COMPUTER SCIENCE 349A, SPRING 2018
ASSIGNMENT #6 - SOLUTIONS - 20 MARKS

1. (a) **(2 points)** The Lagrange interpolating polynomial is

$$\begin{aligned} P(x) &= \frac{(x-h)(x-4h)}{(-h)(-4h)}f(0) + \frac{(x-0)(x-4h)}{(h)(-3h)}f(h) + \frac{(x-0)(x-h)}{(4h)(3h)}f(4h) \\ &= \frac{(x^2-5hx+4h^2)}{4h^2}f(0) - \frac{(x^2-4hx)}{3h^2}f(h) + \frac{(x^2-hx)}{12h^2}f(4h) \end{aligned}$$

   (b) **(4 points)** We now integrate this between 0 and $4h$.

$$\begin{aligned} &\int_0^{4h} P(x)dx \\ &= \int_0^{4h}\left(\tfrac{(x^2-5hx+4h^2)}{4h^2}f(0) - \tfrac{(x^2-4hx)}{3h^2}f(h) + \tfrac{(x^2-hx)}{12h^2}f(4h)\right)dx \\ &= \tfrac{f(0)}{4h^2}\int_0^{4h}(x^2-5hx+4h^2)dx - \tfrac{f(h)}{3h^2}\int_0^{4h}(x^2-4hx)dx + \tfrac{f(4h)}{12h^2}\int_0^{4h}(x^2-hx)dx \\ &= \tfrac{f(0)}{4h^2}\left[\tfrac{x^3}{3} - \tfrac{5hx^2}{2} + 4h^2x\right]_0^{4h} - \tfrac{f(h)}{3h^2}\left[\tfrac{x^3}{3} - \tfrac{4hx^2}{2}\right]_0^{4h} + \tfrac{f(4h)}{12h^2}\left[\tfrac{x^3}{3} - \tfrac{hx^2}{2}\right]_0^{4h} \\ &= \tfrac{f(0)}{4h^2}\left[\tfrac{64h^3}{3} - 40h^3 + 16h^3\right] - \tfrac{f(h)}{3h^2}\left[\tfrac{64h^3}{3} - 32h^3\right] + \tfrac{f(4h)}{12h^2}\left[\tfrac{64h^3}{3} - 8h^3\right] \\ &= \tfrac{f(0)}{4h^2}\left[\tfrac{-8h^3}{3}\right] - \tfrac{f(h)}{3h^2}\left[\tfrac{-32h^3}{3}\right] + \tfrac{f(4h)}{12h^2}\left[\tfrac{40h^3}{3}\right] \\ &= \tfrac{-2h}{3}f(0) + \tfrac{32h}{9}f(h) + \tfrac{10h}{9}f(4h)(\text{ or } \tfrac{h}{9}[-6f(0) + 32f(h) + 10f(4h)]) \end{aligned}$$

   (c) **(2 points)** Here we have $x_0 = 0, x_1 = 0.1, x_2 = 0.4, h = 0.1, f(0) = 1, f(h) = 1.11091, f(4h) = 1.63778$, and thus

$$\begin{aligned} \int_0^{0.4} f(x)dx &\approx \frac{0.1}{9}[-6(1) + 32(1.11091) + 10(1.63778)] \\ &= \frac{0.1}{9}[-6 + 35.54912 + 16.3778] \\ &= 0.510299111 \end{aligned}$$

2. **(6 points)** We want the formula

$$\int_{-1}^{1} f(x)dx = af(-1) + bf(1) + cf'(-1) + df'(1)$$

to hold for polynomials 1, x, x^2, and x^3. Thus, plugging these into the formula we obtain:

For $f(x) = 1 \Rightarrow f'(x) = 0$, we have

$$\int_{-1}^{1} 1dx = [x]_{-1}^{1} = 2 = a(1) + b(1) + c(0) + d(0) = a + b$$

for $f(x) = x \Rightarrow f'(x) = 1$, we have

$$\int_{-1}^{1} x dx = [x^2/2]_{-1}^{1} = 0 = a(-1) + b(1) + c(1) + d(1) = -a + b + c + d$$

for $f(x) = x^2 \Rightarrow f'(x) = 2x$, we have

$$\int_{-1}^{1} x^2 dx = [x^3/3]_{-1}^{1} = 2/3 = a(1) + b(1) + c(-2) + d(2) = a + b - 2c + 2d$$

and for $f(x) = x^3 \Rightarrow f'(x) = 3x^2$, we have

$$\int_{-1}^{1} x^3 dx = [x^4/4]_{-1}^{1} = 0 = a(-1) + b(1) + c(3) + d(3) = -a + b + 3c + 3d$$

Thus, we have 4 equations in 4 unknowns solved by the following system:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 1 \\ 1 & 1 & -2 & 2 \\ -1 & 1 & 3 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 2/3 \\ 0 \end{pmatrix}$$

whose solution is $a = 1$, $b = 1$, $c = 1/3$, and $d = -1/3$ (see the MATLAB output below).

```
>>  A = [1 1 0 0; -1 1 1 1; 1 1 -2 2; -1 1 3 3];
>> A\[2; 0; 2/3; 0]
ans =
    1.0000
    1.0000
    0.3333
   -0.3333
```

Thus, the quadrature formula of the above form with accuracy $d = 3$ is:

$$\int_{-1}^{1} f(x) dx = f(-1) + f(1) + \frac{1}{3} f'(-1) - \frac{1}{3} f'(1)$$

NOTE: They do not have to use MATLAB to solve this system, they may do it by hand.

3. (a) **(4 points)** Here $f(x) = e^{3x} \sin 2x$ and

$$I_{1,1} = \frac{h}{2} [f(x_0) + f(x_1)] = \frac{\pi/4}{2} [f(0) + f(\pi/4)] = \frac{\pi}{8} [0 + 10.55072407] = 4.1432597$$

where $h = \pi/4$. Then, we let $h = \pi/8$ and get

$$\begin{aligned} I_{2,1} &= \frac{h}{2}\left[f(x_0) + 2f(x_1) + f(x_2)\right] = \frac{\pi/8}{2}\left[f(0) + 2f(\pi/8) + f(\pi/4)\right] \\ &= \frac{\pi}{16}\left[0 + 2(2.29681563) + 10.55072407\right] = 2.9735872 \end{aligned}$$

Now, we let $h = \pi/16$ and use

$$\begin{aligned} I_{3,1} &= \frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\right] \\ &= \frac{\pi/16}{2}\left[f(0) + 2f(\pi/16) + 2f(\pi/8) + 2f(3\pi/16) + f(\pi/4)\right] \\ &= \frac{\pi}{32}\left[0 + 2(0.6897) + 2(2.29681563) + 2(5.4085026) + 10.55072407\right] \\ &= 2.6841729 \end{aligned}$$

Now, we use the Romberg Integration formulas to combine these.

$$\begin{aligned} I_{2,2} &= I_{2,1} + \frac{I_{2,1} - I_{1,1}}{3} = 2.9735872 + \frac{2.9735872 - 4.1432597}{3} = 2.5836964 \\ I_{3,2} &= I_{3,1} + \frac{I_{3,1} - I_{2,1}}{3} = 2.6841729 + \frac{2.6841729 - 2.9735872}{3} = 2.5877015 \\ I_{3,3} &= I_{3,2} + \frac{I_{3,2} - I_{2,2}}{15} = 2.5877015 + \frac{2.5877015 - 2.5836964}{15} = 2.5879685 \end{aligned}$$

(b) **(2 points)** So,

$$|\varepsilon_t| = \left|\frac{\int_0^{\pi/4} e^{3x}\sin 2x dx - I_{3,3}}{\int_0^{\pi/4} e^{3x}\sin 2x dx}\right| = \left|\frac{2.5886286 - 2.5879685}{2.5886286}\right| = 0.000254999... \approx 0.025\%$$

and

$$|\varepsilon_a| = \left|\frac{I_{3,3} - I_{2,2}}{I_{3,3}}\right| = \left|\frac{2.5879685 - 2.5836964}{2.5879685}\right| = 0.001650754... \approx 0.165\%$$

As is to be expected the approximate relative error is a conservative approximation of the true relative error.