These are the lecture notes for CSC349A Numerical Analysis taught by Rich Little in the Spring of 2018. They roughly correspond to the material covered in each lecture in the classroom but the actual classroom presentation might deviate significantly from them depending on the flow of the course delivery. They are provided as a reference to the instructor as well as supporting material for students who miss the lectures. They are simply notes to support the lecture so the text is not detailed and they are not thoroughly checked. Use at your own risk. They are complimentary to the handouts. Many thanks to all the guidance and materials I received from Dale Olesky who has taught this course for many years and George Tzanetakis.

1 Ordinary Differential Equations

- Corresponds to Handouts 34-37 and Chapter 25 in the text.
- We did this way back in Lecture 1 when we solved the parachutist problem analytically and numerically.
- These correspond to solving first-order, initial-value ordinary differential equations.
- That is, solving for $y(x)$ given,
$$y'(x) = f(x, y(x)) \text{ and } y(x_0) = y_0$$
- Here, x is the independent variable, y the dependent variable as a function of x, and $f(x, y(x))$ is a function of x and y.
- The goal is to determine $y(x)$ over $a \leq x \leq b$ given $a = x_0, b, y_0$ and $f(x, y(x))$.

Example 1: Analytic Method

Determine $y(x)$ for $0 \leq x \leq 2$ such that
$$y'(x) = y - x^2 + 1$$
subject to $y(0) = 0.5$.

This problem has an analytic solution, namely
$$y(x) = (x + 1)^2 - 0.5e^x$$
.

1.1 Numerical Methods

All numerical methods we will consider for approximating $y(x)$ are called **difference methods**: that is, the continuous, exact solution $y(x)$ is approximated by a finite set of computed values at a set of mesh points $x_0, x_1, ..., x_N$ in $[a, b]$.

For now, we consider only equally-spaced mesh points, and let

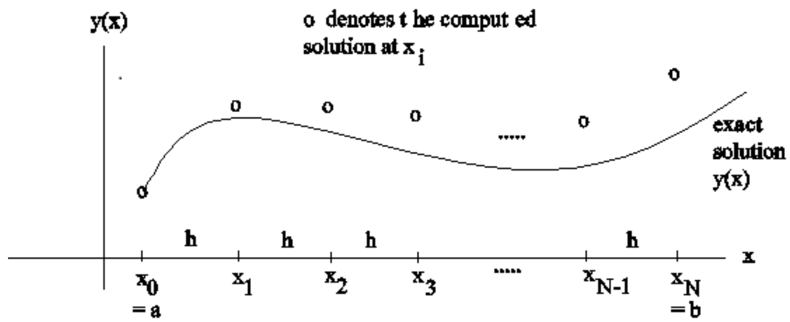
$$x_i = a + ih = x_0 + ih, \text{ for } i = 0, 1, 2, ..., N$$

where

$$h = \frac{b-a}{N}$$

is called the step size.

Notation: At each of the $N+1$ mesh points x_i, the exact solution is denoted by $y(x_i)$ and the approximate solution by y_i.



In general, we derive iterative formulas of the form

$$y_{i+1} = y_i + \phi h$$

where ϕ is the slope of the tangent at x_i.

- Usually, we approximate ϕ. This is done in different ways for different methods.
- In Euler's Method the right hand side of the differential equation itself gives us an approxiamtion of ϕ at each point.
- Recall, $y'(x) = f(x, y(x))$ is the form of the ODEs we are looking at.
- So, at each step i we let $\phi = f(x_i, y(x_i))$.

2 Euler's Method

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2.1 Derivation of Euler's Method

Taylor's Theorem for $y(x)$ (with $n = 1$) expanded about x_i is

$$y(x) = y(x_i) + hy'(x_i) + \frac{h^2}{2}y''(\xi)$$

for some value ξ between x_i and x. Thus, at $x = x_{i+1}$,

$$y(x_{i+1}) = y(x_i) + hy'(x_i) + \frac{h^2}{2}y''(\xi)$$

where $h = x_{i+1} - x_i$.

For small h, this suggests the approximation

$$y(x_{i+1}) \approx y(x_i) + hy'(x_i) = y(x_i) + hf(x_i, y(x_i)),$$

using the differential equation $y'(x) = f(x, y(x))$.

Here, we used exact soultions $y(x_i)$ but at each itereation we generally use previous approximations y_i.

So, Euler's Method approxiamtes each $y(x_{i+1})$ by

$$y_{i+1} = y_i + hf(x_i, y_i)$$

for $i = 0, 1, 2, ..., N$ where $y_0 = y(x_0)$ is the initial condition.

2.2 Example 2: Euler's Method

Consider the initial-value problem

$$y'(x) = y - x^2 + 1, \text{ subject to } y(0) = 0.5.$$

The first few computed approximations y_i and the corresponding exact solutions $y(x_i)$ are as follows:

x_i	y_i	$y(x_i)$	$\lvert y(x_i) - y_i \rvert$
0	0.5	0.5	0
0.2	0.8	0.8292986	0.0292986
0.4	1.152	1.1240877	0.0620877
0.6	1.5504	1.6489406	0.0985406
0.8	1.98848	2.1272295	0.1387495

3 Geometric Interpretation of Euler's Method

$y_0 = y(x_0)$ is the intitial condition. Using this value, we compute

$$y_1 = y_0 + hf(x_0, y_0) = y_0 + hy'(x_0)$$

from which it follows that

$$y'(x_0) = \frac{y_1 - y_0}{h} = \frac{y_1 - y_0}{x_1 - x_0}$$

Geometrically, this says that y_1 is obtained from y_0 by constructing the tangent line to the graph of $y(x)$ at x_0 (which has slope equal to $y'(x_0)$) and going a distance h.

Similarly,

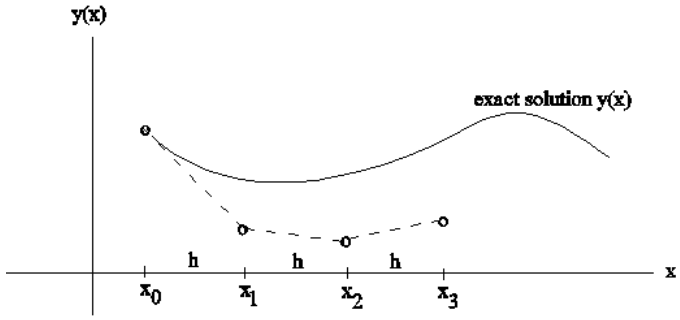
$$y_2 = y_1 + hf(x_1, y_1)$$

so y_2 is determined by constructing a straight line through (x_1, y_1) with slope $f(x_1, y_1)$ (not $f(x_1, y(x_1))$, the exact slope).

Notes on Euler's Method

In general, y_{i+1} is obtained by constructing a straight line through (x_i, y_i) with slope equal to $f(x_i, y_i)$, which is an approximation to $y'(x_i)$.

As y_i depends on y_{i-1}, which in turn depends on y_{i-2} and so on, successive values of y_i tend to be less and less accurate (as the truncation errors accumulate as you go across the interval $[a, b]$).



4 Error Analysis of Euler's Method

Truncation Error

The **total truncation error** in each computed approximation y_{i+1} is composed of two parts:

1. the **local truncation error** is the amount of truncation error that results from a single application of a numerical method (that is, from the computation of y_{i+1} from y_i), and
2. the **global truncation error** contains the accumulated local truncation errors from all of the steps leading up to the computation of y_{i+1}.

4.1 Global Truncation Error

Definition: $|y(x_i) - y_i|$ is called the **global truncation error** at x_i.

Definition: If the global truncation error is $O(h^k)$, the numerical method used to compute the values y_i is said to be of **order** k (or a k^{th} order method).

Definition A numerical method is said to be **convergent** (with respect to the differential equation it approximates) if

$$\lim_{h \to 0} \max_{1 \le i \le N} |y(x_i) - y_i| = 0$$

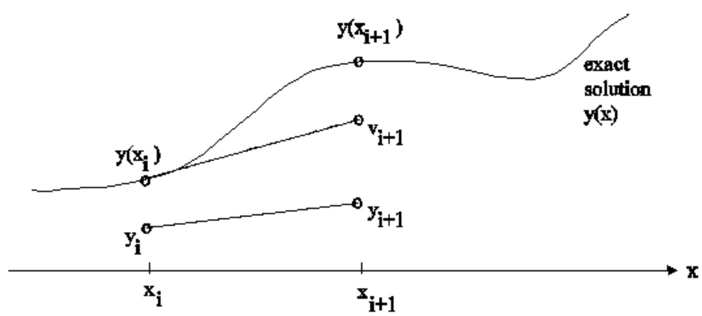
Notes on Global Truncation Error

The order of a method is a measure of the accuracy of the computed approximations, or of the rate of convergence of the computed approximations y_i to the exact solutions $y(x_i)$ as $h \to 0$.

For any fixed value of the step size h, the larger the order k, the more accurate are the computed approximations. .

4.2 Derivation of Local Truncation Error for Euler's Method

Definition: The **local truncation error** at any point x_{i+1} is the amount of truncation error that would result from using a numerical method with the exact value $y(x_i)$ rather than the computed approximation y_i



Recall, Eulers method is

$$y_{i+1} = y_i + hf(x_i, y_i).$$

Using the exact value $y(x_i)$ in this formula instead of the computed approximation y_i, define

$$v_{i+1} = y(x_i) + hf(x_i, y(x_i)). \tag{1}$$

Then the local truncation error at x_{i+1} is equal to

$$|y(x_{i+1}) - v_{i+1}|.$$

4.3 Order of the Local Truncation Error for Euler's Method

We use Taylor's Theorem,

$$y(x_{i+1}) = y(x_i) + hy'(x_i) + \frac{h^2}{2}y''(\xi_i)$$

where $h = x_{i+1} - x_i$ and $\xi_i \in [x_i, x_{i+1}]$. So,

$$y(x_{i+1}) = y(x_i) + hf(x_i, y(x_i)) + \frac{h^2}{2}y''(\xi_i). \qquad (2)$$

Now, (2)-(1) gives,

$$|y(x_{i+1}) - v_{i+1}| = \left|\frac{h^2}{2}y''(\xi_i)\right|.$$

Thus, Euler's Method is $O(h^2)$ locally.

4.4 Relationship between Local and Global Error for Euler's Method

An informal justification for the relationship between the local and global truncation errors for Euler's method is as follows:

The local truncation error in each step of Euler's method is, as shown above, $O(h^2)$.

After N steps of Euler's method, the global truncation error $|y(x_N) - y_N|$ of the final computed approximation y_N at x_N will depend on the N local truncation errors of $y_1, y_2, ..., y_N$.

But the magnitude of N local truncation errors is

$$N \times O(h^2) = \frac{b-a}{h} \times O(h^2) = O(h) \text{ since } h = \frac{b-a}{N}$$

Theorem: (for any method) If the local truncation error is $O(h^{k+1})$, then the global truncation error is $O(h^k)$. That is, the numerical method used to compute the approximate solution has **order** k.

Thus, the global truncation error for Eulers method is $O(h)$, and Eulers method has order 1.

Usually, you cannot compute either the global or local errors exactly. Also, the global error is difficult to approximate directly. But, there are good ways of approximating the local errors which can then be used to approximate the global error.

Disadvantages of Euler's Method

- Euler's method is not sufficiently accurate.
- Because the global truncation error is only $O(h)$, a very small step size h is required to compute highly accurate approximations.

- Euler is not often used.
- Methods with a higher order of accuracy are used in practice.

5 Higher-Order Taylor Series Methods

Higher order methods can be obtained by keeping more terms from the Taylor expansion.

$$
\begin{aligned}
y(x_{i+1}) &= y(x_i) + hy'(x_i) + \frac{h^2}{2}y''(x_i) + \cdots \\
&\qquad + \frac{h^n}{n!}y^{(n)}(x_i) + \frac{h^{n+1}}{(n+1)!}y^{(n+1)}(\xi_i) \\
&= y(x_i) + hf(x_i, y(x_i)) + \frac{h^2}{2}f'(x_i, y(x_i)) + \cdots \\
&\qquad + \frac{h^n}{n!}f^{(n-1)}(x_i, y(x_i)) + O(h^{n+1})
\end{aligned}
$$

5.1 The Taylor Method of Order n

Dropping the $O(h^{n+1})$ remainder term in the above Taylor expansion, gives a numerical method

$$
y_{i+1} \quad = \quad y_i \; + \; hf(x_i, y_i) \; + \; \frac{h^2}{2}f'(x_i, y_i) \; + \; \cdots \; + \; \frac{h^n}{n!}f^{(n-1)}(x_i, y_i)
$$

for any integer $n \geq 1$.

This is called **the Taylor method of order** n (as its local truncation error is $O(h^{n+1})$, and thus its global truncation error is $O(h^n)$).

Eulers method is just the case when $n = 1$.

Example 3: Taylor Method of Order 2

Solve the differential equation $y' = y - x^2 + 1$ with $y(0) = 0.5$ using the Taylor Method of order $n = 2$ with step size $h = 0.2$.

The Taylor Method of order 2 is

$$
y_{i+1} = y_i + hf(x_i, y_i) + \frac{h^2}{2}f'(x_i, y_i)
$$

where

$$f'(x, y(x)) = (y - x^2 + 1)' = y' - 2x = y - x^2 + 1 - 2x$$

So,

$$\begin{aligned} y_{i+1} &= y_i + h(y_i - x_i^2 + 1) + \frac{h^2}{2}(y_i - x_i^2 + 1 - 2x_i) \\ &= y_i + h\left[((y_i - x_i^2 + 1) + \frac{h}{2}(y_i - x_i^2 + 1) - hx_i\right] \\ &= y_i + h\left[(1 + \frac{h}{2})(y_i - x_i^2 + 1) - hx_i\right] \end{aligned}$$

Thus, at $(0, 0.5)$ with $h = 0.2$ we get

$$\begin{aligned} y_1 &= y_0 + h\left[(1 + \frac{h}{2})(y_0 - x_0^2 + 1) - hx_0\right] \\ &= 0.5 + (0.2)\left[(1 + 0.1)(0.5 - 0^2 + 1) - (0.2)(0)\right] \\ &= 0.5 + (0.2)(1.1)(1.5) \\ &= 0.83 \end{aligned}$$

Problems with higher-order Taylor Methods: For $n \geq 2$, use of a Taylor method requires evaluation of the derivatives of the function $f(x, y(x))$ with respect to x. Although the Taylor methods of order n have high accuracy for values of $n = 3, 4$ or 5, they are seldom used in practice because of the difficulty and expense in evaluating the required higher derivatives. Runge-Kutta methods are a class of higher-order methods that are more often used in practice.