COMPUTER SCIENCE 349A

Handout Number 12

MULTIPLE ROOTS AND THE MULTIPLICITY OF A ZERO
(Section 6.5 in both the 7th and 6th ed.)

If Newton's method converges to a zero x_t of $f(x)$, a necessary condition for <u>quadratic convergence</u> is that $f'(x_t) \neq 0$. We now relate this condition on the derivative of $f(x)$ to the <u>multiplicity</u> of the zero x_t.

Definition (not in the textbook)
If x_t is a zero of any analytic function $f(x)$, then there exists a positive integer *m* and a function $q(x)$ such that

$$f(x) = (x - x_t)^m q(x), \quad \text{where } \lim_{x \to x_t} q(x) \neq 0.$$

(In particular, if $q(x_t)$ is defined, note that $q(x_t) \neq 0$.) The value *m* is called the **multiplicity** of the zero x_t.

If $m = 1$, then x_t is called a **simple zero** of $f(x)$.

Example 1
Consider

$$\begin{aligned} f(x) &= x^4 + 9.5x^3 + 18x^2 - 56x - 160 \\ &= (x+4)^3 (x - 2.5) \end{aligned}$$

The zero at $x_t = -4$ has $m = 3$ (here $q(x) = x - 2.5$ and $q(-4) \neq 0$).
The zero at $x_t = 2.5$ has $m = 1$ (here $q(x) = (x+4)^3$ and $q(2.5) \neq 0$).

Example 2
Consider

$$f(x) = e^x - x - 1.$$

Since $f(0) = 0$, $x_t = 0$ is a zero of $f(x)$. This zero has multiplicity $m = 2$ since

$$f(x) = (x-0)^2 q(x) \quad \text{with} \quad q(x) = \frac{e^x - x - 1}{x^2}$$

and (using l'Hospital's rule) $\lim_{x \to 0} q(x) = 0.5 \neq 0$.

Theorem (not in the textbook)

Suppose that $f(x)$ and $f'(x)$ are continuous on some interval $[a, b]$, and that $x_t \in (a, b)$ and $f(x_t) = 0$. Then x_t is a simple zero of $f(x)$ if and only if $f'(x_t) \neq 0$.

Proof.

(i) Suppose first that x_t is a simple zero of $f(x)$. Then

$$f(x) = (x - x_t)\,q(x), \quad \text{where } \lim_{x \to x_t} q(x) \neq 0.$$

Therefore,

$$f'(x) = q(x) + (x - x_t)q'(x)$$

and thus

$$f'(x_t) = q(x_t) \neq 0.$$

(ii) For the converse, suppose that $f'(x_t) \neq 0$. Then by Taylor's Theorem expanded about $a = x_t$,

$$\begin{aligned} f(x) &= f(x_t) + (x - x_t) f'(\xi), \quad \text{for some value } \xi \text{ between } x \text{ and } x_t \\ &= (x - x_t) f'(\xi), \quad \text{since } f(x_t) = 0. \end{aligned}$$

Thus, $q(x) = f'(\xi)$ and $\lim_{x \to x_t} q(x) = \lim_{x \to x_t} f'(\xi) = f'(x_t) \neq 0$. Hence x_t is a simple zero (that is, the multiplicity is $m = 1$).

The following result follows directly from the above Theorem and our previous result about quadratic convergence of Newton's method.

Corollary

If Newton's method converges to a simple zero x_t of $f(x)$, then the order of convergence is 2.

In order to determine whether or not Newton's method converges quadratically to a zero x_t of $f(x)$, you only need to know whether the multiplicity of x_t is 1 or is ≥ 2. The following result is more general than the above Theorem, and enables you determine the exact multiplicity of a zero.

Theorem (not in the textbook)

Suppose that $f(x)$ and its first m derivatives are continuous on some interval $[a, b]$ that contains a zero x_t of $f(x)$. Then the multiplicity of x_t is m if and only if

$$f(x_t) = f'(x_t) = f''(x_t) = \cdots = f^{(m-1)}(x_t) = 0 \quad \text{but} \quad f^{(m)}(x_t) \neq 0.$$

Example 3

Consider

$$f(x) = e^x - x - 1.$$

Since $f(0) = 0$, $x_t = 0$ is a zero of $f(x)$ with multiplicity $m \geq 1$.

Since $f'(x) = e^x - 1$ and $f'(0) = 0$, $x_t = 0$ is a zero of $f(x)$ with multiplicity $m \geq 2$.

Since $f''(x) = e^x$ and $f''(0) \neq 0$, $x_t = 0$ is a zero of $f(x)$ with multiplicity $m = 2$.

The significance of the multiplicity concerning root-finding algorithms (see pages 166-167 of the 7th ed. or pages 164-165 of the 6th ed.)

- Bracketing methods, such as the Bisection method, cannot be used to compute zeros of <u>even</u> multiplicity.

- Newton's method and the Secant method both converge only <u>linearly</u> (order of convergence is $\alpha = 1$) if the multiplicity m is ≥ 2.

- If the multiplicity $m \geq 2$ of a zero is known (in practice this is very unlikely), then the following modification of Newton's method is <u>quadratically convergent</u>:
$$x_{i+1} = x_i - m\frac{f(x_i)}{f'(x_i)}.$$

- A <u>quadratically convergent</u> algorithm for computing a zero x_t of <u>any (unknown) multiplicity</u> of a function $f(x)$ is obtained by applying Newton's method to the new function
$$u(x) = \frac{f(x)}{f'(x)}$$
rather than to $f(x)$. This is true since if $f(x) = (x - x_t)^m q(x)$ and $m \geq 2$, then
$$u(x) = \frac{f(x)}{f'(x)} = \frac{(x - x_t)q(x)}{m\,q(x) + (x - x_t)q'(x)}$$

has a simple zero $(m = 1)$ at x_t. By evaluating $u'(x)$, this new algorithm can be written as

$$x_{i+1} = x_i - \frac{u(x_i)}{u'(x_i)}$$
$$= x_i - \frac{f(x_i) f'(x_i)}{[f'(x_i)]^2 - f(x_i) f''(x_i)}$$

See (6.16) on page 167 of the 7th ed. or on page 165 of the 6th ed.

See also **Example 6.10** on page 167-168 of the 7th ed. (**Example 6.10** on pages 165-166 of the 6th ed.). This example illustrates linear convergence of Newton's method when the multiplicity is 2, and then quadratic convergence of the above formula (6.16) for this same zero.