Combinatorics for Beginners: 1

Chris Dugdale

October 25, 2014

Combinatorics is a subject that shows up in most mathematical areas in one way or another. In this article, we introduce the reader to the combinatorial proof: a method of dealing solving combinatorics problems in a conceptually clear, rather than simply algebraic, means. In this first article, we remind the reader of the standard formula for $\binom{n}{r}$ and provide its proof. We then discuss some examples of 'double counting,' which is a proof by counting a quantity in two different ways. Often, an identity can be proved through mathematical induction or simple algebraic manipulation, however those techniques do not provide as much insight as a proof by double counting, as I hope to make clear.

1 A Reminder

Suppose that we have a collection of n objects that we want to select r objects from. A natural question to ask is: 'how many possible choices do we have.' Since we are not interested in the *order* of selection, we need to make sure that we don't over count. Let $\binom{n}{r}$ denote the number of ways to choose r objects from a collection of n without regard to order. If were interested in order, we would have n possibilities for the first choice, $n-1$ possibilities for the second choice, all the way to $n-(r-1) = n-r+1$ choices for the rth choice. In other words, with order being considered, there are $n \dots (n-r+1)$ possible choices. Note that we have that $n \dots (n-r+1) = n \dots (n-r+1)\frac{(n-r)!}{(n-r)!} = \frac{n!}{(n-r)!}$. But for a given collection of r objects, there are $r!$ ways of arranging them, that is to say $r!$ means of selecting those objects with order being considered. Thus, if we are not interested in the order of selection, then $\frac{n!}{(n-r)!}$ counts each selection $r!$ times, so that we must divide to obtain the number of possible choices where order is not relevent. This gives us $\binom{n}{r} = \frac{n!}{r!(n-r)!}$.

One observation that we might make is that $\binom{n}{r} = \binom{n}{n-r}$, but how are we to prove this? One straight forward way to do this is to use the formula above: $\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n!}{(n-r)!(n-(n-r))!} = \frac{n!}{(n-r)!r!} = \binom{n}{n-r}$. While this is a perfect valid argument, it does not tell us *why* we should believe the result to be true. Another means is as follows. First we count the number of ways to choose r objects from n without order mattering. This gives $\binom{n}{r}$. Another way to do the same thing is to select the items that we are *not* going to be putting into our collection. Since we are wanting to have a collection of r objects, we don't select $n-r$ objects. This gives $\binom{n}{n-r}$. Since we are counting the same quantity, we must have that $\binom{n}{r} = \binom{n}{n-r}$. Now that you have seen that argument, you *understand* the equality, rather than just know it to be true. From a practical standpoint, these proofs are far more memorable, so that you are less likely to forget the result.

2 More Interesting Examples

2.1 The Binomial Theorem

While not strictly a double counting proof, the following proof of the binomial theorem is much nicer than standard induction arguments.

Theorem 1. *Let $n \in \mathbb{N}$. Then, $(x+y)^n = \binom{n}{0}y^n + \binom{n}{1}xy^{n-1} + \ldots + \binom{n}{n}x^n = \sum_{i=0}^{n}\binom{n}{i}x^i y^{n-i}$.*

Proof. First note that after expanding and collecting all like terms, each term is of the form Cx^iy^{n-i}, where C is some positive integer. Moreover, note that C is the total number of times that x^iy^{n-i} appears in the expansion, before it is simplified. Now, if we are to choose an x^iy^{n-i} term, how many ways are there to do so? While, we first choose which factors we select i xs from, and then the factors where y is chosen are selected automatically. Since there are n factors, there are $\binom{n}{i}$ ways of doing this, giving $C = \binom{n}{i}$. Thus, $(x+y)^n = \sum_{i=0}^{n}\binom{n}{i}x^iy^{n-i}$ as required. □

2.2 An Identity

Theorem 2. $\sum_{i=0}^{n}\binom{n}{i} = 2^n$

Proof. A quick way to prove this is to use the binomial theorem as above: $(1+1)^n = 2^n = \sum_{i=0}^{n}\binom{n}{i}$. That's a slick way to do it, but, there is a way that is still better. Suppose that we are interested in counting the total number of ways to select a subset of n objects. One way to do this is to first count the number of ways to choose 0 items, the number of subsets of size 1, the number of subsets of size 2, and so on, then add them up. This method gives $\sum_{i=0}^{n}\binom{n}{i}$. The other method of counting is that for each element of our set, there are two possibilities: it is either in our subset or it isn't. This means that there are 2^n possible subsets. Hence, $\sum_{i=0}^{n}\binom{n}{i} = 2^n$ as required. □

2.3 Pascal's Identity

The following identity is one commonly exploited in inductive proofs.

Theorem 3. $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.

Proof. We're interested in selecting k objects from n. This gives $\binom{n}{k}$ possibilities. Now select one particular item. This item will either be in our selection or it won't be. If it is not, we pick all k items from the $n-1$ items remaining. This gives $\binom{n-1}{k}$. If it is in our selection, we have to select $k-1$ items from the $n-1$ objects remaining. This gives $\binom{n-1}{k-1}$. Thus, we have a total of $\binom{n-1}{k} + \binom{n-1}{k-1}$ possibilities. Thus, $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$. □

Note that the number of ways to select r objects from n with $r > n$ is 0. Also note that the number of ways to select $r \geq 0$ items from 0 items is also 0. Why does this matter?

2.4 Another Identity

This identity is called the Vandermonde identity.

Theorem 4. $\binom{n+m}{r} = \sum_{i=0}^{n} \binom{n}{i}\binom{m}{r-i}$.

Proof. The first way to count is to say that we have $n+m$ items and that we wish to choose r of them. The second way is to divide the $n+m$ items into two groups: one of size m and the other of size n. Now, if i of our items comes from the set of size n $r-i$ must come from the set of size m. Thus, we have that the same quantity is given by $\sum_{i=0}^{n} \binom{n}{i}\binom{m}{r-i}$ as required. □

3 An Example from Number Theory

In this section, we give a formula for the number of divisors of a natural number.

For $n \in \mathbb{N}$, let $\sigma(n)$ denote the number of (positive) divisors of n. Recall that every natural number has a unique factorization into a product of primes: $n = p_1^{e_1} \cdots p_s^{e_s}$ where the p_i are prime and e_i is the number of times that p_i divides n. If a number $m \in \mathbb{N}$ divides n, then m can only be divisible by 1 and possibly other numbers divisible by the primes p_i. Of course, m can not be divisible by p_i more than e_i times. Thus, we can find a formula for $\sigma(n)$ by selecting exponents for the primes p_i in our divisor. If p_i is to divide m, there are e_i possibilities. If p_i doesn't divide n, then the exponent of p_i is 0. This gives $e_i + 1$ possible choices for p_i. Thus the total number of divisors of n is $(e_1+1)(e_2+1)\cdots(e_s+1)$. For the case when $n = 1$, there is only one possibility.

4 Problems

In this section, we include some problems for the reader to work on.

1. Prove combinatorially that $\binom{2n}{2} = 2\binom{n}{2} + n^2$.

2. Prove that $\binom{2n}{n} = \sum_{i=0}^{n} \binom{n}{i}^2$.

3. Prove that $\sum_{i=k}^{n} \binom{i}{k} = \binom{n+1}{k+1}$.
4. Prove that $\sum_{k=j}^{n} \binom{n}{k}\binom{k}{j} = 2^{n-j}\binom{n}{j}$.
5. Prove that $\sum_{k=0}^{n} k\binom{n}{k} = 2^{n-1}n$.
6. Find a simple formula for the number of solutions in nonnegative integers to $x_1 + \ldots + x_n = n$ where n is an integer greater than zero.
7. The following is a bit harder than the others. Count the number of permutations, $(a_1, \ldots, a_12)$ of $(1, 2, 3, \ldots, 12)$ such that $a_1 > a_2 > a_3 > a_4 > a_5 > a_6 < a_7 < a_8 < a_9 < a_10 < a_11 < a_12$.

5 An Open Problem in Number Theory

A proof of the following conjecture from number theory may use some of the ideas featured in this article.

Erdos' Squarefree Conjecture. *Prove that $\binom{2n}{n}$ is never square free for $n \geq 4$.*