These are the lecture notes for CSC349A Numerical Analysis taught by Rich Little in the Spring of 2018. They roughly correspond to the material covered in each lecture in the classroom but the actual classroom presentation might deviate significantly from them depending on the flow of the course delivery. They are provide as a reference to the instructor as well as supporting material for students who miss the lectures. They are simply notes to support the lecture so the text is not detailed and they are not thoroughly checked. Use at your own risk. They are complimentary to the handouts.

1 Overview

In this lecture we reviewed the Newton/Raphson method for root finding as an example of an open method (i.e requires only an initial estimate of the root) in contrast to bracketing methods like Bisection which require an interval to bound the root. This included two derivations of the algorithm, an example of the algorithm and analysis of the error convergence. The exact details are provided below.

2 Newton-Raphson

2.1 Geometric derivation

The real roots of a function $f(x)$ occur when the graph of the function intersects with the x-axis. The main idea behind the Newton/Raphson method for root finding is given an initial approximation x_0 to a zero of $f(x)$ to approximate the graph of $f(x)$ at x_0 by the **tangent line** - essentially linearizing the function in that area. Figure 1 shows this visually.
Let x_1, the intersection of the tangent line with the x-axis, be the next approximation to x_t. The slope of the tangent line is:

$$f'(x_0) = \frac{f(x_0) - 0}{x_0 - x_1} \tag{1}$$

Thus the tangent line intersects the x-axis when

$$f'(x_0)(x_0 - x_1) = f(x_0) \tag{2}$$

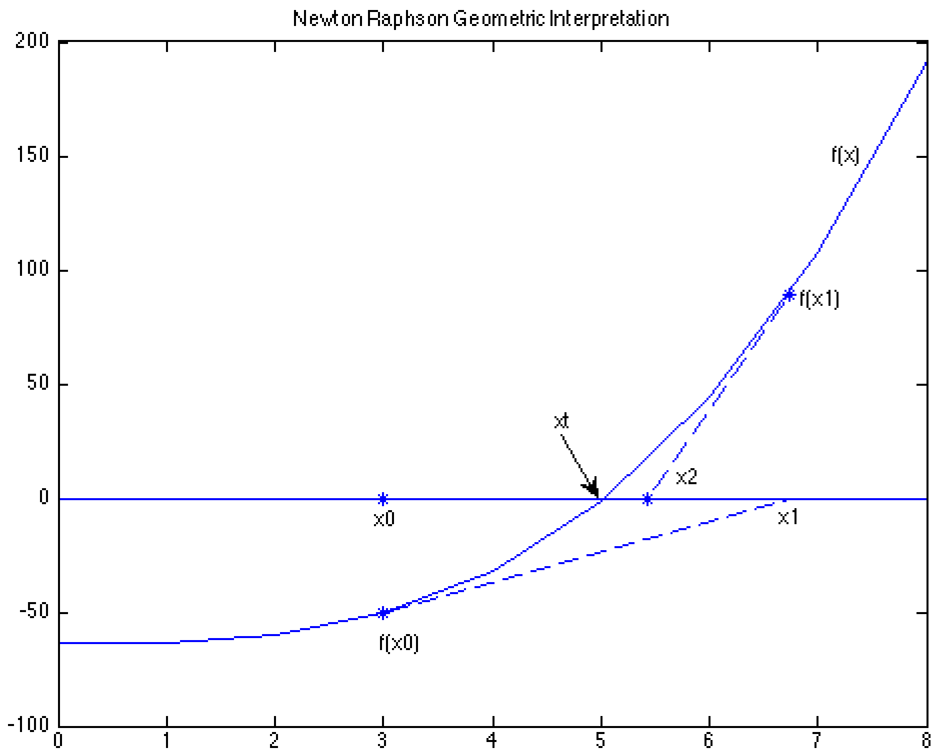


Figure 1: Geometric interpetation of the Newton Raphson method for root finding.

so the next estimate x_1 can be calculated as:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \tag{3}$$

The process can be repeated starting from the new root estimate x_1 with the following equation.

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \tag{4}$$

In general, we calculate a new root estimate x_{i+1} from the previous estimate x_i with the following equation.

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \tag{5}$$

2.2 Example of Newton-Raphson Method

Estimate the root of $f(x) = e^{-x} - x$ employing an initial guess of $x_0 = 0$. Note that $x_t = 0.56714329...$ is the actual root. Also, $f'(x) = -e^{-x} - 1$ and thus

$$x_{i+1} = x_i - \frac{e^{-x_i} - x_i}{-e^{-x_i} - 1} \tag{6}$$

This iterative equation can be applied to compute:

i	x_i	$\varepsilon_t(\%)$
0	0	100
1	0.5	11.8
2	0.566311003	0.147
3	0.567143165	0.0000220
4	0.567143290	$< 10^{-8}$

Notice that the approach rapidly converges on the true root much faster than it would using *Bisection*.

3 Newton method convergence

Consider the Taylor theorem for $f(x)$ with $n = 1$ expanded about x_i (i.e $a = x_i$):

$$f(x) = f(x_i) + (x - x_i)f'(x_i) + \frac{f''(\xi)}{2}(x - x_i)^2 \tag{7}$$

for some value ξ between x and x_i.

The derivation of the Newton/Raphson method gives insight into how fast Newton's method converges:
First we evaluate the Taylor Theorem at $x = x_t$, an exact zero:

$$0 = f(x_t) = f(x_i) + (x_t - x_i)f'(x_i) + \frac{(x_t - x_i)^2}{2}f''(\xi) \tag{8}$$

Newton's method $x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$ can be rewritten as:

$$0 = f(x_i) + (x_{i+1} - x_i)f'(x_i) \tag{9}$$

If we subtract the last two equations then we get:

$$0 = (x_t - x_{i+1})f'(x_i) + \frac{(x_t - x_i)^2}{2}f''(\xi) \tag{10}$$

or if we let $E_{i+1} = x_t - x_{i+1}$ and $E_i = x_t - x_i$ denote the error in x_{i+1}, x_i then we have:

$$0 = E_{i+1}f'(x_i) + \frac{E_i^2}{2}f''(\xi) \tag{11}$$

Thus:

$$\frac{E_{i+1}}{E_i^2} = \frac{-f''(\xi)}{2f'(x_i)} \tag{12}$$

which relates E_{i+1} to E_i.

3.1 Definition (not in textbook)

If a sequence $x_0, x_1, x_2, x_3, \ldots$ converges to x_t that is $\lim_{i\to\infty} x_i = x_t$ and $E_i = x_t - x_i$, then the order of converge of the sequence is α if there are constants $\lambda > 0$ and $\alpha \geq 1$ such that:

$$\lim_{i\to\infty} \frac{|E_{i+1}|}{|E_i|^\alpha} = \lambda \tag{13}$$

In general, λ and α depend on the algorithm used to compute x_i, on $f(x)$, and on the multiplicity of the zero x_t.

Most common case:
$\alpha = 1$ linear convergence
For large i, $|E_{i+1}| \approx \lambda|E_i|$

In this case, successive errors decrease approximately by a constant amount:

$$\begin{aligned}
|E_{i+1}| &\approx \lambda|E_i| \\
|E_{i+2}| &\approx \lambda|E_{i+1}| \approx \lambda^2|E_i| \\
|E_{i+3}| &\approx \lambda|E_{i+2}| \approx \lambda^3|E_i| \\
\text{etc} &
\end{aligned}$$

Errors $|E_{i+1}| \to 0$, that is $\lim_{i\to\infty} x_i = x_t$ only if $0 < \lambda < 1$.

For $\alpha = 2$ we have quadratic convergence. For large i, $|E_{i+1}| \approx \lambda |E_i|^2$ Meaning: after some error $|E_i| < 1$, convergence is rapid as the number of correct significant digits approximately doubles with each iteration e.g if $|E_i| = 10^{-t}$, then $|E_{i+1}| \approx \lambda 10^{-2t}$.

For Newton's method above:

$$\frac{E_{i+1}}{E_i^2} = \frac{-f''(\xi)}{2f'(x_i)}$$

for some ξ between x_i and x_{i+1}.

$$\lim_{i\to\infty} \frac{|E_{i+1}|}{|E_i|^2} = \lim_{i\to\infty} \frac{|f''(\xi)|}{2|f'(x_i)|} = \frac{|f''(x_t)|}{2|f'(x_t)|}$$

which is a constant λ provided that $f'(x_t) \neq 0$.

4 Implementation

The algorithm for Newton's method can be written in pseudocode as follows:

function root = Newton(x_0, ε, `imax`)
$i \leftarrow 1$
output heading
while $i \leq$ `imax`
    $root \leftarrow x_0 - f(x_0)/f'(x_0)$
    output i, root
    if $|1 - x_0/root| < \varepsilon$
        exit
    end if
    $i \leftarrow i + 1$
    $x_0 \leftarrow root$
end while
output "failed to converge"

Note that in general, using $|f(root)| < \varepsilon$ is not a suitable test for convergence (instead of testing approximation error). The reason is that $|f(root)| < \varepsilon$ does not imply that the value *root* is within distance of ε of an exact root x_t.

5 Quadratic convergence of Newton's method

Theorem: If Newton's method is applied to $f(x) = 0$ producing a sequence x_i that converges to a root x_t, and if $f'(x_t) \neq 0$, then then order of convergence is 2.

If $f'(x_t) = 0$ and Newton's method convergences to a root x_t, then we will see later that the order of convergence is NOT quadratic.

Example 1 An illustration of the quadratic convergence of Newton's method. Here $f(x) = cos(x) - x$. This was computed in MATLAB, so at most 16 correct digits are possible. The bold digits are all correct.

i	x_i	no. of correct digits
0	$\frac{\pi}{4} = 0.\mathbf{7}85398$	1
1	$0.\mathbf{739}5361$	3
2	$0.\mathbf{7390851}78$	7
3	$0.\mathbf{73908513321516}10$	14
4	$0.\mathbf{7390851332151606}$	16

Example 2. The following illustrates the possible effect of a poor initial approximation with Newton's method, yet the eventual characteristic quadratic convergence. Here Newton's method is used to compute a root of $x^3 + 4x^2 - 10 = 0$ with $p_0 = -100$. Partial results are as follows.

$p_0 = -100$	
$p_1 = -67.12$	
$p_2 = -45.21$	
...	...
$p_{14} = -2.54$	
$p_{15} = -3.14$	
$p_{16} = -2.80$	
...	...
$p_{21} = 1.9405$	no. of correct digits
$p_{22} = \mathbf{1}.4793$	1
$p_{23} = \mathbf{1.3}711$	2
$p_{24} = \mathbf{1.365}25$	4
$p_{25} = \mathbf{1.365230001}1$	9

I ran this example with a Newton function of my own devising that also recorded $|E_{i+1}|/|E_i|$ at each step to give you an idea of how this example is converging. This value is given in the third column of the output:

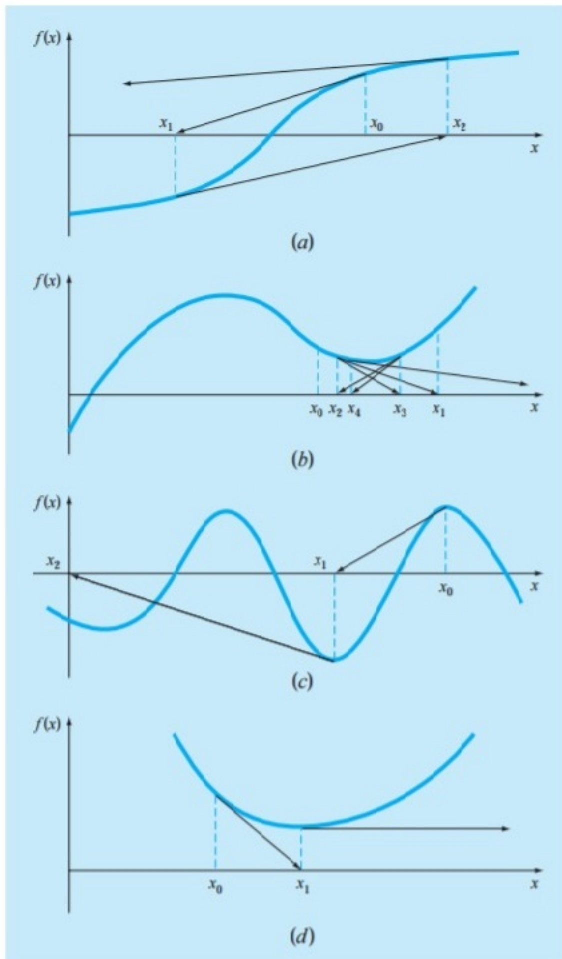
```
>> Newton_wCR(-100,10^(-8),1000,'Ex2','Ex2Prime',xt)
 iteration approximation
      0 -100.0000000000000000
      1 -67.1229452054794540 0.6756574735718571
      2 -45.2106915629508280 0.6800578556444912
      3 -30.6110303374327980 0.6865405829580693
      4 -20.8903135326827250 0.6960020747237687
      5 -14.4275998450253840 0.7096133071622610
      6 -10.1439728120448680 0.7287612751243944
      7 -7.3216506641581773 0.7547769215046272
      8 -5.4823476696099007 0.7882665754466991
      9 -4.3043281697322326 0.8279655150465651
     10 -3.5648242613362786 0.8695658665971102
     11 -3.0994781840095929 0.9056103540867716
     12 -2.7643039450246527 0.9249280749908025
     13 -2.0756744750764016 0.8332428121722965
     14 -2.5401125042410282 1.1349755654997482
     15 -3.1421077624340263 1.1541465967380635
     16 -2.8006794352317459 0.9242505567184756
     17 -2.2742377750642446 0.8736310362342069
     18 -2.6754008707963317 1.1102257580083743
     19 4.7255390285480541 0.8316297903540348
     20 2.9616670020385882 0.4750863630203516
     21 1.9405440180104074 0.3603737627577821
     22 1.4793327062840065 0.1983311582167616
     23 1.3711198338928769 0.0516185931343018
     24 1.3652469470192556 0.0028750630379588
     25 1.3652300135546727 0.0000083015856297
     26 1.3652300134140969 0.0000000000000000

ans =

    1.3652
```

Notice how the convergence rate actually looks linear for the first 20 iterations in this example. This reitereates that the quadratic convergence only occurs once we get close to the actual root. If your first approximation is far away then the convergence is not entirely quadratic.

Your text illustrates four cases where Newton has poor convergence or divergence. I have inserted it here as well.



Theorem Suppose that $f(x)$, $f'(x)$ and $f''(x)$ all exist and are continuous on some interval $[a, b]$, that $x_t \in [a, b]$ is a root of $f(x) = 0$, and that $f'(x_t) \neq 0$. Then there exists a value $\delta > 0$, such that Newton's method converges for all initial approximations $x_0 \in [x_t - \delta, x_t + \delta]$.

Note that in general there is no way to determine such a value δ. This theorem only says that for all such functions $f(x)$, such a value δ exists. Even if the value of δ is extremely small, there is an interval of values around the root x_t such that if x_0 (the initial approximation) lies in this interval, then Newton's method will converge.

Thus the interpretation of the above theorem is that Newton's method always converges if the initial approximation x_0 is sufficiently close to the root x_t.

Appendix

Sir Isaac Newton (1642-1726) - During much of 1665-1666, immediately after Newton had earned his degree from Trinity College, Cambridge, the college was closed due to the black plague. Newton went home to live and think during those two years. The result was the most productive period of mathematical discovery ever reported. During that time he made four of his greatest mathematical discoveries: (1) the binomial theorem, (2) calculus, (3) the law of gravity, and (4) the colour spectrum.

While Newton needs no introduction I was always curious about who Raphson was. I was under the mistaken impression that he was some mathematician in the 20th century who "re-discovered" Newton's method. It turns out that's not the case. He was relatively a contemporary of Newton and a member of the famous Royal Philosophical Society. He supported Newton's claim to the invention of calculus in the famous debate between Newton and Leibnitz about who invented first. One tidbit of minor information that I found particularly interesting was that he also was known for translating one of Newton's books into English. It is easy to forget that at the time all scientific communication was still conducted in Latin even though Latin was a language that had not been spoken for centuries.

In the process of writing these notes I used MATLAB to create the figure illustrating the geometric interpretation of the Newton/Raphson method. For those of you who are curious here is the relevant code. It is quickly written and could easily be cleaned up and some comments added. I also used the ability to add text boxes and text arrow boxes in the Figure window.

```
x = 0:8;
fx = 0.5 * x.^3 -64;
plot(x,fx);
hold;
```

```
x0=3;
fx0 = 0.5 * x0.^3-64;
x1 = x0 - (0.5 * x0.^3-64)/(1.5 * x0.^2);
plot([x0,x1],[fx0,0],'--');
plot(x, zeros(size(x)));
plot(x0,0,'*');
plot(x0,fx0, '*');
x2 = x1 - (0.5 * x1.^3-64)/(1.5 * x1^2);
fx1 = 0.5 * x1.^3-64;
plot(x2,0,'*');
plot([x1,x2],[fx1,0],'--');
plot(x1,fx1,'*')
```