COMPUTER SCIENCE 349A

Handout Number 7

STABILITY OF AN ALGORITHM

Textbook (page 100 of the 7th ed.; page 97 of the 6th): a computation is <u>numerically unstable</u> if the uncertainty of the input values is greatly magnified by the numerical method. The following is a more precise definition.

<u>Definition.</u> An algorithm is said to be <u>stable</u> (for a class of problems) if it determines a computed solution (using floating-point arithmetic) that is close to the exact solution of some (small) perturbation of the given problem.

given problem, specified by data $\{d_i\}$	$\xrightarrow{}$ floating-point computation	computed solution $\{r_i\}$
perturbed problem, data $\{\hat{d}_i\} = \{d_i + \varepsilon_i\}$ with $\left\lvert \dfrac{\varepsilon_i}{d_i} \right\rvert$ small	$\xrightarrow{}$ exact computation	exact solution $\{\hat{r}_i\}$

If there exist data $\hat{d}_i \approx d_i$ (small ε_i for all i) such that $\hat{r}_i \approx r_i$ (for all i), then the algorithm is said to be **stable.**

If there exists <u>no set</u> of data $\{\hat{d}_i\}$ close to $\{d_i\}$ such that $\hat{r}_i \approx r_i$ for all i, then the algorithm is said to be **unstable**.

Meaning of numerical stability: the effect of uncertainty in the input data or of the floating-point arithmetic (the round-off error) is no worse than the effect of slightly perturbing the given problem, and solving the perturbed problem exactly.

Example 1

Approximate e^x when $x = -5.5$ using $b = 10,\ k = 5$ rounding floating-point arithmetic and the Taylor polynomial approximation (expanded about $a = 0$)

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!} \ .$$

The floating-point computation results in the summation of the following terms:

$$
\begin{aligned}
e^{-5.5} \approx\ & 1.0000 \\
& -5.5000 \\
& +15.125 \\
& -27.730 \\
& +38.129 \\
& -41.942 \\
& +38.446 \\
& -30.208 \\
& +20.768 \\
& -12.692 \\
& +6.9803 \\
& -3.4902 \\
& +1.5997
\end{aligned}
$$

and so on.

Using rounding floating-point arithmetic with $b = 10$ and $k = 5$, this sum equals 0.0026363 (or $+0.26363 \times 10^{-2}$) after summing 25 terms (that is, $n = 24$), and no further terms change this sum (as they are all $< 10^{-7}$).

However, the exact value of $e^{-5.5}$ is 0.00408677 (to 6 significant digits), so $f\ell(e^{-5.5})$ has no correct significant digits.

Stability Analysis of the above computation of $f\ell(e^{-5.5})$.

given problem, data $x = -5.5$	$\longrightarrow$ floating-point computation	0.0026363
perturbed problem, $\hat{x} = -5.5 + \varepsilon$ with $\lvert\varepsilon / 5.5\rvert$ small	$\longrightarrow$ exact computation	$e^{-5.5+\varepsilon} = e^{-5.5}e^{\varepsilon} = 0.00408677(1+\varepsilon+\frac{\varepsilon^2}{2}+\cdots)$ If $\lvert\varepsilon / 5.5\rvert$ is "small", then $e^{-5.5+\varepsilon} \approx 0.00408677(1+\varepsilon)$ and this value is very close to 0.00408 for <u>all</u> small values of ε.

That is, there are no small values of ε for which $e^{-5.5+\varepsilon}$ is close to 0.0026363, and thus the computation is **unstable**.

Another way to see this: if $e^{-5.5+\varepsilon} = 0.0026363$, then $\varepsilon = -0.43837\cdots$ and this value of ε is not small relative to -5.5 since $0.43837/5.5 \approx 0.08$ or 8%.

Note

A stable algorithm for computing $e^{-5.5}$ (and, in general, for computing e^x for $x < 0$):

$$\begin{aligned} e^{-5.5} &= \frac{1}{e^{5.5}} \\ &= \frac{1}{1 + 5.5 + \frac{(5.5)^2}{2} + \frac{(5.5)^3}{6} + \ldots} \end{aligned}$$

E.g., using $b = 10$, $k = 5$ floating-point arithmetic, this computation (using 18 terms of the Taylor polynomial approximation) gives a computed solution of 0.0040865 (which is very accurate).

Example 2

Consider the function

$$y(x) = \frac{1-\cos x}{x^2}, \qquad x \neq 0.$$

Due to subtractive cancellation, the evaluation of $y(x)$ will be inaccurate in floating-point arithmetic for values of x close to 0. For example, evaluation of $f\ell(y(0.009))$ using 4 decimal digit, idealized, rounding floating-point arithmetic gives a very inaccurate answer:

$$\begin{aligned} &f\ell(\cos(0.009)) = f\ell(0.9999595\cdots) = 1.000 \text{ or } 0.1000 \times 10^1 \\ &f\ell(1-\cos(0.009)) = f\ell(1.000 - 1.000) = 0.0 \\ &f\ell(0.009 \times 0.009) = 0.000081 \text{ or } 0.8100 \times 10^{-4} \\ &f\ell(y(0.009)) = f\ell(0.0 / 0.000081) = 0.0 \end{aligned}$$

As the correct value of $y(0.009)$ is 0.499996625… , the relative error in the above approximation is 1.0 or 100%.

To show that the above computation of $f\ell(y(0.009))$ is unstable, use a Taylor polynomial approximation in order to obtain a polynomial approximation to $y(x)$ that is very accurate for values of x close to 0. The order $n = 4$ Taylor polynomial approximation for $f(x) = \cos x$ expanded about $a = 0$ is

$$\cos x \approx 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4,$$

which gives the approximation

$$y(x) \approx \frac{1 - \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4\right)}{x^2} = \frac{1}{2} - \frac{1}{24}x^2.$$

Note that this approximation is very accurate for values of x close to 0.

STABILITY ANALYSIS

Given problem, $y(x) = \dfrac{1 - \cos x}{x^2}$, data $x = 0.009$ $\Rightarrow$ computed solution $r = 0.0$

Perturbed problem, $y(0.009 + \varepsilon) = \dfrac{1 - \cos(0.009 + \varepsilon)}{(0.009 + \varepsilon)^2}$ with $\left|\dfrac{\varepsilon}{0.009}\right|$ small $\Rightarrow$ exact value of $y(0.009 + \varepsilon)$ = $\dfrac{1 - \cos(0.009 + \varepsilon)}{(0.009 + \varepsilon)^2}$ is very close to $\dfrac{1}{2} - \dfrac{1}{24}(0.009 + \varepsilon)^2$

using the above Taylor approximation (since $0.009 + \varepsilon$ is close to 0).

If there is any value of ε such that $\left|\dfrac{\varepsilon}{0.009}\right|$ is small and $y(0.009 + \varepsilon) \approx 0.0$, then the computation of $f\ell(y(0.009))$ is stable. Otherwise, it is unstable.

However,

$$\frac{1}{2} - \frac{1}{24}(0.009 + \varepsilon)^2 = 0.499996625 - 0.00075\varepsilon - \frac{\varepsilon^2}{24}$$

and the question now is whether or not there exists a small value of ε (that is, with $\left|\frac{\varepsilon}{0.009}\right|$ small) such that this is ≈ 0.0 .

Clearly $0.499996625 - 0.00075\varepsilon - \frac{\varepsilon^2}{24}$ is approximately equal to 0.4999 for <u>all</u> values of ε such that $\left|\frac{\varepsilon}{0.009}\right|$ is small. As 0.4999 is not close to 0, the computation is <u>unstable</u>.

Example 3

The polynomial

$$\begin{aligned} P(x) &= (x-1)(x-2)(x-3)\cdots(x-20) \\ &= x^{20} - 210x^{19} + (\cdots)x^{18} + (\cdots)x^{17} + \cdots + (20!) \end{aligned}$$

clearly has zeros exactly equal to $1, 2, 3, \ldots, 20$. Let $Q(x)$ be identical to $P(x)$ except that the coefficient of x^{19} is changed from -210 to $-210 + 2^{-23}$. Then some of the 20 zeros of $Q(x)$ are approximately equal to

$$\begin{aligned} &20.8469 \\ &19.502 \pm 1.94i \\ &16.73 \pm 2.81i \end{aligned}$$

Thus, the problem computing the zeros of $P(x)$ is an extremely ill-conditioned problem.

Note: the zeros of $P(x)$ are the roots of the equation $P(x) = 0$.