CSC349A Numerical Analysis

Lecture 9

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Introduction

- The advantage of the Newton method is that it provides quadratic convergence.
- One disadvantage is that it requires knowledge of the derivative $f'(x)$.
- In many applications the derivative might not be known or impossible to derive analytically through calculus.
- In this case it is possible to use a discrete approximation to the derivative. One such approximation is used in the *Secant* method.

Secant derivation

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We can derive the *Secant* method starting from the update equation of the Newton/Raphson method:

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

We can approximate $f'(x_i)$ by a finite divided difference:

$$f'(x_i) = \lim_{x \to x_i} \frac{f(x) - f(x_i)}{x - x_i}$$

using

$$f'(x_i) \approx \frac{f(x_{i-1}) - f(x_i)}{x_{i-1} - x_i}$$

gives:

$$x_{i+1} = x_i - \frac{f(x_i)(x_{i-1} - x_i)}{f(x_{i-1}) - f(x_i)}$$

Secant Geometry

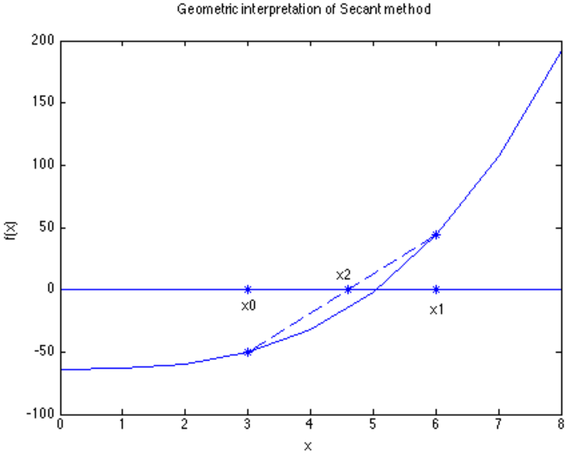

Figure: Geometric interpetation of the Secant method for root finding.

MATLAB Code for figure

```
x = 0:8;
fx = 0.5 * x.^3 -64;
% Plot the function
plot(x,fx);
hold on;

x0 = 3;
fx0 = 0.5 * x0.^3-64;
% Plot zero axis
plot(x, zeros(size(x)));
% plot x0 and f(x2) points
plot(x0,0,'*');
plot(x0,fx0, '*');
```

MATLAB Code for figure

```
% plot x1 and f(x1) points
x1 = 6;
fx1 = 0.5 * x1.^3-64;
plot(x1,0,'*');
plot(x1,fx1, '*');

% plot secant line
plot([x0,x1],[fx0,fx1],'--');
x2 = x1 - fx1 *(x0-x1)/(fx0-fx1);
plot(x2,0, '*');
hold off;
```

Example of Secant Method

Estimate the root of $f(x) = e^{-x} - x$ employing initial guesses of $x_{-1} = 0$ and $x_0 = 1$. The iterative equation can be applied to compute:

i	x_i	$\varepsilon_t(\%)$
-1	0	100
0	1	76
1	0.61270	8.03
2	0.56384	0.58
3	0.56717	0.0048

Notice that the approach converges on the true root faster than *Bisection* but slower than *Newton*.

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Order of convergence of the Secant method

The order of convergence of the Secant method derives from the following limit,

$$\lim_{i\to\infty}\left|\frac{E_{i+1}}{E_i E_{i-1}}\right| = \left|\frac{f''(x_t)}{2f'(x_t)}\right| \tag{1}$$

This gives a relationship **between 3 successive errors**. However, this does indicate the **order** α of the Secant method, which requires that the errors of 2 successive approximations be related by

$$\lim_{i\to\infty}\frac{|E_{i+1}|}{|E_i|^{\alpha}} = \lambda, \quad \text{for some constant } \lambda \tag{2}$$

Secant and Golden Ratio

It can be shown in fact that,

$$\lim_{i\to\infty}\left|\frac{E_{i+1}}{E_i E_{i-1}}\right| = \lim_{i\to\infty}\frac{|E_{i+1}|}{|E_i|^{\alpha}} = \left|\frac{f''(x_t)}{2f'(x_t)}\right| \tag{3}$$

where

$$\alpha = 1 + \frac{1}{\alpha} \implies \alpha^2 - \alpha - 1 = 0 \implies \alpha = \frac{1+\sqrt{5}}{2} \approx 1.618$$

which is the **order of the Secant method**.
Note: this value α is known as the "golden ratio", and occurs in many places in nature as well as many diverse applications.

Bisection convergence

An alternate definition of **linear convergence**:

$$|E_i| \leq c|E_{i-1}| \text{ or } |x_t - x_i| \leq c|x_t - x_{i-1}|$$

for some constant c such that $0 < c < 1$.
Applying this inequality recursively gives

$$|x_t - x_i| \leq c^i |x_t - x_0|$$

For the Bisection method we had (Handout 8, pg. 2):

$$|x_t - x_i| \leq \left(\frac{1}{2}\right)^i \Delta x^0, \text{where } \Delta x^0 = x_u - x_l$$

and $[x_l, x_u]$ is the initial interval. This implies linear convergence with the above definition, and $c = \frac{1}{2}$.

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Introduction

If Newton's method converges to a zero x_t of $f(x)$, a necessary condition for quadratic convergence is that $f'(x_t) \neq 0$. We now relate this condition on the derivative of $f(x)$ to the multiplicity of the zero x_t.

Multiplicity

Theorem

If x_t is a zero of any analytic function $f(x)$, then there exists a positive integer m and a function $q(x)$ such that :

$$f(x) = (x - x_t)^m q(x), \quad \textit{where} \quad \lim_{x \to x_t} q(x) \neq 0$$

(In particular, if $q(x_t)$ is defined, note that $q(x_t) \neq 0$.) The value m is called the **multiplicity** of the zero x_t. If $m = 1$, then x_t is called a **simple zero** of $f(x)$.

Example 1

$$f(x) = x^4 = 9.5x^3 + 18x^2 - 56x - 160 = (x+4)^3(x-2.5)$$

The zero at $x_t = -4$ has $m = 3$ (here $q(x) = x - 2.5$ and $q(-4) \neq 0$).
The zero at $x = 2.5$ has $m = 1$ (here $q(x) = (x+4)^3$ and $q(2.5) \neq 0$).

Example 2

$$f(x) = e^x - x - 1$$

Since $f(0) = 0$, $x_t = 0$ is a zero of $f(x)$. This zero has multiplicity m=2 since $f(x) = (x-0)^2 q(x)$ with $q(x) = \frac{e^x - x - 1}{x^2}$.

Using l'Hospital's rule we have:

$$\lim_{x \to 0} q(x) = \lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{f'(x)}{g'(x)} = \lim_{x \to 0} \frac{e^x - 1}{2x} = \lim_{x \to 0} \frac{e^x}{2} = 0.5$$

Simple zero

Theorem

Suppose that $f(x)$ and $f'(x)$ are continuous on some interval $[a, b]$, and that $x_t \in (a, b)$ and $f(x_t) = 0$. Then x_t is a simple zero of $f(x)$ if and only if $f'(x_t) \neq 0$.

Proof I

Suppose first that x_t is a simple zero of $f(x)$. Then

$$f(x) = (x - x_t)q(x), \text{where } \lim_{x \to x_t} q(x) \neq 0$$

Therefore,

$$f'(x) = q(x) + (x - x_t)q'(x)$$

and thus

$$f'(x_t) = q(x_t) \neq 0$$

Proof II

For the converse, suppose that $f'(x_t) \neq 0$. Then by Taylor's Theorem expansion about $a = x_t$,

$$f(x) = f(x_t) + (x - x_t)f'(\xi) = (x - x_t)f'(\xi) \text{ since } f(x_t) = 0$$

for some value ξ between x and x_t. Thus $q(x) = f'(\xi)$ and $\lim_{x \to x_t} q(x) = \lim_{x \to x_t} f'(\xi) = f'(x_t) \neq 0$. Hence x_t is a simple zero (that is, the multiplicity is $m = 1$).

Corrolary

The following result follows directly from the above Theorem and our previous result about the quadratic convergence of Newton's method.

Corrolary

If Newtons method converges to a simple zero x_t of $f(x)$, then the order of convergence is 2.

In order to determine whether or not Newtons method converges quadratically to a zero x_t of $f(x)$, you only need to know whether the multiplicity of x_t is 1 or is ≥ 2. The following result is more general than the above Theorem, and enables you to determine the exact multiplicity of a zero.

Multiplicity and derivatives

Theorem

Suppose that $f(x)$ and its first m derivatives are continuous on some interval $[a, b]$ that contains a zero x_t of $f(x)$. Then the multiplicity of x_t is m if and only if
$f(x_t) = f'(x_t) = f''(x_t) = \cdots = f^{(m-1)}(x_t) = 0$ *but*
$f^{(m)}(x_t) \neq 0$.

Example

Consider

$$f(x) = e^x - x - 1$$

Since $f(0) = 0$, $x_t = 0$ is a zero of $f(x)$ with multiplicity $m \geq 1$.
Since $f'(x) = e^x - 1$ and $f'(0) = 0$, $x_t = 0$ is a zero of $f(x)$ with multiplicity $m \geq 2$.
Since $f''(x) = e^x$ and $f''(0) \neq 0$, $x_t = 0$ is a zero of $f(x)$ with multiplicity $m = 2$.

Significance of multiplicity

- Bracketing methods, such as the Bisection method, cannot be used to compute zeros of **even** multiplicity.
- Newton's method and the Secant method both converge only linearly (order of convergence is $\alpha = 1$) if the multiplicity m is ≥ 2.

Variant

A quadratically convergent algorithm for computing a zero x_t of any (unknown) multiplicity of a function $f(x)$ is obtained by applying Newtons method to the new function.

$$u(x) = \frac{f(x)}{f'(x)}$$

rather than to $f(x)$. This is true since if $f(x) = (x - x_t)^m q(x)$ and $m \geq 2$, then

$$u(x) = \frac{f(x)}{f'(x)} = \frac{(x - x_t)q(x)}{mq(x) + (x - x_t)q'(x)}$$

has a simple zero $(m = 1)$ at x_t. By evaluating $u'(x)$, this new algorithm can be written as:

$$x_{i+1} = x_i - \frac{u(x_i)}{u'(x_i)} = x_i - \frac{f(x_i)f'(x_i)}{[f'(x_i)]^2 - f(x_i)f''(x_i)}$$