These are the lecture notes for CSC349A Numerical Analysis taught by Rich Little in the Spring of 2018. They roughly correspond to the material covered in each lecture in the classroom but the actual classroom presentation might deviate significantly from them depending on the flow of the course delivery. They are provided as a reference to the instructor as well as supporting material for students who miss the lectures. They are simply notes to support the lecture so the text is not detailed and they are not thoroughly checked. Use at your own risk. They are complimentary to the handouts. Many thanks to all the guidance and materials I received from Dale Olesky who has taught this course for many years and George Tzanetakis.

1 Numerical Differentiation Formulas

In Chapter 4, Taylors Theorem was used to derive a numerical differentiation formula that was used to approximate a derivative in a mathematical model that was developed to determine the terminal velocity of a free-falling body (a parachutist).

Recall Taylor's Theorem (with $n = 2$):

$$f(x) = f(x_0) + (x - x_0)f'(x_0) + \frac{(x - x_0)^2}{2!}f''(x_0) + \frac{(x - x_0)^3}{3!}f'''(\xi_1)$$

where ξ_1 lies between x and x_0.

1.1 Forward Difference Approximation

Taylor's Theorem can also be stated in terms of a step size, $h < 1$, by letting $h = x - x_0$ which implies that $x = x_0 + h$. Thus, Taylor's approxiamtion becomes

$$f(x_0 + h) = f(x_0) + f'(x_0)h + \frac{f''(x_0)}{2}h^2 + \frac{f'''(\xi_1)}{6}h^3 \tag{1}$$

where ξ_1 lies between x_0 and $x_0 + h$.

Solving for $f'(x_0)$ in (1) gives

$$\begin{aligned} f'(x_0)h =& f(x_0 + h) - f(x_0) - \frac{f''(x_0)}{2}h^2 - \frac{f'''(\xi_1)}{6}h^3 \\ f'(x_0) =& \frac{f(x_0 + h) - f(x_0)}{h} - \frac{f''(x_0)}{2}h - \frac{f'''(\xi_1)}{6}h^2 \end{aligned}$$

or

$$f'(x_0) \approx \frac{f(x_0+h)-f(x_0)}{h} \text{ with error } O(h).$$

1.2 Backward Difference Approximation

Consider yet another form of Taylor, where $x = x_0 - h$, thus $-h = x - x_0$ and

$$f(x_0-h) = f(x_0) - f'(x_0)h + \frac{f''(x_0)}{2}h^2 - \frac{f'''(\xi_2)}{6}h^3 \tag{2}$$

where ξ_2 lies between $x_0 - h$ and x_0.

Solving for $f'(x_0)$ in (2) gives

$$\begin{aligned} f'(x_0)h =& f(x_0) - f(x_0-h) + \frac{f''(x_0)}{2}h^2 - \frac{f'''(\xi_2)}{6}h^3 \\ f'(x_0) =& \frac{f(x_0)-f(x_0-h)}{h} + \frac{f''(x_0)}{2}h - \frac{f'''(\xi_2)}{6}h^2 \end{aligned}$$

or

$$f'(x_0) \approx \frac{f(x_0)-f(x_0-h)}{h} \text{ with error } O(h).$$

1.3 Central Difference Approximation

A more accurate approximation to a first derivative can be obtained by taking a linear combination of two Taylor Theorem approximations. Thus, subtracting (2) from (1) givesn and solving for $f'(x_0)$ gives

$$f'(x_0) \approx \frac{f(x_0+h)-f(x_0-h)}{2h} \text{ with error } O(h^2).$$

Example 1: Use forward and central difference approximations to estimate the first derivative of

$$f(x) = -0.1x^4 - 0.15x^3 - 0.5x^2 - 0.25x + 1.25$$

at $x = 0.5$ using step size $h = 0.5$.

1.4 High-Accuracy Differentiation Formulas

Still more accurate approximations to $f'(x_0)$ can be obtained by taking linear combinations of more Taylor Theorem approximations and solving for $f'(x_0)$. For example, Taylor Theorem approximations for each of

$$f(x_0 - 2h), f(x_0 - h), f(x_0 + 2h), f(x_0 + h)$$

can be used to derive the $O(h^4)$ approximation

$$f'(x_0) \approx \frac{1}{12h}\left[f(x_0 - 2h) - 8f(x_0 - h) + 8f(x_0 + h) - f(x_0 + 2h)\right]$$

FIGURES 23.1 to 23.3 on pages 656-657 in the text summarize other such formulas.

We can also use Richardson's Extrapolation to get higher accuracy.

Example of Richardson's Extrapolation

Use Richardson's extrapolation and the approximation formula

$$f'(x_0) \approx \frac{f(x_o + h) - f(x_0 - h)}{2h},$$

which has a truncation error of the form,

$$K_1h^2 + K_2h^4 + K_3h^6 + \cdots,$$

to approximate $f'(2)$ where $f(x) = xe^x$ and $h = 0.2$. The first 4 rows and columns of the Richardson's Extrapolation table are as follows:

	N_1	N_2	N_3	N_4
$h = 0.2$	22.4141607			
$h = 0.1$	22.2287869	22.1669956		
$h = 0.05$	22.1825649	22.1671575	22.1671683	
$h = 0.025$	22.1710169	22.1671676	22.1671683	22.1671683

1.5 Numerical Differentiation Formulas for Higher Derivatives

Similar techniques can be applied to find formulas for higher derivatives. For example, we can derive $f''(x_0)$ by doing the following:

Example: Let $n = 3$ and use the following Taylor Approximations

$$f(x_0 + h) = f(x_0) + hf'(x_0) + \frac{h^2}{2} f''(x_0) + \frac{h^3}{6} f'''(x_0) + \frac{h^4}{24} f^{(4)}(\xi_1)$$

and

$$f(x_0 - h) = f(x_0) - hf'(x_0) + \frac{h^2}{2} f''(x_0) - \frac{h^3}{6} f'''(x_0) + \frac{h^4}{24} f^{(4)}(\xi_2)$$

Adding these gives:

$$f(x_0 + h) + f(x_0 - h) = 2f(x_0) + h^2 f''(x_0) + \frac{h^4}{24} \left[f^{(4)}(\xi_1) + f^{(4)}(\xi_2) \right]$$

and solving for the desired derivative $f''(x_0)$ gives

$$f''(x_0) = \frac{f(x_0 + h) - 2f(x_0) + f(x_0 - h)}{h^2} - \frac{h^2}{24} \left[f^{(4)}(\xi_1) + f^{(4)}(\xi_2) \right]$$

That is,

$$f''(x_0) \approx \frac{f(x_0 + h) - 2f(x_0) + f(x_0 - h)}{h^2}$$

with a truncation error of $O(h^2)$.