COMPUTER SCIENCE 349A

Handout Number 10

QUADRATIC CONVERGENCE OF NEWTON'S METHOD

The following result is proved on p. 153 of the 7th ed. (p. 150 of the 6th ed.)

Theorem If Newton's method is applied to $f(x)=0$ producing a sequence $\{x_i\}$ that converges to a root x_t and if $f'(x_t)\neq 0$, then the order of convergence is 2.

If $f'(x_t)=0$ and Newton's method converges to a root x_t, then we will see later that the order of convergence is NOT quadratic.

Example 1. An illustration of quadratic convergence of Newton's method. Here $f(x)=\cos x - x$. This was computed in MATLAB, so at most 16 correct digits are possible. The underlined digits are all correct.

i	x_i	no. of correct digits
0	$\frac{\pi}{4} = 0.\underline{7}853\ 98$	1
1	$0.\underline{739}5\ 361$	3
2	$0.\underline{7390}\ \underline{851}7\ 81$	7
3	$0.\underline{7390}\ \underline{8513}\ \underline{3215}\ \underline{16}11$	14
4	$0.\underline{7390}\ \underline{8513}\ \underline{3215}\ \underline{1607}$	16

Example 2. The following illustrates the possible effect of a poor initial approximation with Newton's method, yet the eventual characteristic quadratic convergence. Here Newton's method is used to compute a root of $x^3+4x^2-10=0$ with $p_0=-100$. Partial results are as follows.

$p_0 = -100$
$p_1 = -67.12$
$p_2 = -45.21$

$\vdots$

$p_{14} = -2.54$
$p_{15} = -3.14$
$p_{16} = -2.80$

$\vdots$		no. of correct digits
$p_{21} =$	1.9405	
$p_{22} =$	$\underline{1}.4793$	1
$p_{23} =$	$\underline{1.3}711$	2
$p_{24} =$	$\underline{1.3652}5$	4
$p_{25} =$	$\underline{1.3652}\ \underline{3001}\ 1$	9

For some graphical situations where Newton's method can exhibit poor convergence, see page 156 of the 7th ed. (p. 153 of the 6th ed.).

The following result, which is not stated in the textbook but is alluded to on page 155 of the 7th ed. (page 152 of the 6th ed.), gives conditions that guarantee convergence of Newton's method.

Theorem

Suppose that $f(x)$, $f'(x)$ and $f''(x)$ all exist and are continuous on some interval [a, b], that $x_t \in [a,b]$ is a root of $f(x) = 0$, and that $f'(x_t) \neq 0$. Then there exists a value $\delta > 0$ such that Newton's method converges for all initial approximations $x_0 \in [x_t - \delta, x_t + \delta]$.

Note that in general there is no way to determine such a value δ. This theorem only says that for all such functions $f(x)$, such a value δ exists. Even if the value of δ is extremely small, there is an interval of values around the root x_t such that if x_0 (the initial approximation) lies in this interval, then Newton's method will converge.

Thus the interpretation of the above theorem is that Newton's method always converges if the initial approximation x_0 is sufficiently close to the root x_t.