest entry is a_{pk} (that is, $|a_{pk}| = \max_{k \le i \le n} |a_{ik}|$), then rows k and p of the augmented matrix are interchanged. (If $p = k$, then no interchange is done.)

With the partial pivoting strategy, note that all multipliers are ≤ 1 in absolute value.

Note also that a row interchange is a Type 3 elementary row operation (thus, the linear system obtained by applying Gaussian elimination with partial pivoting is equivalent to the given linear system).

The following is an algorithm for **Gaussian Elimination with partial pivoting**:

```
for k = 1, 2, ..., n-1                    (begin forward elimination)
    p ← k                                 (find the pivot)
    for i = k+1, k+2, ..., n
        if |a_ik| > |a_pk|
            p ← i
        end if
    end for
    if p ≠ k                              (interchange rows if necessary)
        for j = k, k+1, ..., n
            temp ← a_kj
            a_kj ← a_pj
            a_pj ← temp
        end for
        temp ← b_k
        b_k ← b_p
        b_p ← temp
    end if
    for i = k+1, k+2, ..., n              (do the elimination on A and b)
        mult ← a_ik / a_kk
        for j = k+1, k+2, ..., n
            a_ij ← a_ij - mult × a_kj
        end for
        b_i ← b_i - mult × b_k
    end for
end for
x_n ← b_n / a_nn                          (begin back substitution)
for i = n-1, n-2, ..., 1
    sum ← b_i
    for j = i+1, i+2, ..., n
        sum ← sum - a_ij × x_j
    end for
    x_i ← sum / a_ii
end for
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NOTE: the above algorithm is similar to that given in the textbook in Figures 9.4 and 9.5 together (on pages 254 and 266 in the 7th ed.; pages 250 and 262 in the 6th ed.).

Example: Gaussian Elimination with partial pivoting.

Suppose that the given augmented matrix is

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 3 & 4 \\ 2 & 1 & -1 & 1 & 1 \\ 3 & -1 & -1 & 2 & -3 \\ -1 & 2 & 3 & -1 & 4 \end{array}\right].$$

<u>Step 1 of the forward elimination</u> with $k=1$: the row index of the pivot is $p \leftarrow 3$, so rows 1 and 3 are interchanged to give

$$\left[\begin{array}{cccc|c} 3 & -1 & -1 & 2 & -3 \\ 2 & 1 & -1 & 1 & 1 \\ 1 & 1 & 0 & 3 & 4 \\ -1 & 2 & 3 & -1 & 4 \end{array}\right].$$

The elimination on rows 2, 3 and 4 gives

$$\left[\begin{array}{cccc|c} 3 & -1 & -1 & 2 & -3 \\ 0 & 5/3 & -1/3 & -1/3 & 3 \\ 0 & 4/3 & 1/3 & 7/3 & 5 \\ 0 & 5/3 & 8/3 & -1/3 & 3 \end{array}\right].$$

<u>Step 2 of the forward elimination</u> with $k=2$: the row index of the pivot is $p \leftarrow 2$. Since $p=k$, no row interchange is done. The elimination on rows 3 and 4 gives

$$\left[\begin{array}{cccc|c} 3 & -1 & -1 & 2 & -3 \\ 0 & 5/3 & -1/3 & -1/3 & 3 \\ 0 & 0 & 3/5 & 13/5 & 13/5 \\ 0 & 0 & 3 & 0 & 0 \end{array}\right]$$

<u>Step 3 of the forward elimination</u> with $k=3$: the row index of the pivot is $p \leftarrow 4$, so rows 3 and 4 are interchanged to give

$$\left[\begin{array}{cccc|c} 3 & -1 & -1 & 2 & -3 \\ 0 & 5/3 & -1/3 & -1/3 & 3 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 3/5 & 13/5 & 13/5 \end{array}\right].$$

The elimination on row 4 gives

$$\left[\begin{array}{cccc|c} 3 & -1 & -1 & 2 & -3 \\ 0 & 5/3 & -1/3 & -1/3 & 3 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 13/5 & 13/5 \end{array}\right],$$

which completes the forward elimination.

Back substitution now gives the solution

$$x = \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix}.$$

Determinant of *A* (see page 263 in the 7th ed.; page 259 in the 6th ed.)

The reduction of A to upper triangular form by Naïve Gaussian elimination uses only the Type 2 elementary row operation

$$E_i \leftarrow E_i - \text{mult} \times E_j .$$

This row operation does not change the value of the determinant of A . So, if no rows are interchanged,

$$\begin{aligned} \det A &= \det\text{ (equivalent upper triangular form of } A \text{)} \\ &= a_{11} a_{22}^{(1)} a_{33}^{(2)} \cdots a_{nn}^{(n-1)} \end{aligned}$$

since the determinant of a triangular matrix is equal to the product of its diagonal entries.

However, if Gaussian elimination with partial pivoting is used to reduce A to upper triangular form, each row interchange causes the determinant to be multiplied by -1. Thus, if m row interchanges are done during the reduction of A to upper triangular form, then

$$\det A = (-1)^m a_{11} a_{22}^{(1)} a_{33}^{(2)} \cdots a_{nn}^{(n-1)}$$