COMPUTER SCIENCE 349A
SAMPLE EXAM QUESTIONS WITH SOLUTIONS
PARTS 3, 5, 6 , 7

PART 3.

3.1 Suppose that a computer program, using the Gaussian elimination algorithm, is to be written to accurately solve a system of linear equations $Ax = b$, where A is an arbitrary $n \times n$ nonsingular matrix. Give two reasons why it is necessary to incorporate a pivoting strategy (such as partial pivoting) into the algorithm.

3.2 Let

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 2 & 2 \end{bmatrix}, \qquad b = \begin{bmatrix} -1 \\ 0.5 \\ 3 \end{bmatrix}$$

and suppose that

$$x = A^{-1}b.$$

Use Naïve Gaussian Elimination (Gaussian elimination without pivoting) to compute x. Do <u>not</u> compute A^{-1}. Show all of your work.

3.3 Consider the following system of linear equations $Ax = b$:

$$\begin{aligned} -2x_2 + 2x_3 &= 4 \\ x_1 - 2x_2 + x_3 &= 3 \\ -2x_1 + 2x_2 + 4x_3 &= 0 \end{aligned}$$

Specify the augmented matrix for this linear system, and use Gaussian elimination with partial pivoting to compute the solution vector x. Show all of your work.

3.4 Suppose that the following MATLAB statement has been executed.

```
A = [-1 2 3 4 ; 5 6 7 8 ; 9 8 7 6 ; 5 4 3 1 ] ;
```

Specify 1 or 2 MATLAB statements that could be used to efficiently compute the second column vector of A^{-1}. Do not compute the entire matrix A^{-1}.

3.5 Let $n \geq 2$, let A denote an $n \times n$ nonsingular, upper triangular matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ & a_{22} & a_{23} & \cdots & a_{2n} \\ & & a_{33} & \cdots & a_{3n} \\ & \mathrm{O} & & \ddots & \vdots \\ & & & & a_{nn} \end{bmatrix},$$

and let $y = (y_1, y_2, y_3, \ldots, y_n)^T$ denote a column vector with n entries. The most efficient way to compute $x = A^{-1}y$ is to use the back-substitution algorithm. Assuming that n, A and y are specified, write a MATLAB function M-file

```
function x = solve ( n , A , y )
```

that will compute $x = A^{-1}y$ using the back-substitution algorithm.

Note: do <u>not</u> use the MATLAB operator \ or the MATLAB function *inv*.

PART 5.

5.1 Consider the following data:

x_i	$f(x_i)$
-1	0
1	1
2	3

Suppose that a function $g(x)$ of the form $g(x) = c_0 + c_1 e^{-x} + c_2 e^{x}$ is to be determined so that $g(x)$ interpolates the above data at the specified points x_i.

Write down a system of linear equations (in matrix/vector form $Ac = b$) whose solution will give the values of the unknowns c_0, c_1 and c_2 that solve this interpolation problem.

Note: Leave your answer in terms of e and powers of e. Do not solve the resultant linear system.

5.2 (a) Give the Lagrange form of the quadratic ($n = 2$) interpolating polynomial $P(x)$ that interpolates $f(x) = e^{-x}$ at $x = 0,\ x = 0.2$ and $x = 0.4$.

Note: Do not numerically evaluate $f(x)$; instead, give your answer in terms of values such as $e^{-0.2}$. Also, do not simplify the expression for $P(x)$.

(b) Using the error term for polynomial interpolation, determine a good upper bound for $\left|P(0.1) - e^{-0.1}\right|$.

Note: Do not determine an upper bound for $\left|P(x) - e^{-x}\right|$ for all $x \in [0, 0.4]$, only for $x = 0.1$.

5.3 Let $P(x)$ denote the (linear) polynomial of degree 1 that interpolates $f(x) = \cos x$ at the points $x_0 = -0.1$ and $x_1 = 0.1$ (where x is in radians). Use the error term of polynomial interpolation to determine an upper bound for $|P(x) - f(x)|$, where $x \in [-0.1, 0.1]$. Do not construct $P(x)$.

5.4 (a) Fill in the blanks below so that the following MATLAB statements could be used to compute the value of $z = P(\pi)$, where $P(x)$ is the piecewise linear interpolating polynomial that interpolates $y = x\ \ell n(x)$ at 31 equally-spaced points

$$x_1 = 1,\ x_2 = 1.1,\ x_3 = 1.2, \ldots, x_{31} = 4.0$$

in the interval $[1, 4]$.

X = linspace (___________ , ____________ , ____________) ;

Y = ____________________________________ ;

z = interp1 (___________ , ____________ , ___________ , 'linear')

(b) Use the error term for polynomial interpolation to determine a good upper bound for the error when $f(x) = x\ \ell n(x)$ is approximated by the above piecewise linear interpolating polynomial $P(x)$, where x is any value in $[3, 3.1]$. That is, determine a value of ε such that

$$\left| x\ \ell n(x) - P(x) \right| \le \varepsilon \qquad \text{whenever } x \in [3, 3.1].$$

5.5 Determine values for the parameters a, b, c, d and e so that

$$Q(x) = \begin{cases} ax^2 + x + b\,, & -1 \le x \le 0 \\ cx^2 + dx + e\,, & 0 \le x \le 1 \end{cases}$$

is a quadratic spline function that interpolates $f(x)$, where

$$f(-1) = 1\,, \quad f(0) = 1\,, \quad f(1) = 1 \ .$$

Show all of the equations that the unknowns must satisfy, and then solve these equations.

5.6 Determine $a_0, b_0, d_0, a_1, b_1, c_1$ and d_1 so that

$$S(x) = \begin{cases} a_0 + b_0 x - 3x^2 + d_0 x^3, & -1 \le x \le 0 \\ a_1 + b_1 x + c_1 x^2 + d_1 x^3, & 0 \le x \le 1 \end{cases}$$

is the natural cubic spline function such that

$$S(-1) = 1, \quad S(0) = 2 \quad \text{and} \quad S(1) = -1.$$

Clearly identify the 8 conditions that the unknowns must satisfy, and then solve for the 7 unknowns.

PART 6.

6.1 Determine the degree of precision of the quadrature formula

$$\frac{3h}{4}\left(3f(h) + f(3h)\right)$$

which is an approximation to $\int_0^{3h} f(x)dx$. Show all of your work.

6.2 The <u>open</u> Newton-Cotes quadrature formula for $n = 3$ is

$$\int_{x_{-1}}^{x_4} f(x)dx \approx \frac{5h}{24}\left[11f(x_0) + f(x_1) + f(x_2) + 11f(x_3)\right].$$

Use one application of this formula on $[-1,1]$ to approximate

$$\int_{-1}^{1} \frac{\cos(x)}{\sqrt{1-x^2}} dx,$$

given the following values:

x	$\cos(x)/\sqrt{1-x^2}$
± 0.9	1.426071
± 0.8	1.161179
± 0.7	1.070993
± 0.6	1.031670
± 0.5	1.013345

x	$\cos(x)/\sqrt{1-x^2}$
± 0.4	1.004960
± 0.3	1.001465
± 0.2	1.000276
± 0.1	1.000017
0.0	1.000000

6.3 Consider approximating $\int_a^b f(x)dx$ using Romberg integration. Denote the Romberg table by

$$\begin{array}{llll} I_{1,1} & & & \\ I_{2,1} & I_{2,2} & & \\ I_{3,1} & I_{3,2} & I_{3,3} & \\ \vdots & \vdots & \vdots & \ddots \end{array}$$

where

$I_{k,1} =$ the trapezoidal rule approximation to $\int_a^b f(x)dx$ using 2^{k-1} subintervals on [a, b] .

Use $I_{1,1}$ and $I_{2,1}$ and Richardson extrapolation to show that $I_{2,2}$ is equal to the Simpson's rule approximation to $\int_a^b f(x)dx$.

6.4 If

$$\int_a^b f(x)dx$$

is approximated by the Composite Simpson's rule (that is, using m applications of Simpson's rule on $[a, b]$ with stepsize $h = \frac{b-a}{2m}$) , then the truncation error is

$$-\frac{(b-a)}{180}h^4 f^{(4)}(\mu) \ ,$$

for some $\mu \in (a, b)$. Use this error term in order to determine the smallest value of m for which the truncation error is guaranteed to be $\le 10^{-8}$ when the Composite Simpson's rule is used to approximate

$$\int_{0.5}^{1.5} \frac{1}{x}\, dx \ .$$

6.5 (a) If Simpson's rule is applied $m \ge 1$ times on subintervals of $[a,b]$, the resulting composite Simpson's rule approximation is of the following form:

$$\int_a^b f(x)dx \approx \frac{h}{c_1}\left(f_0 + c_2 \sum_{j=1}^{p} f_r + c_3 \sum_{j=1}^{q} f_t + f_{2m} \right),$$

where $h = \dfrac{b-a}{2m}$ and $f_k = f(x_k) = f(a+kh)$. Specify the values of the parameters c_1, c_2, c_3, p, q, r and t.

(b) Given that a MATLAB function $f(x)$ has been defined, fill in the blanks in the following MATLAB function *compsimp* so that it will implement the composite Simpson's rule.

```
function approx = compsimp (a , b , m)
h = ___________________ ;
sum1 =  0 ;
for j = 1 : _________
        sum1 = sum1 + f ( _______________  ) ;
end
sum2 = 0 ;
for j = 1 :  _______________
        sum2 = sum2 + f ( ________________  ) ;
end
approx = (  h /  _____  )*( f(a) + ____________ +  ___________ + f(b) ) ;
```

6.6 Construct the Taylor polynomial approximations of order 3 (with their remainder terms simply written as $O(h^4)$) for both of $f(x_0 + h)$ and $f(x_0 + 2h)$ expanded about x_0. Derive a numerical differentiation formula for $f'(x_0)$ (and its truncation error term in a form $O(h^k)$) as follows:

substitute the above Taylor polynomial approximations (with their remainder terms) into the expression

$$-3f(x_0) + 4f(x_0 + h) - f(x_0 + 2h)$$

and solve for $f'(x_0)$.

6.7 It can be shown that

$$\lim_{h \to 0} \left(\frac{2+h}{2-h} \right)^{1/h} = e,$$

where $e = 2.7182818\cdots$. If

$$N(h) = \left(\frac{2+h}{2-h} \right)^{1/h}$$

denotes a formula for approximating the value of e, then it can be shown that the truncation error of this approximation is of the form

$$K_2 h^2 + K_4 h^4 + K_6 h^6 + \cdots$$

for some constants K_i ; that is,

$$e = N(h) + K_2 h^2 + K_4 h^4 + K_6 h^6 + \cdots .$$

Note that

$$N(0.04) = \left(\frac{2+0.04}{2-0.04} \right)^{1/0.04} = 2.718644$$

$$N(0.02) = \left(\frac{2+0.02}{2-0.02} \right)^{1/0.02} = 2.718372$$

Apply Richardson's extrapolation to the above two values in order to obtain an $O(h^4)$ approximation to the value of e.

6.8 Fill in the three blanks in the following Richardson's Extrapolation table, given that the truncation error of the formula $N_1(h)$ is of the form

$$K_1 h^2 + K_2 h^4 + K_3 h^6 + \cdots$$

for some constants K_i.

h	$N_1(h)$		
0.45	2.766013		
0.15	2.723401	________________	
0.05	2.718848	________________	__________________

Show the formulas that you use to compute these three entries.

6.9 (a) Let $h > 0$. Suppose you are given 3 values of a function $f(x)$, namely $f(0), f(h)$ and $f(3h)$. Construct $P(x)$, the Lagrange form of the interpolating polynomial at the three points $x = 0, h$ and $3h$. Then differentiate $P(x)$ in order to obtain a numerical differentiation formula $P'(x)$ for approximating $f'(x)$. (This formula will be a function of x, h and the values $f(0), f(h)$ and $f(3h)$.)

(b) Given the following data:

x	$f(x)$
0	0.12
0.05	0.11
0.15	0.15

use the numerical differentiation formula from (a) to approximate $f'(0)$ by $P'(0)$.

PART 7.

7.1 Consider the initial-value problem

$$y'(x) = \frac{1}{x}\left(y^2 + y\right),$$
$$y(1) = -2$$

(a) Use the Taylor method of order $n = 2$ with $h = 0.1$ to approximate $y(1.1)$. Show all of your work and the iterative formula.

(b) Approximate $y(1.1)$ using $h = 0.1$ and the following second-order Runge-Kutta method:

$$y_{i+1} = y_i + \frac{h}{2}\left(f(x_i, y_i) + f(x_{i+1}, y_i + hf(x_i, y_i))\right)$$

(c) What is the order of the local truncation error of the Runge-Kutta method in (b) (as a function of h)? No justification required.

7.2 Consider the initial-value problem $y'(x) = 1 + (x - y)^2$, $y(2) = 1$.

If the solution to this problem is approximated using Euler's method with a fixed step size of $h = 0.01$ on $[2, 2.04]$, then the following computed approximations y_i are obtained (and the corresponding exact solution is given by $y(x_i)$):

x_i	y_i	$y(x_i)$
2.00	1.0	1.0
2.01	1.02	1.019910
2.02	1.039801	1.039608
2.03	1.059409	1.059126
2.04	1.078829	1.078462

(a) What is the global truncation error at $x = 2.04$? (Give an exact numeric answer.)

(b) Give an expression (in terms of the function f with numeric arguments) for the local truncation error at $x = 2.04$. (Do not attempt to evaluate this expression.)

(c) Fill in the blanks in the following MATLAB statements so that they could be used to invoke the MATLAB ode solver *ode45* to approximate $y(x)$ on $[2, 4]$, displaying the values of all computed approximations y_i and all x_i :

function z = f(x, y)

z = ____ ;

____________ = ode45(_________ , ___________ , ________)

7.3 Consider the initial-value problem $y'(x) = 1 + (x - y)^2, \quad y(2) = 1$.
If the solution to this problem is approximated using the Runge-Kutta method

$$y_{i+1} = y_i + \frac{h}{2}\left(f(x_i, y_i) + f(x_{i+1}, y_i + hf(x_i, y_i))\right)$$

with a fixed step size of $h = 0.01$ on $[2, 2.04]$, then the following computed approximations y_i are obtained (and the corresponding exact solution is given by $y(x_i)$):

x_i	y_i	$y(x_i)$
2.00	1.0	1.0
2.01	1.01990050	1.01990099
2.02	1.03960689	1.03960784
2.03	1.05912483	1.05912621
2.04	1.07845974	1.07846154

(a) What is the global truncation error at $x = 2.04$? (Give an exact numeric answer.)

(b) Give an expression (in terms of the function f with numeric arguments) for the local truncation error at $x = 2.04$. (Do not attempt to evaluate this expression.)

7.4 (a) Give the iterative formula for the Taylor method of order $n = 2$ for approximating the solution of the initial-value problem

$$y'(x) = 1 + \frac{y(x)}{x}, \qquad y(1) = 2 \ .$$

(Determine any required derivatives.)

(b) Complete the specification of the following MATLAB M-file *taylor.m* so that it will compute an approximate solution to the above initial-value problem on [1, 2] using a step size of $h = 0.01$ and the Taylor method of order $n = 2$. Instead of using one-dimensional arrays to store the values x_i and y_i, this M-file uses only two (scalar) variables, *x* and *y* (that is, *y* is initialized to y_0, the computed approximation y_1 is also stored as *y* and is printed, then the computed approximation y_2 is stored as *y* and printed, and so on). Do not print any of the values of *x*.

```
function  taylor
x = 1 ;
y = 2 ;
h = 0.01 ;
for  i  =


end
```

SOLUTIONS

PART 3.

3.1

-- to avoid possible division by 0 (that is, a pivot that is exactly equal to 0)

-- to avoid using pivots that are very small in magnitude, since they may cause the computation to be unstable

3.2 Solve $Ax = b$ for x.

$$m_{21} = -1/2 \qquad \left[\begin{array}{ccc|c} 2 & -1 & 0 & -1 \\ -1 & 0 & 1 & 0.5 \\ 0 & 2 & 2 & 3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 2 & -1 & 0 & -1 \\ 0 & -0.5 & 1 & 0 \\ 0 & 2 & 2 & 3 \end{array}\right]$$

$$m_{32} = 2/(-0.5) = -4$$

$$\left[\begin{array}{ccc|c} 2 & -1 & 0 & -1 \\ 0 & -0.5 & 1 & 0 \\ 0 & 0 & 6 & 3 \end{array}\right]$$

Back-substitution:

$x_3 = 3/6 = 0.5$

$-0.5x_2 + x_3 = 0 \quad \Rightarrow \quad -0.5x_2 = -0.5 \quad \Rightarrow \quad x_2 = 1$

$2x_1 - x_2 = -1 \quad \Rightarrow \quad 2x_1 = 0 \quad \Rightarrow \quad x_1 = 0$

That is, $x = \begin{bmatrix} 0 \\ 1 \\ 0.5 \end{bmatrix}$

3.3

The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & -2 & 2 & 4 \\ 1 & -2 & 1 & 3 \\ -2 & 2 & 4 & 0 \end{array}\right]$$

Interchange rows 1 and 3

$$\left[\begin{array}{ccc|c} -2 & 2 & 4 & 0 \\ 1 & -2 & 1 & 3 \\ 0 & -2 & 2 & 4 \end{array}\right]$$

Eliminate

$$\left[\begin{array}{ccc|c} -2 & 2 & 4 & 0 \\ 0 & -1 & 3 & 3 \\ 0 & -2 & 2 & 4 \end{array}\right]$$

Interchange rows 2 and 3

$$\left[\begin{array}{ccc|c} -2 & 2 & 4 & 0 \\ 0 & -2 & 2 & 4 \\ 0 & -1 & 3 & 3 \end{array}\right]$$

Eliminate

$$\left[\begin{array}{ccc|c} -2 & 2 & 4 & 0 \\ 0 & -2 & 2 & 4 \\ 0 & 0 & 2 & 1 \end{array}\right]$$

Back-substitution

$$x_3 = 1/2$$

$$x_2 = \frac{4 - 2x_3}{-2} = -3/2$$

$$x_1 = \frac{0 - 2x_2 - 4x_3}{-2} = -1/2$$

3.4

A \ [0 1 0 0]’ or A \ [0 ; 1 ; 0 ; 0]

or the two statements

```
b = [ 0 1 0 0 ]’;
A \ b
```

3.5

```
% Use back-substitution to solve Ax = y for x.
function x = solve(n, A, y)
x(n) = y(n) / A(n, n);
for i = n-1 : -1 : 1
        sum = 0;
        for j = i+1 : n
                sum = sum + A(i, j) * x(j);
        end
        x(i) = (y(i) – sum)/A(i, i);
end
```

PART 5.

5.1

$$\begin{bmatrix} 1 & e & e^{-1} \\ 1 & e^{-1} & e \\ 1 & e^{-2} & e^{2} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$$

5.2 (a)

$$P(x) = \frac{(x-0.2)(x-0.4)}{(0-0.2)(0-0.4)} e^{0} + \frac{(x-0)(x-0.4)}{(0.2-0)(0.2-0.4)} e^{-0.2} + \frac{(x-0)(x-0.2)}{(0.4-0)(0.4-0.2)} e^{-0.4}$$

or

$$P(x) = \frac{(x-0.2)(x-0.4)}{0.08}(1) + \frac{x(x-0.4)}{-0.04} e^{-0.2} + \frac{x(x-0.2)}{0.08} e^{-0.4}$$

(b)

$$\text{error} = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^{n} (x - x_i)$$

For $n = 2$,

$$\text{error} = \frac{f'''(\xi)}{6} (x-0)(x-0.2)(x-0.4)$$

For $f(x) = e^{-x}$ and $x = 0.1$,

$$\text{error} = \frac{-e^{-\xi}}{6} (0.1-0)(0.1-0.2)(0.1-0.4).$$

Since $0 \le \xi \le 0.4$ for all values of x,

$$|\text{error}| \le \frac{e^{0}}{6} (0.1)(0.1)(0.3) = 0.0005$$

5.3

$$|P(x) - f(x)| = \left|\frac{f''(\xi)}{2!}(x-0.1)(x+0.1)\right| \text{ with } -0.1 \le \xi \le 0.1$$

$$= \left|\frac{-\cos(\xi)}{2}(x^2 - 0.01)\right|$$

$$\le \frac{\cos(0)}{2} \max_{-0.1\le\xi\le 0.1} |x^2 - 0.01|$$

$$= \frac{1}{2}|0 - 0.01| \text{ since this max is attained at } x = 0$$

$$= 0.005$$

5.4 (a)

```
X = linspace ( 1 , 4 , 31 ) ;

Y = X .* log(X)

z = interp1( X , Y , pi , 'linear' )
```

(b)

$f(x) = x\,\ell n\,x\,, \quad f'(x) = \ell n\,x + 1\,, \quad f''(x) = 1/x\,.$

The error term of polynomial interpolation (n = 1) is

$$f(x) - P(x) = \frac{f''(\xi)}{2!}(x-3)(x-3.1)\,, \quad \text{where } \xi \in [3, 3.1]\ .$$

$$|f(x) - P(x)| \le \frac{1}{2}\left|\frac{1}{\xi}\right| |(x-3)(x-3.1)|$$

Thus,
$$\le \frac{1}{2}\,\frac{1}{3} \max_{3\le x\le 3.1} |(x-3)(x-3.1)|$$

$$\le \frac{1}{6}|(3.05-3)(3.05-3.1)| = 0.00041667$$

5.5 Let

$$Q_0(x) = ax^2 + x + b, \quad Q_1(x) = cx^2 + dx + e.$$

Then

$Q_0(0) = Q_1(0) \quad \Rightarrow \quad b = e$

$Q_0'(0) = Q_1'(0) \quad \Rightarrow \quad 2ax + 1 = 2cx + d \text{ at } x = 0 \quad \Rightarrow \quad d = 1$

$Q_0(-1) = f(-1) \quad \Rightarrow \quad a - 1 + b = 1$

$Q_0(0) = f(0) \text{ and } Q_1(0) = f(0) \quad \Rightarrow \quad b = 1 \text{ and } e = 1$

$Q_1(1) = f(1) \quad \Rightarrow \quad c + d + e = 1$

Solution:

$$d = 1, \quad b = 1, \quad e = 1, \quad a = 1, \quad c = -1$$

Thus,

$$Q(x) = \begin{cases} x^2 + x + 1, & -1 \le x \le 0 \\ -x^2 + x + 1, & 0 \le x \le 1 \end{cases}$$

5.6

$$S_0'(x) = b_0 - 6x + 3d_0x^2 \qquad S_1'(x) = b_1 + 2c_1x + 3d_1x^2$$
$$S_0''(x) = -6 + 6d_0x \qquad S_1''(x) = 2c_1 + 6d_1x$$

The 8 conditions are

$$S_0(-1) = 1 \;\Rightarrow\; a_0 - b_0 - 3 - d_0 = 1 \;\Rightarrow\; a_0 - b_0 - d_0 = 4$$
$$S_1(0) = 2 \;\Rightarrow\; a_1 = 2$$
$$S_1(1) = -1 \;\Rightarrow\; a_1 + b_1 + c_1 + d_1 = -1 \;\Rightarrow\; b_1 + c_1 + d_1 = -3$$
$$S_1(0) = S_0(0) \;\Rightarrow\; a_1 = a_0 \;\Rightarrow\; a_0 = 2$$
$$S_1'(0) = S_0'(0) \;\Rightarrow\; b_1 = b_0$$
$$S_1''(0) = S_0''(0) \;\Rightarrow\; 2c_1 = -6 \;\Rightarrow\; c_1 = -3$$
$$S_0''(-1) = 0 \;\Rightarrow\; -6 - 6d_0 = 0 \;\Rightarrow\; d_0 = -1$$
$$S_1''(1) = 0 \;\Rightarrow\; 2c_1 + 6d_1 = 0 \;\Rightarrow\; -6 + 6d_1 = 0 \;\Rightarrow\; d_1 = 1$$

From the first condition, $b_0 = a_0 - d_0 - 4 = -1$

From the fifth condition, $b_1 = b_0 \;\Rightarrow\; b_1 = -1$

PART 6.

6.1

$$f(x) = 1 \qquad \int_0^{3h} dx = 3h \qquad \frac{3h}{4}(3+1) = 3h$$
$$f(x) = x \qquad \int_0^{3h} x\,dx = \frac{9h^2}{2} \qquad \frac{3h}{4}(3h+3h) = \frac{9h^2}{2}$$
$$f(x) = x^2 \qquad \int_0^{3h} x^2 dx = 9h^3 \qquad \frac{3h}{4}(3h^2 + 9h^2) = 9h^3$$
$$f(x) = x^3 \qquad \int_0^{3h} x^3 dx = \frac{81h^4}{4} \qquad \frac{3h}{4}(3h^3 + 27h^3) = \frac{45h^4}{2} \ne \frac{81h^4}{4}$$

thus degree of precision is 2

6.2

$h = 2/5$, $x_0 = -0.6$, $x_1 = -0.2$, $x_2 = 0.2$, $x_3 = 0.6$

$$I \approx \frac{5}{24}\frac{2}{5}\left(11f(-0.6) + f(-0.2) + f(0.2) + 11f(0.6)\right)$$

$$= \frac{1}{12}\left(11(1.031670) + 1.000276 + 1.000276 + 11(1.031670)\right) \quad \leftarrow \text{leave your answer in this form}$$

$$= 2.058108$$

6.3

$$I_{22} = \frac{4I_{21} - I_{11}}{3} \quad \text{or} \quad I_{21} + \frac{I_{21} - I_{11}}{3}$$

$$= \frac{4\left[\frac{b-a}{4}\left(f_0 + 2f_1 + f_2\right)\right] - \frac{b-a}{2}\left(f_0 + f_2\right)}{3}$$

$$= \frac{b-a}{6}\left[2\left(f_0 + 2f_1 + f_2\right) - \left(f_0 + f_2\right)\right]$$

$$= \frac{b-a}{6}\left[f_0 + 4f_1 + f_2\right]$$

$$= \frac{h}{3}\left(f_0 + 4f_1 + f_2\right) \quad \text{with } h = \frac{b-a}{2}$$

6.4

If $f(x) = 1/x$, then $f^{(4)}(x) = 24/x^5$. Thus

$$\max_{0.5<\mu<1.5} \left|f^{(4)}(\mu)\right| = \frac{24}{(0.5)^5} = 768.$$

So

$$|\text{error}| = \left|-\frac{b-a}{180}h^4 f^{(4)}(\mu)\right| = \frac{1}{180}\frac{1}{(2m)^4}\left|f^{(4)}(\mu)\right| \le \frac{1}{2880m^4}(768).$$

Therefore, $|\text{error}| \le 10^{-8}$ implies that $\frac{768}{2880m^4} \le 10^{-8}$, which gives

$$m^4 \ge \frac{768 \times 10^8}{2880} = 26666666.66\cdots \text{ or } m \ge \left(\frac{768\times 10^8}{2880}\right)^{1/4} = 71.86.$$

NOTE: leave your answer in this $\Uparrow$ form as calculators are not allowed

6.5 (a)

$$\int_a^b f(x)dx \approx \frac{h}{3}\left(f_0 + 2\sum_{j=1}^{m-1} f_{2j} + 4\sum_{j=1}^{m} f_{2j-1} + f_{2m} \right)$$

That is,

$$c_1 = 3,\ c_2 = 2,\ c_3 = 4,\ p = m-1,\ q = m,\ r = 2j,\ t = 2j-1$$

(b)

```
function approx = compsimp (a , b, m)
h = ( b - a ) / ( 2 * m ) ;
sum1 = 0 ;
for j = 1 : m-1
        sum1 = sum1 + f ( a + 2 * j * h ) ;
end
sum2 = 0 ;
for j = 1 : m
        sum2 = sum2 + f ( a + (2*j-1) * h ) ;
end
approx = ( h / 3 ) * ( f(a) + 2*sum1 + 4*sum2 + f(b) ) ;
```

6.6

$$f(x_0 + h) = f(x_0) + hf'(x_0) + \frac{h^2}{2} f''(x_0) + \frac{h^3}{6} f'''(x_0) + O(h^4)$$

$$f(x_0 + 2h) = f(x_0) + 2hf'(x_0) + \frac{4h^2}{2} f''(x_0) + \frac{8h^3}{6} f'''(x_0) + O(h^4)$$

Thus

$$\begin{aligned}
&-3f(x_0) + 4f(x_0 + h) - f(x_0 + 2h) \\
&\quad = -3f(x_0) \\
&\qquad + 4f(x_0) + 4hf'(x_0) + 2h^2 f''(x_0) + \frac{2h^3}{3} f'''(x_0) + O(h^4) \\
&\qquad - f(x_0) - 2hf'(x_0) - 2h^2 f''(x_0) - \frac{4}{3}h^3 f'''(x_0) + O(h^4) \\
&\quad = 2hf'(x_0) - \frac{2}{3}h^3 f'''(x_0) + O(h^4).
\end{aligned}$$

Therefore,

$$f'(x_0) = \frac{-3f(x_0) + 4f(x_0 + h) - f(x_0 + 2h)}{2h} + \frac{h^2}{3} f'''(x_0) + O(h^3)$$

6.7

If $h = 0.04$, then $h/2 = 0.02$ and the given approximations (with their truncation errors) are

$$e = N(0.04) + K_2h^2 + K_4h^4 + K_6h^6 + \cdots$$

and

$$e = N(0.02) + K_2(h/2)^2 + K_4(h/2)^4 + K_6(h/2)^6 + \cdots$$

Multiplying the second equation by 4, and subtracting gives

$$3e = 4N(0.02) - N(0.04) + O(h^4)$$

$$e = \frac{4N(0.02) - N(0.04)}{3} + O(h^4) \text{ or}$$

$$e = N(0.02) + \frac{N(0.02) - N(0.04)}{3} + O(h^4)$$

Thus, the $O(h^4)$ approximation to e is

$$2.718372 + \frac{2.718372 - 2.718644}{3} = 2.718281333...$$

6.8

The three answers are

2.7180745

2.7182789 2.7182815

The justification is as follows.

Entries in column 2:

(a) $M = N_1(h) + K_1h^2 + K_2h^4 + \cdots$

(b) $M = N_1(h/3) + K_1\left(\frac{h}{3}\right)^2 + K_2\left(\frac{h}{3}\right)^4 + \cdots$

Calculate 9 * (b) – (a):

$$8M = 9N_1(h/3) - N_1(h) + O(h^4)$$

which implies that

$$M = \frac{9N_1(h/3) - N_1(h)}{8} \quad \text{or} \quad M = N_1(h/3) + \frac{N_1(h/3) - N_1(h)}{8}.$$

Thus, the two required values in the second column of the table are computed as follows:

$$2.723401 + \frac{2.723401 - 2.766013}{8} = 2.7180745$$

$$2.718848 + \frac{2.718848 - 2.723401}{8} = 2.7182789$$

For the entry in the third column:

(c) $\quad M = N_2(h) + K_1' h^4 + K_2' h^6 + \cdots$

(d) $\quad M = N_2(h/3) + K_1' \left(\frac{h}{3}\right)^4 + K_2' \left(\frac{h}{3}\right)^6 + \cdots$

Calculate 81*(d)-(c):

$$80M = 81N_2(h/3) - N_2(h) + O(h^6)$$

which implies that

$$M = \frac{81N_2(h/3) - N_2(h)}{80} \quad \text{or} \quad M = N_2(h/3) + \frac{N_2(h/3) - N_2(h)}{80}$$

Thus, the required value in the third column is

$$2.7182789 + \frac{2.7182789 - 2.7180745}{80} = 2.7182815$$

6.9 (a)

$$P(x) = \frac{(x-h)(x-3h)}{(0-h)(0-3h)} f(0) + \frac{(x-0)(x-3h)}{(h-0)(h-3h)} f(h) + \frac{(x-0)(x-h)}{(3h-0)(3h-h)} f(3h)$$

$$= \frac{x^2 - 4hx + 3h^2}{3h^2} f(0) - \frac{x^2 - 3hx}{2h^2} f(h) + \frac{x^2 - hx}{6h^2} f(3h)$$

Thus,

$$P'(x) = \frac{2x - 4h}{3h^2} f(0) - \frac{2x - 3h}{2h^2} f(h) + \frac{2x - h}{6h^2} f(3h)$$

(b) At $x = 0$,

$$P'(0) = \frac{-4}{3h} f(0) + \frac{3}{2h} f(h) - \frac{1}{6h} f(3h) \ .$$

With $h = 0.05$ and the given data,

$$f'(0) \approx P'(0) = \frac{-4}{3(0.05)}(0.12) + \frac{3}{2(0.05)}(0.11) - \frac{1}{6(0.05)}(0.15) = -3.2 + 3.3 - 0.5 = -0.4$$

PART 7.

7.1 (a)

$$f(x, y(x)) = \frac{1}{x}\left(y^2 + y\right)$$

so $$f'(x,\, y(x)) = \frac{1}{x}\left(2y\,y' + y'\right) + \left(\frac{-1}{x^2}\right)\left(y^2 + y\right) = \frac{1}{x^2}\left(y^2 + y\right)\left(2y + 1 - 1\right) = \frac{2y^2}{x^2}(y+1)$$

The iterative formula for the Taylor method of order 2 is

$$\begin{aligned} y_{i+1} &= y_i + hf(x_i, y_i) + \frac{h^2}{2} f'(x_i, y_i) \\ &= y_i + \frac{h}{x_i}\left(y_i^2 + y_i\right) + \frac{h^2}{2}\frac{2y_i^2}{x_i^2}\left(y_i + 1\right) \\ &= y_i + \frac{hy_i}{x_i}\left(y_i + 1\right) + \frac{h^2 y_i^2}{x_i^2}\left(y_i + 1\right) \end{aligned}$$

So

$$\begin{aligned} y_1 &= y_0 + \frac{hy_0}{x_0}\left(y_0 + 1\right) + \frac{h^2 y_0^2}{x_0^2}\left(y_0 + 1\right) \\ &= -2 + \frac{(0.1)(-2)}{(1)}(-2+1) + \frac{(0.01)(4)}{(1)^2}(-2+1) \\ &= -1.84 \end{aligned}$$

(b)

$$\begin{aligned} y_1 &= y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_0 + hf(x_0, y_0))\right] \\ &= -2 + \frac{0.1}{2}\left[f(1,-2) + f(1.1,-2 + (0.1)f(1,-2))\right] \quad ** \\ &= -2 + (0.05)\left[\frac{1}{1}(4-2) + f(1.1,-2 + (0.1)(2))\right] \\ &= -2 + (0.05)\left[2 + \frac{1}{1.1}\left((-1.8)^2 - 1.8\right)\right] \\ &= -2 + (0.05)(2 + 1.309090) \\ &= -1.8345 \end{aligned}$$

Line $**$ is OK for your answer if calculators are not allowed.

(c)

$O(h^3)$

7.2 (a)

$$\left| y(2.04) - y_4 \right| = \left| 1.078462 - 1.078829 \right| = 0.000367$$

(b)

Use the exact value $y(2.03) = 1.059126$ to define

$$v = y(2.03) + h\,f(2.03, 1.059126) = 1.059126 + (0.01)\left(1 + (2.03 - 1.059126)^2\right) \quad **$$
$$= 1.059126 + (0.01)(1.942596) = 1.078552$$

Line ** is OK for your answer if calculators are not allowed.

Then

$$\text{local t.e.} = |y(2.04) - v| = |1.078462 - v| \quad **$$
$$= |1.078462 - 1.078552| = 0.000090$$

Line ** is OK for your answer if calculators are not allowed.

(c)

```
function z = f( x , y)

z = 1 + (x – y)^2

[ x , y ] = ode45 ( @f , [ 2 , 4 ] , 1 )
```

NOTE. z = 1 + (x – y).^2 is OK, but the .^ is not required.
@f can also be ‘ f ‘ .
[2 , 4] can also be [2 4] .

7.3 (a)

$$|y(2.04) - y_4| = |1.07846154 - 1.07845974| = 0.00000180$$

(b)

Use the exact value $y(2.03) = 1.05912621$ to define

$$v = y(2.03) + \frac{h}{2}\left(f(2.03, y(2.03)) + f(2.04, y(2.03) + h\ f(2.03, y(2.03)))\right) \quad **$$
$$= 1.05912621 + 0.005\left(f(2.03, 1.05912621) + f(2.04, 1.05912621 + 0.01\,f(2.03, 1.05912621)\ )\right)$$
$$= 1.05912621 + 0.005\left(1.94259592 + f(2.04, 1.07855217\right)$$
$$= 1.07846110$$

Line ** is OK for your answer if calculators are not allowed.

Then

$$\text{local t.e.} = |y(2.04) - v| = |1.07846154 - v| \quad **$$
$$= |1.07846154 - 1.07846110| = 0.00000044$$

Line ** is OK for your answer if calculators are not allowed.

7.4 (a)

$$y''(x) = \frac{x\,y'(x) - y(x)}{x^2} = \frac{x\left[1 + y(x)/x\right] - y(x)}{x^2} = \frac{1}{x}$$

The iterative formula is

$$y_{i+1} = y_i + h\left[1 + \frac{y_i}{x_i}\right] + \frac{h^2}{2}\left[\frac{1}{x_i}\right]$$

(b)

```
function  taylor
x = 1 ;
y = 2 ;
h = 0.01 ;
for  i  =  1 : 100
        y = y + h * (1 + y / x ) + h ^ 2 / (2 * x )
        x = x + h ;
end
```