These are the lecture notes for CSC349A Numerical Analysis taught by Rich Little in the Spring of 2018. They roughly correspond to the material covered in each lecture in the classroom but the actual classroom presentation might deviate significantly from them depending on the flow of the course delivery. They are provided as a reference to the instructor as well as supporting material for students who miss the lectures. They are simply notes to support the lecture so the text is not detailed and they are not thoroughly checked. Use at your own risk. They are complimentary to the handouts. Many thanks to all the guidance and materials I received from Dale Olesky who has taught this course for many years and George Tzanetakis.

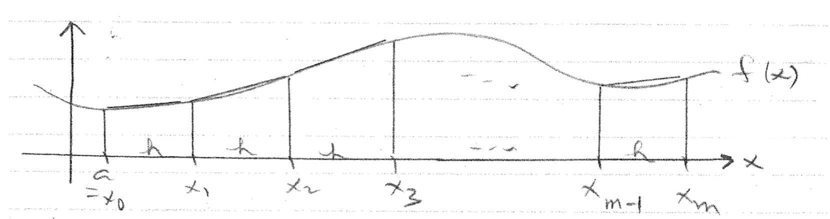
1 Composite Newton-Cotes Formulas

This section corresponds to sections 21.1.2 and 21.2.2 of the text.

Objective: We want the **truncation error** $\to 0$ as the **number of quadrature points** $\to \infty$**.** Note: this does not happen in general as n, the order of the interpolating polynomial, $\to \infty$. **Solution:** We use composite (multiple-application) quadrature formulas.

1.1 Trapezoidal rule

Main idea: for $m \geq 1$, apply a closed N-C formula (with n small) m times on $[a, b]$.
Example: Trapezoidal rule ($n = 1$)
For any $m \geq 1$, let $h = \frac{b-a}{m}$, subdivide $[a, b]$ into m subintervals of length h, and apply the trapezoidal rule on each subinterval.



Composite trapezoidal rule

$$\int_a^b f(x)dx = \int_{x_0}^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + \cdots + \int_{x_{m-1}}^{x_m} f(x)dx$$
$$\approx \int_{x_0}^{x_1} P_0(x)dx + \int_{x_1}^{x_2} P_1(x)dx + \cdots + \int_{x_{m-1}}^{x_m} P_{m-1}(x)dx$$
$$= \frac{h}{2}\left[f(x_0) + f(x_1)\right] + \frac{h}{2}\left[f(x_1) + f(x_2)\right] + \cdots + \frac{h}{2}\left[f(x_{m-1}) + f(x_m)\right]$$
$$= h\left[\frac{f(x_0)}{2} + \sum_{i=1}^{m-1} f(x_i) + \frac{f(x_m)}{2}\right]$$

This is called the composite trapezoial rule.
Truncation Error

$$\begin{aligned} E_t &= -\frac{h^3}{12}f''(\xi_1) - \frac{h^3}{12}f''(\xi_2) - \cdots - -\frac{h^3}{12}f''(\xi_m) \\ &= -\frac{h^3}{12}\left[f''(\xi_1) + f''(\xi_2) + \cdots + f''(\xi_m)\right] \end{aligned}$$

where $x_{i-1} \leq \xi_i \leq x_i$.
We know that:

$$\min_{1\leq i\leq m} f''(\xi_i) \leq \frac{f''(\xi_1) + f''(\xi_2) + \cdots + f''(\xi_m)}{m} \leq \max_{1\leq i\leq m} f''(\xi_i)$$

If $f''(x)$ is continuous on $[a,b]$, then there exists a value $\mu \in [a,b]$ such that:

$$f''(\mu) = \frac{f''(\xi_1) + f''(\xi_2) + \cdots + f''(\xi_n)}{m}$$

This is called the intermediate value theorem.

$$E_t = -\frac{h^3}{12}[mf''(\mu)] = -\frac{(b-a)}{12}h^2 f''(\mu)$$

since $h = \frac{b-a}{m}$.
Important point

$$\lim_{m\to\infty} E_t = \lim_{h\to 0} E_t = 0$$

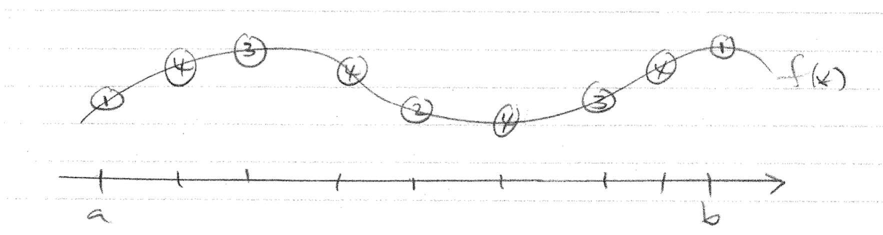
provided that $f''(x)$ is continuous on $[a, b]$.
(there is no comparable result as $n \to \infty$, where n is the degree of the interpolating polynomial)

Usual implementation of composite trapezoidal:

- Initialize $m = 1$
- Repeatedly double m (m=1,2,4,8,16,32,...)
- Until two consecutive approximations are sufficiently close

The reason for using these values of m is that they permit re-use of the function evaluations from previous evaluations

i.e all values $f(x_i)$ computed for $m = k$ can be re-used for $m = 2k$.



1.2 Composite Simpson's Rule

Each application of Simpson's rule requires 2 subintervals on the interval of integration and 3 quadrature points. Thus, m applications of Simpson's rule on $[a, b]$ require that $[a, b]$ be subdivided into $2m$ subintervals using $2m + 1$ quadrature points. Each subinterval then is of length

$$h = \frac{b-a}{2m}$$

Thus, at the jth subinterval we have the three quadrature points x_{2j-2}, x_{2j-1}, and x_{2j}, and

$$\int_{x_{2j-2}}^{x_{2j}} f(x)dx \approx \frac{h}{3}\left[f(x_{2j-2}) + 4f(x_{2j-1}) + f(x_{2j})\right]$$

When $m = 1$ (regular Simpson's rule) we have $2(1) + 1 = 3$ quadrature points and 2 subintervals each of length $h = \frac{b-a}{2}$.

When $m = 2$, we apply Simpson's rule twice. We need $2(2) + 1 = 5$ quadrature points to create 4 subintervals each of length $h = \frac{b-a}{4}$.

Here,

$$\begin{aligned}
&\int_a^b f(x)dx \\
&\approx \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right] + \frac{h}{3}\left[f(x_2) + 4f(x_3) + f(x_4)\right] \\
&= \frac{h}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\right]
\end{aligned}$$

In general, when $m \geq 1$, the composite Simpson's rule approximation is

$$\begin{aligned}
&\int_a^b f(x)dx \\
&\qquad \approx \frac{h}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \\
&\qquad\quad \cdots + 2f(x_{2m-2}) + 4f(x_{2m-1}) + f(x_{2m})] \\
&\qquad = \frac{h}{3}\left[f(x_0) + 4\sum_{j=1}^{m} f(x_{2j-1}) + 2\sum_{j=1}^{m-1} f(x_{2j}) + f(x_{2m})\right]
\end{aligned}$$

Truncation Error

$$\begin{aligned}
E_t &= -\frac{h^5}{90}f^{(4)}(\xi_1) - \frac{h^5}{90}f^{(4)}(\xi_2) - \cdots - -\frac{h^5}{90}f^{(4)}(\xi_m) \\
&= -\frac{h^5}{90}\left[f^{(4)}(\xi_1) + f^{(4)}(\xi_2) + \cdots + f^{(4)}(\xi_m)\right] \\
&= -\frac{h^5}{90}[mf^{(4)}(\mu)]
\end{aligned}$$

where $a \leq \mu \leq b$ and $f^{(4)}(x)$ is continous. So,

$$E_t = -\frac{(b-a)h^4}{180}f^{(4)}(\mu)$$

since $h = \frac{b-a}{2m}$.

2 Newton-Cotes Formulas with Unequal Segments

By way of example, consider the following 11 unequally spaced data points:

x	$f(x)$	x	$f(x)$
0.0	0.2	0.44	2.842985
0.12	1.309729	0.54	3.507297
0.22	1.305241	0.64	3.181929
0.32	1.743393	0.70	2.363
0.36	2.074903	0.80	0.232
0.40	2.456		

We could interpolate with a polynomial of degree 10 but we know that should be avoided. Instead, we look for patterns in the lengths of the subintervals and apply the appropriate method over those subintervals. See Handout 29.

2.1 Summary of Examples

Below is a summary of the six examples of integrating $f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$ over the range $a = 0$ to $b = 0.8$, which has a true value of 1.640533:

Method	Quadrature	E_t	E_a
Trapezoid	0.1728	1.467733	2.56
1/3 Simpson	1.367467	0.2730667	0.2730667
3/8 Simpson	1.519170	0.1213630	0.1213630
Comp. Trapezoid	1.0688	0.57173	0.64
Comp. 1/3 Simpson	1.623467	0.017067	0.017067
Unequal Segments	1.603641	0.036892	unknown

2.2 Notes

The composite Simpsons rule is very stable: roundoff errors do not disastrously accumulate as $m \to \infty$. As $m \to \infty$, the accumulated effect of the summation errors will eventually make the total roundoff error large. However, sufficient accuracy can usually be obtained without using such large values of m, so algorithms for quadrature are stable. The roundoff error analysis for other quadrature formulas is similar.

3 Open Newton-Cotes formula

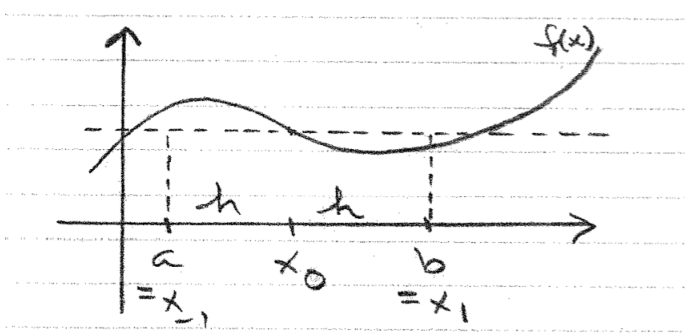
The goal is to integrate an interpolating polynomial with all quadrature points x_i in the open interval (a, b).

$$h = \frac{b-a}{n+2}, \quad n \geq 0$$

Construct an interpolating polynomial $P_n(x) = \sum_{i=0}^{n} L_i(x)f(x_i)$ of degree n through $x_0, x_1, \ldots, x_n$. Then

$$\int_a^b f(x)dx \approx \int_a^b P_n(x)dx = \sum_{i=0}^{n} \left(\int_a^b L_i(x)dx \right) f(x_i)$$

Note: $f(x)$ is not evaluated at $a = x_1$ or $b = x_{n+1}$



Case n=0: midpoint rule

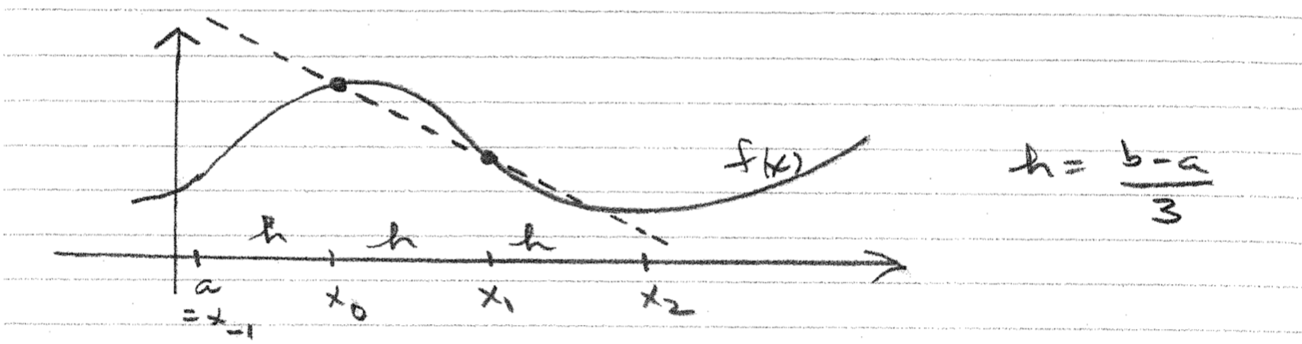
$$\int_a^b f(x)dx \approx 2hf(x_0), \quad h = \frac{b-a}{2}$$

Case n=1

$$\int_a^b f(x)dx \approx \int_{x_1}^{x_2} \left[\frac{x-x_1}{x_0-x_1} f(x_0) + \frac{x-x_0}{x_1-x_0} f(x_1) \right] dx$$

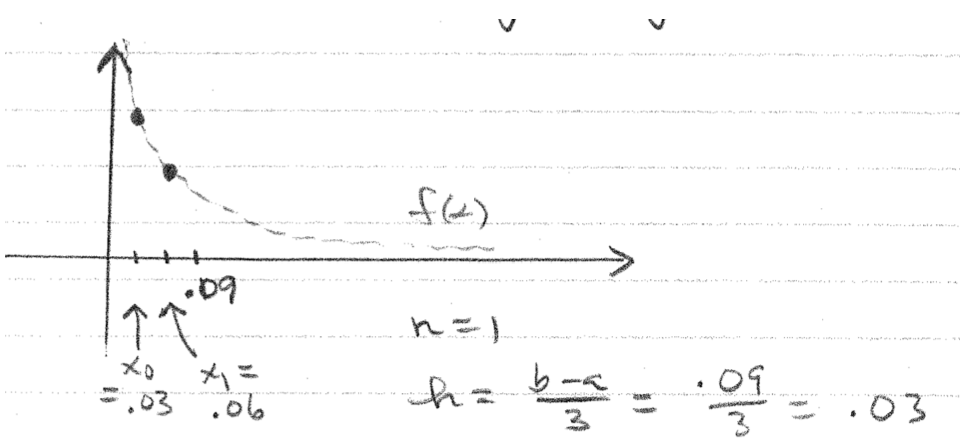
To integrate this more easily, use a exchange of variable $t = \frac{x-x_0}{h}$ then:

$$\begin{aligned}
&x - x_0 = th \\
&x - x_1 = x - (x_0 + h) = (x - x_0) - h = (t-1)h \\
&dx = hdt
\end{aligned}$$



$$\begin{aligned}\int_a^b f(x)dx &\approx \int_{-1}^{2} \left[\frac{(t-1)h}{-h} f(x_0) + \frac{th}{h} f(x_1)\right] hdt \\ &= -hf(x_0)\int_{-1}^{2}(t-1)dt + hf(x_1)\int_{-1}^{2} tdt \\ &= \frac{3h}{2}\left[f(x_0) + f(x_1)\right]\end{aligned}$$

Use of open quadrature formulas: If $f(x)$ has a singularity.



$$h = \frac{b-a}{3} = \frac{.09}{3} = .03$$

$$\int_0^{0.09} f(x)dx \approx \frac{3h}{2}\left[f(x_0) + f(x_1)\right] = \frac{3(.03)}{2}\left[f(.03) + f(.06)\right]$$

The truncation error term is $\frac{3}{4}h^3 f''(\ xi)$, for some $\xi \in (0, 0.06)$