COMPUTER SCIENCE 349A

Handout Number 13

HORNER'S ALGORITHM
(NESTED MULTIPLICATION, SYNTHETIC DIVISION)

Given a polynomial $f(x) = \sum_{i=0}^{n} a_i x^i$ and a value x_0, this algorithm is used to efficiently evaluate $f(x_0)$ and $f'(x_0)$. To illustrate the basic idea, consider the case $n = 4$:

(1) $$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$$

can be rewritten in the form

(2) $$f(x) = a_0 + x \times (a_1 + x \times (a_2 + x \times (a_3 + x \times a_4))).$$

Evaluation of (1) at x_0 requires 7 multiplications and 4 additions, whereas (2) requires only 4 multiplications and 4 additions.

The general case (for a polynomial of order *n*):
- form (1) requires $2n-1$ multiplications and *n* additions,
- form (2) requires *n* multiplications and *n* additions.

An algorithm to evaluate $f(x_0)$, assuming that $f(x) = \sum_{i=0}^{n} a_i x^i$ is written in the **"nested" form**, as in (2):

Let $b_n = a_n$. Then compute

$$\left.\begin{aligned} b_{n-1} &= a_{n-1} + b_n x_0 \\ b_{n-2} &= a_{n-2} + b_{n-1} x_0 \\ &\ \vdots \\ b_0 &= a_0 + b_1 x_0 \end{aligned}\right\} \qquad \begin{gathered} b_k = a_k + b_{k+1} x_0 \\ \text{for } k = n-1, n-2, \ldots, 1, 0 \end{gathered}$$

Then $b_0 = f(x_0)$.

NOTE that execution of this algorithm requires <u>exactly</u> *n* multiplications and *n* additions.

Development of an algorithm to evaluate $f'(x_0)$:

Let $b_n, b_{n-1}, \ldots, b_0$ be defined as above, and now define

$$Q(x) = b_1 + b_2 x + b_3 x^2 + \cdots + b_n x^{n-1}.$$

Then

$$\begin{aligned}
&(x - x_0)Q(x) + b_0 \\
&\quad = (x - x_0)(b_1 + b_2 x + \cdots + b_n x^{n-1}) + b_0 \\
&\quad = (b_1 x + b_2 x^2 + \cdots + b_n x^n) - (b_1 x_0 + b_2 x_0 x + \cdots + b_n x_0 x^{n-1}) + b_0 \\
&\quad = (b_0 - b_1 x_0) + (b_1 - b_2 x_0)x + \cdots + (b_{n-1} - b_n x_0)x^{n-1} + b_n x^n \\
&\quad = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} + a_n x^n \\
&\qquad\qquad \text{using the definition of the } b_k\text{'s} \\
&\quad = f(x).
\end{aligned}$$

Differentiating this with respect to x gives

$$f'(x) = Q(x) + (x - x_0)Q'(x),$$

which implies that

$$f'(x_0) = Q(x_0).$$

Thus, to evaluate $f'(x_0)$, one first needs to evaluate $f(x_0)$ as above, which gives the coefficients $b_n, b_{n-1}, \ldots, b_0$, and then evaluate $Q(x_0)$. The most efficient way to evaluate $Q(x_0)$, of course, is to use the nested form for the polynomial $Q(x)$. The following algorithm evaluates both $f(x_0)$ and $f'(x_0) = Q(x_0)$, using <u>nested multiplication</u> to evaluate both of the polynomials.

HORNER'S ALGORITHM

Given values $a_0, a_1, \ldots, a_n$ and x_0, compute:

$$\begin{aligned}
b_n &= a_n & c_n &= b_n \\
b_{n-1} &= a_{n-1} + b_n x_0 & c_{n-1} &= b_{n-1} + c_n x_0 \\
b_{n-2} &= a_{n-2} + b_{n-1} x_0 & c_{n-2} &= b_{n-2} + c_{n-1} x_0 \\
&\vdots & &\vdots \\
b_1 &= a_1 + b_2 x_0 & c_1 &= b_1 + c_2 x_0 \\
b_0 &= a_0 + b_1 x_0 & &
\end{aligned}$$

Then

$$b_0 = f(x_0) \text{ and } c_1 = f'(x_0).$$

EXAMPLE

Let $n = 4$ and

$$f(x) = x^4 - 2x^3 + 2x^2 - 3x + 4.$$

Using Horner's algorithm to evaluate $f(1)$ and $f'(1)$:

$$\begin{array}{ll}
b_4 = 1 & c_4 = 1 \\
b_3 = -2 + (1)(1) = -1 & c_3 = -1 + (1)(1) = 0 \\
b_2 = 2 + (-1)(1) = 1 & c_2 = 1 + (0)(1) = 1 \\
b_1 = -3 + (1)(1) = -2 & c_1 = -2 + (1)(1) = -1 \\
b_0 = 4 + (-2)(1) = 2 &
\end{array}$$

giving $f(1) = b_0 = 2$ and $f'(1) = c_1 = -1$.

Note that the explicit form of $f'(x)$, namely

$$f'(x) = 4x^3 - 6x^2 + 4x - 3,$$

is not obtained; only the <u>value</u> of $f'(1)$ is computed. Since $Q(x)$ depends on the value of x_0, which is equal to 1 above, all computations must be re-done in order to evaluate $f'(x)$ at a different value of *x*.