# COMPUTER SCIENCE 349A
# SAMPLE EXAM QUESTIONS WITH SOLUTIONS
# PARTS 1, 2

## PART 1.

1.1 (a) Define the term "ill-conditioned problem".

(b) Give an example of a polynomial that has ill-conditioned zeros.

1.2 Consider evaluation of

$$f(x) = \frac{1}{1 - \tanh(x)}, \qquad \text{where } \tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

If $f(x)$ is to be evaluated in floating-point arithmetic (e.g., $k = 4$ decimal digit, idealized, rounding floating-point), for each of the following ranges of values of $x$, specify whether the computed floating-point result will be accurate or inaccurate.

(a) $x$ is large and positive (for example, $x > 4$ if $k = 4$)

(b) $x$ is close to 0 (for example, $|x| \le 0.001$ if $k = 4$)

(c) $x$ is large and negative (for example, $x < -4$ if $k = 4$)

1.3 Consider

$$g(h) = \frac{\sin(1+h) - \sin(1)}{h}, \qquad h \ne 0$$

where the arguments for sin are in radians. When $|h|$ is close to 0, evaluation of $g(h)$ is inaccurate in floating-point arithmetic. In (a) and (d) below, use 4 decimal digit, idealized, rounding floating-point arithmetic. If $x$ is a floating-point number, assume that $f\ell(\sin x)$ is determined by rounding the exact value of $\sin x$ to 4 significant digits.

(a) Evaluate $f\ell(g(h))$ for $h = 0.00351$ . Note that $\sin(1.003) = 0.843088\cdots$, $\sin(1.004) = 0.843625\cdots$ and $\sin(1) = 0.841470\cdots$ .

(b) Taylor's Theorem can be expressed in two equivalent forms: given any fixed value $x_0$ ,

$$f(x) = f(x_0) + (x - x_0) f'(x_0) + \frac{(x - x_0)^2}{2!} f''(x_0) + \frac{(x - x_0)^3}{3!} f'''(x_0) + \cdots$$

or, using a change of variable (replacing $x$ by $x_0 + h$, so that $h = x - x_0$ is the independent variable),

$$f(x_0 + h) = f(x_0) + h\, f'(x_0) + \frac{h^2}{2!} f''(x_0) + \frac{h^3}{3!} f'''(x_0) + \cdots .$$

Using the latter form of Taylor's Theorem (without the remainder term), determine the quadratic (in $h$) Taylor polynomial approximation to $\sin(1+h)$. Note: leave your answer in terms of $\cos(1)$ and $\sin(1)$; do not evaluate these numerically.

(c) Use the Taylor polynomial approximation from (b) to obtain a polynomial approximation, say $p(h)$, to $g(h)$.

(d) Show that $p(h)$ is much better than $g(h)$ for floating-point evaluation when $|h|$ is close to 0 by evaluating $f\ell(p(0.00351))$. Note that $\sin(1) = 0.841470\cdots$ and $\cos(1) = 0.540302\cdots$ .

1.4 If $a, b, c, d, e, f$ have known values, then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$$

is a system of 2 linear equations in the 2 unknowns $x$ and $y$. If $ad - bc \neq 0$, then the solution is

$$x = \frac{de - bf}{ad - bc} \quad \text{and} \quad y = \frac{af - ce}{ad - bc} .$$

Consider the linear system

$$\begin{bmatrix} 0.96 & -1.23 \\ 4.91 & -6.29 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -0.27 \\ -1.38 \end{bmatrix}.$$

Show that the problem of computing the solution $\begin{bmatrix} x \\ y \end{bmatrix}$ is ill-conditioned.

1.5 (a) For what values of the real variable $x$, where $x > 1$, is the following expression subject to subtractive cancellation that will produce a very inaccurate result (in terms of relative error) using floating-point arithmetic?

$$f(x) = \sqrt{x} - \sqrt{x-1}, \qquad \text{where } \; x > 1.$$

(b) How should $f(x)$ be evaluated in floating-point arithmetic in order to avoid the subtractive cancellation in (a)?

1.6 Let

$$f(x) = \frac{(\sin x - e^x) + 1}{x^2}, \qquad x \neq 0.$$

Note: $x$ is in radians (for the sine function).

(a) In the following, use 4 decimal digit, idealized, chopping floating-point arithmetic. If $w$ is a floating-point number, compute $fl(e^w)$ and $fl(\sin w)$ by chopping the exact value of $e^w$ and $\sin w$, respectively, to 4 significant digits.
For the sine function, $w$ is in radians.

Evaluate $fl(f(0.123))$. Note that $\sin(0.123) = 0.122690\cdots$ and $e^{0.123} = 1.13088\cdots$.

(b) To 4 significant digits, the exact value of $f(0.123)$ is $-0.5416$, so the computation in (a) is inaccurate. In order to obtain a better formula for approximating $f(x)$ when $x$ is close to 0, use the fourth order Taylor polynomial approximations for $e^x$ and $\sin x$ (both expanded about $x_0 = 0$) in order to obtain a quadratic polynomial approximation for $f(x)$.

Note: if you know these required Taylor expansions, it is not necessary to show their derivations.

(c) Use the polynomial approximation for $f(x)$ from (b) (which is accurate when $x$ is close to 0) to show that the computation of $fl(f(0.123))$ in (a) is unstable.

Note: consider the perturbed problem with $\hat{x} = 0.123 + \varepsilon$, where $\dfrac{|\varepsilon|}{0.123}$ is small.

1.7 (a) Use 4 decimal digit, idealized, chopping floating-point arithmetic in the following. If $w$ is a floating-point number, approximate $fl(w^{1/4})$ by chopping the exact value of $w^{1/4}$ to 4 significant decimal digits. The evaluation of

$$g(x) = \frac{x^{1/4} - 1}{x - 1}$$

is inaccurate in floating-point arithmetic when $x$ is approximately equal to 1. Verify this by evaluating $f\ell(\ g(1.015))$. Note that the exact value of $1.015^{1/4}$ is $1.003729\cdots$. (Using real arithmetic, the exact value of $g(1.015)$ is $0.2486059\cdots$.)

(b) Determine the second order ($n = 2$) Taylor polynomial approximation for $f(x) = x^{1/4}$ expanded about $x_0 = 1$. Include the remainder term. Leave this polynomial in terms of expressions involving powers of $x - 1$. (Do not multiply out these powers of $x - 1$.)

(c) Substitute the polynomial approximation from (b), without the remainder term, into the formula for $g(x)$, and simplify in order to obtain <u>a polynomial approximation</u> for $g(x)$. (This polynomial approximation is accurate using floating-point arithmetic when $x$ is close to 1.)

Note: leave this polynomial in terms of expressions involving $x - 1$.

(d) Determine a good upper bound for the truncation error of the Taylor polynomial approximation in (b) when $0.95 \le x \le 1.06$ by bounding the remainder term. Give at least 4 correct significant digits.

1.8 Using idealized, <u>rounding</u> floating-point arithmetic (base 10, precision $k = 4$), the evaluation of

$$f\ell(\ f\ell(w * x) - f\ell(y * z)\ )$$

for

$$w = 16.00 \qquad x = 43.61 \qquad y = 12.31 \qquad z = 56.68$$

gives a result of $0.1000$, whereas the exact value is $0.0292$. The relative error of this computed result is 242%. Using the definition of stability given in class, show (by using only a perturbation of $y$) that the above floating-point computation is stable.

1.9 One of the two zeros of a quadratic polynomial $ax^2 + bx + c$ can be computed using either the formula

(i) $$\frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

or

(ii) $$\frac{-2c}{b + \sqrt{b^2 - 4ac}}.$$

For each of the specified polynomials in the table below, place an X in the appropriate box to indicate which of these formulas is more accurate in precision $k = 4$ floating-point arithmetic, or if they are both accurate. Put one X in each row of the table. (No justification for your answers is required.)

| polynomial | (i) is more accurate | (ii) is more accurate | both (i) and (ii) are accurate |
|---|---|---|---|
| $0.01x^2 + 100x + 0.01$ | | | |
| $100x^2 + 100x - 100$ | | | |
| $0.01x^2 - 100x - 0.01$ | | | |
| $0.01x^2 - 100x + 0.01$ | | | |
| $-0.01x^2 + 100x + 0.01$ | | | |
| $100x^2 - x + 0.01$ | | | |

## PART 2.

2.1 Let $c$ denote any positive number.

(a) Apply Newton's method to

$$f(x) = x^2 - \frac{1}{c}$$

in order to determine an iterative formula for computing $1/\sqrt{c}$.

(b) For arbitrary $c > 0$, let $x_0$ be the initial approximation to $1/\sqrt{c}$, let $\{x_1, x_2, x_3, \cdots\}$ be the sequence of computed approximations to $1/\sqrt{c}$ using the iterative formula from (a), and let

$$e_i = x_i - \frac{1}{\sqrt{c}}, \qquad \text{for } i = 0, 1, 2, 3, \ldots$$

Show (using algebra) that

$$e_i = \frac{e_{i-1}^2}{2x_{i-1}}.$$

(Note: from this, it follows that $\lim_{i\to\infty} \frac{|e_i|}{|e_{i-1}|^2} = \lim_{i\to\infty} \frac{1}{2x_{i-1}} = \frac{\sqrt{c}}{2}$ , proving that the iterative formula in (a) is quadratically convergent.)

2.2 (a) If Newton's method is used to compute an approximation to a zero of

$$P(x) = x^5 + 5x^4 - 40x^2 - 80x - 48$$

using the initial approximation $x_0 = -1$, convergence is obtained to the zero $x_t = -2$ of $P(x)$. If this computation is carried out, what is the order of convergence? Justify your answer.

(b) Give 1 or 2 MATLAB statements that could be used to compute all of the zeros of the polynomial

$$P(x) = x^5 + 5x^4 - 40x^2 - 80x - 48$$

using the MATLAB function *roots*.

2.3 (a) Let $R$ denote any positive number. Apply Newton's method to

$$f(x) = x^2 - \frac{R}{x}$$

in order to determine an <u>iterative formula</u> for computing $\sqrt[3]{R}$. Simplify the formula so that it is in the form

$$x_i \times \left( \frac{g(x_i)}{h(x_i)} \right)$$

where $g(x_i)$ and $h(x_i)$ are simple polynomials in $x_i$.

(b) Consider the case $R = 2$. Given some initial value $x_0$, if the iterative formula in (a) converges to $\sqrt[3]{2}$, what will be the order of convergence? Very briefly justify your answer, referring to any results from your class notes or the textbook.

2.4 (a) With regard to an algorithm for computing a root $x_t$ of $f(x) = 0$, what is the definition of order of convergence?

(b) The following sequence of values is converging to a root at $x_t = 1.895494$. What is the order of convergence?

| $i$ | $x_i$ |
|---|---|
| 0 | 1.80000 |
| 1 | 1.85078 |
| 2 | 1.87375 |
| 3 | 1.88476 |
| 4 | 1.89016 |
| 5 | 1.89284 |
| 6 | 1.89417 |
| 7 | 1.89483 |
| 8 | 1.89516 |
| 9 | 1.89533 |
| 10 | 1.89541 |

(c) Could the computed approximations in (b) have been computed using Newton's method? Justify your answer.

2.5 (a) Show how to evaluate

$$P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

using nested multiplication.

(b) Give pseudocode for the computation in (a).

2.6 A good approximation to one of the zeros of

$$P(x) = x^4 + x^3 - 6x^2 - 7x - 7$$

is $x_0 = 2.64$. If $x_0$ is used as an approximation to a zero of $P(x)$, use synthetic division (that is, Horner's algorithm) to determine the associated deflated polynomial. Show all of your calculations. Note: do <u>not</u> do any computations with Newton's method.

2.7 (a) Fill in the 7 blanks in the following MATLAB code so that the function M-files f.m and secant.m could be used to compute one zero of $f(x) = \sin x - e^{-x}$ using the Secant method. The function M-file secant.m has the following input parameters:

initial approximations x0 and x1
maximum number of iterations N

error tolerance tol (tests relative error)

and it prints each successive computed approximation to a zero of $f(x)$. If the function doesn't converge within N iterations, then an error message is printed.

The M-file f.m :

```
function y = f(x)
y = ____________________________________ ;
```

The M-file secant.m :

```
function root = secant ( x0, x1, N, tol )
i = 2;
q0 = f(x0);
q1 = f(x1);
while i <= N
      root = _________________________________________ ;
            fprintf('i = %g',i),fprintf(' approximation = %18.10f\n',root)
      if ____________________ < tol
            return
      end
      i = i+1;
      x0 = ______________________ ;
      q0 = ______________________ ;
      x1 = ______________________ ;
      q1 = _______________________ ;
end
fprintf('method failed to converge in %g',N),fprintf(' iterations\n')
```

(b) If the above MATLAB M-files f.m and secant.m are used to compute one zero of

$$f(x) = \sin x - e^{-x}$$

with initial approximations x0 = 0 and x1 = 1, N = 20 and tol = $10^{-6}$, then a computed approximation of root = 0.5885327440 is obtained. What is the order of convergence for this computation of this zero of $f(x)$? Briefly justify your answer using results given in class.

2.8 Use Taylor's Theorem to derive Newton's method for computing a root of $f(x) = 0$.

2.9 The volume of liquid in a spherical tank of radius *R* filled to a depth *h* is

$$V = \pi h^2 (3R - h)/3 .$$

Give one MATLAB statement that uses the MATLAB built-in function *fzero* to compute the depth to which a tank of radius 3 must be filled so that the volume is 30. In this statement, specify an appropriate interval that contains the answer and that can be used with *fzero*.

## SOLUTIONS

## PART 1.

1.1 (a) A problem is ill-conditioned if its exact solution can change greatly with a small change in the data defining the problem.

(b) $P(x) = (x-1)^5$. Polynomial roots with multiplicity greater than one are ill-conditioned.

1.2 (a) inaccurate, since tanh(x) is approximately equal to 1.
(b) accurate (assuming that tanh(x) is computed accurately), since tanh(x) is approximately equal to 0. (Note that $f\ell(\tanh(x))$ may not be inaccurate if x is close to 0.)
(c) accurate, since tanh(x) is approximately equal to $-1$.

1.3 (a)

$$f\ell(1+h) = f\ell(1+0.00351) = f\ell(1.00351) = 1.004$$
$$f\ell(\sin(1+h)) = f\ell(1.004) = f\ell(0.843625\cdots) = 0.8436$$
$$f\ell(\sin(1)) = f\ell(0.841470\cdots) = 0.8415$$
$$f\ell(\sin(1+h)-\sin(1)) = f\ell(0.8436-0.8415) = 0.0021 \quad \text{or} \quad 0.002100 \quad \text{or} \quad 0.2100\times 10^{-2}$$
$$f\ell((\sin(1+h)-\sin(1))/h) = f\ell(0.0021/0.00351) = f\ell(0.598290\cdots) = 0.5983$$

Note: exact value of $g(h)$ is 0.538824... and the relative error in the above computed approximation is $\left|1-\dfrac{0.5983}{0.538824}\right| \approx 0.11$ or 11%.

(b)

$$\sin(1+h) \approx \sin(1) + h\cos(1) + \frac{h^2}{2}(-\sin(1))$$

(c)

$$p(h) = \frac{\left(\sin(1) + h\cos(1) - \dfrac{h^2}{2}\sin(1)\right) - \sin(1)}{h} = \cos(1) - \frac{h}{2}\sin(1)$$

(d)

$$fl(\cos(1)) = fl(0.540302\cdots) = 0.5403$$
$$fl(h/2) = fl(0.00351/2) = 0.001755 \quad \text{or} \quad 0.1755\times 10^{-2}$$
$$fl(\sin(1)) = fl(0.841471\cdots) = 0.8415$$
$$fl((h/2)\times\sin(1)) = fl(0.001755\times 0.8415) = fl(0.0014768325) = 0.001477 \quad \text{or} \quad 0.1477\times 10^{-2}$$
$$fl(\cos(1)-(h/2)\sin(1)) = fl(0.5403-0.001477) = fl(0.538823) = 0.5388$$

which has all 4 significant digits correct.

1.4 Almost any perturbation of the 6 constants in the data $\begin{bmatrix}0.96 & -1.23\\ 4.91 & -6.29\end{bmatrix}, \begin{bmatrix}-0.27\\ -1.38\end{bmatrix}$ will do: for example,

$$\begin{bmatrix}0.961 & -1.23\\ 4.89 & -6.29\end{bmatrix}\begin{bmatrix}\hat{x}\\ \hat{y}\end{bmatrix} = \begin{bmatrix}-0.27\\ -1.38\end{bmatrix} \text{ has exact solution } \approx \begin{bmatrix}-0.03\\ 0.196\end{bmatrix}$$

whereas the given system has solution $\begin{bmatrix}1\\ 1\end{bmatrix}$.

1.5 (a) For $x$ sufficiently large and positive. (Note that there is no problem when $x \approx 1$ since then $fl(f(x)) \approx 1$, which is accurate.)

(b) $$f(x) = \left(\sqrt{x}-\sqrt{x-1}\right)\times\frac{\sqrt{x}+\sqrt{x-1}}{\sqrt{x}+\sqrt{x-1}} = \frac{1}{\sqrt{x}+\sqrt{x-1}}$$

1.6 (a)

$$fl(\sin x) = fl(0.122690...) = 0.1226$$
$$fl(e^x) = fl(1.13088...) = 1.130$$
$$fl(\sin x - e^x) = fl(0.1226 - 1.130) = fl(-1.0074) = -1.007$$
$$fl(\sin x - e^x + 1) = fl(-1.007+1) = -.007 \text{ or } -0.007000$$
$$fl(x^2) = fl(0.015129) = 0.01512$$
$$fl(f(x)) = fl(-0.007/0.01512) = fl(-0.462962) = -0.4629$$

(b)

$$\sin x \approx x - \frac{x^3}{6}, \qquad e^x \approx 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$$

$$f(x) \approx \frac{x - \frac{x^3}{6} - \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}\right) + 1}{x^2}$$

$$= \frac{-\frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{24}}{x^2}$$

$$= -\frac{1}{2} - \frac{x}{3} - \frac{x^2}{24}$$

(c)

given problem $\rightarrow$ computed solution

$x = 0.123$ $\qquad -0.4629$

perturbed problem $\rightarrow$ $f(\hat{x}) = \dfrac{\sin\hat{x} - e^{\hat{x}} + 1}{\hat{x}^2}$

$\hat{x} = 0.123 + \varepsilon$

And

$$f(\hat{x}) \approx -\frac{1}{2} - \frac{\hat{x}}{3} - \frac{\hat{x}^2}{24} \quad \text{if } \hat{x} \text{ is close to } 0$$

$$= -\frac{1}{2} - \frac{0.123+\varepsilon}{3} - \frac{(0.123+\varepsilon)^2}{24}$$

$$= -0.54163 - 0.34358\varepsilon + O(\varepsilon^2)$$

$$\approx -0.5416 \quad \text{for all } \varepsilon \text{ such that } \left|\frac{\varepsilon}{0.123}\right| \text{ is small.}$$

Since this is not close to $-0.4629$ for <u>all</u> small $\varepsilon$, the computation is unstable.

1.7 (a)

$fl(x^{1/4}) = fl(1.003729\cdots) = 1.003$

$fl(x^{1/4} - 1) = fl(1.003 - 1.000) = 0.003$ or $0.3000\times10^{-2}$

$fl(x - 1) = fl(1.015 - 1.000) = 0.015$ or $0.1500\times10^{-1}$

$fl(g(x)) = fl(0.003/0.015) = 0.2$ or $0.2000\times10^{0}$

(b)

$$f(x) = x^{1/4} \qquad f(1) = 1$$

$$f'(x) = \frac{1}{4}x^{-3/4} \qquad f'(1) = 1/4$$

$$f''(x) = -\frac{3}{16}x^{-7/4} \qquad f''(1) = -3/16$$

$$f'''(x) = \frac{21}{64}x^{-11/4}$$

Thus

$$f(x) = f(1) + f'(1)\,(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(\xi(x))}{3!}(x-1)^3$$

$$= 1 + \frac{1}{4}(x-1) - \frac{3}{32}(x-1)^2 + \frac{7}{128}\left[\xi(x)\right]^{-11/4}(x-1)^3$$

(c)

$$g(x) = \frac{x^{1/4}-1}{x-1} \approx \frac{1+\frac{1}{4}(x-1)-\frac{3}{32}(x-1)^2-1}{x-1} = \frac{1}{4} - \frac{3}{32}(x-1)$$

(d)

$$\left|\frac{7}{128}\left[\xi(x)\right]^{-11/4}(x-1)^3\right| \le \frac{7}{128}\max_{0.95\le\xi(x)\le1.06}\left[\xi(x)\right]^{-11/4}\max_{0.95\le x\le1.06}(x-1)^3$$

$$\le \frac{7}{128}\,\frac{1}{(0.95)^{11/4}}\,(1.06-1)^3\ , \quad \text{which is approx. } 0.0000136$$

1.8 To show stability, find a value $\varepsilon$ for which the exact value of

$$(16.00)\,(43.61) - (12.31+\varepsilon)\,(56.68)$$

is approx. equal to $0.1$ By continuity, any such value must be approx. equal to the value $\varepsilon$ such that

$$(16.00)\,(43.61) - (12.31+\varepsilon)\,(56.68) = 0.1$$

Solving for $\varepsilon$ gives

$$\varepsilon = \frac{16\times43.61-0.1}{56.68} - 12.31 = -0.0012491$$

Thus, for example, if $y = 12.31$ is perturbed to $\hat{y} = y + \varepsilon = 12.31 - 0.00125$ (or you could use $\varepsilon = -0.0012491$), then the exact value of

$$w\times x - \hat{y}\times z = 0.10005\ ,$$

which is very close to 0.1 . And since $\left|\frac{\varepsilon}{12.31}\right| = \frac{0.00125}{12.31}$ is small, the computation in (a) is stable.

1.9

| polynomial | (i) is more accurate | (ii) is more accurate | both (i) and (ii) are accurate |
|---|---|---|---|
| $0.01x^2 + 100x + 0.01$ | | $X$ | |
| $100x^2 + 100x - 100$ | | | $X$ |
| $0.01x^2 - 100x - 0.01$ | $X$ | | |
| $0.01x^2 - 100x + 0.01$ | $X$ | | |
| $-0.01x^2 + 100x + 0.01$ | | $X$ | |
| $100x^2 - x + 0.01$ | | | $X$ |

# PART 2.

2.1 (a)

$$f(x) = x^2 - 1/c$$
$$f'(x) = 2x$$
$$x_i = x_{i-1} - \frac{x_{i-1}^2 - 1/c}{2x_{i-1}} = \frac{x_{i-1}^2 + 1/c}{2x_{i-1}} \quad \text{or} \quad \frac{1}{2}\left(x_{i-1} + \frac{1}{cx_{i-1}}\right)$$

(b)

$$\begin{aligned}
e_i &= x_i - 1/\sqrt{c} \\
&= \frac{x_{i-1}^2 + 1/c}{2x_{i-1}} - \frac{1}{\sqrt{c}} \quad \text{from (a)} \\
&= \frac{\sqrt{c}x_{i-1}^2 + 1/\sqrt{c} - 2x_{i-1}}{2x_{i-1}\sqrt{c}} \\
&= \frac{\sqrt{c}\left(x_{i-1}^2 - \frac{2x_{i-1}}{\sqrt{c}} + \frac{1}{c}\right)}{2x_{i-1}\sqrt{c}} \\
&= \frac{\left(x_{i-1} - 1/\sqrt{c}\right)^2}{2x_{i-1}} = \frac{e_{i-1}^2}{2x_{i-1}}
\end{aligned}$$

2.2 (a)

$$P'(x) = 5x^4 + 20x^3 - 80x - 80, \text{ so } P'(-2) = 80 - 160 + 160 - 80 = 0.$$

Thus, the zero at $p = -2$ has multiplicity $m \geq 2$, which implies that Newton's method has linear convergence (that is, the order of convergence is 1).

(b)

```
roots ( [ 1  5  0  -40   -80  -48 ] )
```

or

```
p = [ 1  5  0  -40   -80  -48 ] ;
roots ( p )
```

2.3 (a)

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} = x_i - \frac{x_i^2 - \frac{R}{x_i}}{2x_i + \frac{R}{x_i^2}} = x_i - \left(\frac{x_i^3 - R}{x_i}\right)\left(\frac{x_i^2}{2x_i^3 + R}\right)$$

$$= x_i\left(1 - \frac{x_i^3 - R}{2x_i^3 + R}\right) = x_i\left(\frac{2x_i^3 + R - x_i^3 + R}{2x_i^3 + R}\right) = x_i\left(\frac{x_i^3 + 2R}{2x_i^3 + R}\right)$$

(b)

$$f'(x) = 2x + \frac{R}{x^2} = 2x + \frac{2}{x^2}, \quad \text{so at the root } x = \sqrt[3]{2},$$

$$f'(\sqrt[3]{2}) = 2\sqrt[3]{2} + \frac{2}{2^{2/3}} \neq 0 \quad \Rightarrow \quad \sqrt[3]{2} \text{ is a simple zero}$$

(that is, the multiplicity of the root is 1) and thus Newton's method converges quadratically (the order of convergence is $\alpha = 2$)

2.4 (a) The order of convergence is $\alpha$ if there exist constants $\lambda > 0$ and $\alpha \geq 1$ such that

$$\lim_{i \to \infty} \frac{|x_{i+1} - x_t|}{|x_i - x_t|^{\alpha}} = \lambda$$

(b) By inspection, the order of convergence is 1 (linear convergence).

(c) Yes, these approximations could have been computed with Newton's method if the root $x_t$ has multiplicity $m \geq 2$, since in this case, the order of convergence of Newton's method is only 1.

2.5 (a)

$$P(x) = a_0 + x\left(a_1 + x\left(a_2 + x\left(\cdots + x\left(a_{n-1} + xa_n\right)\right)\right)\right)$$

(b)

$b_n \leftarrow a_n$

for $k = n-1, n-2, \ldots, 1, 0$

$\quad b_k \leftarrow a_k + b_{k+1}x$

2.6

$b_4 = 1$

$b_3 = a_3 + b_4 x_0 = 1 + (1)(2.64) = 3.64$

$b_2 = a_2 + b_3 x_0 = -6 + (3.64)(2.64) = 3.6096$

$b_1 = a_1 + b_2 x_0 = -7 + (3.6096)(2.64) = 2.529344$

$b_0 = a_0 + b_1 x_0 = -7 + (2.529344)(2.64) = -0.322532$

The deflated polynomial is $x^3 + 3.64x^2 + 3.6096x + 2.529344$

2.7 (a)

```
y = sin(x) – exp(-x) ;

root = x1 – q1 * (x1 – x0)/(q1 – q0) ;

if  abs(1 - x1 / root) < tol

x0 = x1 ;
q0 = q1 ;
x1 = root ;
q1 = f(root) ;
```

(b)

Order of convergence is 1.618 since this is the order of convergence of the Secant method for a simple zero (multiplicity = 1). The multiplicity is 1 because $f'(root) \neq 0$, since clearly $\cos(root) > 0$ and $e^{-root} > 0$.

2.8 The linear ($n = 1$) Taylor Theorem expansion for $f(x)$ expanded about $x_0$ is

$$f(x) = f(x_0) + (x - x_0)f'(x_0) + \frac{(x - x_0)^2}{2!} f''(\xi),$$

for some value $\xi$ between $x$ and $x_0$. If we take $x = x_t$ (where $x_t$ is a root of the equation $f(x) = 0$), then from above

$$f(x_t) = f(x_0) + (x_t - x_0) f'(x_0) + \frac{(x_t - x_0)^2}{2!} f''(\xi)$$

$$= 0, \quad \text{since } x_t \text{ is a zero of } f(x).$$

On dropping the remainder term, this gives

$$f(x_0) + (x_t - x_0) f'(x_0) \approx 0,$$

which implies that $x_t \approx x_0 - \dfrac{f(x_0)}{f'(x_0)}$. This suggests computing

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)},$$

which is the first step of Newton's method, and then continue iterating with

$$x_i = x_{i-1} - \frac{f(x_{i-1})}{f'(x_{i-1})}.$$

2.9 fzero ( ‘pi * x^2 * (9-x) - 90’, [0 3] )