COMPUTER SCIENCE 349A

Handout Number 6

CONDITION OF A PROBLEM

Definition A problem whose (exact) solution can change greatly with small changes in the data defining the problem is called ill-conditioned.

data $\{d_i\}$ defining a problem	$\xrightarrow{\text{exact arithmetic}}$	exact solution $\{r_i\}$
perturbed problem, data $\{\hat{d}_i\} = \{d_i + \varepsilon_i\}$ with $\left\lvert \frac{\varepsilon_i}{d_i} \right\rvert$ small	$\xrightarrow{\text{exact arithmetic}}$	exact solution $\{\hat{r}_i\}$

If there exist small ε_i such that $\{\hat{r}_i\}$ are not close to $\{r_i\}$, then the problem is **ill-conditioned**.

If $\{\hat{r}_i\} \approx \{r_i\}$ for all small ε_i, then the problem is **well-conditioned**.

Example Consider the 3×3 system of linear equations $Hx = b$:

$$\begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 11/6 \\ 13/12 \\ 47/60 \end{bmatrix}.$$

Using the exact $\{d_i\}$ given above in the matrix H and the vector b, the exact solution is

$$x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

However, if the entries of H and b are rounded to 3 significant decimal digits to give the following perturbed problem $\hat{H}\hat{x} = \hat{b}$:

$$\begin{bmatrix} 1.00 & 0.500 & 0.333 \\ 0.500 & 0.333 & 0.250 \\ 0.333 & 0.250 & 0.200 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} = \begin{bmatrix} 1.83 \\ 1.08 \\ 0.783 \end{bmatrix}$$

then the exact solution (to 5 significant digits) is

$$\hat{x} = \begin{bmatrix} 1.0895 \\ 0.48797 \\ 1.4910 \end{bmatrix}.$$

Thus, the problem of solving $Hx = b$ is ill-conditioned.

Note. The condition of a problem has nothing to do with floating-point arithmetic or round-off error; it is defined in terms of exact computation. However, if a problem is ill-conditioned, it will be difficult (or impossible) to solve accurately using floating-point arithmetic.

Another approach to analyzing the condition of a problem if the first derivative of the quantity $f(x)$ being computed can be determined: see pages 100-101 of the 7th edition; pages 97-98 of the 6th.

By the Taylor polynomial approximation of order $n = 1$ for $f(x)$ expanded about $\tilde{x}$,

$$f(x) \approx f(\tilde{x}) + f'(\tilde{x})(x - \tilde{x}),$$

which implies that

$$\frac{f(x) - f(\tilde{x})}{f(\tilde{x})} \approx \frac{\tilde{x} f'(\tilde{x})}{f(\tilde{x})} \left(\frac{x - \tilde{x}}{\tilde{x}} \right).$$

If $\tilde{x}$ is some small perturbation of x, then the left hand side above is the relative change in $f(x)$ as x is perturbed to $\tilde{x}$. Thus,

$$\text{relative change in } f(x) \approx \left(\frac{\tilde{x} f'(\tilde{x})}{f(\tilde{x})} \right) \times \text{relative change in } x$$

The quantity $\dfrac{\tilde{x} f'(\tilde{x})}{f(\tilde{x})}$ is called a **condition number** for the computation of $f(x)$. If this number is "large", then $f(x)$ is ill-conditioned; if this number is "small", then $f(x)$ is well-conditioned.

Example 4.7 (page 101 of the 7th edition; page 98 of the 6th)

For values of x close to $\pi/2$, the computation of

$$f(x) = \tan x$$

is ill-conditioned. The condition number is

$$\frac{\tilde{x} f'(\tilde{x})}{f(\tilde{x})} = \frac{\tilde{x}\left(1/\cos^2 \tilde{x}\right)}{\tan \tilde{x}},$$

and for example, for $\tilde{x} = \pi/2 + 0.01\pi/2$, this condition number is

$$\frac{\tilde{x} f'(\tilde{x})}{f(\tilde{x})} = \frac{\tilde{x}\left(1/\cos^2 \tilde{x}\right)}{\tan \tilde{x}} = \frac{1.01(\pi/2)(4053.2)}{-63.66} = -101.01 \ .$$

Interpretation: for values of x close to $\pi/2$, the relative change

$$\frac{\tan(x) - \tan(\tilde{x})}{\tan(\tilde{x})} \text{ is approximately } -101 \times \frac{x - \tilde{x}}{\tilde{x}}.$$

This means that the computation of $f(x) = \tan x$ for values of x close to $\pi/2$ is **ill-conditioned**. For a well-conditioned problem,

$$\frac{f(x) - f(\tilde{x})}{f(x)} \approx \frac{x - \tilde{x}}{\tilde{x}},$$

that is, the condition number is approximately equal to 1.