CSC349A Numerical Analysis

Lecture 18

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Introduction

- Chapter 22 - Accurate and efficient algorithms for approximating $\int_a^b f(x)dx$ when $f(x)$ is known at any $x_i \in [a, b]$.
- We will look at **Richardson's Extrapolation** (22.2.1) and **Romberg Integration** (22.2) in this lecture.
- This corresponds with Handouts 31 and 32.
- Richardson's extrapolation is the technique of combining two different numerical approximations that depend on a parameter (usually a step size h) in order to obtain a new approximation having a smaller truncation error.

Richardson's Extrapolation I

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- Let M denote some value to be computed, for example,

$$f'(x) \text{ or } f''(x) \text{ or } \int_a^b f(x)dx.$$

- Let $N_1(h)$ denote an approximation to M, dependent on h.
- Suppose that the form of the truncation error is a known infinite series in powers of h.

Richardson's Extrapolation II

For example,

$$\underbrace{M}_{\substack{\text{exact}\\ \text{value}}} = \underbrace{N_1(h)}_{\substack{\text{computed}\\ \text{approx.}}} + \underbrace{K_1 h^2 + K_2 h^4 + K_3 h^6 + \cdots}_{\text{turncation error } O(h^2)} \tag{1}$$

- where the values K_i are some (possibly unknown) constants,
- and the parameter h can be any positive value,
- but as $h \to 0$, the truncation error $\to 0$; that is, $N_1(h) \to M$.

Richardson's Extrapolation III

- Now, consider a new approximation with a different stepsize, say $h/2$,

$$M = N_1\left(\frac{h}{2}\right) + K_1\frac{h^2}{4} + K_2\frac{h^4}{16} + K_3\frac{h^6}{64} + \cdots \qquad (2)$$

- In order to obtain an $O(h^4)$ approximation to M, we compute $4 \times (2) - (1)$, which gives,

$$M = \underbrace{N_1\left(\frac{h}{2}\right) + \frac{N_1\left(\frac{h}{2}\right) - N_1(h)}{3}}_{\text{New approxiamtion } N_2(h)} \underbrace{-\frac{K_2}{4}h^4 - \frac{5K_3}{16}h^6 - \cdots}_{\text{turncation error } O(h^4)} \qquad (3)$$

Richardson's Extrapolation IV

- This is Richardson's Extrapolation - combine two $O(h^2)$ approximations to get a better $O(h^4)$ approximation with the simple computation,

$$N_2(h) = N_1\left(\frac{h}{2}\right) + \frac{N_1\left(\frac{h}{2}\right) - N_1(h)}{3}$$

- We can extend this to get an $O(h^6)$ approximation, then an $O(h^8)$ approximation, etc.
- To do so, just keep taking successively smaller stepsizes, but they must decrease by the same factor.

Richardson's Extrapolation V

- If we continue from equation (3), we can derive a simple formula for all of these approximations.
- Let (3) be rewritten as,

$$M = N_2(h) + K_2' h^4 + K_3' h^6 + \cdots \tag{4}$$

- Then, let $h = \frac{h}{4}$, giving,

$$M = N_1\left(\frac{h}{4}\right) + K_1 \frac{h^2}{4^2} + K_2 \frac{h^4}{4^4} + K_3 \frac{h^6}{4^6} + \cdots \tag{5}$$

Richardson's Extrapolation VI

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- Combine $4 \times (5) - (2)$, getting,

$$M = \underbrace{N_1\left(\frac{h}{4}\right) + \frac{N_1\left(\frac{h}{4}\right) - N_1\left(\frac{h}{2}\right)}{3}}_{\text{New approxiamtion } N_2\left(\frac{h}{2}\right)} \underbrace{-\frac{K_2}{4}\frac{h^4}{2^4} - \frac{5K_3}{16}\frac{h^6}{2^6} - \cdots}_{\text{turncation error } O(h^4)} \tag{6}$$

- Then, combine $16 \times (6) - (4)$, this gives,

$$M = \underbrace{N_2\left(\frac{h}{2}\right) + \frac{N_2\left(\frac{h}{2}\right) - N_2\left(h\right)}{15}}_{\text{New approxiamtion } N_3\left(h\right)} + \underbrace{K_3'' h^6 + K_4'' h^8 + \cdots}_{\text{turncation error } O(h^6)} \tag{7}$$

Richardson's Extrapolation Table

$O(h^2)$	$O(h^4)$	$O(h^6)$	$O(h^8)$
$N_1(h)$			
$N_1\left(\frac{h}{2}\right)$	$N_2(h)$		
$N_1\left(\frac{h}{4}\right)$	$N_2\left(\frac{h}{2}\right)$	$N_3(h)$	
$N_1\left(\frac{h}{8}\right)$	$N_2\left(\frac{h}{4}\right)$	$N_3\left(\frac{h}{2}\right)$	$N_4(h)$

- In general, entries are computed by,

$$N_j\left(\frac{h}{2^i}\right) = N_{j-1}\left(\frac{h}{2^{i+1}}\right) + \frac{N_{j-1}\left(\frac{h}{2^{i+1}}\right) - N_{j-1}\left(\frac{h}{2^i}\right)}{4^{j-1}-1},$$

where $j \geq 2$ and the truncation error is $O(h^{2j})$.

Advantages of Richardson's Extrapolation

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- Using Richardson's extrapolation has the following advantages:
  1. obtain high accuracy with little computation
  2. doesn't require very small values of h to get high accuracy, so roundoff error is not a concern
- We now look to apply Richardson's extrapolation to one of our integration formulas.

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Romberg Integration I

Romberg integration is the application of Richardson's extrapolation to composite trapezoid rule approximations.

- Let $I_{k,1}$ denote the composite trapezoid rule approximation to $\int_a^b f(x)dx$ with 2^{k-1} subintervals, using $h = \frac{b-a}{2^{k-l}}$.
- Then, letting $f_i = f(x_i)$,

$$I_{1,1} = \frac{h}{2}\left[f_0 + f_1\right], \text{ where } h = b - a$$
$$I_{2,1} = \frac{h}{2}\left[f_0 + 2f_1 + f_2\right], \text{ where } h = \frac{b-a}{2}$$
$$I_{3,1} = \frac{h}{2}\left[f_0 + 2f_1 + 2f_2 + 2f_3 + f_4\right], \text{ where } h = \frac{b-a}{4}$$

etc.

Romberg Integration II

- Turns out that the error term for the composite trapezoid rule, $-\frac{b-a}{12}h^2 f''(\mu)$ with $a < \mu < b$, has a series expansion of the form,

$$K_1 h^2 + K_2 h^4 + K_3 h^6 + \cdots ,$$

where each K_i is a constant independent of h, thus R.E. applies.

- For example, letting $h = b - a$,

$$\int_a^b f(x)dx = I_{1,1} + K_1 h^2 + K_2 h^4 + K_3 h^6 + \cdots$$

$$\int_a^b f(x)dx = I_{2,1} + K_1 \frac{h^2}{4} + K_2 \frac{h^4}{16} + K_3 \frac{h^6}{64} + \cdots$$

etc.

Romberg Integration Table

Thus, we have the comparable Romberg table

$O(h^2)$	$O(h^4)$	$O(h^6)$	$O(h^8)$
$I_{1,1}$			
$I_{2,1}$	$I_{2,2}$		
$I_{3,1}$	$I_{3,2}$	$I_{3,3}$	
$I_{4,1}$	$I_{4,2}$	$I_{4,3}$	$I_{4,4}$

- In general, entries are computed by,

$$I_{k,j} = I_{k,j-1} + \frac{I_{k,j-1} - I_{k-1,j-1}}{4^{j-1} - 1},$$

for $k = 2, 3, 4, \dots$ and $j = 2, 3, \dots, k$.

Example of Romberg Integration

- **Example 2:** Approximate $\int_1^3 \frac{1}{x} dx$ using Romberg integration.
- The Romberg table is as follows:

	$I_{k,1}$	$I_{k,2}$	$I_{k,3}$	$I_{k,4}$
$h = 2$	1.333333			
$h = 1$	1.166667	1.111111		
$h = 0.5$	1.116667	1.100000	1.099259	
$h = 0.25$	1.103211	1.098726	1.098641	1.098631
$h = 0.125$	1.099768	1.098620	1.098613	1.098613

- with the final solution $I_{5,5} = 1.098613$.

Notes on Romberg

1. The entries in the Romberg table are computed row-by-row, stopping when two successive diagonal entries in the table are sufficiently close together:

$$\left| \frac{I_{n,n} - I_{n-1,n-1}}{I_{n,n}} \right| < \varepsilon.$$

2. There is a convergence theorem: If

$$\lim_{n \to \infty} I_{n,1} = \int_a^b f(x) dx$$

, then

$$\lim_{n \to \infty} I_{n,n} = \int_a^b f(x) dx.$$