COMPUTER SCIENCE 349A

Handout Number 11

ORDER OF CONVERGENCE OF THE SECANT METHOD AND BISECTION METHOD

The **Secant method iterative formula** is

$$x_{i+1} = x_i - f(x_i)\frac{x_i - x_{i-1}}{f(x_i) - f(x_{i-1})}.$$

Therefore, if x_t denotes an exact zero of $f(x)$,

$$x_{i+1} - x_t = (x_i - x_t) - f(x_i)\frac{x_i - x_{i-1}}{f(x_i) - f(x_{i-1})}$$

$$= (x_i - x_t)(x_{i-1} - x_t)\left[\frac{1}{\left[\dfrac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}\right]}\right]\left[\frac{\dfrac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} - \dfrac{f(x_i) - f(x_t)}{x_i - x_t}}{x_{i-1} - x_t}\right]$$

By the Mean Value Theorem, the first large square bracket term above is equal to $\dfrac{1}{f'(\xi_i)}$, for some value ξ_i between x_i and x_{i-1}.

The term in the second large square bracket above is equal to $\dfrac{f''(\eta_i)}{2}$, for some value η_i in an interval spanned by x_i, x_{i-1} and x_t as this is a "second divided difference" -- see page 97 of the 7th ed. (page 94 of the 6th ed.).

Thus, with the usual notation that $E_k = x_k - x_t$ denotes the error of the *k*-th approximation x_k, the above can be written as

$$E_{i+1} = E_i E_{i-1}\left[\frac{1}{f'(\xi_i)}\right]\left[\frac{f''(\eta_i)}{2}\right]$$

or

$$\frac{E_{i+1}}{E_i E_{i-1}} = \frac{f''(\eta_i)}{2f'(\xi_i)}.$$

Thus

(1) $$\lim_{i\to\infty}\left|\frac{E_{i+1}}{E_iE_{i-1}}\right| = \lim_{i\to\infty}\left|\frac{f''(\eta_i)}{2f'(\xi_i)}\right| = \left|\frac{f''(x_t)}{2f'(x_t)}\right|.$$

This gives a relationship **between 3 successive errors** of the computed approximations using the Secant method. However, this does not indicate the **order** α of the Secant method, which requires that the errors of 2 successive approximations be related by

(2) $$\lim_{i\to\infty}\frac{|E_{i+1}|}{|E_i|^{\alpha}} = \lambda, \text{ for some constant } \lambda.$$

The following analysis shows how to rewrite (1) in a form (2). From (1), it follows that for sufficiently large values of *i*,

(3) $$|E_{i+1}| \approx C|E_i||E_{i-1}|, \text{ for some constant } C.$$

From (2), it follows that for sufficiently large values of *i*,

(4) $$|E_{i+1}| \approx \lambda|E_i|^{\alpha} \text{ and } |E_i| \approx \lambda|E_{i-1}|^{\alpha}.$$

Substituting (4) into (3) gives

$$\lambda|E_i|^{\alpha} \approx C|E_i|\left[\frac{|E_i|}{\lambda}\right]^{\frac{1}{\alpha}},$$

which implies that

$$\lambda^{1+\frac{1}{\alpha}}|E_i|^{\alpha} \approx C|E_i|^{1+\frac{1}{\alpha}}.$$

From this it follows that

$$\alpha = 1+\frac{1}{\alpha} \quad\Rightarrow\quad \alpha^2-\alpha-1=0 \quad\Rightarrow\quad \alpha=\frac{1+\sqrt{5}}{2}\approx 1.618,$$

which is the **order of the Secant method**.

Note: this value α is known as the "golden ratio", and occurs in many diverse applications.

ORDER OF CONVERGENCE OF THE BISECTION METHOD

An alternate definition of **linear convergence** (similar to the definition given in class except that it does not involve a limit) is

$$|E_i| \le c|E_{i-1}| \quad \text{or} \quad |x_t - x_i| \le c|x_t - x_{i-1}|$$

for some constant c such that $0 < c < 1$.

Applying this inequality recursively gives

$$|x_t - x_i| \le c^i |x_t - x_0|.$$

From Page 2 of Handout Number 8, for the Bisection Method we had

$$|x_t - x_i| \le \left(\frac{1}{2}\right)^i \Delta x^0, \text{ where } \Delta x^0 = x_u - x_\ell$$

and $[x_\ell, x_u]$ is the initial interval. This implies linear convergence with the above definition, and $c = \frac{1}{2}$.