These are the lecture notes for CSC349A Numerical Analysis taught by Rich Little in the Spring of 2018. They roughly correspond to the material covered in each lecture in the classroom but the actual classroom presentation might deviate significantly from them depending on the flow of the course delivery. They are provided as a reference to the instructor as well as supporting material for students who miss the lectures. They are simply notes to support the lecture so the text is not detailed and they are not thoroughly checked. Use at your own risk.

1 Introduction

Truncation errors occur when some exact mathematical procedure is replaced by a finite approximation.

Examples:

- Approximation of a function by a finite number of terms in its Taylor series e.g.
$$e^x \approx 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$
- Approximation of a derivative by a finite difference:
$$\frac{dv}{dt}\Big|_{t=t_i} \approx \frac{v(t_{i+1}) - v(t_i)}{t_{i+1} - t_i}$$
- Approximation of a definite integral by a finite sum of function values
$$\int_a^b f(x)dx \approx \frac{b-a}{2}\left[f(a) + f(b)\right]$$

Taylor's Theorem is the fundamental tool for deriving and analyzing numerical approximation formulas in this course. It states that any "smooth" function (one with a sufficient number of derivatives) can be approximated by a polynomial, and it includes an error (remainder) term that indicates how accurate the polynomial approximation is. Taylor's theorem also provides a means to estimate the value of a function $f(x)$ at some point x_{i+1} using the values of $f(x)$ and its derivatives at some nearby point x_i.

Taylor's Theorem

Let $n \geq 0$ and let a be any constant. If $f(x)$ and its first $n+1$ derivatives are continuous on some interval containing x and a, then:

$$f(x) = f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+\ldots+\frac{f^n(a)}{n!}(x-a)^n+R_n$$

where the remainder (or error) term is:

$$R_n = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$$

and ξ is some value between x and a. Note that:

$$P_n(x) = f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+\ldots+\frac{f^n(a)}{n!}(x-a)^n$$

is a polynomial of degree n in x, and is called the **Taylor polynomial approximation of degree** n for f(x) expanded about a. R_n is the **truncation error** of this polynomial approximation to $f(x)$.

2 Taylor Example 1

Determine the Taylor polynomial approximation of order $n = 3$ for $f(x) = \ln(x+1)$ expanded about $a = 0$ (McLaurin series when $a = 0$).

$$\begin{aligned}
f(x) &= \ln(x+1), \quad f(0) = 0 && (1)\\
f'(x) &= \tfrac{1}{x+1}, \quad f'(0) = 1 && (2)\\
f''(x) &= \tfrac{-1}{(x+1)^2}, \quad f''(0) = -1 && (3)\\
f'''(x) &= \tfrac{2}{(x+1)^3}, \quad f'''(0) = 2 && (4)
\end{aligned}$$

Thus:

$$\begin{aligned}
\ln(x+1) &= f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+\ldots+\frac{f^n(a)}{n!}(x-a)^n\\
&= f(0)+f'(0)+\frac{f''(0)}{2}x^2+\frac{f'''(0)}{6}x^3\\
&= x-\frac{x^2}{2}+\frac{x^3}{3}
\end{aligned}$$

How accurate is this polynomial approximation?

Let $P_3(x) = x - \frac{x^2}{2} + \frac{x^3}{3}$. Since $f^{(4)}(x) = \frac{-6}{(x+1)^4}$, for each value of x there exists a value ξ such that:

$$\ln(x+1) - P_3(x) = R_n = \frac{f^{(4)}(\xi)}{4!}x^4 = \frac{-x^4}{4(\xi+1)^4}$$

where ξ is some number between x and $a = 0$.

For example, if $x = 0.25$ then

$$ln(1.25) \approx 0.25 - \frac{(0.25)^2}{2} + \frac{(0.25)^3}{3} = 0.223958333\ldots$$

and the trunaction error of this approximation is:

$$R_3 = \frac{-(0.25)^4}{4(\xi+1)^4} = \frac{-1}{1024(\xi+1)^4}$$

for some value of ξ such that $0 \leq \xi \leq 0.25$.

It is not possible to determine the value of ξ that gives the exact value of R_3, but it is possible to determine an **upper bound** for this truncation error:

$$|R_3| = \left|\frac{-1}{1024(\xi_1)^4}\right| \leq \frac{1}{1024(0+1)^4} = 0.0009765625$$

This gives a guaranteed upper bound for the truncation error in the above approximation to ln(1.25). Note that the actual (absolute) error is:

$$|E_t| = |0.22314335 - 0.22395833| = 0.00081478$$

Rather than just using the Taylor polynomial approximation to estimate the value of a function at one specified point, it is more common to use the polynomial approximation **for an entire interval of values x**. In such a case, it is also desirable to be able to determine the accuracy (that is an upper bound for the error). For example, suppose that

$$\ln(x+1) \approx x - \frac{x^2}{2} + \frac{x^3}{3}$$

for any value $x \in [0, 0.5]$. Then for any value of $x \in [0, 0.5]$,

$$|\ln(x+1) - P_3(x)| = \left|\frac{-x^4}{4(\xi+1)^4}\right| \leq \left|\frac{(0.5)^4}{4(0+1)^4}\right| = 0.015625$$

since $\max_{0 \le x \le 0.5} |x^4| = (0.5)^4$ and, as ξ lies between 0 and x, $\max_{0 \le \xi \le 0.5} \frac{1}{|(\xi+1)^4|} = 1$.

To illustrate Taylor's approximation I plotted this functon $f(x) = ln(x+1)$ against three different approximations, $P_1(x) = x$, $P_2(x) = x - \frac{x^2}{2}$, and $P_3(x) = x - \frac{x^2}{2} + \frac{x^3}{3}$. The commands and plot are shown below:

```
>> x=[-1:0.05:2.5];
>> fx = log(x+1);
>> Px1 = x;
>> Px2 = x - x.^2/2;
>> Px3 = Px2 + x.^3/3;
>> plot(x,fx,'r',x,Px1,'b',x,Px2,'g',x,Px3,'c')
```

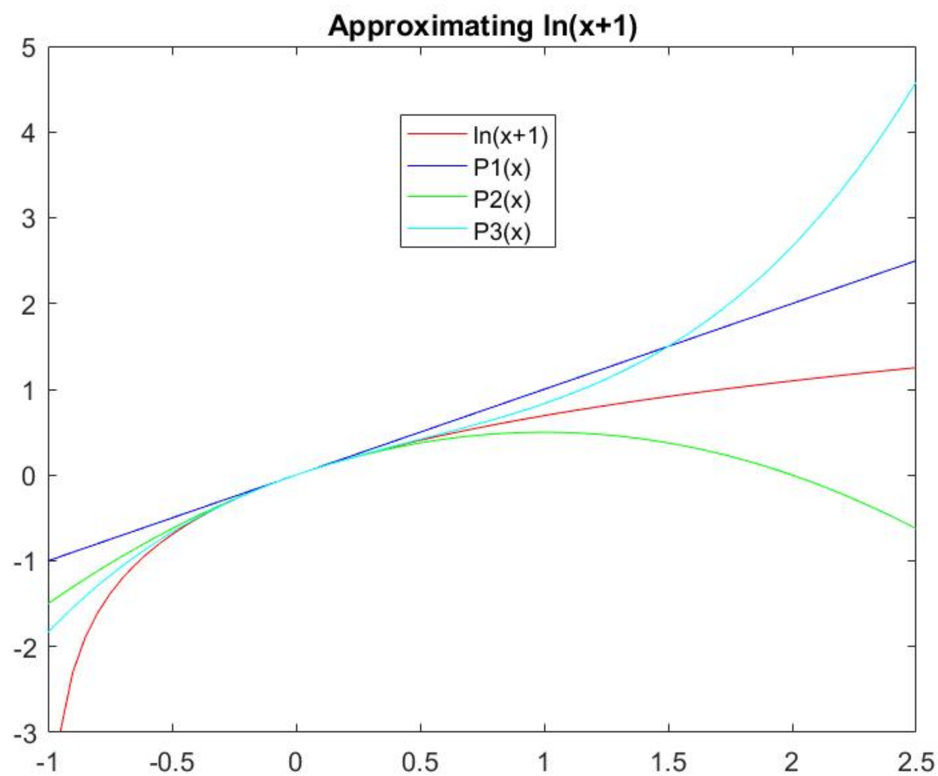

Figure 1: Comparison of degrees of Approximation

3 Taylor Example 2

Let $x_{i+1} = x_i + h$ so that $h = x_{i+1} - x_i$. Then Taylor's theorem for $f(x)$ expanded about x_i, and evaluated at $x = x_{i+1}$ is:

$$f(x_{i+1}) = f(x_i) + f'(x_i)h + \frac{f''(x_i)}{2!}h^2 + \frac{f'''(x_i)}{3!}h^3 + \ldots + \frac{f^n(x_i)}{n!}h^n + R_n$$

Letting $n = 1$, this gives

$$f(x_{i+1}) = f(x_i) + f'(x_i)h + R_1$$

which implies that:

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h} - \frac{R_1}{h}$$

where

$$\frac{R_1}{h} = \frac{1}{h}\frac{f''(\xi)}{2}h^2 = \frac{f''(\xi)}{2}h$$

This gives the first derivative approximation

$$f'(x_i) \approx \frac{f(x_{i+1}) - f(x_i)}{h}$$

that was used in Chapter 1; it also gives the trunaction error of this finite difference approximation to the derivative, namely $-\frac{R_1}{h} = -\frac{f''(\xi)}{2}h$. As this is some constant times h, we say that this truncation error is $O(h)$.

4 Taylor Example 3

The Taylor polynomial approximation for $f(x) = e^x$ expanded about $a = 0$ is

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!}$$

It is clear from:

$$R_n = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$$

that the truncation error of any Taylor polynomial approximation is small when x is close to a (note that $R_n = 0$ when $x = a$) and will increase as x gets further away from a. Also, as n increases, the Taylor polynomial approximations become better and better approximations to $f(x)$, provided of course that $f^{(n+1)}(x)$ is bounded on some interval containing x and a.

5 Additional material

In addition to the material in the handouts I briefly mentioned that there is an alternative form for the remainder R_n which can also be defined as:

$$R_n = \int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)dt \tag{5}$$

This form is called the *integral form*. The form we covered in class with the unkown value ξ is referred to as the *derivative* or *Lagrange form* of the remainder named after the famous mathematician *Joseph-Louis Lagrange (1736-1813)* who characterized the remainder term and realized the fundamental importance of Taylor's theorem in calculus. The derivation is based on the first theorem of mean for integrals which states that if a function f is continuous and integrable on an interval containing α and x, then there exists a point ξ between α and x such that:

$$\int_\alpha^x g(t)dt = g(\xi)(x-\alpha) \tag{6}$$