These are the lecture notes for CSC349A Numerical Analysis taught by Rich Little in the Spring of 2018. They roughly correspond to the material covered in each lecture in the classroom but the actual classroom presentation might deviate significantly from them depending on the flow of the course delivery. They are provided as a reference to the instructor as well as supporting material for students who miss the lectures. They are simply notes to support the lecture so the text is not detailed and they are not thoroughly checked. Use at your own risk. They are complimentary to the handouts. Many thanks to all the guidance and materials I received from Dale Olesky who has taught this course for many years and George Tzanetakis.

1 Richardson's Extrapolation

Richardson's extrapolation is the technique of combining two different numerical approximations that depend on a parameter (usually a step size h) in order to obtain a new approximation having a smaller truncation error. Let M denote some value to be computed, for example,

$$f'(x) \texttt{ or } f''(x) \texttt{ or } \int_a^b f(x)dx.$$

Let $N_1(h)$ denote a formula (that depends on a parameter h that can take on different values) for computing an approximation to M, and suppose that the form of the truncation error of this formula is a known infinite series in powers of h. For example, the most common case is that the truncation error is $O(h^2)$ and is an infinite series with only even powers of h, that is,

$$\underbrace{M}_{\substack{\text{exact}\\\text{value}}} = \underbrace{N_1(h)}_{\substack{\text{computed}\\\text{approx.}}} + \underbrace{K_1h^2 + K_2h^4 + K_3h^6 + \cdots}_{\text{turncation error } O(h^2)} \tag{1}$$

where the values K_i are some (possibly unknown) constants. The parameter h can be any positive value, but as $h \to 0$, the truncation error $\to 0$; that is, $N_1(h) \to M$.

Now, consider a new approximation with a different stepsize, say $h/2$,

$$M = N_1\left(\frac{h}{2}\right) + K_1\frac{h^2}{4} + K_2\frac{h^4}{16} + K_3\frac{h^6}{64} + \cdots \tag{2}$$

In order to obtain an $O(h^4)$ approximation to M, we compute $4 \times (2) - (1)$, which gives,

$$M = \underbrace{N_1\left(\frac{h}{2}\right) + \frac{N_1\left(\frac{h}{2}\right) - N_1(h)}{3}}_{\text{New approxiamtion } N_2(h)} \underbrace{- \frac{K_2}{4}h^4 - \frac{5K_3}{16}h^6 - \cdots}_{\text{turncation error } O(h^4)} \tag{3}$$

This is Richardson's Extrapolation - combine two $O(h^2)$ approximations to get a better $O(h^4)$ approximation with the simple computation,

$$N_2(h) = N_1\left(\frac{h}{2}\right) + \frac{N_1\left(\frac{h}{2}\right) - N_1(h)}{3}$$

We can extend this to get an $O(h^6)$ approximation, then an $O(h^8)$ approximation, etc. To do so, just keep taking successively smaller stepsizes, but they must decrease by the same factor. If we continue from equation (3), we can derive a simple formula for all of these approximations. Let (3) be rewritten as,

$$M = N_2(h) + K_2' h^4 + K_3' h^6 + \cdots \tag{4}$$

Then, let $h = \frac{h}{4}$, giving,

$$M = N_1\left(\frac{h}{4}\right) + K_1\frac{h^2}{4^2} + K_2\frac{h^4}{4^4} + K_3\frac{h^6}{4^6} + \cdots \tag{5}$$

Combine $4 \times (5) - (2)$, getting,

$$M = \underbrace{N_1\left(\frac{h}{4}\right) + \frac{N_1\left(\frac{h}{4}\right) - N_1\left(\frac{h}{2}\right)}{3}}_{\text{New approxiamtion } N_2\left(\frac{h}{2}\right)} \underbrace{- \frac{K_2}{4}\frac{h^4}{2^4} - \frac{5K_3}{16}\frac{h^6}{2^6} - \cdots}_{\text{turncation error } O(h^4)} \tag{6}$$

Then, combine $16 \times (6) - (4)$, this gives,

$$M = \underbrace{N_2\left(\frac{h}{2}\right) + \frac{N_2\left(\frac{h}{2}\right) - N_2(h)}{15}}_{\text{New approxiamtion } N_3(h)} + \underbrace{K_3'' h^6 + K_4'' h^8 + \cdots}_{\text{turncation error } O(h^6)} \tag{7}$$

Richardson's Extrapolation Table

$O(h^2)$	$O(h^4)$	$O(h^6)$	$O(h^8)$
$N_1(h)$			
$N_1\left(\frac{h}{2}\right)$	$N_2(h)$		
$N_1\left(\frac{h}{4}\right)$	$N_2\left(\frac{h}{2}\right)$	$N_3(h)$	
$N_1\left(\frac{h}{8}\right)$	$N_2\left(\frac{h}{4}\right)$	$N_3\left(\frac{h}{2}\right)$	$N_4(h)$

In general, entries are computed by,

$$N_j\left(\frac{h}{2^i}\right) = N_{j-1}\left(\frac{h}{2^{i+1}}\right) + \frac{N_{j-1}\left(\frac{h}{2^{i+1}}\right) - N_{j-1}\left(\frac{h}{2^i}\right)}{4^{j-1}-1},$$

where $j \geq 2$ and the truncation error is $O(h^{2j})$.

Advantages of Richardson's Extrapolation

- Using Richardson's extrapolation has the following advantages:
  1. obtain high accuracy with little computation
  2. doesn't require very small values of h to get high accuracy, so roundoff error is not a concern
- We now look to apply Richardson's extrapolation to one of our integration formulas.

2 Romberg Integration

Romberg integration is the application of Richardson's extrapolation to composite trapezoid rule approximations. Let $I_{k,1}$ denote the composite trapezoid rule approximation to $\int_a^b f(x)dx$ with 2^{k-1} subintervals, using $h = \frac{b-a}{2^{k-l}}$. Then, letting $f_i = f(x_i)$,

$$\begin{aligned}
I_{1,1} &= \frac{h}{2}\left[f_0 + f_1\right], \text{ where } h = b-a \\
I_{2,1} &= \frac{h}{2}\left[f_0 + 2f_1 + f_2\right], \text{ where } h = \frac{b-a}{2} \\
I_{3,1} &= \frac{h}{2}\left[f_0 + 2f_1 + 2f_2 + 2f_3 + f_4\right], \text{ where } h = \frac{b-a}{4}
\end{aligned}$$

etc.

Turns out that the error term for the composite trapezoid rule, $-\frac{b-a}{12}h^2 f''(\mu)$ with $a < \mu < b$, has a series expansion of the form,

$$K_1 h^2 + K_2 h^4 + K_3 h^6 + \cdots ,$$

where each K_i is a constant independent of h, thus R.E. applies. For example, letting $h = b - a$,

$$\int_a^b f(x)dx = I_{1,1} + K_1 h^2 + K_2 h^4 + K_3 h^6 + \cdots$$
$$\int_a^b f(x)dx = I_{2,1} + K_1 \frac{h^2}{4} + K_2 \frac{h^4}{16} + K_3 \frac{h^6}{64} + \cdots$$

etc.

Romberg Integration Table Thus, we have the comparable Romberg table

$O(h^2)$	$O(h^4)$	$O(h^6)$	$O(h^8)$
$I_{1,1}$			
$I_{2,1}$	$I_{2,2}$		
$I_{3,1}$	$I_{3,2}$	$I_{3,3}$	
$I_{4,1}$	$I_{4,2}$	$I_{4,3}$	$I_{4,4}$

In general, entries are computed by,

$$I_{k,j} = I_{k,j-1} + \frac{I_{k,j-1} - I_{k-1,j-1}}{4^{j-1} - 1},$$

for $k = 2, 3, 4, \ldots$ and $j = 2, 3, \ldots, k$.

Example of Romberg Integration Approximate $\int_1^3 \frac{1}{x}dx$ using Romberg integration. The Romberg table is as follows:

	$I_{k,1}$	$I_{k,2}$	$I_{k,3}$	$I_{k,4}$
$h = 2$	1.333333			
$h = 1$	1.166667	1.111111		
$h = 0.5$	1.116667	1.100000	1.099259	
$h = 0.25$	1.103211	1.098726	1.098641	1.098631
$h = 0.125$	1.099768	1.098620	1.098613	1.098613

with the final solution $I_{5,5} = 1.098613$.

Notes on Romberg

1. The entries in the Romberg table are computed row-by-row, stopping when two successive diagonal entries in the table are sufficiently close together:
$$\left|\frac{I_{n,n} - I_{n-1,n-1}}{I_{n,n}}\right| < \varepsilon.$$

2. There is a convergence theorem: If
$$\lim_{n\to\infty} I_{n,1} = \int_a^b f(x)dx$$
, then
$$\lim_{n\to\infty} I_{n,n} = \int_a^b f(x)dx.$$