COMPUTER SCIENCE 349A
Handout Number 31

RICHARDSON'S EXTRAPOLATION (Section 22.2.1)

-- the technique of combining two different numerical approximations that depend on a parameter (usually a stepsize *h*) in order to obtain a new approximation having a smaller truncation error.

Let *M* denote some value to be computed, for example,

$$f'(x_0) \text{ or } f''(x_0) \text{ or } \int_a^b f(x)dx.$$

Let $N_1(h)$ denote a formula (that depends on a parameter *h* that can take on different values) for computing an approximation to *M*, and suppose that the form of the truncation error of this formula is a known infinite series in powers of *h*. For example, the most common case is that the truncation error is $O(h^2)$ and is an infinite series with only even powers of *h*, that is,

(1) $$\underbrace{M}_{\substack{\text{exact}\\ \text{value}}} = \underbrace{N_1(h)}_{\substack{\text{computed}\\ \text{approximation}\\ \text{using stepsize } h}} + \underbrace{K_1h^2 + K_2h^4 + K_3h^6 + \cdots}_{\text{truncation error is } O(h^2)}$$

where the values K_i are some (possibly unknown) constants. The parameter *h* can be any positive value, but as $h \to 0$, the truncation error $\to 0$; that is, $N_1(h) \to M$.

If (1) holds, then using a stepsize of $h/2$,

(2) $$M = N_1\left(\frac{h}{2}\right) + K_1\frac{h^2}{4} + K_2\frac{h^4}{16} + K_3\frac{h^6}{64} + \cdots .$$

In order to obtain an $O(h^4)$ approximation to *M*, we need to determine a linear combination of equations (1) and (2) in which the $O(h^2)$ terms cancel out; this will occur if we compute

$$4 \times (2) - (1),$$

which gives

$$4M - M = 4N_1\left(\frac{h}{2}\right) - N_1(h) - \frac{3K_2h^4}{4} - \frac{15K_3h^6}{16} - \cdots ,$$

or (by solving for *M*)

$$(3) \qquad M = \underbrace{N_1\left(\frac{h}{2}\right) + \frac{N_1\left(\frac{h}{2}\right) - N_1(h)}{3}}_{\substack{\text{this new approximation} \\ \text{to } M \text{ is called } N_2(h)}} \; \underbrace{-\frac{K_2}{4}h^4 - \frac{5K_3}{16}h^6 - \cdots}_{\substack{\text{the truncaton error of} \\ N_2(h) \text{ is } O(h^4)}} .$$

The above computation illustrates <u>the basic idea of Richardson's Extrapolation</u>: an $O(h^2)$ formula is used with two different stepsizes h and $h/2$ to compute two approximations $N_1(h)$ and $N_1\left(\frac{h}{2}\right)$ to some desired value *M*; then the simple computation

$$N_2(h) = N_1\left(\frac{h}{2}\right) + \frac{N_1\left(\frac{h}{2}\right) - N_1(h)}{3}$$

results in an approximation to *M* that has $O(h^4)$ accuracy.

Note. In the textbook, instead of stepsizes *h* and $h/2$, the two different stepsizes are denoted by h_1 and h_2.

By further decreasing the stepsize, the extrapolation procedure can be continued to give approximations to *M* that have accuracy $O(h^6)$, $O(h^8)$, and so on. Provided that the successive <u>stepsizes decrease by the same factor</u>, for example,

$$h, \frac{h}{2}, \frac{h}{4}, \frac{h}{8}, \cdots \quad \text{or} \quad h, \frac{h}{3}, \frac{h}{9}, \frac{h}{27}, \cdots ,$$

then a simple formula can be determined for computing all of these approximations to *M* and the computation can be automated.

Summarizing the above, we have:

$$\begin{array}{lll} N_1(h) & & \\ & \ddots & \\ N_1\left(\frac{h}{2}\right) & \cdots & N_2(h) = N_1\left(\frac{h}{2}\right) + \dfrac{N_1\left(\frac{h}{2}\right) - N_1(h)}{3} \end{array}$$

and (3) can be written as

$$(4) \qquad M = N_2(h) + K_2' h^4 + K_3' h^6 + \cdots .$$

for some constants K_i'. If the formula N_1 is now used with a stepsize $h/4$, this approximation to *M* satisfies

$$(5) \qquad M = N_1\left(\frac{h}{4}\right) + K_1\frac{h^2}{4^2} + K_2\frac{h^4}{4^4} + K_3\frac{h^6}{4^6} + \cdots ,$$

so equations (2) and (5) can be combined to give another $O(h^4)$ approximation to *M*: calculating $4\times(5)-(2)$ gives

$$4M - M = 4N_1\left(\frac{h}{4}\right) - N_1\left(\frac{h}{2}\right) + \text{ some } O(h^4) \text{ terms}$$

or (analogous to (3) and (4)), on solving for *M* this gives

$$(6) \qquad M = \underbrace{N_1\left(\frac{h}{4}\right) + \frac{N_1\left(\frac{h}{4}\right) - N_1\left(\frac{h}{2}\right)}{3}}_{\text{call this } N_2\left(\frac{h}{2}\right)} \underbrace{-\frac{K_2}{4}\frac{h^4}{2^4} - \frac{5K_3}{16}\frac{h^6}{2^6} - \cdots}_{\substack{\text{write this as} \\ K_2'\frac{h^4}{16} + K_3'\frac{h^6}{64} + \cdots}}$$

Now, equations (4) and (6) can be combined to give an $O(h^6)$ approximation to *M*: calculating $16\times(6)-(4)$ gives

$$16M - M = 16N_2\left(\frac{h}{2}\right) - N_2(h) \quad + \quad \text{some } O(h^6) \text{ terms}$$

which can be written as

$$(7) \qquad M = \underbrace{N_2\left(\frac{h}{2}\right) + \frac{N_2\left(\frac{h}{2}\right) - N_2(h)}{15}}_{\text{call this } N_3(h)} + \underbrace{K_3''h^6 + K_4''h^8 + \cdots}_{\text{truncation error is } O(h^6)}$$

for some constants K_i''. The above procedure can be continued, resulting in the following **Richardson's Extrapolation Table** (which can have as many rows and columns in it as you want):

$O(h^2)$	$O(h^4)$	$O(h^6)$	$O(h^8)$
$N_1(h)$			
$N_1\left(\frac{h}{2}\right)$	$N_2(h)$		
$N_1\left(\frac{h}{4}\right)$	$N_2\left(\frac{h}{2}\right)$	$N_3(h)$	
$N_1\left(\frac{h}{8}\right)$	$N_2\left(\frac{h}{4}\right)$	$N_3\left(\frac{h}{2}\right)$	$N_4(h)$

Entries in this table are computed row-by-row as follows:

$$N_j\left(\frac{h}{2^i}\right) = N_{j-1}\left(\frac{h}{2^{i+1}}\right) + \frac{N_{j-1}\left(\frac{h}{2^{i+1}}\right) - N_{j-1}\left(\frac{h}{2^i}\right)}{4^{j-1}-1}, \quad \text{for } j \geq 2,$$

and this approximation to M has truncation error of $O(h^{2j})$.

Example 1

Use Richardson's Extrapolation and the formula

$$f'(x_0) \approx \frac{f(x_0+h) - f(x_0-h)}{2h} ,$$

which has a truncation error term of the form in (1) above

$$K_1 h^2 + K_2 h^4 + K_3 h^6 + \cdots ,$$

to approximate $f'(2)$ when $f(x) = x\,e^x$. Start with $h = 0.2$.

In terms of the notation used above, here we have $M = f'(2)$ and $N_1(h) = \frac{f(2+h) - f(2-h)}{2h}$. The first 4 rows and columns of the Richardson's Extrapolation table are as follows:

	N_1	N_2	N_3	N_4
$h = 0.2$	22.41416066			
$h/2 = 0.1$	22.22878688	22.16699562		
$h/4 = 0.05$	22.18256486	22.16715752	22.16716831	
$h/8 = 0.025$	22.17101693	22.16716762	22.16716830	$\underbrace{22.16716830}_{\text{all digits correct}}$

Example 2.

It can be shown that

$$\lim_{h \to 0} \left(\frac{2+h}{2-h} \right)^{1/h} = e,$$

where $e = 2.718281828\cdots$. If

$$N_1(h) = \left(\frac{2+h}{2-h} \right)^{1/h}$$

denotes a formula for approximating the value of *e* for any fixed value of *h*, then it can be shown that the truncation error of this approximation is of the form

$$K_2 h^2 + K_4 h^4 + K_6 h^6 + \cdots$$

for some constants K_i ; that is,

$$e = N_1(h) + K_2 h^2 + K_4 h^4 + K_6 h^6 + \cdots \ .$$

Problem. Using $h = 0.04$ and stepsizes $h/2$ and $h/4$, compute an $O(h^6)$ approximation to the value of *e* using Richardson's extrapolation.

Solution. Compute the first column of the Richardson's extrapolation table using the given numerical formula:

$$N_1(0.04) = \left(\frac{2+0.04}{2-0.04} \right)^{1/0.04} = 2.7186443772$$

$$N_1(0.02) = \left(\frac{2+0.02}{2-0.02} \right)^{1/0.02} = 2.7183724448$$

$$N_1(0.01) = \left(\frac{2+0.01}{2-0.01} \right)^{1/0.01} = 2.7183044812$$

(Note that these approximations each have about 4 correct significant digits.)

Apply Richardson's extrapolation to the above two values in order to obtain two $O(h^4)$ approximations to the value of *e*.

$$N_2(h) = 2.7183724448 + \frac{2.7183724448 - 2.7186443772}{3} = 2.7182818007$$

$$N_2(h/2) = 2.7183044812 + \frac{2.7183044812 - 2.7183724448}{3} = 2.7182818267$$

Combine these two values to obtain an $O(h^6)$ approximation to the value of *e*:

$$N_3(h) = 2.7182818267 + \frac{2.7182818267 - 2.7182818007}{15} = 2.7182817990$$

Note. This approximation has 8 correct significant digits.

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Advantages of Richardson's Extrapolation:

-- obtain high accuracy with little computation

-- doesn't require very small values of *h* to get high accuracy, so roundoff error is not a concern. Note that the formulas used in the two examples above would have a loss of significant digits for very small values of *h*, and would give inaccurate results in floating-point arithmetic.