CSC349A Numerical Analysis

Lecture 17

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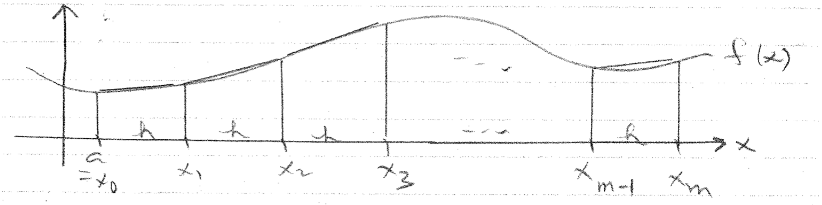
Introduction

- Corresponds to sections 21.1.2 and 21.2.2 of the text
- **Objective:** We want the **truncation error** $\rightarrow 0$ as the **number of quadrature points** $\rightarrow \infty$**.**
- Note: this does not happen in general as n, the order of the interpolating polynomial, $\rightarrow \infty$.
- **Solution:** We use composite (multiple-application) quadrature formulas.

Trapezoidal rule

Main idea: for $m \geq 1$, apply a closed N-C formula (with n small) m times on $[a, b]$.
Example: Trapezoidal rule ($n = 1$)
For any $m \geq 1$, let $h = \frac{b-a}{m}$, subdivide $[a, b]$ into m subintervals of length h, and apply the trapezoidal rule on each subinterval.



Composite trapezoidal rule

$$
\begin{aligned}
\int_a^b f(x)dx &= \int_{x_0}^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + \cdots + \int_{x_{m-1}}^{x_m} f(x)dx \\
&\approx \int_{x_0}^{x_1} P_0(x)dx + \int_{x_1}^{x_2} P_1(x)dx + \cdots + \int_{x_{m-1}}^{x_m} P_{m-1}(x)dx \\
&= \frac{h}{2}\left[f(x_0) + f(x_1)\right] + \frac{h}{2}\left[f(x_1) + f(x_2)\right] + \cdots + \frac{h}{2}\left[f(x_{m-1}) + f(x_m)\right] \\
&= h\left[\frac{f(x_0)}{2} + \sum_{i=1}^{m-1} f(x_i) + \frac{f(x_m)}{2}\right]
\end{aligned}
$$

This is called the composite trapezoial rule.

Truncation Error

$$E_t = -\frac{h^3}{12}f''(\xi_1) - \frac{h^3}{12}f''(\xi_2) - \cdots - -\frac{h^3}{12}f''(\xi_m)$$
$$= -\frac{h^3}{12}\left[f''(\xi_1) + f''(\xi_2) + \cdots + f''(\xi_m)\right]$$

where $x_{i-1} \leq \xi_i \leq x_i$.

Truncation Error II

We know that:

$$\min_{1 \le i \le m} f''(\xi_i) \le \frac{f''(\xi_1) + f''(\xi_2) + \cdots + f''(\xi_m)}{m} \le \max_{1 \le i \le m} f''(\xi_i)$$

If $f''(x)$ is continuous on $[a, b]$, then there exists a value $\mu \in [a, b]$ such that:

$$f''(\mu) = \frac{f''(\xi_1) + f''(\xi_2) + \cdots + f''(\xi_n)}{m}$$

This is called the intermediate value theorem.

$$E_t = -\frac{h^3}{12}[mf''(\mu)] = -\frac{(b-a)}{12}h^2 f''(\mu)$$

since $h = \frac{b-a}{m}$.

Important point

$$\lim_{m \to \infty} E_t = \lim_{h \to 0} E_t = 0$$

provided that $f''(x)$ is continuous on $[a, b]$.
(there is no comparable result as $n \to \infty$, where n is the degree of the interpolating polynomial)

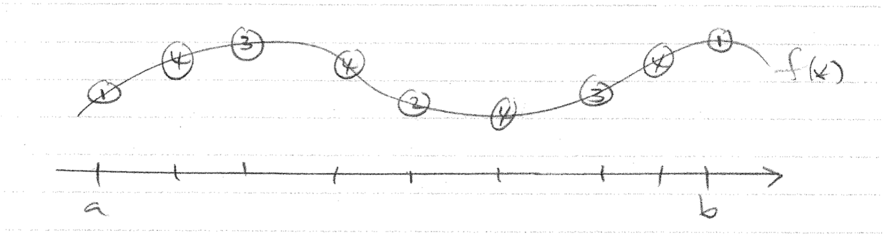
Implementation

Usual implementation of composite trapezoidal:

- Initialize $m = 1$
- Repeatedly double m (m=1,2,4,8,16,32,...)
- Until two consecutive approximations are sufficiently close

Reusing function evaluations

The reason for using these values of m is that they permit re-use of the function evaluations from previous evaluations i.e all values $f(x_i)$ computed for $m = k$ can be re-used for $m = 2k$.



Composite Simpson's Rule

- Each application of Simpson's rule requires 2 subintervals on the interval of integration and 3 quadrature points.
- Thus, m applications of Simpson's rule on $[a, b]$ require that $[a, b]$ be subdivided into $2m$ subintervals using $2m + 1$ quadrature points.
- Each subinterval then is of length

$$h = \frac{b - a}{2m}$$

Composite Simpson's Rule II

Thus, at the *jth* subinterval we have the three quadrature points x_{2j-2}, x_{2j-1}, and x_{2j}, and

$$\int_{x_{2j-2}}^{x_{2j}} f(x)dx \approx \frac{h}{3}\left[f(x_{2j-2}) + 4f(x_{2j-1}) + f(x_{2j})\right]$$

When $m = 1$ (regular Simpson's rule) we have $2(1) + 1 = 3$ quadrature points and 2 subintervals each of length $h = \frac{b-a}{2}$.

Composite Simpson's Rule ($m = 2$)

When $m = 2$, we apply Simpson's rule twice. We need $2(2) + 1 = 5$ quadrature points to create 4 subintervals each of length $h = \frac{b-a}{4}$.

Here,

$$\begin{aligned}
&\int_a^b f(x)dx \\
&\approx \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right] + \frac{h}{3}\left[f(x_2) + 4f(x_3) + f(x_4)\right] \\
&= \frac{h}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\right]
\end{aligned}$$

General Composite Simpson's Rule

In general, when $m \geq 1$, the composite Simpson's rule approximation is

$$\begin{aligned}
&\int_a^b f(x)dx \\
&\qquad \approx \frac{h}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \\
&\qquad\qquad \cdots + 2f(x_{2m-2}) + 4f(x_{2m-1}) + f(x_{2m})] \\
&\qquad = \frac{h}{3}\left[f(x_0) + 4\sum_{j=1}^{m} f(x_{2j-1}) + 2\sum_{j=1}^{m-1} f(x_{2j}) + f(x_{2m})\right]
\end{aligned}$$

Truncation Error

$$
\begin{aligned}
E_t &= -\frac{h^5}{90} f^{(4)}(\xi_1) - \frac{h^5}{90} f^{(4)}(\xi_2) - \cdots - -\frac{h^5}{90} f^{(4)}(\xi_m) \\
&= -\frac{h^5}{90} \left[f^{(4)}(\xi_1) + f^{(4)}(\xi_2) + \cdots + f^{(4)}(\xi_m) \right] \\
&= -\frac{h^5}{90} [m f^{(4)}(\mu)]
\end{aligned}
$$

where $a \leq \mu \leq b$ and $f^{(4)}(x)$ is continous. So,

$$
E_t = -\frac{(b-a)h^4}{180} f^{(4)}(\mu)
$$

since $h = \frac{b-a}{2m}$.

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Newton-Cotes Formulas with Unequal Segments

By way of example, consider the following 11 unequally spaced data points:

x	$f(x)$	x	$f(x)$
0.0	0.2	0.44	2.842985
0.12	1.309729	0.54	3.507297
0.22	1.305241	0.64	3.181929
0.32	1.743393	0.70	2.363
0.36	2.074903	0.80	0.232
0.40	2.456		

We could interpolate with a polynomial of degree 10 but we know that should be avoided. Instead, we look for patterns in the lengths of the subintervals and apply the appropriate method over those subintervals.

Summary of Examples

Below is a summary of the six examples of integrating $f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$ over the range $a = 0$ to $b = 0.8$ which has a true value of 1.640533:

Method	Quadrature	E_t	E_a
Trapezoid	0.1728	1.467733	2.56
1/3 Simpson	1.367467	0.2730667	0.2730667
3/8 Simpson	1.519170	0.1213630	0.1213630
Comp. Trapezoid	1.0688	0.57173	0.64
Comp. 1/3 Simpson	1.623467	0.017067	0.017067
Unequal Segments	1.603641	0.036892	unknown

Notes

- The composite Simpsons rule is very stable: roundoff errors do not disastrously accumulate as $m \to \infty$.
- As $m \to \infty$, the accumulated effect of the summation errors will eventually make the total roundoff error large.
- However, sufficient accuracy can usually be obtained without using such large values of m, so algorithms for quadrature are stable.
- The roundoff error analysis for other quadrature formulas is similar.

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Open Newton-Cotes

The goal is to integrate an interpolating polynomial with all quadrature points x_i in the open interval (a, b).

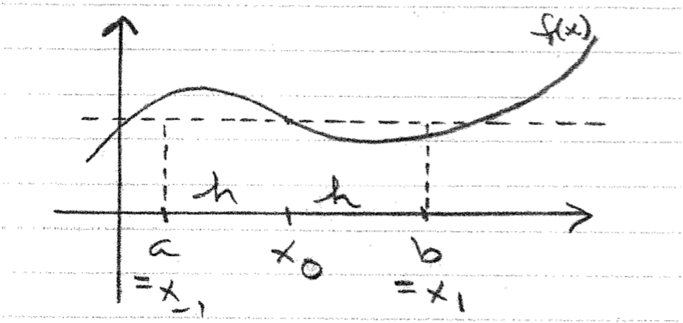
$$h = \frac{b-a}{n+2}, \quad n \geq 0$$

Construct an interpolating polynomial $P_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i)$ of degree n through $x_0, x_1, \ldots, x_n$. Then

$$\int_a^b f(x)dx \approx \int_a^b P_n(x)dx = \sum_{i=0}^{n} \left(\int_a^b L_i(x)dx \right) f(x_i)$$

Note: $f(x)$ is not evaluated at $a = x_{-1}$ or $b = x_{n+1}$

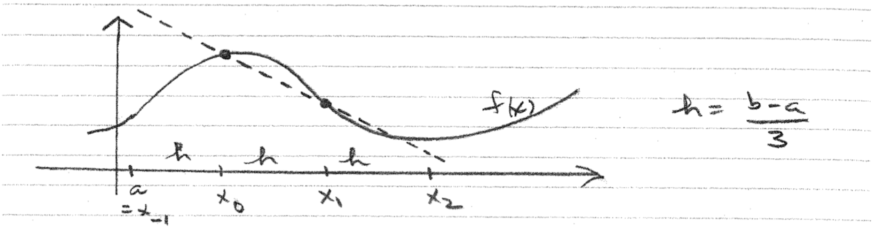
Midpoint rule



Case n=0: midpoint rule

$$\int_a^b f(x)dx \approx 2hf(x_0), \quad h = \frac{b-a}{2}$$

Case n=1



$$\int_a^b f(x)dx \approx \int_{x_{-1}}^{x_2} \left[\frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1) \right] dx$$

Integration for n=1

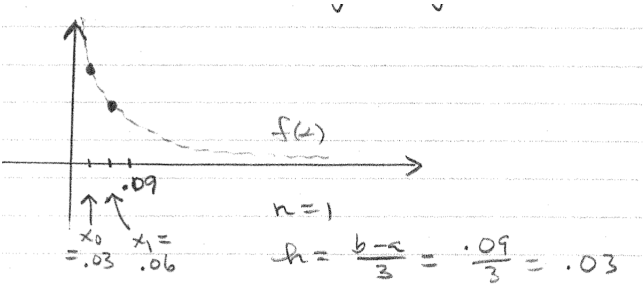
To integrate this more easily, use a exchange of variable $t = \frac{x - x_0}{h}$ then:

$$
\begin{aligned}
x - x_0 &= th \\
x - x_1 &= x - (x_0 + h) = (x - x_0) - h = (t-1)h \\
dx &= hdt
\end{aligned}
$$

$$
\begin{aligned}
\int_a^b f(x)dx \approx & \int_{-1}^{2} \left[\frac{(t-1)h}{-h} f(x_0) + \frac{th}{h} f(x_1) \right] hdt \\
&= -hf(x_0) \int_{-1}^{2} (t-1)dt + hf(x_1) \int_{-1}^{2} tdt \\
&= \frac{3h}{2} \left[f(x_0) + f(x_1) \right]
\end{aligned}
$$

Using open quadrature formulas

Use of open quadrature formulas: If $f(x)$ has a singularity.



Using open quadrature formulas II

$$h = \frac{b-a}{3} = \frac{.09}{3} = .03$$

$$\int_0^{0.09} f(x)dx \approx \frac{3h}{2}\left[f(x_0) + f(x_1)\right] = \frac{3(.03)}{2}\left[f(.03) + f(.06)\right]$$

The truncation error term is $\frac{3}{4}h^3 f''(\xi)$, for some $\xi \in (0.03, 0.06)$