COMPUTER SCIENCE 349A, SPRING 2018
ASSIGNMENT #3 - 20 MARKS - SOLUTION

Question #1 - 8 marks.

(a) Using $n = 3$ and $a = -1$ we get,

$$\begin{aligned}
f(x) &= \ln(x+2) & f(-1) &= \ln(1) = 0 \\
f'(x) &= \frac{1}{(x+2)} & f'(-1) &= \frac{1}{(-1+2)} = 1 \\
f''(x) &= \frac{-1}{(x+2)^2} & f''(-1) &= \frac{-1}{1^2} = -1 \\
f'''(x) &= \frac{2}{(x+2)^3} & f'''(-1) &= 2 \\
f^{(4)}(x) &= \frac{-6}{(x+2)^4} & f^{(4)}(\xi) &= \frac{-6}{(\xi+2)^4}
\end{aligned}$$

Thus, the Taylor expansion for $f(x)$ is

$$f(x) = (x+1) - \frac{(x+1)^2}{2} + \frac{(x+1)^3}{3} - \frac{(x+1)^4}{4(\xi+2)^4}$$

(b)

$$\begin{aligned}
f(-1.06) &= \ln(0.94) \\
&\approx (-1.06+1) - \frac{(-1.06+1)^2}{2} + \frac{(-1.06+1)^3}{3} \\
&= 0.6 - \frac{0.0036}{2} - \frac{0.000216}{3} \\
&= -0.0618720
\end{aligned}$$

(c) $|E_t| = |-0.0618754 + 0.0618720| = 0.0000034$ or 0.34×10^{-5}

(d) Here,

$$E_t = \frac{-(x+1)^4}{4(\xi+2)^4} = \frac{-(-1.06+1)^4}{4(\xi+2)^4} = \frac{-0.00001296}{4(\xi+2)^4}$$

where $-1.06 \le \xi \le -1$. So, when $\xi = -1.06$ we maximize the error bound giving us,

$$|E_t| \le \frac{0.00001296}{4(-1.06+2)^4} = \frac{0.00001296}{4(0.94)^4} = \frac{0.00001296}{3.12299584} = 0.000004149 = 0.4149 \times 10^{-5}$$

Question #2 - 6 marks.

Let

$$g(x) = \begin{cases} \frac{(1+\cos x)}{(x-\pi)^2} & x \neq \pi \\ 0.5 & x = \pi \end{cases}$$

Note that the correct value of $g(3.16)$ is 0.499985..., so this computed approximation has a relative error of approximately 18%.

(a) Here, when $a = \pi$,

$$\cos x \approx -1 + \frac{(x-\pi)^2}{2} - \frac{(x-\pi)^4}{24}$$

so, when $x \neq \pi$,

$$\begin{aligned} g(x) &\approx \frac{1 + \left(-1 + \frac{(x-\pi)^2}{2} - \frac{(x-\pi)^4}{24}\right)}{(x-\pi)^2} \\ &= \frac{1}{2} - \frac{(x-\pi)^2}{24} \end{aligned}$$

(b) Stability analysis: We were given that if $x = 3.16$, then

$$fl(g(3.16)) = 0.5908$$

using 4 decimal digit, idealized, chopping floating-point arithmetic.

Now, we must show that there does not exist an ε with $|\varepsilon/3.16|$ small such that $g(3.16+\varepsilon) = 0.5908$.

$$\begin{aligned} g(3.16+\varepsilon) &\approx \frac{1}{2} - \frac{(3.16+\varepsilon-\pi)^2}{24} \\ &= 0.5 - \frac{(0.018407346+\varepsilon)^2}{24} \\ &= 0.5 - \frac{0.000326649}{24} - \frac{0.036814692\varepsilon}{24} - \frac{\varepsilon^2}{24} \\ &= 0.499986389 - 0.001533945\varepsilon - \frac{\varepsilon^2}{24} \\ &= 0.499986389 - 0.001533945\varepsilon - 0.04166667\varepsilon^2 \end{aligned}$$

From here student's can go one of two ways generally.

<u>**Method 1:**</u>

Note that $g(3.16+\varepsilon) \approx 0.49999$ for any ε such that $\left|\frac{\varepsilon}{3.16}\right| < 0.01$ and the distance between 0.49999 and 0.5908 is near 18% as stated in the question. Thus, no ε exists meaning the computation is unstable.

Method 2:

At this point we note that $-\frac{(\varepsilon)^2}{24}$ is negligible when $|\varepsilon/3.16|$ less than 1%. Thus, we need an ε that makes

$$\begin{aligned} 0.5908 &\approx 0.499986389 - 0.001533945\varepsilon \\ \varepsilon &\approx 59 \end{aligned}$$

But then, for this ε it would be true that $|\varepsilon/0.123| \approx 18.67\cdots \approx 1867\%$. Therefore, $g(x)$ is unstable near $x = 3.16$.

(c) Here, we are given that

$$fl(g(1.41)) = 0.3871$$

using 4 decimal digit, idealized, chopping floating-point arithmetic. Now, consider $\hat{x} = 1.41 + 0.01(1.141) = 1.4241$. Then,

$$g(\hat{x}) = g(1.4241) = \frac{1 + \cos(1.4241)}{(1.4241 - \pi)^2} = 0.3885613$$

The relative error between this exact value of the perturbed data compared to the floating point evaluation of the original data is

$$\varepsilon_t = \frac{|0.3871 - 0.3885613|}{0.3871} = 0.003774993 \approx 0.38\%$$

Therefore, there exists an ε such that $\left|\frac{\varepsilon}{1.41}\right|$ is small where the relative error between $fl(g(1.41))$ and $fl(g(1.41 + \varepsilon))$ is similarly small. This means that this calculation is stable.

Question #3 - 6 Marks

(a) The Bisect function is as follows:

```
function [ root ] = Bisect( xl, xu, eps, imax, f )
    i=1;
    fl=f(xl);
    fprintf ( ' iteration approximation \n')
    while i <= imax
        xr = (xl + xu)/2;
        fprintf ( ' %6.0f %18.8f \n', i, xr )
        fr = f(xr);
        if (fr == 0) || (((xu - xl)/abs(xu + xl)) < eps)
            root = xr;
            return;
        end
        i = i + 1;
```

```
        if fl*fr < 0
            xu = xr;
        else
            xl = xr;
            fl = fr;
        end
    end
    fprintf ( ' failed to converge in %g iterations\n', imax )
end
```

(b) The function:

```
function [ fh ] = Ass3Q3b( h )
    fh = (pi*h^2*(3*(4.1) - h))./3 - 45;
end
```

The call and output:

```
>> root = Bisect(0,4.1,1e-4,20,@Ass3Q3b)
 iteration approximation
      1         2.05000000
      2         1.02500000
      3         1.53750000
      4         1.79375000
      5         1.92187500
      6         1.98593750
      7         2.01796875
      8         2.03398437
      9         2.04199219
     10         2.04599609
     11         2.04799805
     12         2.04699707
     13         2.04749756
     14         2.04724731
     15         2.04737244

root =

    2.0474
```

(c) The function:

```
function [ fm ] = Ass3Q3c( m )
    fm = (9.81)*m/(13.5)*(1-exp(-13.5*10/m))-40;
end
```

The call and output:

```
>> root = Bisect(1,100,1e-4,20,@Ass3Q3c)
 iteration approximation
      1        50.50000000
      2        75.25000000
      3        62.87500000
      4        56.68750000
      5        59.78125000
      6        61.32812500
      7        62.10156250
      8        62.48828125
      9        62.29492188
     10        62.19824219
     11        62.14990234
     12        62.12573242
     13        62.11364746
     14        62.10760498

root =

   62.1076
```