CSC349A Numerical Analysis

Lecture 6

Rich Little

University of Victoria

2018

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Introduction

In analyzing the effects of roundoff errors in a computation to solve a problem, the concepts of "stability" and "condition" distinguish between whether the algorithm (the procedure for computing a solution to the problem) is satisfactory, or if the problem is such that no algorithm can be expected to reasonably solve the problem. The concepts involved are:

- **stable/unstable algorithm**
- **well-conditioned/ill-conditioned problem**

Condition

Definition

A problem whose (exact) solution can change greatly with small changes in the data defining the problem is called **ill-conditioned**.

Note: The condition of a problem has nothing to do with floating-point arithmetic or round-off error; it is defined in terms of exact computation. However, if a problem is ill-conditioned, it will be difficult (or impossible) to solve accurately using floating-point arithmetic.

Condition Analysis

Original problem with exact arithmetic :

$$\text{data} \quad \{d_i\} \rightarrow \quad \text{exact solution}\{r_i\}$$

Perturbed problem with exact arithmetic:

$$\text{data} \quad \{\hat{d}_i\} = \{d_i + \varepsilon_i\} \rightarrow \quad \text{exact solution}\{\hat{r}_i\}; \text{ where} \left|\frac{\varepsilon_i}{d_i}\right| \text{ small.}$$

If there exist small ε_i such that $\{\hat{r}_i\}$ are not close to $\{r_i\}$, then the problem is **ill-conditioned**.
If $\{\hat{r}_i\} \approx \{r_i\}$ for **all** small ε_i, then the problem is **well-conditioned**.

Condition number derivation

The **condition number** is another approach to analyzing the condition of a problem if the first derivative of the quantity $f(x)$ being computed can be determined. By the Taylor polynomial approximation of order $n = 1$ for $f(x)$ expanded around $\tilde{x}$ we have:

$$f(x) \approx f(\tilde{x}) + f'(\tilde{x})(x - \tilde{x})$$

which implies that:

$$\frac{f(x) - f(\tilde{x})}{f(\tilde{x})} \approx \frac{\tilde{x}f'(\tilde{x})}{f(\tilde{x})} \left(\frac{x - \tilde{x}}{\tilde{x}} \right)$$

If $\tilde{x}$ is some small pertubation of x, then the left hand side above is the *relative change* in $f(x)$ as x is perturbed to $\tilde{x}$.

Condition number

Thus,

$$\text{relative change in } f(x) \approx \left(\frac{\tilde{x} f'(\tilde{x})}{f(\tilde{x})} \right) \times \; \text{relative change in } x$$

The quantity $\frac{\tilde{x} f'(\tilde{x})}{f(\tilde{x})}$ is called a **condition number** for the computation of $f(x)$. If this number is "large", then $f(x)$ is ill-conditioned; if this number is "small", then $f(x)$ is well-conditioned.

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Stability

A computation is **numerically unstable** if the uncertainty of the input values is greatly magnified by the numerical method.

Definition

An algorithm is said to be **stable** (for a class of problems) if it determines a computed solution (using floating-point arithmetic) that is close to the exact solution of some (small) perturbation of the given problem.

Meaning of numerical stability: the effect of uncertainty in the input data or of the floating-point artihmetic (the round-off error) is no worse that the effect of slightly perturbing the given problem, and solving the perturbed problem exactly.

Stability analysis

Suppose that for the original problem we have using floating-point computation:

$$\text{data} \quad \{d_i\} \rightarrow \quad \text{computed solution}\{r_i\}$$

and we create a perturbed problem using exact computation:

$$\text{data} \quad \{\hat{d}_i + \varepsilon_i\} \rightarrow \quad \text{exact solution}\{\hat{r}_i\}$$

where $|\frac{\varepsilon_i}{d_i}|$ is small.
If there exist data $\hat{d}_i \approx d_i$ (small ε_i for all i) such that $\hat{r}_i \approx r_i$ for all i, then the algorithm is said to be **stable**.
If there exists **no set** of data $\{\hat{d}_i\}$ close to $\{d_i\}$ such that $\hat{r}_i \approx r_i$ for all i, then the algorithm is said to be **unstable**