CSC349A Numerical Analysis

Lecture 14

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2018

Table of Contents I

Introduction

Our next topic is the study of how a given function can be approximated by another function from a specified class of functions. The given function may be discrete or continuous. Typically the approximating function exhibits some desired properties such as:

1. Continuity
2. Easily differentiated
3. Easily integrated
4. Easily evaluated

Introduction II

Common classes of approximating functions:

1. Polynomials
2. Piecewise polynomials (splines)
3. Trigonometric sums (fourier series)

We will also study criteria for what constitutes a "good" approximating function.

Table of Contents I

Polynomial interpolation

Recall that the general formula for an nth-order polynomial is

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n$$

For $n + 1$ distinct data points there is one and only one order n (or less) polynomial that passes through them all. That is,

- only one line that passes through two points
- only one parabola that passes through three points, etc

Polynomial Interpolation consists of determining the unique nth-order polynomial that fits the $n + 1$ data points in question.

- Although the polynomial is unique there are different methods for finding it and different formats for expressing it.

Polynomial interpolation II

Formally: Let $y = f(x)$ be any given function. For any value of $n \geq 0$ and any given values $x_0, x_1, \ldots, x_n$, let $y_i = f(x_i)$. The **polynomial interpolation problem** is to determine a polynomial $P(x)$ of degree less than or equal to n for which:

$$P(x_i) = y_i \quad \text{for } i = 0, 1, \ldots, n$$

- The set of $n+1$ data point (x_i, y_i) may be the only functional values known (that is, $f(x)$ is a **discrete function**, which could occur for example with experimental data), or
- $f(x)$ maybe be a known **continuous function**, and the $n+1$ data points (x_i, y_i) are a finite set of values with $y_i = f(x_i)$ (samples).

Terminology

- If z is some value between 2 of the given values x_i and if $P(z)$ is computed as an approximation to $f(z)$, then this approximation is said to be determined by polynomial **interpolation**.
- On the other hand, if z lies outside of the interval containing all of the values x_i and if $P(z)$ is computed as an approximation to $f(z)$, then this approximation is said to be determined by polynomial **extrapolation**.

Polynomial interpolation vs. Taylor approximation

- Note that an **interpolating polynomial** and the **Taylor polynomial** both determine polynomial approximations to $f(x)$. However, in general they are very different approximations to $f(x)$.
- Note that an interpolating polynomial uses the information:

$$y_0 = f(x_0), y_1 = f(x_1), \ldots, y_n = f(x_n)$$

to determine the polynomial approximation.
- Whereas, the Taylor polynomial uses the information:

$$f(x_0), f'(x_0), \ldots, f^{(n)}(x_0)$$

to determine the polynomial approximation.

Table of Contents I

Lagrange Interpolating Polynomial

Given $(x_i, f(x_i)), 0 \leq i \leq n$, with all x_i distinct, consider the function:

$$\begin{aligned} P(x) &= \sum_{i=0}^{n} L_i(x) f(x_i) \\ &= L_0(x) f(x_0) + L_1(x) f(x_1) + \cdots + L_n(x) f(x_n) \end{aligned}$$

where

$$\begin{aligned} L_i(x) &= \frac{(x - x_0)(x - x_1)\ldots(x - x_{i-1})(x - x_{i+1})\ldots(x_i - x_n)}{(x_i - x_0)(x_i - x_1)\ldots(x_i - x_{i-1})(x_i - x_{i+1})\ldots(x_i - x_n)} \\ &= \prod_{j=0, j\neq i}^{n} \frac{x - x_j}{x_i - x_j}, \quad \text{for } i = 0, 1, 2, \ldots, n \end{aligned}$$

Lagrange Interpolating Polynomial II

Since each function $L_i(x)$ is a polynomial of order n and $f(x_i)$ is a constant, $P(x)$ is a polynomial of order $\leq n$.
Also, since

$$L_i(x_i) = 1 \text{ and } L_i(x_j) = 0 \text{ if } j \neq i,$$

it follows that:

$$P(x_i) = f(x_i), \quad \text{for } i = 0, 1, 2, \ldots, n$$

that is, $P(x)$ is a an interpolating polynomial for the given data. It is called the **Lagrange interpolating polynomial**.

Examples

Example 1 Evaluate ln(2) using Lagrange polynomial interpolation, given that

$$\ln 1 = 0$$
$$\ln 4 = 1.386294$$
$$\ln 6 = 1.791760$$

Example 2 See Handout 20 - Complete elliptic integral function

$$K(k) = \int_0^{\pi/2} \frac{dz}{\sqrt{1 - k^2 \sin^2 z}}$$

where

$\sin^{-1} k$	$K(k)$
65°	2.3088
66°	2.3439
67°	2.3809

Table of Contents I

Finding the coefficients of the interpolating polynomial

Although the Langrange polynomial is well-suited to solving intermediate values it does not give you a polynomial in simple form

$$P(x) = a_0 + a_1 x + \cdots + a_n x^n$$

The coefficients of such an interpolating polynomial can be determined by solving a system of linear equations.

Finding the coefficients of the interpolating polynomial

Given a function $f(x)$ and distinct points $x_0, x_1, ..., x_n$, let $P(x)$ be the polynomial of degree $\leq n$ for which $P(x_i) = f(x_i)$ for $i = 0, 1, ..., n$.
Then,

$$\begin{aligned} a_0 + a_1 x_0 + \cdots + a_n x_0^n &= f(x_0) \\ a_0 + a_1 x_1 + \cdots + a_n x_1^n &= f(x_1) \\ &\vdots \\ a_0 + a_1 x_n + \cdots + a_n x_n^n &= f(x_n) \end{aligned}$$

Finding the coefficients of the interpolating polynomial

In matrix form, solve

$$\begin{bmatrix} 1 & x_0 & \dots & x_0^n \\ 1 & x_1 & \dots & x_1^n \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & \dots & x_n^n \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_n) \end{bmatrix}$$

So, if $n = 2$, we let $P(x) = a_0 + a_1 x + a_2 x^2$ and solve

$$\begin{bmatrix} 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \end{bmatrix}$$

Example Let $f(x) = \sin x$, $x_0 = 0.2$, $x_1 = 0.5$, and $x_2 = 1$ and find the interpolating polynomial.

Note

An interpolating polynomial can be specified in many different forms.

1. For example the form can be $a(x-x_2)^2+b(x-x_2)+c$ or
2. using the Lagrange form for $n=2$:

$$\begin{aligned}
P(x) &= L_0(x)f(x_0)+L_1(x)f(x_1)+L_2(x)f(x_2) \\
&= \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}f(x_0)+\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}f(x_1) \\
&+ \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}f(x_2)
\end{aligned}$$

   or
3. simply as $P(x)=Ax^2+Bx+C$.

We will show that all of these forms are identical as the interpolating polynomial is unique.

Table of Contents I

Uniqueness

Theorem: Given any $n+1$ distinct points $x_0, x_1, \ldots, x_n$ and any $n+1$ values $f(x_0), f(x_1), \ldots, f(x_n)$, there exists a unique polynomial $P(x)$ of degree $\leq n$ such that

$$P(x_i) = f(x_i) \text{ for } i = 0, 1, ..., n$$

Proof:
Existence: by construction of the Lagrange interpolating polynomial

Uniqueness proof

Uniqueness:
Suppose there exist two polynomials $P(x)$ and $Q(x)$ of degree $\leq n$ such that:

$$P(x_i) = Q(x_i) = f(x_i), 0 \leq i \leq n$$

Consider the function

$$R(x) = P(x) - Q(x)$$

which is also a polynomial of degree $\leq n$. But $R(x_i) = 0$ for $0 \leq i \leq n$. That is $R(x)$ has $n + 1$ distinct zeros. This implies that $R(x) = 0$ for all x and therefore $P(x) = Q(x)$.

Error term of polynomial interpolation

Theorem:
Let $x_0, x_1, \ldots x_n$ be any distinct points in $[a, b]$. Let $f(x) \in C^{n+1}[a, b]$ and let $P(x)$ interpolate $f(x)$ at x_i.
Then for each $\hat{x} \in [a, b]$, there exists a value ξ in (a, b) such that

$$f(\hat{x}) = P(\hat{x}) + \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^{n} (\hat{x} - x_i)$$

for example for $n = 3$

$$f(\hat{x}) = P(\hat{x}) + \frac{f^{(4)}(\xi)}{24} (\hat{x} - x_0)(\hat{x} - x_1)(\hat{x} - x_2)(\hat{x} - x_3)$$

The limitation of this error bound for polynomial interpolation is the need to find an upper bound for $f^{(n+1)}(x)$ on $[a, b]$.