1. Using the Cramer's rule obtain the solutions to the following set of equations:

$$2x_1 + x_2 - x_3 = 0$$
$$x_1 + x_3 = 4$$
$$x_1 + x_2 + x_3 = 0$$

$$2x_1 + x_2 - x_3 = 0 \qquad a_1x_1 + b_1x_2 + c_1x_3 = d_1$$
$$x_1 + 0x_2 + x_3 = 4 \qquad a_2x_1 + b_2x_2 + c_2x_3 = d_2$$
$$x_1 + x_2 + x_3 = 0 \qquad a_3x_1 + b_3x_2 + b_3x_3 = d_3$$

$$x_1 = \frac{D_{x_1}}{D} \qquad x_2 = \frac{D_{x_2}}{D} \qquad x_3 = \frac{D_{x_3}}{D}$$

So,

$$D = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$D = 2\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} - 1\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + (-1)\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$
$$= 2(0-1) - (1-1) - (1-0)$$
$$= 2(-1) - (0) - (1)$$
$$= -2 - 1$$
$$= -3$$

$$\underline{\underline{D = -3}}$$

$$D_{x_1} = \begin{bmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ 4 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$D_{x_1} = 0\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} - 1\begin{bmatrix} 4 & 1 \\ 0 & 1 \end{bmatrix} + (-1)\begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$$

$$D_{x_1} = 0(0-1) - (4-0) - (4-0)$$

$$D_{x_1} = 0 - 4 - 4$$

$$\underline{\underline{D_{x_1} = -8}}$$

$$D_{x_2} = \begin{bmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 4 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$D_{x_2} = 2\begin{bmatrix} 4 & 1 \\ 0 & 1 \end{bmatrix} - 0\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + (-1)\begin{bmatrix} 1 & 4 \\ 1 & 0 \end{bmatrix}$$

$$= 2(4-0) - 0(1-1) - (0-4)$$

$$= 2(4) - 0(0) - (-4)$$

$$= 8 + 4$$

$$= 12$$

$$\underline{\underline{D_{x_2} = 12}}$$

$$D_{x_3} = \begin{bmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 4 \\ 1 & 1 & 0 \end{bmatrix}$$

$$D_{x_3} = 2\begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix} - 1\begin{bmatrix} 1 & 4 \\ 1 & 0 \end{bmatrix} + 0\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$= 2(0-4) - 1(0-4) + 0(1-0)$$

$$= 2(-4) - 1(-4) + 0(1)$$

$$= -8 + 4 + 0$$

$$= -4$$

$$\underline{\underline{D_{x_3} = -4}}$$

So,

$D = -3$, $D_{x_1} = -8$, $D_{x_2} = 12$, $D_{x_3} = -4$

$$x_1 = \frac{D_{x_1}}{D} \qquad x_2 = \frac{D_{x_2}}{D} \qquad x_3 = \frac{D_{x_3}}{D}$$

$$x_1 = \frac{-8}{-3} \qquad x_2 = 12/-3 \qquad x_3 = \frac{-4}{-3}$$

$$\underline{\underline{x = 8/3}} \qquad \underline{\underline{x_2 = -4}} \qquad \underline{\underline{x_3 = 4/3}}$$

<u>Sanity check:</u>

$x_1 + x_3 = 4$

$8/3 + 4/3 = 4$

$x_1 + x_2 + x_3 = 0$

$8/3 + (-4) + 4/3 = 0$

$$12/3 = 4$$
$$4 = 4 \checkmark$$

$$12/3 - 4 = 0$$
$$4 - 4 = 0$$
$$0 = 0 \checkmark$$

2. Solve the following using the inverse matrix method:

$$6x - y = 0$$
$$2x - 4y = 1$$

$$\begin{pmatrix} 6 & -1 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{6(-4)-(-1)(2)}\begin{pmatrix} -4 & 1 \\ -2 & 6 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{-24+2}\begin{pmatrix} 0+1 \\ 0+6 \end{pmatrix}$$

$$= -1/22\begin{pmatrix} 1 \\ 6 \end{pmatrix}$$

$$= \begin{pmatrix} -1/22 \\ -3/11 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1/22 \\ -3/11 \end{pmatrix}$$

$$x = -1/22 \qquad y = -3/11$$

Sanity check

$$2x - 4y = 1$$

$$2\left(-\tfrac{1}{22}\right) - 4\left(-\tfrac{3}{11}\right) = 1$$

$$-\tfrac{2}{22} + \tfrac{12}{11} = 1$$

$$-\tfrac{1}{11} + \tfrac{12}{11} = 1$$

$$\tfrac{11}{11} = 1$$

$$\underline{\underline{1 = 1}} \checkmark$$

3.

(a) Solve the following using Gauss elimination:

$$x_1 + x_2 + x_3 = 2$$
$$2x_1 + 3x_2 + 4x_3 = 3$$
$$x_1 - 2x - x_3 = 1$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 3 & 4 & 3 \\ 1 & -2 & -1 & 1 \end{array}\right] \xrightarrow{R_1 - R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 3 & 4 & 3 \\ 0 & 3 & 2 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 3 & 4 & 3 \\ 0 & 3 & 2 & 1 \end{array}\right] \xrightarrow{2R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -1 & -2 & 1 \\ 0 & 3 & 2 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -1 & -2 & 1 \\ 0 & 3 & 2 & 1 \end{array}\right] \xrightarrow{3R_2 + R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -1 & -2 & 1 \\ 0 & 0 & -4 & 4 \end{array}\right]$$

So,

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -1 & -2 & 1 \\ 0 & 0 & -4 & 4 \end{array}\right]$$

$$\begin{aligned} x_1 + x_2 + x_3 &= 2 \\ -x_2 - 2x_3 &= 1 \\ -4x_3 &= 4 \end{aligned}$$

$-4x_3 = 4$

$x_3 = 4/-4$

$\underline{\underline{x_3 = -1}}$

$-x_2 - 2x_3 = 1$

$-x_2 - 2(-1) = 1$

$-x_2 + 2 = 1$

$x_2 = 2 - 1$

$\underline{\underline{x_2 = 1}}$

$x_1 + x_2 + x_3 = 2$

$x_1 + 1 + (-1) = 2$

$x_1 + 1 - 1 = 2$

$\underline{\underline{x_1 = 2}}$

Sanity check

$x_1 + x_2 + x_3 = 2$

$2 + 1 + (-1) = 2$

$2 + 1 - 1 = 2$

$\underline{\underline{2 = 2}}$ ✓

$x_1 - 2x_2 - x_3 = 1$

$2 - 2(1) - (-1) = 1$

$2 - 2 + 1 = 1$

$0 + 1 = 1$

$\underline{\underline{1 = 1}}$ ✓

(b) Find the inverse of the matrix from (a) using the same method and step.

$$x_1 + x_2 + x_3 = 2$$
$$2x_1 + 3x_2 + 4x_3 = 3$$
$$x_1 - 2x - x_3 = 1$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 1 & -2 & -1 \end{bmatrix} = A$$

$$|A| = 1\begin{bmatrix} 3 & 4 \\ -2 & -1 \end{bmatrix} - 1\begin{bmatrix} 2 & 4 \\ 1 & -1 \end{bmatrix} + 1\begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix}$$

$$= 1(-3+8) - (-2-4) + (-4-3)$$

$$= 5 + 6 - 7$$

$$\underline{|A| = 4}$$

$$\begin{bmatrix} \begin{vmatrix} 3 & 4 \\ -2 & -1 \end{vmatrix} & \begin{vmatrix} 2 & 4 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 \\ -2 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 \\ 3 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} \end{bmatrix}$$

$$\begin{bmatrix} +(-3+8) & -(-2-4) & +(-4-3) \\ -(-1+2) & +(-1-1) & -(-2-1) \\ +(4-3) & -(4-2) & +(3-2) \end{bmatrix}$$

$$\begin{bmatrix} 5 & +6 & -7 \\ -1 & -2 & +3 \\ 1 & -2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 5 & -1 & 1 \\ 6 & -2 & -2 \\ -7 & 3 & 1 \end{bmatrix}$$

So,

$$\frac{1}{4}\begin{bmatrix} 5 & -1 & 1 \\ 6 & -2 & -2 \\ -7 & 3 & 1 \end{bmatrix} = A^{-1}$$

Sanity check

$$\boxed{AA^{-1} = I}$$

$$\frac{1}{4}\begin{bmatrix} 5 & -1 & 1 \\ 6 & -2 & -2 \\ -7 & 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 1 & -2 & -1 \end{bmatrix}$$

$$\frac{1}{4}\begin{bmatrix} 5(1)+(-1)(2)+(1)(1) & 5(1)+3(-1)+1(-2) & 5(1)+4(-1)+1(-1) \\ 6(1)+2(-2)+1(-2) & 6(1)+3(-2)+(-2)(-2) & 6(1)+4(-2)+(-2)(-1) \\ 1(-7)+3(2)+1(1) & 1(-7)+3(3)+1(-2) & 1(-7)+3(4)+1(-1) \end{bmatrix}$$

$$\frac{1}{4}\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

4. Write the following series in summation form:

$$\frac{(\ln 1)^2}{1 \times 3} - \frac{(\ln 2)^2}{2 \times 2} + \frac{(\ln 3)^2}{3 \times 5} - \cdots + \frac{(\ln 27)^2}{27 \times 29}$$

$$\sum_{p=1}^{27} \frac{(-1)(\ln p)^2}{p(p+2)}$$