COMPUTER SCIENCE 349A
Handout Number 34

ORDINARY DIFFERENTIAL EQUATIONS

Part 7 of the textbook: see pages 697-706 of the 6th ed. or pages 699-708 of the 7th ed. for Motivation and Mathematical Background. Recall that the mathematical model of a free-falling body developed in Chapter 1 involved a first order differential equation, and a simple numerical method was developed there for solving that differential equation.

Chapter 25 Runge-Kutta methods

The problem considered is a first-order initial value problem, which is always assumed to be in the form

$$y'(x) = f(x, y(x)), \quad \text{subject to the initial condition} \quad y(x_0) = y_0.$$

Here x is the independent variable, the dependent variable y is a function of x, and f is a given function of both x and $y(x)$ that specifies the derivative of the function y.

Given constants $a = x_0$, b and y_0 and a function $f(x, y(x))$, the problem is to determine $y(x)$ for $a \le x \le b$.

Example

Determine $y(x)$ for $0 \le x \le 2$ such that

$$y'(x) = y - x^2 + 1, \quad \text{subject to } \; y(0) = 0.5 \;.$$

This problem has an analytic solution, namely $y(x) = (x+1)^2 - 0.5e^x$.

Numerical methods

All numerical methods we will consider for approximating $y(x)$ are called "difference methods" (or discrete variable methods): that is, the continuous, exact solution $y(x)$ is approximated by a finite set of computed values at a set of mesh points (or grid points) $x_0, x_1, x_2, \dots, x_N$ in $[a, b]$.

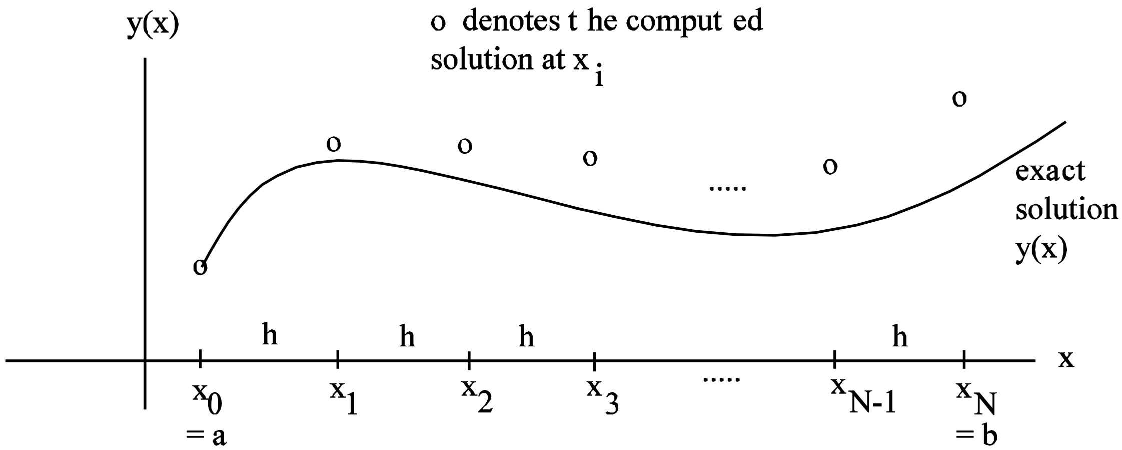
For now, we consider only equally-spaced mesh points, and let

$$x_i = a + ih = x_0 + ih, \quad i = 0, 1, 2, \dots, N,$$

where

$$h = \frac{b-a}{N}$$

is called the step size.



At each of the $N+1$ mesh points x_i, the exact solution is denoted by $y(x_i)$, and the computed approximation is denoted by y_i.

Euler's method (Section 25.1, page 708 of the 6th ed. or page 710 of the 7th ed.)
-- seldom used, but illustrates the derivation and use of numerical methods.

Derivation from Taylor's Theorem:
Taylor's Theorem for $y(x)$ (with $n=1$) expanded about x_i is

$$y(x) = y(x_i) + h\,y'(x_i) + \frac{h^2}{2}\,y''(\xi)$$

for some value ξ between x_i and x. Thus, at $x = x_{i+1}$,

$$y(x_{i+1}) = y(x_i) + h\,y'(x_i) + \frac{h^2}{2}\,y''(\xi)$$

where $h = x_{i+1} - x_i$.

For small values of *h*, this suggests the approximation

$$\begin{aligned} y(x_{i+1}) &\approx y(x_i) + h y'(x_i) \\ &= y(x_i) + h\,f(x_i, y(x_i)), \quad . \end{aligned}$$

using the differential equation $y'(x) = f(x, y(x))$.

Euler's method is obtained from this truncated Taylor polynomial approximation by replacing the exact solution $y(x_{i+1})$ by its numerical approximation y_{i+1}, and similarly by replacing $y(x_i)$ by y_i: given an initial condition

$$y_0 = y(x_0),$$

let

$$y_{i+1} = y_i + h\,f(x_i, y_i), \quad \text{for } i = 0, 1, 2, 3, \ldots, N-1.$$

Example

Consider the initial-value problem

$$y'(x) = y - x^2 + 1, \quad \text{subject to } y(0) = 0.5 .$$

Here $x_0 = 0$ and $y_0 = 0.5$. Let $h = 0.2$. Using Euler's method with $i = 0$ gives

$$y_1 = y_0 + h\,f(x_0, y_0) = 0.5 + (0.2)\,f(0, 0.5) = 0.5 + (0.2)(0.5 - 0 + 1) = 0.8$$

The first few computed approximations y_i and the corresponding exact solutions $y(x_i)$ are as follows:

x_i	y_i	$y(x_i)$	$\lvert y(x_i) - y_i \rvert$
0	0.5	0.5	0.0
0.2	0.8	0.8292986	0.0292986
0.4	1.152	1.2140877	0.0620877
0.6	1.5504	1.6489406	0.0985406
0.8	1.98848	2.1272295	0.1387495

Geometric interpretation of Euler's method:

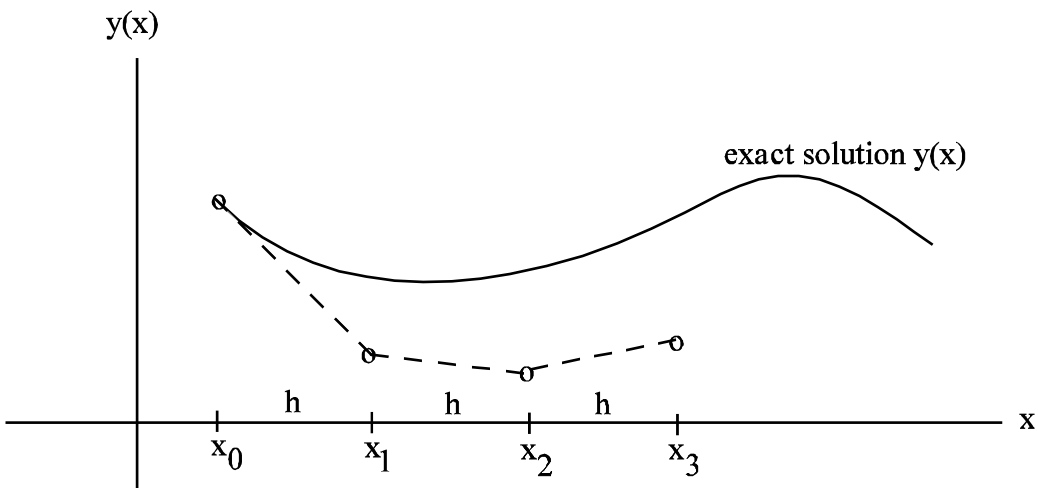
$y_0 = y(x_0)$ is the given initial condition. Using this value, we compute

$$\begin{aligned} y_1 &= y_0 + h\,f(x_0, y_0) \\ &= y_0 + h\,y'(x_0) \end{aligned}$$

from which it follows that

$$\frac{y_1 - y_0}{h} = y'(x_0) .$$

Geometrically, this says that y_1 is obtained from y_0 by constructing the tangent line to the graph of $y(x)$ at x_0 (which has slope equal to $y'(x_0)$) and going a distance *h*.



Similarly,

$$y_2 = y_1 + h\,f(x_1, y_1)\,,$$

so y_2 is determined by constructing a straight line through (x_1, y_1) with slope $f(x_1, y_1)$. Note, however, that this is not the slope of the tangent line to the graph of $y(x)$ at x_1, since $f(x_1, y_1)$ is only an approximation to $y'(x_1)$ (since the computed approximation y_1 is not equal to the exact value $y(x_1)$).

In general, y_{i+1} is obtained by constructing a straight line through (x_i, y_i) with slope equal to $f(x_i, y_i)$, which is an approximation to $y'(x_i)$. As y_i depends on y_{i-1}, which in turn depends on y_{i-2} and so on, successive values of y_i tend to be less and less accurate (as the truncation errors accumulate as you go across the interval $[a, b]$).