# CSC349A Numerical Analysis

## Lecture 21

Rich Little

University of Victoria

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# Runge-Kutta Methods

Advantage of Taylor methods of order $n$

- global truncation error of $O(h^n)$ insures high accuracy (even for $n = 3, 4$ or 5)

Disadvantage

- high order derivatives of $f(x, y(x))$ may be difficult and expensive to evaluate.

**Runge-Kutta methods** are higher order formulas (they can have any order $\geq 1$) that require function evaluations only of $f(x, y(x))$, and not of any of its derivatives.

# Taylor Polynomial with 2 Variables

This is accomplished using the Taylor polynomial for a function of 2 variables:

$$
\begin{aligned}
f(x+h, y+k) = f(x,y) + hf_x(x,y) + kf_y(x,y) \\
+ \frac{h^2}{2} f_{xx}(x,y) + hkf_{xy}(x,y) + \frac{k^2}{2} f_{yy}(x,y) \\
+ \frac{h^3}{6} f_{xxx}(x,y) + \frac{h^2 k}{2} f_{xxy}(x,y) + \frac{hk^2}{2} f_{xyy}(x,y) + \frac{k^3}{6} f_{yyy}(x,y) \\
+ \cdots
\end{aligned}
$$

where $f_x \equiv \frac{\partial f}{\partial x}$, $f_{xy} \equiv \frac{\partial^2 f}{\partial x \partial y}$, etc.

# Taylor Polynomial with 2 Variables

The derivation of Runge-Kutta methods and an understanding of why they work requires the Taylor polynomial for a function of 2 variables, but this Taylor polynomial is not required to use these methods to numerically approximate the solution of a differential equation.
It is only needed in the derivation and we will derive the second-order Runge-Kutta methods a little later

# General Form of RK Methods

Runge-Kutta methods are so-called one-step methods (as also are Eulers method and all Taylor methods): that is, they are of the form

$$y_{i+1} = y_i + h\Phi(x_i, y_i, h)$$

for some (possibly very complicated) function $\Phi$.
That is, each computed approximation $y_{i+1}$ is computed using only the value $y_i$ at the previous grid point, along with the values of $x_i$, the step size $h$, and of course the function $f(x, y(x))$ that specifies the differential equation.

# General Form of RK Methods of Order $m$

A Runge-Kutta method of order $m$ is of the form:

$$y_{i+1} = y_i + \sum_{j=1}^{m} a_j k_j$$

where the $a_j$ are constants and the $k_j$ are functions of the form,

$$k_1 = hf(x_i, y_i)$$

$$k_j = hf\left(x_i + \alpha_j h, y_i + \sum_{l=1}^{j-1} \beta_{jl} k_l\right), \text{ for } 2 \leq j \leq m$$

**Examples:** Derive the general forms for $m = 1, 2, 3$, and 4.

# The Goal given General Form of Order $m$

Each of these formuals defines a whole class of Runge-Kutta methods of order $m$.
Our goal is to take the formula for any fixed value of $m \geq 1$, and determine values for the parameters:

- $\{a_1\}$ when $m = 1$
- $\{a_1, a_2, \alpha_2, \beta_{21}\}$ when $m = 2$
- $\{a_1, a_2, a_3, \alpha_2, \alpha_3, \beta_{21}, \beta_{31}, \beta_{32}\}$ when $m = 3$
- $\{a_1, a_2, a_3, a_4, \alpha_2, \alpha_3, \alpha_4, \beta_{21}, \beta_{31}, \beta_{32}, \beta_{41}, \beta_{42}, \beta_{43}\}$ when $m = 4$

so that the resulting Runge-Kutta method has as high an order as possible (i.e., its local truncation error is as small as possible).

# Deriving RK Methods of Order $m$

This is accomplished by choosing the unknown parameters $\{a_i\}, \{\alpha_i\}$, and $\{\beta_{ij}\}$ so that the Runge-Kutta formula

$$y_{i+1} = y_i + \sum_{j=1}^{m} a_j k_j$$

is identical to the Taylor series expansion

$$y(x_{i+1}) = y(x_i) + hy'(x_i) + \frac{h^2}{2} y''(x_i) + \frac{h^3}{6} y'''(x_i) + \cdots$$

to as many terms as possible.

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# Derivation of First Order RK Method

**Case $\mathbf{m = 1}$**
The only Runge-Kutta method of first order is when $a_1 = 1$.
That is, Eulers method

$$y_{i+1} = y_i + hf(x_i, y_i)$$

For each value of $m \geq 2$, there are an infinite number of Runge-Kutta formulas, each one having local truncation error $O(h^{m+1})$ and thus global truncation error $O(h^m)$.

# Derivation of second order R-K

**Case m = 2**

$$\begin{aligned} y(x_{i+1}) &= y(x_i) + a_1 hf(x_i, y_i) \\ &\quad + a_2 hf(x_i + \alpha_2 h, y_i + \beta_{21} hf(x_i, y_i)) \end{aligned}$$

Using the Taylor exapnsion for $f(x_i + \alpha_2 h, y_i + \beta_{21} hf(x_i, y_i))$, we get

$$\begin{aligned} y(x_{i+1}) &= y(x_i) + a_1 hf(x_i, y_i) + a_2 h[f(x_i, y_i) + \alpha_2 hf_x(x_i, y_i) \\ &\quad + \beta_{21} hf(x_i, y_i) f_y(x_i, y_i) + O(h^2)] \\ &= y(x_i) + [a_1 + a_2] hf(x_i, y_i) + h^2[a_2 \alpha_2 f_x(x_i, y_i) \\ &\quad + a_2 \beta_{21} f(x_i, y_i) f_y(x_i, y_i)] + O(h^3) \end{aligned}$$

# Derivation of Second Order RK Method

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But, also by Taylors Theorem

$$
\begin{aligned}
y(x_{i+1}) =& y(x_i) + hy'(x_i) + \frac{h^2}{2}y''(x_i) + O(h^3) \\
=& y(x_i) + hf(x_i, y_i) + \frac{h^2}{2}f'(x_i, y_i) + O(h^3) \\
=& y(x_i) + hf(x_i, y_i) + \frac{h^2}{2}[f_x(x_i, y_i) + f(x_i, y_i)f_y(x_i, y_i)] \\
&+ O(h^3)
\end{aligned}
$$

These two are equal only when

$$
a_1 + a_2 = 1, a_2\alpha_2 = 1/2, a_2\beta_{21} = 1/2
$$

# Examples

1. **Heun's Method:** Let $a_1 = a_2 = 1/2, \alpha_2 = \beta_{21} = 1$, which gives

$$y_{i+1} = y_i + \frac{h}{2}\left[f(x_i, y_i) + f(x_i + h, y_i + hf(x_i, y_i))\right]$$

2. **Midpoint Method:** Let $a_1 = 0, a_2 = 1, \alpha_2 = \beta_{21} = 1/2$, which gives

$$y_{i+1} = y_i + hf(x_i + \frac{h}{2}, y_i + \frac{h}{2}f(x_i, y_i))$$

3. Use Heun's Method to solve $y' = 4e^{0.8x} - 0.5y$ from $x = 0$ to $x = 4$ with $h = 1$ and $y_0 = 2$.

# Derivation of Third Order RK Methods

**Case m = 3** It can be shown that any solution of a certain system of 6 nonlinear equations in 8 unknowns gives a third-order Runge-Kutta Method.
One common solution is

$$a_1 = \frac{1}{6}, a_2 = \frac{2}{3}, a_3 = \frac{1}{6}, \alpha_2 = \frac{1}{2}, \alpha_3 = 1, \beta_{21} = \frac{1}{2}, \beta_{31} = -1, \beta_{32} = 2$$

which gives the third-order Runge-Kutta method

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 4k_2 + k_3)$$

where

$$\begin{aligned} k_1 &= hf(x_i, y_i) \\ k_2 &= hf\left(x_i + \frac{h}{2}, y_i + \frac{k_1}{2}\right) \\ k_3 &= hf(x_i + h, y_i - k_1 + 2k_2) \end{aligned}$$

# Derivation of Fourth Order RK Methods

**Case m = 4** The 13 Runge-Kutta parameters are obtained by solving 11 nonlinear equations in 13 unknowns. One solution is called the **"classical" Runge-Kutta method**, which has global truncation error of $O(h^4)$:

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where

$$\begin{aligned}
k_1 &= hf(x_i, y_i) \\
k_2 &= hf\left(x_i + \frac{h}{2}, y_i + \frac{k_1}{2}\right) \\
k_3 &= hf\left(x_i + \frac{h}{2}, y_i + \frac{k_2}{2}\right) \\
k_4 &= hf(x_i + h, y_i + k_3)
\end{aligned}$$