CSC349A Numerical Analysis

Lecture 16

Rich Little

University of Victoria

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Introduction

The process of determining areas e.g. area of a circle by inscribed and superscribed polygons. This term is used to avoid confusion with the numeric integration of differential equations.

Problem: approximate the value of

$$\int_a^b f(x)dx$$

where $f(x)$ is such that it cannot be integrated analytically or it is known at only a finite set of points.

The main idea

Approximate $f(x)$ by an interpolating polynomial $P(x)$, and approximate $\int_a^b f(x)dx$ by $\int_a^b P(x)dx$
Suppose $P_n(x)$ is the Lagrange form of the interpolating polynomial:

$$P_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i)$$

then

$$\int_a^b f(x)dx \approx \int_a^b \left[\sum_{i=0}^{n} L_i(x) f(x_i)\right] dx = \sum_{i=0}^{n} \left[\int_a^b L_i(x)dx\right] f(x_i)$$

which is of the form $\sum_{i=0}^{n} a_i f(x_i)$.

Quadrature formula

So, our approximation is of the form

$$\int_a^b f(x)dx \approx \sum_{i=0}^{n} a_i f(x_i).$$

Such an approximation is called a **quadrature formula**, and a_i are the **quadrature coefficients** and x_i are the **quadrature points**, the points at which $f(x)$ is sampled to approximate $\int_a^b f(x)dx$.

Types

Types of quadrature formulas:

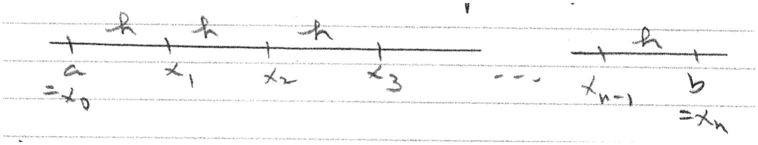
- Newton-Cotes closed
- Newton-Cotes open
- Gaussian (omit)

Any quadrature formulate derived by integrating an interpolating polynomial at equally-spaced quadrature points is called a **Newton-Cotes** formula.

Gaussian formulas obtain high accuracy by using optimally-chosen, unequally-spaced quadrature points.

Newton-Cotes closed formulas

Subdivide [a,b] into n subintervals of length $h = \frac{b-a}{n}$.



$$x_{i+1} - x_i = h, x_i = x_0 + ih$$

If $P_n(x)$ interpolates $f(x)$ at $a = x_0, x_1, x_2, \ldots, b = x_n$ and

$$\int_a^b f(x)dx \approx \int_a^b P_n(x)dx$$

then the resulting quadrature formula is called a **Newton-Cotes** closed formula.

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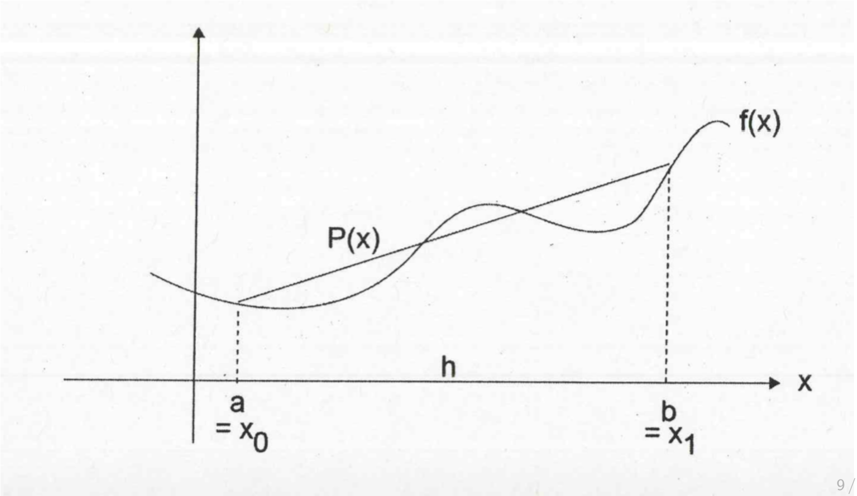
Introduction

The case $n = 1$:



Quadrature formula

The quadrature formula for approximating $\int_a^b f(x)dx$ is obtained by integrating P(x):

$$
\begin{aligned}
\int_a^b f(x)dx &\approx \int_{x_0}^{x_1} P(x)dx \\
&= \left[\int_{x_0}^{x_1} \frac{x - x_1}{x_0 - x_1} f(x_0)dx\right] + \left[\int_{x_0}^{x_1} \frac{x - x_0}{x_1 - x_0} f(x_1)dx\right] \\
&= \frac{f(x_0)}{x_0 - x_1}\left[\frac{x^2}{2} - x_1 x\right]_{x_0}^{x_1} + \frac{f(x_1)}{x_1 - x_0}\left[\frac{x^2}{2} - x_0 x\right]_{x_0}^{x_1} \\
&= \frac{x_1 - x_0}{2} f(x_0) + \frac{x_1 - x_0}{2} f(x_1) \\
&= \frac{h}{2}[f(x_0) + f(x_1)], \ \text{since } h = x_1 - x_0
\end{aligned}
$$

Trapezoid rule

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This is the **trapezoid rule**. Its error term can be obtained by integrating the error term of the Lagrange form of the interpolating polynomial, which for $n = 1$ is

$$f(x) - P(x) = \frac{f''(\xi)}{2}(x - x_0)(x - x_1)$$

where ξ is in the interval $[a, b]$.

Truncation Error

Integrating this gives:

$$
\begin{aligned}
\int_a^b f(x)dx - \int_{x_0}^{x_1} P(x)dx &= \int_a^b f(x)dx - \frac{h}{2}[f(x_0) - f(x_1)] \\
&= \int_a^b \frac{f''(\xi)}{2}(x - x_0)(x - x_1)dx \\
&= \frac{f''(\xi)}{2} \int_a^b (x - x_0)(x - x_1)dx
\end{aligned}
$$

since $f''(\xi)$ is a constant.

Truncation Error

Now, let $t = \frac{x-x_0}{h}$, then $dx = hdt$, $x - x_0 = th$, and $x - x_1 = (t-1)h$.
Also, when $x = x_0$ then $t = 0$ and when $x = x_1$, then $t = 1$.
Thus, we now have

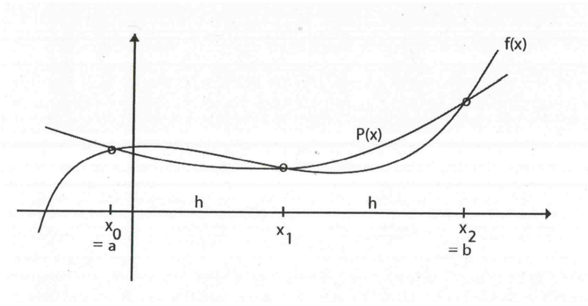
$$\begin{aligned}
\int_a^b f(x)dx - \int_{x_0}^{x_1} P(x)dx &= \frac{f''(\xi)}{2}\int_0^1 h^2 t(t-1)(hdt) \\
&= h^3 \frac{f''(\xi)}{2}\int_0^1 t(t-1)dt \\
&= -\frac{h^3}{12} f''(\xi)
\end{aligned}$$

for some value ξ between a and b.

Qudratic case

For the case $n = 2$ the quadratic interpolating polynomial is:

$$P(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1) + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2)$$



Qudrature formula for n=2

As in the case $n = 1$, the quadrature formula for approximating $\int_a^b f(x)dx$ is obtained by integrating $P(x)$: $\int_a^b f(x)dx \approx \int_{x_0}^{x_1} P(x)dx$.
This gives:

$$\int_a^b f(x)dx \approx \frac{h}{3}(f(x_0) + 4f(x_1) + f(x_2))$$

where now $h = \frac{b-a}{2}$. This is called **Simpson's rule** or **Simpson's 1/3 rule**, and its **truncation error** is given by:

$$\int_a^b f(x)dx - \int_{x_0}^{x_2} P(x)dx = -\frac{h^5}{90}f^{(4)}(\xi), \text{ for some } \xi \in [a, b]$$

Quadrature formula for n=3

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The Newton-Cotes closed quadrature formula for $n = 3$, in which $f(x)$ is approximated by a cubic polynomial that interpolates at four equally-spaced points, is:

$$\int_a^b f(x)dx \approx \frac{3h}{8}(f(x_0)+3f(x_1)+3f(x_2)+f(x_3)), \text{ where } h = \frac{b-a}{3}$$

The truncation error for this is

$$E_t = \frac{-3}{80}h^5 f^{(4)}(\xi)$$

for some $\xi \in [a, b]$.

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Degree of Precision of a Quadrature Formula

- The **degree of precision** of a quadrature formula is a measure of its accuracy or power.
- It is an integer number that indicates the degree (or order) of the set of all polynomials that the quadrature formula will integrate exactly.
- The larger the degree of precision, the more accurate or powerful is the quadrature formula because it will integrate exactly a larger set of polynomials, and this is a very good indicator that it will therefore integrate non-polynomial functions more accurately.

Defintion

If a quadrature formula $\sum_{i=0}^{n} a_i f(x_i)$ computes the exact value of $\int_a^b f(x)dx$ whenever $f(x)$ is a polynomial of degree d, but

$$\sum_{i=0}^{n} a_i f(x_i) \neq \int_a^b f(x)dx$$

for some polynomial $f(x)$ of degree $d+1$, then the **degree of precision** of the quadrature formula is d.

Note

If $f(x)$ is a polynomial of degree d, then $f(x) = \sum_{i=0}^{d} c_i x^i$ and

$$\int_a^b f(x)dx = \sum_{i=0}^{d} c_i \left[\int_a^b x^i dx \right]$$

Consequently, a quadrature formula computes $\int_a^b f(x)dx$ exactly if and only if it computes each of

$$\int_a^b dx, \int_a^b xdx, \ldots, \int_a^b x^d dx$$

exactly.

Theorem

Therefore, the degree of precision is d if and only if the quadrature formula computes the exact value of the integral when

$$f(x) = 1, x, x^2, ..., x^d$$

and it is not exact when $f(x) = x^{d+1}$.

Example 1: Determine the degree of precision for the Trapezoidal Rule.

Another way

- The degree of precision is also clear if you know the error term for the quadrature formula.
- For example, since the error term for the Trapezoidal Rule is $\frac{-h^3}{12} f''(\xi)$, for some value $\xi \in [a, b]$,
- this error term is exactly equal to 0 if and only if $f''(x) = 0$ for all $x \in [a, b]$.
- This is true if and only if $f(x) = c_0 + c_1 x$.
- That is, the Trapezoidal Rule computes the exact value of the integral of a polynomial $f(x)$ if and only if $f(x)$ is a polynomial of degree ≤ 1 (that is, the degree of precision is $d = 1$).
- **Example 2:** Determine the degree of precision for Simpon's 1/3 Rule.

Note on the Precision of Simpson's 1/3 Rule

- The degree of precision $d = 3$ for Simpsons $1/3$ rule is larger than expected since it is obtained by integrating a quadratic interpolating polynomial.
- This can also be found using the error term $\frac{-h^5}{90} f^{(4)}(\xi)$.
- This means that if $P(x)$ is a quadratic polynomial that interpolates any cubic polynomial

  $$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$$

  at the points a, b and $(a+b)/2$, then

  $$\int_a^b f(x)dx = \int_a^b P(x)dx$$

  exactly.

Note on the Precision of All Even Values of n

- This larger-than-expected value of the degree of precision occurs for Newton-Cotes closed quadrature formulas **for all even values of** n.
- This is not so of the odd values of n. Note the error term for the Cubic Quadrature case: $\frac{-3}{80}h^5 f^{(4)}(\xi)$.

n	degree of precision
1 (trap rule)	1
2 (1/3 rule)	3
3 (3/8 rule)	3
4	5
5	5
6	7