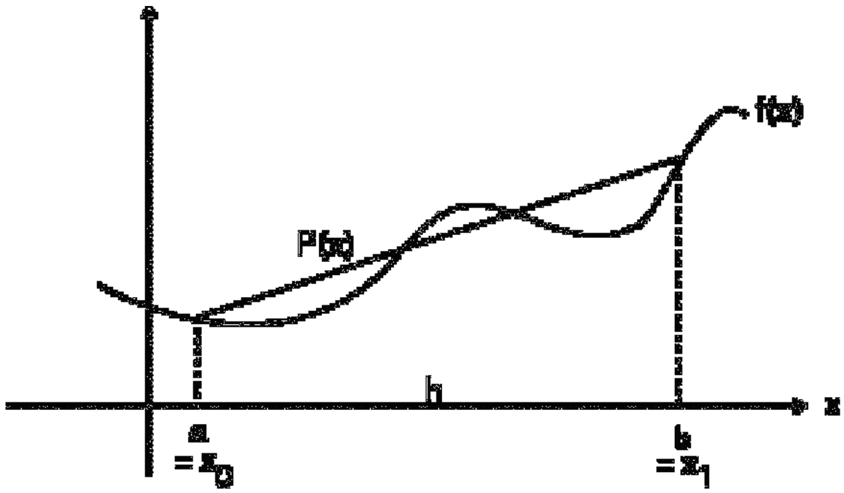
COMPUTER SCIENCE 349A
Handout Number 26

NEWTON-COTES CLOSED QUADRATURE FORMULAS

The case $n = 1$ (Section 21.1 of the textbook):



Here $h = b - a$. The (linear) interpolating polynomial is

$$P(x) = \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1) \ .$$

The quadrature formula for approximating $\int_a^b f(x)dx$ is obtained by integrating $P(x)$:

$$\begin{aligned}
\int_a^b f(x)dx &\approx \int_{x_0}^{x_1} P(x)dx \\
&= \left[\int_{x_0}^{x_1} \frac{x - x_1}{x_0 - x_1} f(x_0)dx\right] + \left[\int_{x_0}^{x_1} \frac{x - x_0}{x_1 - x_0} f(x_1)dx\right] \\
&= \frac{f(x_0)}{x_0 - x_1}\left[\frac{x^2}{2} - x_1 x\right]_{x_0}^{x_1} + \frac{f(x_1)}{x_1 - x_0}\left[\frac{x^2}{2} - x_0 x\right]_{x_0}^{x_1} \\
&= \frac{x_1 - x_0}{2} f(x_0) + \frac{x_1 - x_0}{2} f(x_1) \\
&= \frac{h}{2}\left[f(x_0) + f(x_1)\right], \text{ since } h = x_1 - x_0 .
\end{aligned}$$

This is the **trapezoidal rule**. Its error term can be obtained by integrating the error term of the Lagrange form of the interpolating polynomial, which for $n = 1$ is

$$f(x) - P(x) = \frac{f''(\xi(x))}{2}(x - x_0)(x - x_1).$$

Note that ξ does depend on x; for each value of x in $[a, b]$, there is a different value ξ for which the above expression gives the value of $f(x) - P(x)$. Integrating this gives

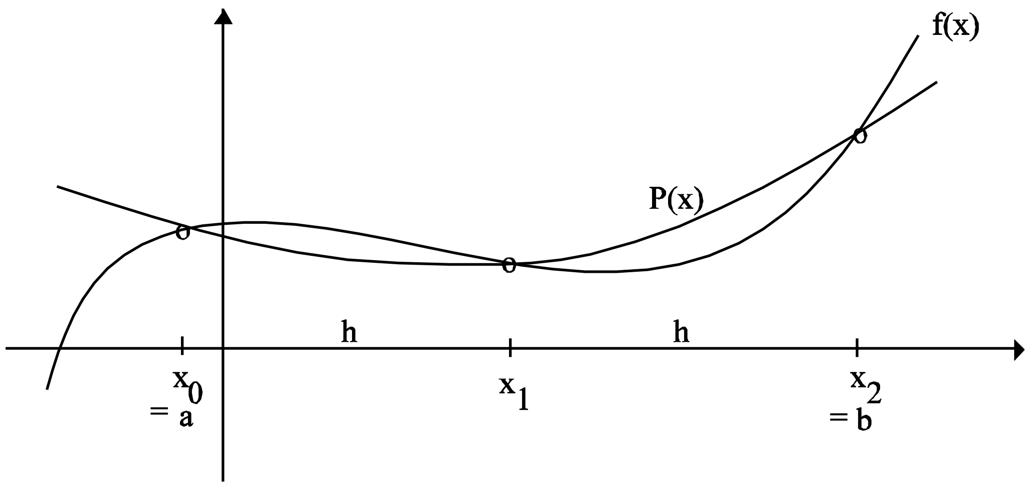
$$\begin{aligned}
\int_a^b f(x)dx - \int_{x_0}^{x_1} P(x)dx &= \int_a^b f(x)dx - \frac{h}{2}\left[f(x_0) + f(x_1)\right] \\
&= \int_a^b \frac{f''(\xi(x))}{2}(x - x_0)(x - x_1)dx \\
&= \frac{f''(\eta)}{2}\int_a^b (x - x_0)(x - x_1)dx, \quad \text{for some constant} \\
&\qquad \eta \text{ between } a \text{ and } b, \text{ by the mean-value} \\
&\qquad \text{theorem for integrals} \\
&= \frac{f''(\eta)}{2}\int_0^1 h^2 t(t-1)(h\,dt), \quad \text{with the change of} \\
&\qquad \text{variable } x = x_0 + th, \text{ so that } x - x_0 = th \\
&\qquad \text{and } x - x_1 = (t-1)h \\
&= h^3 \frac{f''(\eta)}{2}\int_0^1 t(t-1)dt \\
&= -\frac{h^3}{12} f''(\eta), \quad \text{for some value } \eta \text{ between } a \text{ and } b.
\end{aligned}$$

See (21.6) on page 605 of the 6th ed. or page 607 of the 7th ed., where $h = b - a$. This error term is the **truncation error** when $\int_a^b f(x)dx$ is approximated by $\int_a^b P(x)dx$.

The case $n = 2$ (Section 21.2.1 on page 613 of the 6th ed. or page 615 of the 7th ed.):

The quadratic interpolating polynomial is

$$P(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1) + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2) \; .$$



As in the case $n=1$, the quadrature formula for approximating $\int_a^b f(x)dx$ is obtained by integrating $P(x)$: $\int_a^b f(x)dx \approx \int_{x_0}^{x_2} P(x)dx$

This gives

$$\int_a^b f(x)dx \approx \frac{h}{3}\big(f(x_0)+4f(x_1)+f(x_2)\big),$$

where now $h=\frac{b-a}{2}$. This is called **Simpson's rule** (or as in the textbook, Simpson's 1/3 rule), and its **truncation error** is given by

$$\int_a^b f(x)dx - \int_{x_0}^{x_2} P(x)dx = -\frac{h^5}{90} f^{(4)}(\xi), \quad \text{for some value } \xi \in [a, b].$$

See pages 614-615 of the 6th ed. or pages 616-617 of the 7th ed.

The Newton-Cotes closed quadrature formula for $n=3$, in which $f(x)$ is approximated by a cubic polynomial that interpolates it at four equally-spaced points, is

$$\int_a^b f(x)dx \approx \frac{3h}{8}\big(f(x_0)+3f(x_1)+3f(x_2)+f(x_3)\big), \quad \text{where } h=\frac{b-a}{3},$$

which is called the **Simpson's 3/8 rule**. See Section 21.2.3 on page 618 of the 6th ed. or page 620 of the 7th ed.