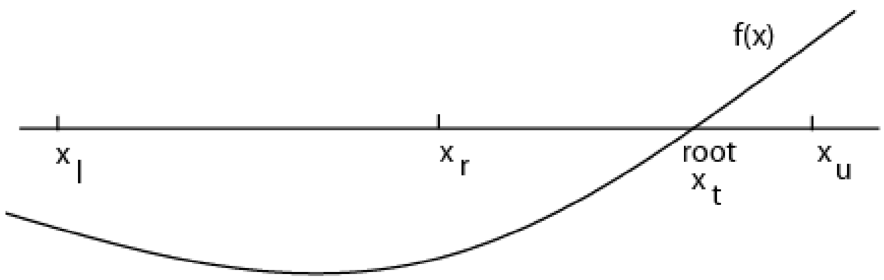
COMPUTER SCIENCE 349A

Handout Number 8

THE BISECTION METHOD (Section 5.2: pp. 127-135 7th ed; pp. 124-131 6th ed)

-- can be used to compute a zero of any function $f(x)$ that is continuous on an interval $[x_\ell, x_u]$ for which $f(x_\ell) \times f(x_u) < 0$.

Consider x_ℓ and x_u as **two initial approximations** to a zero, say x_t, of $f(x)$. The new approximation is the midpoint of the interval $[x_\ell, x_u]$, which is $x_r = \frac{x_\ell + x_u}{2}$.



If $f(x_r) = 0$, then x_r is the desired zero of $f(x)$. Otherwise, a new interval $[x_\ell, x_u]$ that is half the length of the previous interval is determined as follows.

If $f(x_\ell) \times f(x_r) < 0$ then $[x_\ell, x_r]$ contains a zero, so set $x_u \leftarrow x_r$. Otherwise, $f(x_u) \times f(x_r) < 0$ (necessarily) and $[x_r, x_u]$ contains a zero, so set $x_\ell \leftarrow x_r$.

The above procedure is repeated, continually halving the interval $[x_\ell, x_u]$, until $[x_\ell, x_u]$ is sufficiently small, at which time the midpoint $x_r = \frac{x_\ell + x_u}{2}$ will be arbitrarily close to a zero of $f(x)$.

Convergence criterion

As this is an iterative algorithm that computes a sequence of approximations

$$x_1, x_2, x_3, \ldots, x_{i-1}, x_i, \ldots$$

to a zero x_t, recall from Section 3.3 that

$$|\varepsilon_a| = \left|\frac{\text{current approx - previous approx}}{\text{current approx}}\right| = \left|\frac{x_i - x_{i-1}}{x_i}\right| = \left|1 - \frac{x_{i-1}}{x_i}\right|$$

is a good approximation to the actual relative error $|\varepsilon_t|$ in x_i, and can be used to determine the accuracy of x_i.

Note that each approximation x_i is equal to $\frac{x_u + x_\ell}{2}$ and the previous approximation x_{i-1} is either x_ℓ or x_u. Therefore,

$$|x_i - x_{i-1}| = \frac{x_u - x_\ell}{2} \text{ and thus } |\varepsilon_a| = \frac{|x_i - x_{i-1}|}{|x_i|} = \frac{\frac{x_u - x_\ell}{2}}{\left|\frac{x_u + x_\ell}{2}\right|} = \frac{x_u - x_\ell}{|x_u + x_\ell|}.$$

See (5.3) on page 132 in the 7^{th} ed.; page 129 in the 6^{th} ed.

How many iterations *n* are required to obtain a desired accuracy?

Suppose you want the <u>absolute error</u> $< \varepsilon$, and that the length of the initial interval $[x_\ell, x_u]$ is Δx^0.

approximation	absolute error
$x_1 = \frac{x_\ell + x_u}{2}$	$\lvert x_t - x_1\rvert \le \frac{\Delta x^0}{2}$
x_2	$\lvert x_t - x_2\rvert \le \frac{\Delta x^0}{4}$
x_3	$\lvert x_t - x_3\rvert \le \frac{\Delta x^0}{8}$
$\vdots$	$\vdots$
x_n	$\lvert x_t - x_n\rvert \le \frac{\Delta x^0}{2^n}$

Therefore,

$$\frac{\Delta x^0}{2^n} \le \varepsilon \quad \text{implies that} \quad 2^n \ge \frac{\Delta x^0}{\varepsilon} \quad \text{and} \quad n \ge \log_2\left(\frac{\Delta x^0}{\varepsilon}\right)$$

(see (5.5) on page 132 of the 7^{th} ed. or page 129 of the 6^{th} ed.) or

$$n \ln 2 \ge \ln(\Delta x^0) - \ln(\varepsilon) \quad \text{and} \quad n \ge \frac{\ln(\Delta x^0) - \ln(\varepsilon)}{\ln 2}.$$

Example

If initially $x_u - x_\ell = \Delta x^0 = 1$ and $\varepsilon = 10^{-5}$, then the above formula gives $n \geq 16.61$. Thus, 17 iterations would guarantee that the absolute error of the computed approximation to a zero x_t of $f(x)$ is $< 10^{-5}$.

An algorithm for the Bisection method

```
function root = Bisect ( x_ℓ, x_u, ε, imax)
i ← 1
f_ℓ ← f(x_ℓ)
while i ≤ imax
    x_r ← (x_ℓ + x_u)/2        [or    x_ℓ + (x_u − x_ℓ)/2]
    f_r ← f(x_r)
    if  f_r = 0 or (x_u − x_ℓ)/|x_u + x_ℓ| < ε then
        root ← x_r
        exit
    end if
    i ← i + 1
    if  f_ℓ × f_r < 0 then
        x_u ← x_r
    else
        x_ℓ ← x_r
        f_ℓ ← f_r
    end if
end while
root = 'failed to converge'
```

Example

Use the Bisection method to solve Problem 5.17 on page 143 in the 7th ed. (Problem 5.17 on page 140 in the 6th ed.). The volume of liquid in a spherical tank is given by

$$V = \frac{\pi h^2 (3R - h)}{3}$$

where h is the depth of water in the tank and R is the radius of the tank. If $R = 3$, to what depth must the tank be filled so that it contains $30 m^3$ of water?

Solution

Compute a root h of the equation $f(h)=0$, where $f(h)=\dfrac{\pi h^2(9-h)}{3}-30$. Since $f(1)\approx -21.6224$ and $f(3)\approx 26.5487$, the Bisection method can be used with $x_\ell=1$ and $x_u=3$. If the above algorithm is run in MATLAB with $\varepsilon=10^{-3}$, then the following results are obtained.

iteration	approximation
1	2.0000
2	2.5000
3	2.2500
4	2.1250
5	2.0625
6	2.0313
7	2.0156
8	2.0234
9	2.0273
10	2.0254

Note: the exact answer is $2.02690\cdots$.

Advantage of the Bisection method relative to other methods: if $f(x)$ is continuous and if appropriate initial values x_ℓ and x_u can be found, then the method is **guaranteed to converge**

Disadvantages

- converges slowly (requires more iterations than other methods)
- it may be difficult to find appropriate initial values
- it cannot be used to compute a zero x_t of **even multiplicity** of a function $f(x)$; that is, if

  $f(x)=(x-x_t)^m g(x)$ where m is a positive even integer and $g(x_t)\neq 0$