COMPUTER SCIENCE 349A
Handout Number 15

NAÏVE GAUSSIAN ELIMINATION (Section 9.2)

Notation for a system of n linear equations in n unknowns: $Ax = b$, where

A is an $n \times n$ nonsingular matrix

and

x and b are (column) vectors with n entries.

Equivalently,

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n
\end{aligned}
$$

Given data A and b, the problem is to determine x.

Examples and motivation: see, for example, the Case Studies in Chapter 12 of the textbook.

Theoretical basis for the Gaussian elimination algorithm. You can apply any of the following three elementary row operations to $Ax = b$:

(i) multiply any equation E_i by a nonzero constant λ

(ii) replace equation E_i by $E_i + \lambda E_j$

(iii) interchange any two equations E_i and E_j

in order to reduce A to **upper triangular form**. This triangular system is easily solved for x using back-substitution.

The Gaussian elimination algorithm consists of two parts:

(i) **forward elimination** -- the reduction of the coefficient matrix A to upper triangular form

(ii) **back-substitution** -- solving this "reduced" upper triangular system for x.

Numeric Example with $n = 3$.

The given linear system is

$$\begin{aligned} x_1 + x_2 - x_3 &= -2 \\ 2x_1 - x_2 + 3x_3 &= 14 \\ -x_1 - 2x_2 + x_3 &= 3 \end{aligned} \qquad \text{or} \qquad \begin{bmatrix} 1 & 1 & -1 \\ 2 & -1 & 3 \\ -1 & -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 14 \\ 3 \end{bmatrix}.$$

The **augmented matrix** $[A \mid b]$ is

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 2 & -1 & 3 & 14 \\ -1 & -2 & 1 & 3 \end{array}\right];$$

it contains all of the data for this linear system and is all that needs to be stored in memory.

In this example, the coefficient matrix *A* can be reduced to upper triangular form using only the second type of elementary row operation.

The forward elimination:

Using the <u>multipliers</u> $m_{21} = \dfrac{a_{21}}{a_{11}} = \dfrac{2}{1}$ and $m_{31} = \dfrac{a_{31}}{a_{11}} = \dfrac{-1}{1}$, do two elementary row operations

$$\begin{aligned} E_2 &\leftarrow E_2 - m_{21}E_1 \\ E_3 &\leftarrow E_3 - m_{31}E_1 \end{aligned}$$

to obtain

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 0 & -3 & 5 & 18 \\ 0 & -1 & 0 & 1 \end{array}\right].$$

(Note: in memory, these arrays will overwrite *A* and *b*, so we'll still refer to these arrays as *A* and *b*.)

Next, using the multiplier $m_{32} = \dfrac{a_{32}}{a_{22}} = \dfrac{-1}{-3}$, do one elementary row operation

$$E_3 \leftarrow E_3 - m_{32}E_2$$

to obtain

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 0 & -3 & 5 & 18 \\ 0 & 0 & -5/3 & -5 \end{array}\right],$$

completing the reduction of A to upper triangular form.

The back-substitution:

$$x_3 = \frac{-5}{\left(\frac{-5}{3}\right)} = 3$$

$$x_2 = \frac{18 - (5)(3)}{-3} = -1$$

$$x_1 = \frac{-2 - (1)(-1) - (-1)(3)}{1} = 2$$

that is, the solution is $x = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$.

In order <u>to derive an algorithm</u> for the above numeric procedure, let's consider the <u>symbolic analogue</u> of the above numeric example (still when $n = 3$).

Given the augmented matrix

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right],$$

compute the multipliers $m_{21} = \dfrac{a_{21}}{a_{11}}$ and $m_{31} = \dfrac{a_{31}}{a_{11}}$, and do the two elementary row operations

$$E_2 \leftarrow E_2 - m_{21}E_1$$
$$E_3 \leftarrow E_3 - m_{31}E_1$$

to obtain

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a_{22} - m_{21}a_{12} & a_{23} - m_{21}a_{13} & b_2 - m_{21}b_1 \\ 0 & a_{32} - m_{31}a_{12} & a_{33} - m_{3113} & b_3 - m_{31}b_1 \end{array}\right] = \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} & b_2^{(1)} \\ 0 & a_{32}^{(1)} & a_{33}^{(1)} & b_3^{(1)} \end{array}\right].$$

This first step for the general $n \times n$ case is given in equations (9.14) on page 249 of the 6th ed. or page 253 of the 7th ed. (where a prime is used instead of the superscript 1 to indicate the new entries in this first derived system).

Now compute the multiplier $m_{32} = \dfrac{a_{32}^{(1)}}{a_{22}^{(1)}}$ and do the elementary row operation

$$E_3 \leftarrow E_3 - m_{32}E_2$$

to obtain

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} & b_2^{(1)} \\ 0 & 0 & a_{33}^{(1)} - m_{32}a_{23}^{(1)} & b_3^{(1)} - m_{32}b_2^{(1)} \end{array}\right] = \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} & b_2^{(1)} \\ 0 & 0 & a_{33}^{(2)} & b_3^{(2)} \end{array}\right].$$

The back-substitution is then

$$x_3 = \frac{b_3^{(2)}}{a_{33}^{(2)}}$$

$$x_2 = \frac{b_2^{(1)} - a_{23}^{(1)}x_3}{a_{22}^{(1)}}$$

$$x_1 = \frac{b_1 - a_{12}x_2 - a_{13}x_3}{a_{11}}$$

Notes.

1. When programming this, the computed entries overwrite those in *A* and *b*, so the superscripts are not necessary.

2. It is not necessary to compute and store any of the 0 entries that are created in the reduction of *A* to upper triangular form (as they are not used in the back-substitution). Frequently the multipliers are stored in the lower triangular part of A; that is, m_{21} overwrites a_{21}, m_{31} overwrites a_{31}, and so on.

3. The entries $a_{11}, a_{22}^{(1)}, a_{33}^{(2)}, \ldots, a_{n-1,n-1}^{(n-2)}$ are called the **pivots**. They are the denominators in the computation of the multipliers. These $n-1$ values must be nonzero, or the forward elimination will fail.

THE "NAÏVE" GAUSSIAN ELIMINATION ALGORITHM

(*forward elimination*)

for $i = 1, 2, \ldots, n-1$
    for $j = i+1, i+2, \ldots, n$
        mult $\leftarrow a_{ji} / a_{ii}$
        for $k = i+1, i+2, \ldots, n$
            $a_{jk} \leftarrow a_{jk} - \text{mult} \times a_{ik}$
        end for
        $b_j \leftarrow b_j - \text{mult} \times b_i$
    end for
end for

(*back substitution*)

$x_n \leftarrow b_n / a_{nn}$
for $i = n-1, n-2, \ldots, 1$
    sum $\leftarrow b_i$
    for $j = i+1, i+2, \ldots, n$
        sum $\leftarrow \text{sum} - a_{ij} \times x_j$
    end for
    $x_i \leftarrow \text{sum} / a_{ii}$
end for

Notes.

1. The above algorithm (which is very similar to the pseudocode on page 250 of the 6th ed. or page 254 of the 7th ed.) is also called Gaussian Elimination without pivoting. It uses only elementary row operations of the second type.

2. The above algorithm will fail (divide by 0) if any of the pivots a_{ii} in line 3 is zero. This will happen, for example, for the following (nonsingular) coefficient matrices:

$$A = \begin{bmatrix} 0 & 1 & 1 \\ -2 & 3 & 1 \\ 2 & 0 & 1 \end{bmatrix}; \quad \text{the pivot } a_{11} = 0.$$

$$A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & -2 & 3 \\ 2 & 0 & 4 \end{bmatrix}; \quad \text{the pivot } a_{22}^{(1)} = 0.$$