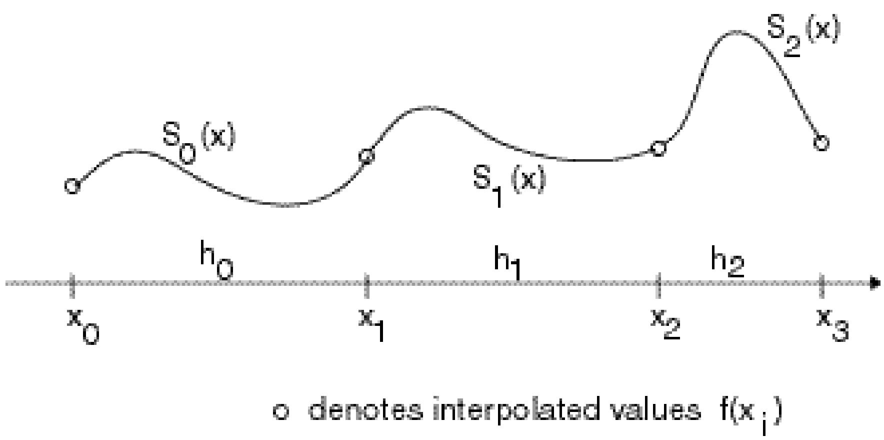
COMPUTER SCIENCE 349A
Handout Number 25

CUBIC SPLINE INTERPOLATION

Example: the case $n = 3$.



Condition (b) in the definition of a cubic spline interpolant implies that

$$
\begin{aligned}
S_0(x_0) &= f(x_0) \\
S_1(x_1) &= f(x_1) \\
S_2(x_2) &= f(x_2) \\
S_2(x_3) &= f(x_3)
\end{aligned}
$$

Condition (c) implies that

$$
\begin{aligned}
&S_1(x_1) = S_0(x_1) \text{ or from above } f(x_1) = S_0(x_1) \\
&S_2(x_2) = S_1(x_2) \text{ or from above } f(x_2) = S_1(x_2)
\end{aligned}
$$

Condition (d) implies that

$$
\begin{aligned}
S_1'(x_1) &= S_0'(x_1) \\
S_2'(x_2) &= S_1'(x_2)
\end{aligned}
$$

Condition (e) implies that

$$
\begin{aligned}
S_1''(x_1) &= S_0''(x_1) \\
S_2''(x_2) &= S_1''(x_2)
\end{aligned}
$$

Computation of a cubic spline

An algorithm is given in the text on page 519 in the 6th ed. or page 521 in the 7th ed., which is based on the derivation given on pages 516-518 in the 6th ed. or pages 518-520 in the 7th ed. The computation of the coefficients of a cubic spline can be reduced to solving a system of $n-1$ linear equations in $n-1$ unknowns. We will not consider this algorithm and derivation from the textbook. Instead, we will use the software provided in MATLAB (which uses an algorithm similar to that in the textbook) to compute a cubic spline.

One important consideration is the form of the cubic polynomials, which is as follows (when $n = 3$):

$$S_0(x) = a_0 + b_0(x - x_0) + c_0(x - x_0)^2 + d_0(x - x_0)^3$$
$$S_1(x) = a_1 + b_1(x - x_1) + c_1(x - x_1)^2 + d_1(x - x_1)^3$$
$$S_2(x) = a_2 + b_2(x - x_2) + c_2(x - x_2)^2 + d_2(x - x_2)^3$$

The problem is to determine the unknowns

$$a_0, b_0, c_0, d_0, a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2$$

so that conditions (b), (c), (d), (e) and one of the conditions in (f), as given in Handout #24, hold. Note that the first part of condition (b) implies that

$$a_0 = f(x_0)$$
$$a_1 = f(x_1)$$
$$a_2 = f(x_2)$$

The computation of the remaining unknowns is more complicated, and in general requires the solution of a system of linear equations.

SPLINE EXAMPLES from the posted Sample Exam Questions

5.5 Determine values for the parameters a, b, c, d and e so that

$$Q(x) = \begin{cases} ax^2 + x + b\,, & -1 \le x \le 0 \\ cx^2 + dx + e\,, & 0 \le x \le 1 \end{cases}$$

is a quadratic spline function that interpolates $f(x)$, where

$$f(-1) = 1\,, \quad f(0) = 1\,, \quad f(1) = 1 \;.$$

Show all of the equations that the unknowns must satisfy, and then solve these equations.

SOLUTION

Let

$$Q_0(x) = ax^2 + x + b, \quad Q_1(x) = cx^2 + dx + e.$$

Then

$Q_0(0) = Q_1(0) \quad \Rightarrow \quad b = e$

$Q_0'(0) = Q_1'(0) \quad \Rightarrow \quad 2ax + 1 = 2cx + d$ at $x = 0 \quad \Rightarrow \quad d = 1$

$Q_0(-1) = f(-1) \quad \Rightarrow \quad a - 1 + b = 1$

$Q_0(0) = f(0)$ and $Q_1(0) = f(0) \quad \Rightarrow \quad b = 1$ and $e = 1$

$Q_1(1) = f(1) \quad \Rightarrow \quad c + d + e = 1$

Solution:

$$d = 1, \quad b = 1, \quad e = 1, \quad a = 1, \quad c = -1$$

Thus,

$$Q(x) = \begin{cases} x^2 + x + 1\,, & -1 \le x \le 0 \\ -x^2 + x + 1\,, & 0 \le x \le 1 \end{cases}$$

5.6 Determine a_0, b_0, d_0, a_1, b_1, c_1 and d_1 so that

$$S(x) = \begin{cases} a_0 + b_0 x - 3x^2 + d_0 x^3, & -1 \le x \le 0 \\ a_1 + b_1 x + c_1 x^2 + d_1 x^3, & 0 \le x \le 1 \end{cases}$$

is the natural cubic spline function such that

$$S(-1) = 1, \quad S(0) = 2 \quad \text{and} \quad S(1) = -1.$$

Clearly identify the 8 conditions that the unknowns must satisfy, and then solve for the 7 unknowns.

SOLUTION

$$S_0'(x) = b_0 - 6x + 3d_0 x^2 \qquad S_1'(x) = b_1 + 2c_1 x + 3d_1 x^2$$
$$S_0''(x) = -6 + 6d_0 x \qquad S_1''(x) = 2c_1 + 6d_1 x$$

The 8 conditions are

$$S_0(-1) = 1 \quad \Rightarrow \quad a_0 - b_0 - 3 - d_0 = 1 \quad \Rightarrow \quad a_0 - b_0 - d_0 = 4$$
$$S_1(0) = 2 \quad \Rightarrow \quad a_1 = 2$$
$$S_1(1) = -1 \quad \Rightarrow \quad a_1 + b_1 + c_1 + d_1 = -1 \quad \Rightarrow \quad b_1 + c_1 + d_1 = -3$$
$$S_1(0) = S_0(0) \quad \Rightarrow \quad a_1 = a_0 \quad \Rightarrow \quad a_0 = 2$$
$$S_1'(0) = S_0'(0) \quad \Rightarrow \quad b_1 = b_0$$
$$S_1''(0) = S_0''(0) \quad \Rightarrow \quad 2c_1 = -6 \quad \Rightarrow \quad c_1 = -3$$
$$S_0''(-1) = 0 \quad \Rightarrow \quad -6 - 6d_0 = 0 \quad \Rightarrow \quad d_0 = -1$$
$$S_1''(1) = 0 \quad \Rightarrow \quad 2c_1 + 6d_1 = 0 \quad \Rightarrow \quad -6 + 6d_1 = 0 \quad \Rightarrow \quad d_1 = 1$$

From the first condition, $b_0 = a_0 - d_0 - 4 = -1$

From the fifth condition, $b_1 = b_0 \quad \Rightarrow \quad b_1 = -1$

Thus

$$S(x) = \begin{cases} 2 - x - 3x^2 - x^3, & -1 \le x \le 0 \\ 2 - x - 3x^2 + x^3, & 0 \le x \le 1 \end{cases}$$