These are the lecture notes for CSC349A Numerical Analysis taught by Rich Little in the Spring of 2018. They roughly correspond to the material covered in each lecture in the classroom but the actual classroom presentation might deviate significantly from them depending on the flow of the course delivery. They are provided as a reference to the instructor as well as supporting material for students who miss the lectures. They are simply notes to support the lecture so the text is not detailed and they are not thoroughly checked. Use at your own risk. They are complimentary to the handouts. Many thanks to all the guidance and materials I received from Dale Olesky who has taught this course for many years and George Tzanetakis.

1 Numerical Integration (Quadrature)

The process of determining areas e.g. area of a circle by inscribed and superscribed polygons. This term is used to avoid confusion with the numeric integration of differential equations.
Problem: approximate the value of

$$\int_a^b f(x)dx$$

where $f(x)$ is such that it cannot be integrated analytically or it is known at only a finite set of points.
Procedure: approximate $f(x)$ by an interpolating polynomial $P(x)$, and approximate $\int_a^b f(x)dx$ by $\int_a^b P(x)dx$
Suppose $P_n(x)$ is the Lagrange form of the interpolating polynomial:

$$P_n(x) = \sum_{i=0}^{n} L_i(x)f(x_i)$$

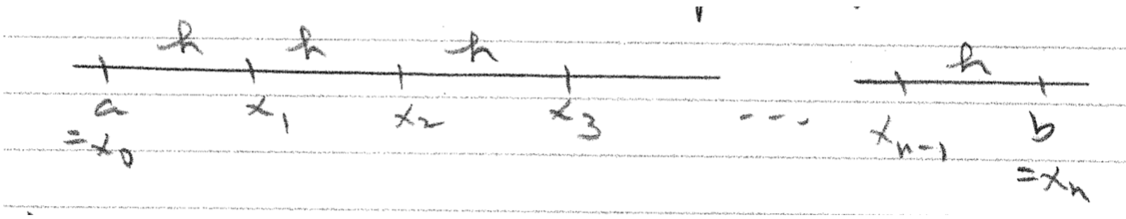
then

$$\int_a^b f(x)dx \approx \int_a^b \left[\sum_{i=0}^{n} L_i(x)f(x_i)\right] dx = \sum_{i=0}^{n} \left[\int_a^b L_i(x)dx\right] f(x_i)$$

which is of the form $\sum_{i=0}^{n} a_i f(x_i)$.

Such an approximation is called a quadrature formulate, and a_i are the **quadrature coefficients** and x_i are the quadrature points, the points at which $f(x)$ is sampled to approximate $\int_a^b f(x)dx$.

Types of quadrature formulas:



- Newton-Cotes closed
- Newton-Cotes open
- Gaussian (omit)

Any quadrature formulate derived by integrating an interpolating polynomial at equally-spaced quadrature points is called a **Newton-Cotes** formula.

Gaussian formulas obtain high accuracy by using optimally-chosen, unequally-spaced quadrature points.

1.1 Netwo-Cotes closed formulas

For any $n \geq 1$, subdivide [a,b] into n subintervals of length $h = \frac{b-a}{n}$.

Thus

$$x_{i+1} - x_i = h$$
$$x_i = x_0 + ih$$

If $P_n(x)$ interpolates $f(x)$ at $a = x_0, x_1, x_2, \ldots, b = x_n$ and

$$\int_a^b f(x)dx \approx \int_a^b P_n(x)dx$$

then the resulting quadrature formula is called a **Newton-Cotes** closed formula.

1.2 Newton-Cotes closed quadrature formulas - n=1

The case $n = 1$:
Here $h = b - a$. The (linear) interpolating polynomial is:

$$P(x) = \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1)$$

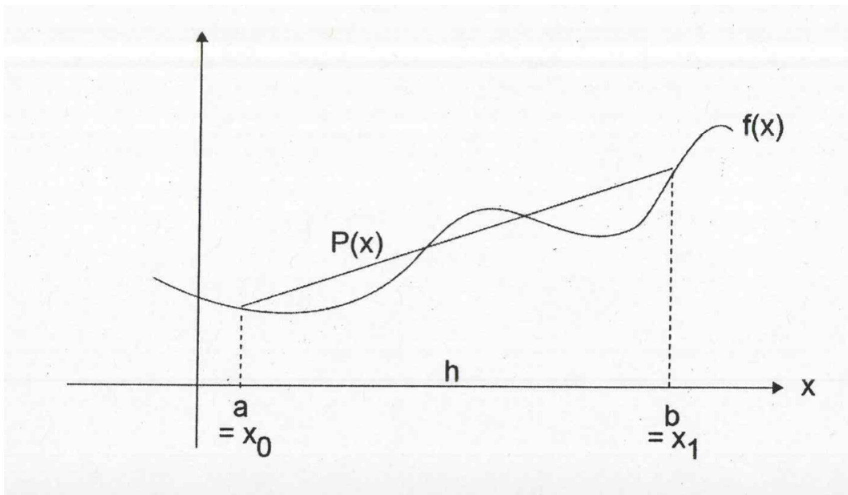


Figure 1: Linear interpolating polynomial

The quadrature formula for approximating $\int_a^b f(x)dx$ is obtained by integrating P(x):

$$\begin{aligned}
\int_a^b f(x)dx &\approx \int_{x_0}^{x_1} P(x)dx \\
&= \left[\int_{x_0}^{x_1} \frac{x-x_1}{x_0-x_1} f(x_0)dx\right] + \left[\int_{x_0}^{x_1} \frac{x-x_0}{x_1-x_0} f(x_1)dx\right] \\
&= \frac{f(x_0)}{x_0-x_1}\left[\frac{x^2}{2} - x_1 x\right]_{x_0}^{x_1} + \frac{f(x_1)}{x_1-x_0}\left[\frac{x^2}{2} - x_0 x\right]_{x_0}^{x_1} \\
&= \frac{x_1-x_0}{2} f(x_0) + \frac{x_1-x_0}{2} f(x_1) \\
&= \frac{h}{2}[f(x_0)+f(x_1)], \text{ since } h = x_1 - x_0
\end{aligned}$$

This is the **trapezoid rule**. Its error term can be obtained by integrating the error term of the Lagrange form of the interpolating polynomial, which for $n=1$ is

$$f(x) - P(x) = \frac{f''(\xi)}{2}(x-x_0)(x-x_1)$$

where ξ is in the interval $[a,b]$. Integrating this gives:

$$\begin{aligned}\int_a^b f(x)dx - \int_{x_0}^{x_1} P(x)dx &= \int_a^b f(x)dx - \frac{h}{2}[f(x_0) - f(x_1)]\\ &= \int_a^b \frac{f''(\xi)}{2}(x-x_0)(x-x_1)dx\\ &= \frac{f''(\xi)}{2}\int_a^b (x-x_0)(x-x_1)dx\end{aligned}$$

since $f''(\xi)$ is a constant. Now, let $t = \frac{x-x_0}{h}$, then $dx = hdt$, $x - x_0 = th$, and $x - x_1 = (t-1)h$. Also, when $x = x_0$ then $t = 0$ and when $x = x_1$, then $t = 1$. Thus, we now have

$$\begin{aligned}\int_a^b f(x)dx - \int_{x_0}^{x_1} P(x)dx &= \frac{f''(\xi)}{2}\int_0^1 h^2 t(t-1)(hdt)\\ &= h^3\frac{f''(\xi)}{2}\int_0^1 t(t-1)dt\\ &= -\frac{h^3}{12}f''(\xi)\end{aligned}$$

for some value ξ between a and b.

This error term is the *truncation error* when $\int_a^b f(x)dx$ is approximated by $\int_a^b P(x)dx$.

For the case $n = 2$ the quadratic interpolating polynomial is:

$$P(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}f(x_2)$$

As in the case $n = 1$, the quadrature formula for approximating $\int_a^b f(x)dx$ is obtained by integrating $P(x) : \int_a^b f(x)dx \approx \int_{x_0}^{x_1} P(x)dx$.

This gives:

$$\int_a^b f(x)dx \approx \frac{h}{3}(f(x_0) + 4f(x_1) + f(x_2))$$

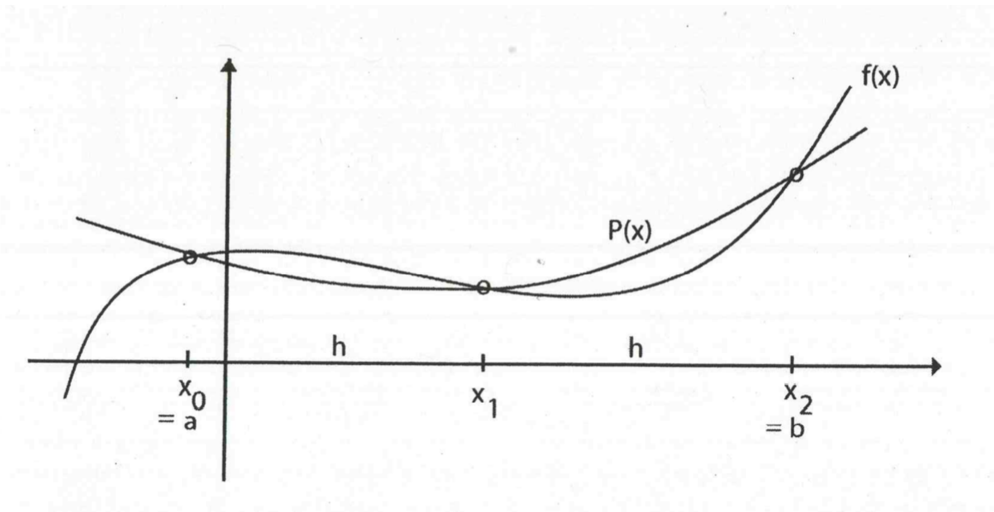


Figure 2: Quadratic interpolating polynomial

where now $h = \frac{b-a}{2}$. This is called **Simpson's rule** or **Simpson's 1/3 rule**, and its **truncation error** is given by:

$$\int_a^b f(x)dx - \int_{x_0}^{x_2} P(x)dx = -\frac{h^5}{90} f^{(4)}(\xi), \text{ for some } \xi \in [a,b]$$

The Newton-Cotes closed quadrature formula for $n = 3$, in which $f(x)$ is approximated by a cubic polynomial that interpolates at four equally-spaced points, is:

$$\int_a^b f(x)dx \approx \frac{3h}{8}(f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)), \text{ where } h = \frac{b-a}{3}$$

The truncation error for this is

$$E_t = \frac{-3}{80} h^5 f^{(4)}(\xi)$$

for some $\xi \in [a,b]$.

2 Degree of Precision of a Quadrature Formula

The **degree of precision** of a quadrature formula is a measure of its accuracy or power. It is an integer number that indicates the degree (or order) of the set of all polynomials that the quadrature formula will integrate exactly. The larger the degree of precision, the more accurate or powerful is the quadrature formula because it will integrate exactly a larger set of polynomials, and this is a very good indicator that it will therefore integrate non-polynomial functions more accurately.

Defintion If a quadrature formula $\sum_{i=0}^{n} a_i f(x_i)$ computes the exact value of $\int_a^b f(x)dx$ whenever $f(x)$ is a polynomial of degree d, but

$$\sum_{i=0}^{n} a_i f(x_i) \neq \int_a^b f(x)dx$$

for some polynomial $f(x)$ of degree $d+1$, then the **degree of precision** of the quadrature formula is d.

Note If $f(x)$ is a polynomial of degree d, then $f(x) = \sum_{i=0}^{d} c_i x^i$ and

$$\int_a^b f(x)dx = \sum_{i=0}^{d} c_i \left[\int_a^b x^i dx\right]$$

Consequently, a quadrature formula computes $\int_a^b f(x)dx$ exactly if and only if it computes each of

$$\int_a^b dx, \int_a^b x dx, ..., \int_a^b x^d dx$$

exactly.

Theorem Therefore, the degree of precision is d if and only if the quadrature formula computes the exact value of the integral when

$$f(x) = 1, x, x^2, ..., x^d$$

and it is not exact when $f(x) = x^{d+1}$.

Example 1: Determine the degree of precision for the Trapezoidal Rule.

2.1 Another way

The degree of precision is also clear if you know the error term for the quadrature formula. For example, since the error term for the Trapezoidal Rule is

$\frac{-h^3}{12} f''(\xi)$, for some value $\xi \in [a, b]$, this error term is exactly equal to 0 if and only if $f''(x) = 0$ for all $x \in [a, b]$. This is true if and only if $f(x) = c_0 + c_1 x$. That is, the Trapezoidal Rule computes the exact value of the integral of a polynomial $f(x)$ if and only if $f(x)$ is a polynomial of degree ≤ 1 (that is, the degree of precision is $d = 1$).

Example 2: Determine the degree of precision for Simpon's 1/3 Rule.

2.2 Note on the Precision of Simpson's 1/3 Rule

The degree of precision $d = 3$ for Simpsons 1/3 rule is larger than expected since it is obtained by integrating a quadratic interpolating polynomial. This means that if $P(x)$ is a quadratic polynomial that interpolates any cubic polynomial

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$$

at the points a, b and $(a + b)/2$, then

$$\int_a^b f(x)dx = \int_a^b P(x)dx$$

exactly. This larger-than-expected value of the degree of precision occurs for Newton-Cotes closed quadrature formulas **for all even values of** n. This is not so of the odd values of n. (Note the error term for the Cubic Quadrature case: $\frac{-3}{80} h^5 f^{(4)}(\xi)$.) In summary,

n	degree of precision
1 (trap rule)	1
2 (1/3 rule)	3
3 (3/8 rule)	3
4	5
5	5
6	7