COMPUTER SCIENCE 349A

Handout Number 2

Measures of error (pages 56-59 of the 6th edition of the textbook; pages 59-62 of the 7th edition)

If p denotes the true (exact) value of some quantity, and p^* denotes some approximation to p, then

$$|E_t| = |p - p^*|$$

is called the **absolute error**, and

$$|\varepsilon_t| = \frac{|p - p^*|}{|p|} = \left|1 - \frac{p^*}{p}\right| \qquad \text{(if } p \neq 0\text{)}$$

is called the **relative error**.

Absolute error is not a meaningful measure of error unless you know the magnitude of p, the quantity you are approximating. For example,

if $p = 1234321$ and $p^* = 1234000$, then $|E_t| = 321$ seems large, although p^* is quite accurate and agrees with p to 4 significant digits;
if $p = 0.001234$ and $p^* = 0.001111$, then $|E_t| = 0.000123$ seems small, although p^* is not very accurate and agrees with p to only 1 significant digit.

Relative error, which is always meaningful, in fact indicates the number of correct significant digits in an approximation p^*.

Example

Consider $p = \pi = 3.14159265\cdots$

approximations p^* to p	number of correct significant digits	relative error
3.1	2	0.013
3.14	3	0.00051
3.141	4	0.00019
3.1415	5	0.000029

Definition of the number *n* of significant digits in an approximation p^* to a value *p*:

If $n \geq 0$ is the largest integer such that

$$|\varepsilon_t| = \frac{|p - p^*|}{|p|} < 5 \times 10^{-n},$$

then p^* approximates *p* to ***n* significant digits**.

Thus, if you know the magnitude of the relative error, then you know how many correct significant digits your approximation has.

Note: in order to compute the relative error, you need to know the true (exact) value *p*. In any real application, the exact answer *p* will be unknown.

Approximation of the relative error in an iterative algorithm

In this course and in many applications, **iterative algorithms** are used to compute a sequence

$$p_1,\ p_2,\ p_3, \ldots,\ p_{i-1},\ p_i, \ldots$$

of approximations to a value *p*. If *p* is unknown, then the relative error in any current approximation p_i is approximated using the previous approximation p_{i-1}:

$$|\varepsilon_a| = \frac{|p_i - p_{i-1}|}{|p_i|} = \left|1 - \frac{p_{i-1}}{p_i}\right|.$$

See (3.5) on page 57 of the 6th edition; page 60 of the 7th edition.

A **result** given on page 58 of 6th edition (page 61 of the 7th): if $|\varepsilon_a| < 0.5 \times 10^{-n}$, then the approximation p_i is accurate to at least *n* significant digits.

EXAMPLE 3.2 (pages 58-59 of the 6th edition; pages 61-62 of the 7th))

Compute a sequence of approximations to e^x using the first few terms in the infinite series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

For example, let

$$p_1 = 1$$
$$p_2 = 1 + x$$
$$p_3 = 1 + x + \frac{x^2}{2}$$
$$p_4 = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$

and so on.

Results using $x = 0.5$ (note that the true value $p = e^{0.5} = 1.648721\cdots$ is used to compute $|\varepsilon_t|$):

i	p_i	$\lvert\varepsilon_t\rvert = \frac{\lvert e^{0.5} - p_i\rvert}{\lvert e^{0.5}\rvert}$	$\lvert\varepsilon_a\rvert = \frac{\lvert p_i - p_{i-1}\rvert}{\lvert p_i\rvert}$
1	1	0.393	
2	1.5	0.0902	0.333
3	1.625	0.0144	0.0769
4	1.645833	0.00175	0.0127
5	1.6484375	0.000172	0.00158
6	1.6486979	0.0000142	0.000158

Note that the value of $|\varepsilon_t|$, which can only be computed if you know the true (exact) answer *p*, indicates the number of correct significant digits in each computed approximation p_i: for example,

$0.0144 < 5 \times 10^{-2}$ and $p_3 = 1.625$ has 2 correct significant digits

$0.000172 < 5 \times 10^{-4}$ and $p_5 = 1.6484375$ has 4 correct significant digits

In practice, if a sequence of approximations $\{p_i\}$ to some <u>unknown</u> value *p* is computed using an iterative algorithm (we will use several such algorithms in this course), then the exact relative errors $|\varepsilon_t|$ cannot be computed. However, the relative error in each approximation p_i can be approximated by computing $|\varepsilon_a|$. As given in (3.6) and (3.7) on page 58 of the 6th or page 61 of the 7th ed. if $|\varepsilon_a| < 0.5 \times 10^{-n}$, then p_i is accurate to <u>at least</u> *n* significant digits. Note that this holds true for the results in the above table. For example, when $i = 6$,

$$|\varepsilon_a| = 0.000158 < 0.5 \times 10^{-3},$$

implying that $p_6 = 1.6486979$ has at least 3 correct significant digits (in fact, it has 4).