These are the lecture notes for CSC349A Numerical Analysis taught by Rich Little in the Spring of 2018. They roughly correspond to the material covered in each lecture in the classroom but the actual classroom presentation might deviate significantly from them depending on the flow of the course delivery. They are provide as a reference to the instructor as well as supporting material for students who miss the lectures. They are simply notes to support the lecture so the text is not detailed and they are not thoroughly checked. Use at your own risk. They are complimentary to the handouts. Many thanks to all the guidance and materials I received from Dale Olesky who has taught this course for many years and George Tzanetakis.

1 Approximation Theory/Curve Fitting

Our next topic is the study of how a given function can be approximated by another function from a sspecified class of functions. The given function may be discrete or continuous.

Typically the approximating function exhibits some desired properties such as:

1. Continuity
2. Easily differentiated
3. Easily integrated
4. Easily evaluated

Common classes of approximating functions:

1. Polynomials
2. Piecewise polynomials (splines)
3. Trigonometric sums (fourier series)

We will also study criteria for what constitutes a “good” approximating function.

2 Polynomial interpolation

Recall that the general formula for an nth-order polynomial is

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n$$

For $n + 1$ distinct data points there is one and only one order n (or less) polynomial that passes through them all. That is,

- only one line that passes through two points
- only one parabola that passes through three points, etc

Polynomial Interpolation consists of determining the unique nth-order polynomial that fits the $n + 1$ data points in question.

- Although the polynomial is unique there are different methods for finding it and different formats for expressing it.

Formally:
Let $y = f(x)$ be any given function. For any value of $n \geq 0$ and any given values $x_0, x_1, \ldots, x_n$, let $y_i = f(x_i)$. The **polynomial interpolation problem** is to determine a polynomial $P(x)$ of degree less or equal to n for which:

$$P(x_i) = y_i \quad \text{for } i = 0, 1, \ldots, n$$

The set of $n + 1$ data point (x_i, y_i) maybe the only functional values known (that is, $f(x)$ is a **discrete function**, which could occur for example with experimental data), or $f(x)$ maybe be a known **continuous function**, and the $n + 1$ data points (x_i, y_i) are a finite set of values with $y_i = f(x_i)$.

If z is some value between 2 of the given value x_i and if $P(z)$ is computed as an approximation to $f(z)$, then this approximation to $f(z)$ is said to be determined by polynomial **interpolation**.

On the other hand, if z lies outside of the interval containing all of the value x_i and if $P(z)$ is computed as an approximation to $f(z)$, then this approximation to f(z) is said to be determined by polynomial **extrapolation**.

Note that an **interpolating polynomial** and the **Taylor polynomial** both determine polynomial approximations to $f(x)$. However, in general they are very different approximations to $f(x)$. Note that an interpolating polynomial uses the information:

$$y_0 = f(x_0), y_1 = f(x_1), \ldots, y_n = f(x_n)$$

to determine the polynomial approximation, whereas the Taylor polynomial uses the information:

$$f(x_0), f'(x_0), \ldots, f^{(n)}(x_0)$$

to determine the polynomial approximation.

3 Lagrange Interpolating Polynomial

Given $(x_i, f(x_i)), 0 \leq i \leq n$, with all x_i distinct, consider the function:

$$\begin{aligned} P(x) &= \sum_{i=0}^{n} L_i(x) f(x_i) \\ &= L_0(x) f(x_0) + L_1(x) f(x_1) + \cdots + L_n(x) f(x_n) \end{aligned}$$

where

$$\begin{aligned} L_i(x) &= \frac{(x - x_0)(x - x_1) \ldots (x - x_{i-1})(x - x_{i+1}) \ldots (x_i - x_n)}{(x_i - x_0)(x_i - x_1) \ldots (x_i - x_{i-1})(x_i - x_{i+1}) \ldots (x_i - x_n)} \\ &= \prod_{j=0, j\neq i}^{n} \frac{x - x_j}{x_i - x_j}, \quad \text{for } i = 0, 1, 2, \ldots, n \end{aligned}$$

Examples
When $n = 1$,

$$\begin{aligned} P(x) &= \sum_{i=0}^{1} L_i(x) f(x_i) \\ &= L_0(x) f(x_0) + L_1(x) f(x_1) \\ &= \prod_{j=0, j\neq 0}^{1} \frac{x - x_j}{x_0 - x_j} f(x_0) + \prod_{j=0, j\neq 1}^{1} \frac{x - x_j}{x_1 - x_j} f(x_1) \\ &= \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1) \end{aligned}$$

When $n = 2$,

$$\begin{aligned} P(x) &= \sum_{i=0}^{2} L_i(x) f(x_i) \\ &= L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2) \end{aligned}$$

where each $L_i(x)$ is at most a second degree polynomial given by,

$$\begin{aligned} L_0(x) &= \prod_{j=0, j\neq 0}^{2} \frac{x - x_j}{x_0 - x_j} = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} \\ L_1(x) &= \prod_{j=0, j\neq 1}^{2} \frac{x - x_j}{x_1 - x_j} = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} \\ L_2(x) &= \prod_{j=0, j\neq 2}^{2} \frac{x - x_j}{x_2 - x_j} = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} \end{aligned}$$

Therefore,

$$\begin{aligned} P(x) = &\frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1) \\ &+ \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2) \end{aligned} \tag{1}$$

Note here that $L_0(x_0) = 1$ and both $L_0(x_1) = L_0(x_2) = 0$. Similarly for L_1 and L_2.

Since each function $L_i(x)$ is a polynomial of order n and $f(x_i)$ is a constant, $P(x)$ is a polynomial of order $\leq n$. Since

$$L_i(x_i) = 1 \text{ and } L_i(x_j) = 0 \text{ if } j \neq i,$$

it follows that:

$$P(x_i) = f(x_i), \quad \text{for } i = 0, 1, 2, \ldots, n$$

that is, $P(x)$ is a an interpolating polynomial for the given data. It is called the **Lagrange interpolating polynomial**.

Example 1 Evaluate ln(2) using Lagrange polynomial interpolation, given that

$$\begin{aligned}\ln 1 &= 0\\ \ln 4 &= 1.386294\\ \ln 6 &= 1.791760\end{aligned}$$

Here, $x_0 = 1, x_1 = 4, x_2 = 6$ and $f(x_0) = 0, f(x_1) = 1.386294, f(x_2) = 1.791760$. Substituting these into equation (1) above gives,

$$\begin{aligned}P(x) &= \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}f(x_1)\\ &\quad + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}f(x_2)\\ &= \frac{(x-4)(x-6)}{(1-4)(1-6)}0 + \frac{(x-1)(x-6)}{(4-1)(4-6)}(1.386294)\\ &\quad + \frac{(x-1)(x-4)}{(6-1)(6-4)}(1.791760)\end{aligned}$$

and thus,

$$\begin{aligned}P(2) &= \frac{(2-4)(2-6)}{(1-4)(1-6)}0 + \frac{(2-1)(2-6)}{(4-1)(4-6)}(1.386294)\\ &\quad + \frac{(2-1)(2-4)}{(6-1)(6-4)}(1.791760)\\ &= \frac{-4}{3(-2)}(1.386294) - \frac{2}{5(2)}(1.791760)\\ &= 0.565844\end{aligned}$$

I ran the following script in MATLAB to illustrate the above example. The plot produced by this is given in Figure 1.

```
x = [0 : 0.01 : 8];
y = log(x);
L0x = 0;
L1x = -0.231049.*(x.^2-7.*x+6);
L2x = 0.179176 .* (x.^2-5.*x+4);
Px = L1x+L2x;
```

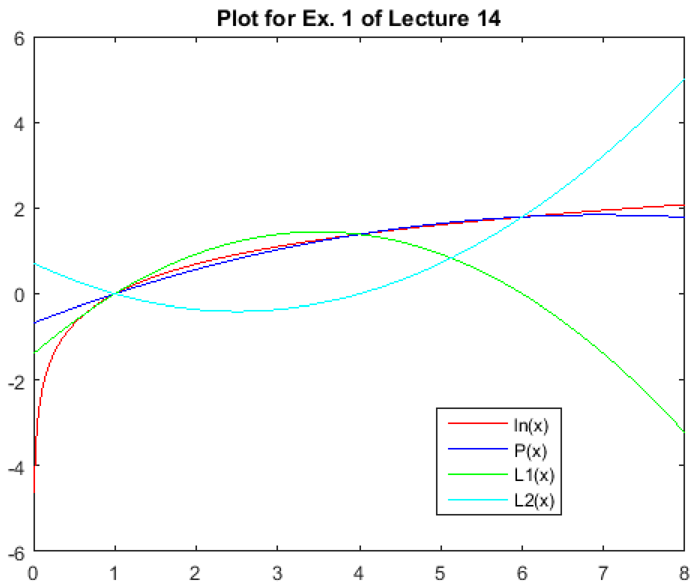

Figure 1: Plot of the Lagrange polynomial for $\ln(x)$ plus the individual $L_i(x)$ polynomials used in its construction

```
plot(x,y)
plot(x,y,x,Px)
plot(x,y,x,Px,x,L0x,x,L1x,x,L2x)
plot(x,y,'r',x,Px,'b',x,L0x,'y',x,L1x,'g',x,L2x,'c')
```

Example 2 See Handout 20 - Complete elliptic integral function

$$K(k) = \int_0^{\pi/2} \frac{dz}{\sqrt{1-k^2\sin^2 z}}$$

where

$\sin^{-1} k$	$K(k)$
$65°$	2.3088
$66°$	2.3439
$67°$	2.3809

3.1 Finding the coefficients of the interpolating polynomial

Although the Langrange polynomial is well-suited to solving intermediate values it does not give you a polynomial in simple form

$$P(x) = a_0 + a_1x + \cdots + a_nx^n$$

The coefficients of such an interpolating polynomial can be determined by solving a system of linear equations.

Given a function $f(x)$ and distinct points $x_0, x_1, ..., x_n$, let $P(x)$ be the polynomial of degree $\leq n$ for which $P(x_i) = f(x_i)$ for $i = 0, 1, ..., n$.

Then,

$$\begin{aligned} a_0 + a_1x_0 + \cdots + a_nx_0^n &= f(x_0) \\ a_0 + a_1x_1 + \cdots + a_nx_1^n &= f(x_1) \\ &\vdots \\ a_0 + a_1x_n + \cdots + a_nx_n^n &= f(x_n) \end{aligned}$$

In matrix form, solve

$$\begin{bmatrix} 1 & x_0 & \dots & x_0^n \\ 1 & x_1 & \dots & x_1^n \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & \dots & x_n^n \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_n) \end{bmatrix}$$

So, if $n = 2$, we let $P(x) = a_0 + a_1x + a_2x^2$ and solve

$$\begin{bmatrix} 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \end{bmatrix}$$

Example Let $f(x) = \sin x$, $x_0 = 0.2$, $x_1 = 0.5$, and $x_2 = 1$ and find the interpolating polynomial.

An interpolating polynomial can be specified in many different forms. For example the form can be $a(x - x_2)^2 + b(x - x_2) + c$ or using the Lagrange form for $n = 2$:

$$\begin{aligned} P(x) &= L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2) \\ &= \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)}f(x_0) + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)}f(x_1) + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}f(x_2) \end{aligned}$$

or simply as $P(x) = Ax^2 + Bx + C$.

We will show that all of these forms are identical as the interpolating polynomial is unique.

3.2 Uniqueness

Theorem: Given any $n+1$ distinct points $x_0, x_1, \dots, x_n$ and any $n+1$ values $f(x_0), f(x_1), \dots, f(x_n)$, there exists a unique polynomial $P(x)$ of degree $\leq n$ such that

$$P(x_i) = f(x_i) \text{ for } 0 \leq i \leq n$$

Proof:
Existence: by construction of the Lagrange interpolating polynomial
Uniqueness:

Suppose there exist two polynomials $P(x)$ and $Q(x)$ of degree $\leq n$ such that:

$$P(x_i) = Q(x_i) = f(x_i), 0 \leq i \leq n$$

Consider the function

$$R(x) = P(x) - Q(x)$$

which is also a polynomial of degree $\leq n$. But $R(x_i) = 0$ for $0 \leq i \leq n$. That is $R(x)$ has $n+1$ distinct zeros. This implies that $R(x) = 0$ for all x and therefore $P(x) = Q(x)$.

3.3 Error term of polynomial interpolation

Theorem:
Let $x_0, x_1, \dots x_n$ be any distinct points in $[a, b]$. Let $f(x) \in C^{n+1}[a, b]$ and let $P(x)$ interpolate $f(x)$ at x_i.

Then for each $\hat{x} \in [a, b]$, there exists a value ξ in (a, b) such that

$$f(\hat{x}) = P(\hat{x}) + \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^{n} (\hat{x} - x_i)$$

for example for $n = 3$

$$f(\hat{x}) = P(\hat{x}) + \frac{f^{(4)}(\xi)}{24} (\hat{x} - x_0)(\hat{x} - x_1)(\hat{x} - x_2)(\hat{x} - x_3)$$

The limitation of this error bound for polynomial interpolation is the need to find an upper bound for $f^{(n+1)}(x)$ on $[a, b]$.