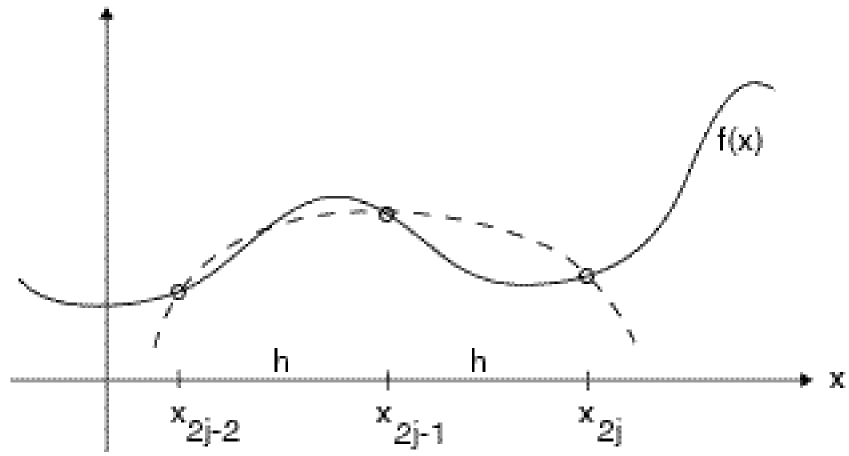
COMPUTER SCIENCE 349A
Handout Number 28

COMPOSITE (Multiple-Application) SIMPSON'S RULE (Section 21.2.2)

Each application of Simpson's rule requires 2 subintervals on the interval of integration and 3 quadrature points, say x_{2j-2}, x_{2j-1} and x_{2j}:



$$\int_{x_{2j-2}}^{x_{2j}} f(x)dx \approx \frac{h}{3}\left[f(x_{2j-2}) + 4f(x_{2j-1}) + f(x_{2j})\right]$$

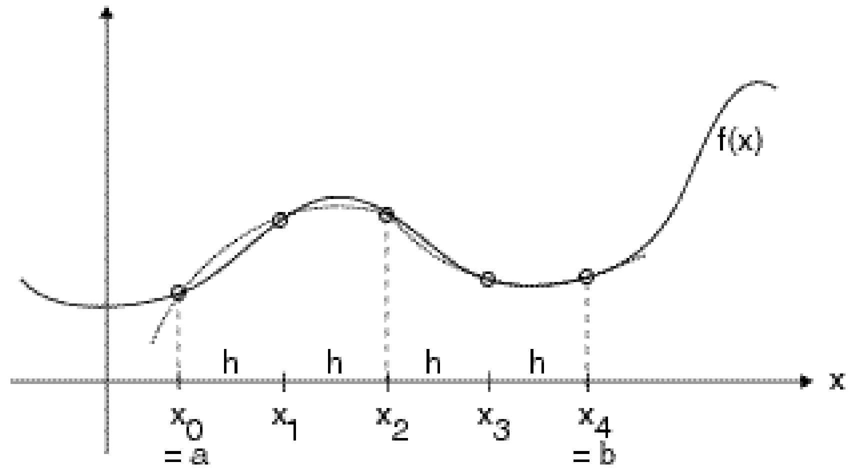
Thus, <u>m applications of Simpson's rule</u> on $[a,b]$ require that $[a,b]$ be subdivided into an <u>even</u> number of subintervals. In the following, we will assume that $[a,b]$ is subdivided into $2m$ subintervals, each of length

$$h = \frac{b-a}{2m} \ ,$$

and the corresponding $2m+1$ quadrature points will be denoted by $x_0, x_1, x_2, \cdots, x_{2m}$.

The case $m=1$: just one application of Simpson's rule (or, we could call this the "noncomposite" Simpson's rule).

The case $m=2$: two applications of Simpson's rule require 5 quadrature points and 4 subintervals of $[a,b]$, each of length $h = \frac{b-a}{4}$.

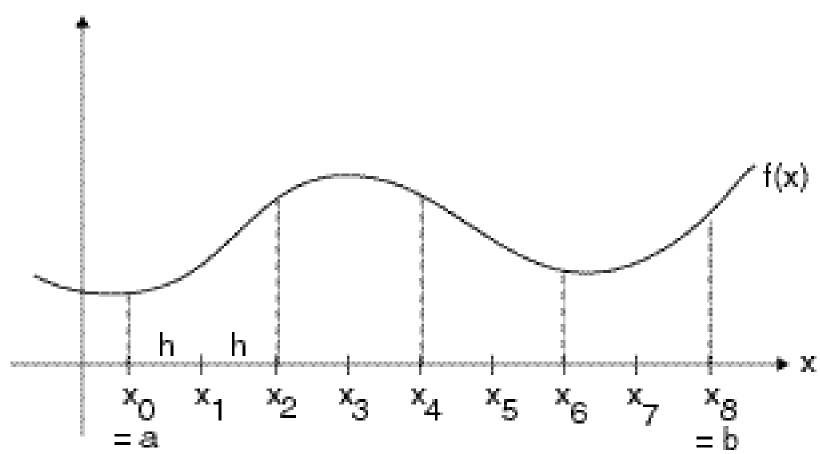


Letting f_i denote $f(x_i)$, this approximation becomes

$$\int_a^b f(x)dx \approx \frac{h}{3}[f_0 + 4f_1 + f_2] + \frac{h}{3}[f_2 + 4f_3 + f_4]$$

$$= \frac{h}{3}[f_0 + 4f_1 + 2f_2 + 4f_3 + f_4]$$

The case $m = 4$: four applications of Simpson's rule require 9 quadrature points and 8 subintervals of $[a, b]$, each of length $h = \dfrac{b-a}{8}$.



This composite Simpson's rule approximation is

$$\int_a^b f(x)dx \approx \frac{h}{3}[f_0+4f_1+f_2]+\frac{h}{3}[f_2+4f_3+f_4]+\frac{h}{3}[f_4+4f_5+f_6]+\frac{h}{3}[f_6+4f_7+f_8]$$
$$=\frac{h}{3}[f_0+4f_1+2f_2+4f_3+2f_4+4f_5+2f_6+4f_7+f_8].$$

In general, for $m \geq 1$, the <u>composite Simpson's rule</u> approximation is

$$\int_a^b f(x)dx \approx \frac{h}{3}[f_0+4f_1+2f_2+4f_3+2f_4+\cdots+2f_{2m-2}+4f_{2m-1}+f_{2m}]$$
$$=\frac{h}{3}\left[f_0+2\sum_{j=1}^{m-1} f_{2j}+4\sum_{j=1}^{m} f_{2j-1}+f_{2m}\right],$$

where

$$h=\frac{b-a}{2m}.$$

<u>Note</u>. See (21.18) on p. 616 of the 6th ed. or p. 618 of the 7th ed., where n is equal to $2m$.

<u>Usual implementation</u> (in order to re-use function evaluations of $f(x)$, and thus make the computation efficient): initialize $m=1$, and then double m at each iteration. Continue iterating until two successive approximations are sufficiently close together.

<u>Truncation error</u>

$$E_{2m}(f)=-\frac{h^5}{90}\left[f^{(4)}(\xi_1)+f^{(4)}(\xi_2)+\cdots+f^{(4)}(\xi_m)\right]$$
$$=-\frac{h^5}{90}\left[m f^{(4)}(\mu)\right], \quad \text{where } a<\mu<b, \text{ assuming } f^{(4)}(x) \text{ is continuous}$$
$$=-\frac{(b-a)h^4}{180} f^{(4)}(\mu), \quad \text{since } h=\frac{b-a}{2m} \text{ implies that } m=\frac{b-a}{2h}.$$

(This is (21.19) on p. 617 of the 6th ed. or p. 619 of the 7th ed., where n is equal to $2m$.)

Thus, if $f^{(4)}(x)$ is continuous on $[a,b]$, it follows that <u>the composite Simpson's rule approximations converge</u> to $\int_a^b f(x)dx$, that is, that $\lim_{m\to\infty} E_{2m}(f)=0$. (Note that this is in the absence of roundoff errors, which we have not taken into account; typically it is the case that the accumulated roundoff error is much smaller than the truncation error.)