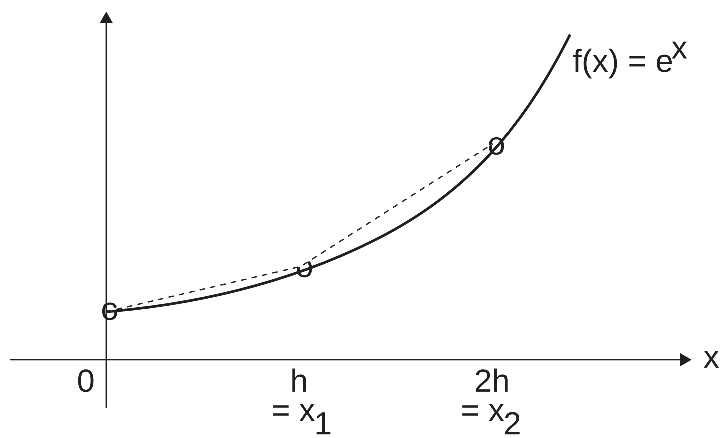
**COMPUTER SCIENCE 349A**

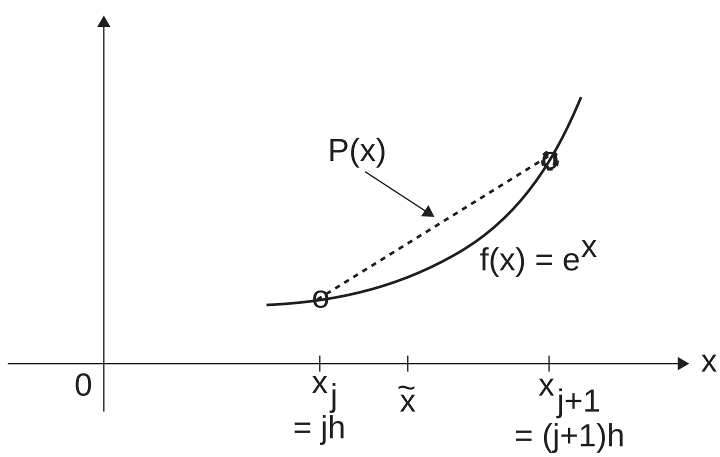
**Handout Number 22**

**USE OF THE ERROR TERM OF POLYNOMIAL INTERPOLATION**

**PROBLEM**: produce a table of values of $e^x$ for $x = 0,\ h,\ 2h,\ 3h,\ \ldots,\ 1$, such that linear interpolation between successive values in the table will yield an approximation with an absolute error $< 10^{-6}$.




Let $x_k = kh$, let $\tilde{x} \in [0, 1]$ and suppose that $j$ is such that $x_j \le \tilde{x} \le x_{j+1}$.




So, our problem can be stated as follows: if we estimate $e^{\tilde{x}}$ by $P(\tilde{x})$, determine a value of $h$ so that the error in this approximation, which is $\left| e^{\tilde{x}} - P(\tilde{x}) \right|$, is $< 10^{-6}$ for all values of $\tilde{x}$ in $[0, 1]$.

**Solution**: The error term for polynomial interpolation with $n = 1$ (linear interpolation) gives

$$f(\tilde{x}) - P(\tilde{x}) = \frac{f''(\xi)}{2}(\tilde{x} - x_j)(\tilde{x} - x_{j+1}),$$

and for our problem we have $f(x) = e^x$, $x_j = jh$ and $x_{j+1} = (j+1)h$. From above,

$$(*) \qquad \left| f(\tilde{x}) - P(\tilde{x}) \right| \le \frac{1}{2} \max_{x_j \le \xi \le x_{j+1}} \left| e^{\xi} \right| \max_{jh \le \tilde{x} \le (j+1)h} \left| (\tilde{x} - jh)\left(\tilde{x} - (j+1)h\right) \right| .$$

The right-hand side of this expression is an upper bound for the absolute value of the error of the linear interpolation, and we need to determine $h$ so that this upper bound is $< 10^{-6}$ for all choices of $\tilde{x}$ and $\xi$. To do this, we need to determine the values of the two maximums above.

Clearly,

$$\max_{x_j \le \xi \le x_{j+1}} \left| e^{\xi} \right| = e \;,$$

since for $\tilde{x} \in [0,1]$, the value of $\xi$ can be any value in $[0,1]$.

To determine the other maximum, use calculus: let

$$\begin{aligned} g(\tilde{x}) &= (\tilde{x} - jh)\left(\tilde{x} - (j+1)h\right) \\ &= \tilde{x}^2 - (2j+1)h\tilde{x} + j(j+1)h^2, \end{aligned}$$

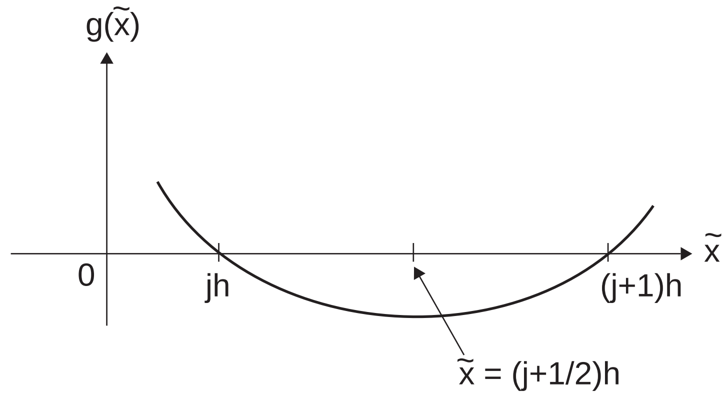
and set $g'(\tilde{x}) = 0$, which gives

$$g'(\tilde{x}) = 2\tilde{x} - (2j+1)h = 0,$$

which implies that

$$\tilde{x} = \left(j + \frac{1}{2}\right)h \;.$$

Alternatively (without using calculus), clearly the maximum of this quadratic in $\tilde{x}$ must occur at the midpoint between its two zeros, and the midpoint between $\tilde{x} = jh$ and $\tilde{x} = (j+1)h$ is $\tilde{x} = \left(j + \frac{1}{2}\right)h$.




So $\max\limits_{jh \le \tilde{x} \le (j+1)h} |g(\tilde{x})|$ occurs when $\tilde{x} = \left(j + \frac{1}{2}\right)h$, which gives a maximum of

$$\left| g\left( (j + \frac{1}{2})h \right) \right| = \left| \frac{-h^2}{4} \right| .$$

Therefore, from (*) above,

$$| f(\tilde{x}) - P(\tilde{x}) | \le \frac{1}{2}(e)\left(\frac{h^2}{4}\right) = \frac{eh^2}{8} .$$

As we want this upper bound on the error of the linear interpolation to be $< 10^{-6}$, we require that $\frac{eh^2}{8} < 10^{-6}$, which implies that $h < 0.00172$. Thus, for example, a good choice for *h* would be 0.001 .

**Conclusion**: if you generated and stored a table of 1001 values of $e^x$ and did linear interpolation between any two adjacent values in the table to determine an approximation to $e^x$ for any value of *x* that is not in this table, then this approximation is guaranteed to have an absolute error $< 10^{-6}$.

| $x$ | $e^x$ |
|---|---|
| 0 | 1.00000000 |
| 0.001 | 1.00100050 |
| 0.002 | 1.00200200 |
| $\vdots$ | |
| 0.123 | 1.13088442 |
| 0.124 | 1.13201587 |
| $\vdots$ | |
| 0.999 | 2.71556491 |
| 1 | 2.71828183 |