Q1. (a) Given the function $f'(x) = 2x^{3} + 4$ find:

(i) $f(7)$ (ii) $f(-0.5)$

(i) $f(x) = 2x^3 + 4$

$f(7) = 2(7)^3 + 4$

$= 2(343) + 4$

$= 686 + 4$

$= 690$

$\therefore$

$\underline{\underline{f(7) = 690}}$

(ii) $f(x) = 2x^3 + 4$

$f(-1/2) = 2(-1/2)^3 + 4$

$= 2(-1/8) + 4$

$= -1/4 + 4$

$= -1/4 + 16/4$

$= 15/4$

$\therefore$

$\underline{\underline{f(-1/2) = 15/4}}$

(b) Calculate f(4x + 5) when f(x) = x^2. write down the corresponding expression for f(4x + 2) - f(x).`

$f(x) = x^2$

$f(4x+5) = (4x+5)^2$

$= (4x+5)(4x+5)$

$= 16x^2 + 20x + 20x + 25$

$= 16x^2 + 40x + 25$

$\therefore$ $\underline{\underline{f(4x+5) = 16x^2 + 40x + 25}}$

$f(4x+2) - f(x)$

This might have been a mistake in the question!! I'm using $f(4x+5)$ from previous section.

$16x^2 + 40x + 25 - x^2$

$15x^2 + 40x + 25$

$\underline{\underline{3x^2 + 8x + 5}}$

(c) if f(x) = 4x + 2 and g(x) = x + 3 find g(f(x)).

$f(x) = 4x + 2$

$g(x) = x + 3$

So

$g(f(x)) = 4x + 2 + 3$

$= \underline{\underline{4x + 5}}$

(d) consider the function given by g(x) = x^2+5, -2 <= x <= 3.

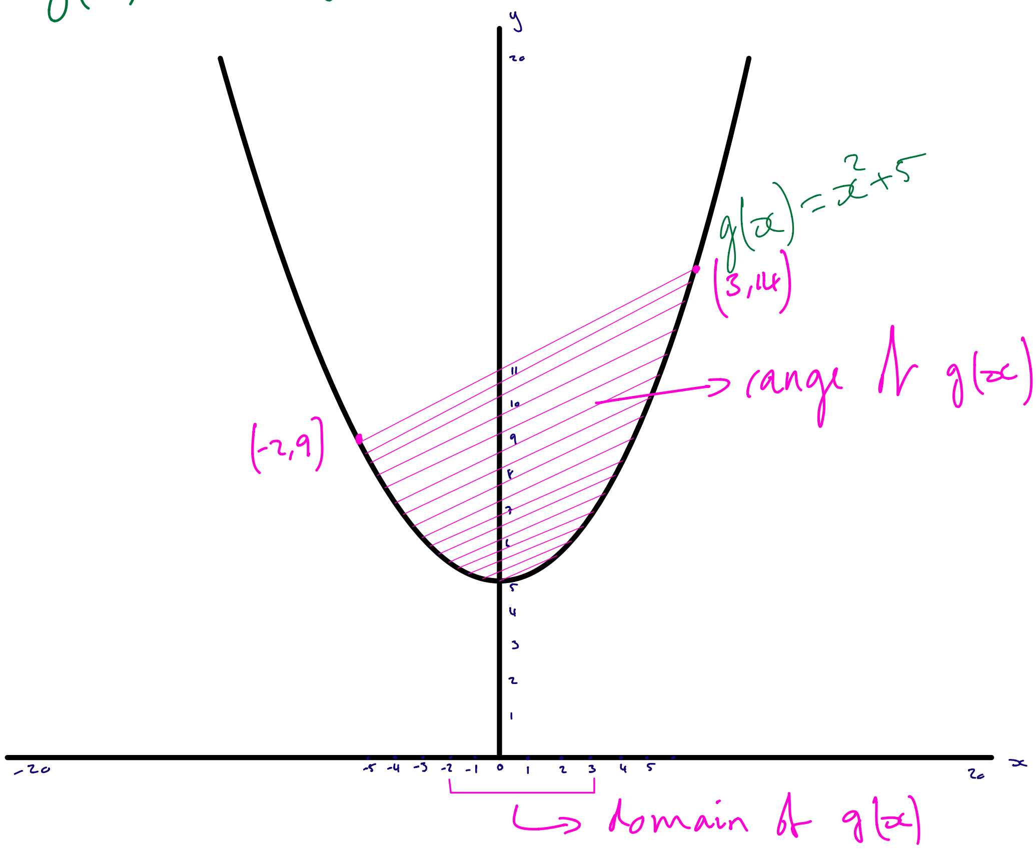
(i) State the domain of the function.

The domain of the function is all the numbers between -2 and 3, inclusive of both.

$\therefore$ domain $= -2, -1, 0, 1, 2, 3.$

(ii) Plot the graph of the function.

$g(x) = x^2 + 5$



(iii) Deduce the range of the function from the graph. state the input and output table.

$g(x) = x^2 + 5$

domain(x)	-2	-1	0	1	2	3
range(y)	9	6	5	6	9	14

$g(-2) = (-2)^2 + 5$
$\underline{\underline{=9}}$

$g(3) = 3^2 + 5$
$= 9 + 5$
$\underline{\underline{= 14}}$

Q2. (a) Find the inverse of each of the following functions:

(i) $f(x) = 7 + 5x$

$y = 5x + 7$

$y - 7 = 5x$

$y - 7/5 = x$

$\therefore$

$f^{-1}(x) = (x-7)/5$

(ii) $f(x) = 3x$

$y = 3x$

$y - 3 = x$

$\therefore$

$f^{-1}(x) = x - 3$

Q3 (a). For the matrices A = $\begin{bmatrix} 1 & 5 & -1 \\ 3 & 0 & 4 \\ 1 & 1 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 & 3 \\ 1 & -2 & 1 \\ 2 & 3 & -2 \end{bmatrix}$ find, where possible:

(i) A^T – rows becomes the columns given matrix A.

$$A = \begin{bmatrix} 1 & 5 & -1 \\ 3 & 0 & 4 \\ 1 & 1 & -2 \end{bmatrix} \rightarrow A^T = \begin{bmatrix} 1 & 3 & 1 \\ 5 & 0 & 1 \\ -1 & 4 & -2 \end{bmatrix}$$

(ii) AB

$$A = \begin{bmatrix} 1 & -5 & 1 \\ 3 & 0 & 4 \\ 1 & 1 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & -1 & 3 \\ 1 & -2 & 1 \\ 2 & 3 & -2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & -5 & 1 \\ 3 & 0 & 4 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 3 \\ 1 & -2 & 1 \\ 2 & 3 & -2 \end{bmatrix} =$$

$$\begin{bmatrix} (1)(1)+1(-5)+1(2) & 1(-1)+(-5)(-2)+1(3) & 1(3)+1(-5)+1(-2) \\ 3(1)+0(1)+4(2) & 3(-1)+0(-2)+4(3) & 3(3)+0(1)+4(-2) \\ 1(1)+1(1)+2(-2) & 1(-1)+1(-2)+3(-2) & 1(3)+1(1)+(-2)(-2) \end{bmatrix}$$

$$= \begin{bmatrix} 1-5+2 & -1+10+3 & 3-5-2 \\ 3+8 & -3+12 & 9-8 \\ 1+1-4 & -1-2-6 & 3+1+4 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 12 & -4 \\ 11 & 9 & 1 \\ -2 & -9 & 8 \end{bmatrix}$$

$$\therefore \quad AB = \begin{bmatrix} -2 & 12 & -4 \\ 11 & 9 & 1 \\ -2 & -9 & 8 \end{bmatrix}$$

(b) Use expansion along the first row to find

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}.$$

$A_{11} = 2, \quad A_{12} = 1, \quad A_{13} = 1$

$$A_{11} = +\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = 4 - 3 = \underline{\underline{1}}$$

$$A_{12} = -\begin{vmatrix} 1 & 3 \\ 3 & 2 \end{vmatrix} = -(2-9) = -(-7) = \underline{\underline{7}}$$

$A_{13} = + \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = 1 - 6 = \underline{\underline{-5}}$

$\Delta = 2(1) + 1(7) + 1(-5) = 2 + 7 - 5 = 2 + 2 = \underline{\underline{4}}$

Q4 (a) Find the inverse of the following matrices using determinants if possible:

(i) $A = \begin{bmatrix} -1 & -1 \\ -2 & -2 \end{bmatrix}$

$|A| = (-1)(-2) - (-2)(-1)$

$= 2 - 2$

$\underline{\underline{= 0}}$

$\therefore$ A have no inverse as it's determinant is zero.

(ii) $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$|B| = 1(1) - 0(1)$

$= 1 - 0$

$\underline{\underline{= 1}}$

$\therefore$ B have an inverse

So:

$B^{-1} = 1/1 \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$\underline{\underline{= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}}$

(b) (i) Using Cramar's rule obtain the solution to the following set of equations:

$$2x + y = 8$$
$$3x - 4y = 5$$

$$\begin{bmatrix} 2 & 1 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 5 \end{bmatrix}$$

$$x = \frac{D_x}{D}, \quad y = \frac{D_y}{D}$$

$$D = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} = x_1 y_2 - y_1 x_2$$

$$D = \begin{bmatrix} 2 & 1 \\ 3 & -4 \end{bmatrix} = 2(-4) - 1(3) = -8 - 3 = \underline{\underline{-11}}$$

$$D_x = \begin{bmatrix} c_1 & y_1 \\ c_2 & y_2 \end{bmatrix} = c_1 y_2 - y_1 c_2$$

$$D_x = \begin{bmatrix} 8 & 1 \\ 5 & -4 \end{bmatrix} = 8(-4) - 1(5) = -32 - 5 = \underline{\underline{-37}}$$

$$D_y = \begin{bmatrix} x_1 & c_1 \\ x_2 & c_2 \end{bmatrix} = x_1 c_2 - c_1 x_2$$

$$D_y = \begin{bmatrix} 2 & 8 \\ 3 & 5 \end{bmatrix} = 2(5) - 3(8) = 10 - 24 = \underline{\underline{-14}}$$

$D = -11, \quad D_x = -37, \quad D_y = -14$

$\therefore$

$x = D_x/D$ $\qquad$ $y = D_y/D$

$x = -37/-11$ $\qquad$ $y = -14/-11$

$\underline{\underline{x = 37/11}}$ $\qquad$ $\underline{\underline{y = 14/11}}$

Sanity check

$2x + y = 8$

$y = 8 - 2x$ (1)

$3x - 4y = 5$ (2)

sub (1) into (2)

$3x - 4(8 - 2x) = 5$

$3x - 32 + 8x = 5$

$11x = 5 + 32$

$11x = 37$

$\underline{\underline{x = 37/11}}$

sub x into (1)

$y = 8 - 2x$

$= 8 - 2(37/11)$

$= 8 - 74/11$

$$= 88/11 - 74/11$$

$$= 88 - 74$$

$$y = 14/11$$

$$\therefore \quad x = 37/11 \text{ and } y = 14/11 \quad \checkmark\checkmark$$

Q5. (a) Solve the following system of equations using inverse matrix method (AX = B and X = A^{-1} B):

$$2x + 3y = 3$$

$$5x + 4y = 10$$

$$\begin{bmatrix} 2 & 3 \\ 5 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 10 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2(4) - 3(5)} \begin{bmatrix} 4 & -3 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 10 \end{bmatrix}$$

$$= \frac{1}{8 - 15} \begin{bmatrix} 12 - 30 \\ -15 + 20 \end{bmatrix}$$

$$= -1/7 \begin{bmatrix} -18 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 18/7 \\ -5/7 \end{bmatrix}$$

$$\therefore \quad x = 18/7 \qquad y = -5/7$$

Sanity check

$$2x + 3y = 3$$
$$3y = 3 - 2x$$
$$y = 1 - 2/3\,x \quad (1)$$
$$5x + 4y = 10 \quad (2)$$

sub (1) into (2)

$$5x + 4(1 - 2/3\,x) = 10$$
$$5x + 4 - 8/3\,x = 10$$
$$5x - 8/3\,x = 10 - 4$$
$$15/3\,x - 8/3\,x = 6$$
$$7/3\,x = 6$$
$$x = 6(3/7)$$
$$x = 18/7$$

sub x into (2)

$$5(18/7) + 4y = 10$$
$$4y = 10 - 90/7$$
$$4y = 70/7 - 90/7$$
$$4y = -20/7$$
$$y = -20/7\,(1/4)$$

$$y = -20/28$$

$$y = -5/7$$

$\therefore x = 18/7 \quad y = -5/7$ //

(b) Diffie-Hellman key exchange is based on the difficulty of solving the Discrete Logarithm Problem. For instance, Alice and Bob agree on a prime number p = 7 and a number (base) g = 4. This is they public aspect of the system. Using this algorithm, find the $k_{private}$.

Client (Alice)

mod = p = 7

base = g = 4

Server (Bob)

Secret # = a = 4

Shared value = A

$A = g^a \mod p$

$A = 4^4 \mod 7$

$= 256 \mod 7$

$= 4$

Secret # = b = 3

Shared value = B

$B = g^b \mod p$

$B = 4^3 \mod 7$

$= 64 \mod 7$

$= 1$

A = 4

B = 1

After working out their secret key, then client and server exchange / share their secret key with each other.

$$\text{shared-key} = B^a \bmod p$$
$$= 1^4 \bmod 7$$
$$= 1$$

$$\text{shared-key} = A^b \bmod p$$
$$= 4^3 \bmod 7$$
$$= 64 \bmod 7$$
$$= 1$$

$$\therefore \text{ Their } K_{private} = 1$$

Q6. (a) Solve the following using Gauss elimination method:

$$2x_1 + x_2 - x_3 = 0$$
$$x_1 + x_3 = 4$$
$$x_1 + 2x_2 + x_3 = 0$$

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 0 \\ 1 & 0 & 1 & 4 \\ 1 & 2 & 1 & 0 \end{array}\right]$$

$R_2 - R_3$

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 0 \\ 1 & 0 & 1 & 4 \\ 0 & -2 & 0 & -4 \end{array}\right]$$

$2(R_2)$

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 0 \\ 2 & 0 & 2 & 8 \\ 0 & -2 & 0 & -4 \end{array}\right]$$

$R_1 - R_2$

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 0 \\ 0 & 1 & -3 & -8 \\ 0 & -2 & 0 & -4 \end{array}\right]$$

$2(R_2)$

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 0 \\ 0 & 2 & -6 & -16 \\ 0 & -2 & 0 & -4 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 0 \\ 0 & 2 & -6 & -16 \\ 0 & 0 & -6 & -20 \end{array}\right] \quad R_2 + R_3$$

$\therefore$

$$2x_1 + x_2 - x_3 = 0 \quad (3)$$

$$2x_2 - 6x_3 = -16 \quad (2)$$

$$-6x_3 = -20 \quad (1)$$

(1)

$$-6x_3 = -20$$

$$x_3 = -20/-6$$

$$x_3 = 10/3$$

(2)

$$2x_2 - 6x_3 = -16$$

$$2x_2 - 6(10/3) = -16$$

$$2x_2 - 20 = -16$$

$$2x_2 = 4$$

$$x_2 = 2$$

(3)

$$2x_1 + x_2 - x_3 = 0$$

$$2x_1 + 2 - 10/3 = 0$$

$$2x_1 + 6/3 - 10/3 = 0$$

$$2x_1 - 4/3 = 0$$

$$2x_1 = 4/3$$

$$x_1 = 1/2(4/3)$$

$$x_1 = 2/3$$

Sanity check

$$2x_1 + x_2 - x_3 = 0$$

$$2(2/3) + 2 - 10/3 = 0$$

$$4/3 + 6/3 - 10/3 = 0$$

$$10/3 - 10/3 = 0$$

$$0 = 0 \quad \checkmark$$

$$x_1 + x_3 = 4$$

$$2/3 + 10/3 = 4$$

$$12/3 = 4$$

$$4 = 4 \quad \checkmark$$

(b) Find the inverse of the matrix from (a) using the same method and steps.

$$\begin{aligned} 2x_1 + x_2 - x_3 &= 0 \\ x_1 + x_3 &= 4 \\ x_1 + 2x_2 + x_3 &= 0 \end{aligned}$$

$$\left[\begin{array}{ccc|ccc} 2 & 1 & -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\Big\downarrow \; R_3 = R_2 - R_3$$

$$\left[\begin{array}{ccc|ccc} 2 & 1 & -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -2 & 0 & 0 & 1 & -1 \end{array}\right]$$

$$\Big\downarrow \; R_2 = R_1 - 2R_2$$

$$\left[\begin{array}{ccc|ccc} 2 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -3 & 1 & -2 & 0 \\ 0 & -2 & 0 & 0 & 1 & -1 \end{array}\right]$$

$$\Big\downarrow \; R_3 = 2R_2 + R_3$$

$$\left[\begin{array}{ccc|ccc} 2 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -3 & 1 & -2 & 0 \\ 0 & 0 & -6 & 2 & -3 & -1 \end{array}\right]$$

$$\Big\downarrow \; R_1 = 3R_1 - R_2$$

$$\left[\begin{array}{ccc|ccc} 6 & 2 & 0 & 2 & 2 & 0 \\ 0 & 1 & -3 & 1 & -2 & 0 \\ 0 & 0 & -6 & 2 & -3 & -1 \end{array}\right]$$

$$\Big\downarrow\ R_2 = 2R_2 - R_3$$

$$\left[\begin{array}{ccc|ccc} 6 & 2 & 0 & 2 & 2 & 0 \\ 0 & 2 & 0 & 0 & -1 & 1 \\ 0 & 0 & -6 & 2 & -3 & -1 \end{array}\right]$$

$$\Big\downarrow\ R_1 = R_1 - R_2$$

$$\left[\begin{array}{ccc|ccc} 6 & 0 & 0 & 2 & 3 & -1 \\ 0 & 2 & 0 & 0 & -1 & 1 \\ 0 & 0 & -6 & 2 & -3 & -1 \end{array}\right]$$

$$\Big\downarrow\ \begin{array}{l} R_1 = \tfrac{1}{6} R_1 \\ R_2 = \tfrac{1}{2} R_2 \\ R_3 = -\tfrac{1}{6} R_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/3 & 1/2 & -1/6 \\ 0 & 1 & 0 & 0 & -1/2 & 1/2 \\ 0 & 0 & 1 & -1/3 & 1/2 & 1/6 \end{array}\right]$$

$$\therefore\quad A^{-1} = \begin{bmatrix} 1/3 & 1/2 & -1/6 \\ 0 & -1/2 & 1/2 \\ -1/3 & 1/2 & 1/6 \end{bmatrix}$$

Sanity check $\qquad AA^{-1} = I$

$$\begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1/3 & 1/2 & -1/6 \\ 0 & -1/2 & 1/2 \\ -1/3 & 1/2 & 1/6 \end{bmatrix} = I$$

$$\begin{bmatrix} 2/3 + 1/3 & 2/2 - 1/2 - 1/2 & -2/6 + 1/2 - 1/6 \\ 1/3 - 1/3 & 1/2 + 1/2 & -1/6 + 1/6 \\ 1/3 - 1/3 & 1/2 - 2/2 + 1/2 & -1/6 + 2/2 + 1/6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I \quad \checkmark$$

Q7. (a) (i) Write out all the terms of the series $\displaystyle\sum_{p=1}^{6} \frac{3^{p-1}}{p!\,(17-p)}$

$p=1 \qquad \dfrac{3^{1-1}}{1!\,(17-1)} = \dfrac{3^0}{1(16)} = \underline{\underline{1/16}}$

$p=2 \qquad \dfrac{3^{2-1}}{2!\,(17-2)} = \dfrac{3^1}{2(15)} = \dfrac{3}{30} = \underline{\underline{1/10}}$

$p=3 \qquad \dfrac{3^{3-1}}{3!\,(17-3)} = \dfrac{3^2}{6(14)} = 9/84 = \underline{\underline{3/28}}$

$p=4 \quad \frac{3^{4-1}}{4!(17-4)} = \frac{3^3}{24(13)} = 27/312 = \underline{\underline{9/104}}$

$p=5 \quad \frac{3^{5-1}}{5!(17-5)} = \frac{3^4}{120(12)} = 81/1440 = \underline{\underline{9/160}}$

$p=6 \quad \frac{3^{6-1}}{6!(17-6)} = \frac{3^5}{720(11)} = 243/7920 = \underline{\underline{27/880}}$

$\therefore$ terms = 1/16, 1/10, 3/28, 9/104, 9/160, 27/880.

(ii) Write the simple formula for the nth Fibonacci number for n >= 2. Write the first 10 element of this sequence (including F10).

$$F_n = F_{n-1} + F_{n-2}$$

The next term in the sequence is the privious two terms added.

$F_0 = 0$

$F_1 = 1$

$n=2$

$F_2 = F_1 + F_0 = 1 + 0 = \underline{\underline{1}}$

$n = 3$

$F_3 = F_2 + F_1 = 1 + 1 = \underline{\underline{2}}$

$n=4$ $\quad F_4 = F_3 + F_2 = 2 + 1 = \underline{\underline{3}}$

$n=5$ $\quad F_5 = F_4 + F_3 = 3 + 2 = \underline{\underline{5}}$

$n=6$ $\quad F_6 = F_5 + F_4 = 5 + 3 = \underline{\underline{8}}$

$n=7$

$F_7 = F_6 + F_5 = 8 + 5 = \underline{\underline{13}}$

$n = 8$

$F_8 = F_7 + F_6 = 13 + 8 = \underline{\underline{21}}$

$n=9$

$F_9 = F_8 + F_7 = 21 + 13 = \underline{\underline{34}}$

$n=10$

$F_{10} = F_9 + F_8 = 34 + 21 = \underline{\underline{55}}$

$\therefore$ 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55.

is the 1st 10 elements, including F_{10}.

(b) Is the following harmonic sequence convergent? explain why.

$$a_n = \frac{n+2}{n(n-1)}$$

$n=1$ $\quad a_1 = \frac{1+2}{1(1-1)} = \frac{3}{0}$ Undefined.

$n=2$ $\quad a_2 = \frac{4}{2(2-1)} = \frac{4}{2} = 2$

$n=3$ $\quad a_3 = \frac{3+2}{3(3-1)} = \frac{5}{6} = 5/6$

$n=4$ $\quad a_4 = \frac{4+2}{4(4-1)} = \frac{6}{12} = 1/2$

$n=5$ $\quad a_5 = \frac{5+2}{5(5-1)} = 7/20 = 7/20$

$n=6$ $\quad a_6 = \frac{6+2}{6(6-1)} = 8/30 = 4/15$

$2, 5/6, 1/2, 7/20, 4/15$

$2, 0.8\dot{3}, 0.5, 0.35, 0.2\dot{6}$ ---- approaching zero.

Therefore, the sequence above is converging because its tending/approaching zero.