COMPUTER SCIENCE 349A

Handout Number 9

An illustration of order of convergence when $\alpha = 1$ and $\alpha = 2$. The limit of each sequence is $x_t = 1.3652\ 3001\ 3414\ 097$.

Note: in the following computed approximations, the underlined digits are correct.

Example 1

Computed approximations	Linear convergence $(\alpha = 1)$
$x_0 = \underline{1}.5$	
$x_1 = \underline{1}.2869\ 53768$	$\lvert x_t - x_1 \rvert / \lvert x_t - x_0 \rvert = 0.5808$
$x_2 = \underline{1}.4025\ 40804$	$\lvert x_t - x_2 \rvert / \lvert x_t - x_1 \rvert = 0.4767$
$x_3 = \underline{1.3}454\ 58374$	$\lvert x_t - x_3 \rvert / \lvert x_t - x_2 \rvert = 0.5299$
$x_4 = \underline{1.3}751\ 70253$	$\lvert x_t - x_4 \rvert / \lvert x_t - x_3 \rvert = 0.5028$
$x_5 = \underline{1.36}00\ 94193$	$\lvert x_t - x_5 \rvert / \lvert x_t - x_4 \rvert = 0.5167$
$x_6 = \underline{1.36}78\ 46968$	$\lvert x_t - x_6 \rvert / \lvert x_t - x_5 \rvert = 0.5095$
$x_7 = \underline{1.36}38\ 87004$	$\lvert x_t - x_7 \rvert / \lvert x_t - x_6 \rvert = 0.5132$
$x_8 = \underline{1.365}9\ 16734$	$\lvert x_t - x_8 \rvert / \lvert x_t - x_7 \rvert = 0.5113$
$x_9 = \underline{1.36}48\ 78217$	$\lvert x_t - x_9 \rvert / \lvert x_t - x_8 \rvert = 0.5123$
$x_{10} = \underline{1.365}4\ 10062$	$\lvert x_t - x_{10} \rvert / \lvert x_t - x_9 \rvert = 0.5118$

$\downarrow$

constant ≈ 0.51

that is, $\displaystyle \lim_{i\to\infty} \frac{\lvert E_{i+1} \rvert}{\lvert E_i \rvert} = \text{constant } \lambda$

Note that the above ratios $\lvert x_t - x_{i+1} \rvert / \lvert x_t - x_i \rvert$ can be computed only if you know the exact zero x_t. In practice these ratios are never computed; they are given here to illustrate the definition of "order of convergence" (that is, how these ratios determine the kind of slow convergence in the first column above, where the number of correct significant digits increases by some constant amount with each successive iteration).

Example 2

Computed approximations	Quadratic convergence ($\alpha = 2$)
$x_0 = \underline{1}.5$	
$x_1 = \underline{1.3}733\ 3333\ 3333\ 333$	$\lvert x_t - x_1 \rvert / \lvert x_t - x_0 \rvert^2 = 0.4461$
$x_2 = \underline{1.3652}\ 6201\ 4874\ 627$	$\lvert x_t - x_2 \rvert / \lvert x_t - x_1 \rvert^2 = 0.4874$
$x_3 = \underline{1.3652\ 3001\ 3}916\ 147$	$\lvert x_t - x_3 \rvert / \lvert x_t - x_2 \rvert^2 = 0.4902$

$$\downarrow$$

constant ≈ 0.49

that is, $\lim_{i \to \infty} \frac{\lvert E_{i+1} \rvert}{\lvert E_i \rvert^2} = \text{constant } \lambda$

and this reflects the approximate doubling of the number of correct significant digits with each iteration