Logarithms

TAKE QUIZZES

A **logarithm** is the inverse of the exponential function. Specifically, the logarithm returns the value of the exponent required to raise the *base* to the specified value.

For example, the base-2 logarithm of 64 is 6, because $2^6 = 64$. In general, we have the following:

> DEFINITION
>
> z is the base-x logarithm of y if and only if $x^z = y$. In typical notation:
>
> $$\log_x y = z \implies x^z = y.$$

Contents

Properties of Logarithms - Basic

First, we must know the basic structure of a logarithm (abbreviated log for convenience). $\log_a b = c$ can be rewritten as $a^c = b$. a is called the base, c is the exponent, and b is the result thingy. Also, *log* without a base is shorthand for the common log of base 10. Now that we know this, we can manipulate logs.

In Math	In English	Example
$\log_a b + \log_a c = \log_a bc$	When you add logs with the same base, you can merge into one log and multiply their result thingies.	$\log_2 5 + \log_2 6 = \log_2 30$
$\log_a b - \log_a c = \log_a \left(\frac{b}{c}\right)$	The opposite of above.	$\log_2 18 - \log_2 6 = \log_2 3$
$\log_a (b^c) = c \cdot \log_a b$	When your result thingy has an exponent, you can move it to the front of the log.	$\log_3 (5^{-4}) = -4 \cdot \log_3 5$

$\log_a b = \dfrac{\log_c b}{\log_c a}$	You can rearrange any log by making a fraction, with log of the result thingy in the numerator and log of the base in the denominator.	$\log_2 \pi = \dfrac{\log \pi}{\log 2}$
$log_a b = \dfrac{1}{\log_b a}$	If you want to switch the base of the log a with the result thing b, then you take the reciprocal	$\log_4 e = \dfrac{\log_4 4}{\log_4 e} = \dfrac{1}{\log_e 4}$

Other properties can be derived from these basic ones, especially when noting that these properties are inversable.

EXAMPLE

Simplify as much as possible:

$$\log_2 \left(\frac{32}{9}\right)^2.$$

Try to follow the steps and identify what properties were used:

$$\begin{aligned} 2 \cdot \log_2 \left(\frac{32}{9}\right) &= 2 \cdot (\log_2 32 - \log_2 9) \\ &= 2 \cdot (\log_2 2^5 - \log_2 3^2) \\ &= 2 \cdot (5 \cdot \log_2 2 - 2 \cdot \log_2 3) \\ &= 2 \cdot (5(1) - 2 \cdot \log_2 3) \\ &= 10 - 4 \cdot \log_2 3. \end{aligned}$$

Note that $\log_2 3$ can't be simplified further. Line 1 used the second property, Line 2 put things into exponential form, Line 3 used the third property, and Lines 5 and 6 did basic simplification.

EXAMPLE

Simplify $2\log_4 \sqrt{5} + \frac{1}{2}\log_2 625 - \log_2 \frac{1}{5}$.

Again, try to follow the steps of the solution:

$$\begin{aligned} 2\log_{(2^2)}\left(5^{\frac{1}{2}}\right) + \frac{1}{2}\log_2\left(5^4\right) - \log_2\left(5^{-1}\right) &= 2\,\frac{\log 5^{\frac{1}{2}}}{\log 2^2} + \frac{1}{2}(4)(\log_2 5) + \log_2 5 \\ &= 2\,\frac{\frac{1}{2}\log 5}{2\log 2} + 2\log_2 5 + \log_2 5 \\ &= \frac{1}{2}\,\frac{\log 5}{\log 2} + 3\log_2 5 \\ &= \frac{1}{2}\log_2 5 + 3\log_2 5 \\ &= \frac{7}{2}\log_2 5. \end{aligned}$$

The first line shows that it is (usually) best to convert numbers so that they are integers to a power. Note that lines 4 reverses the process of the fourth property.

Take quiz: Properties of Logarithms - Basic

Worked examples using properties

1. $\log_a a = 1$

EXAMPLE

Find the value of $\log_4 4$.

Using the property $\log_a a = 1$, we get $\log_4 4 = 1$.

2. $\log_a (b^c) = c\log_a b$

EXAMPLE

Find the value of $\log_2 16$.

We have

$$\begin{aligned} \log_2 16 &= \log_2 2^4 && (16 = 2^4) \\ &= 4\log_2 2 && (\log a^b = b\log a) \\ &= 4. && (\text{by property 1}) \end{aligned}$$

3. $\log_a (b \times c) = \log_a b + \log_a c$

EXAMPLE

Find the value of $\log 90$ **assuming** $\log 3 = 0.47$.

We have

$$\begin{aligned} \log 90 &= \log(9 \times 10) && (90 = 9 \times 10) \\ &= \log 9 + \log 10 && (\log_a (b \times c) = \log_a b + \log_a c) \\ &= 2\log 3 + 1 && (\text{property 2 and 1}) \\ &= 2 \times 0.47 + 1 \\ &= 0.94 + 1 \\ &= 1.94. \end{aligned}$$

4. $\log_a \frac{b}{c} = \log_a b - \log_a c$

EXAMPLE

Evaluate $\log 0.27$ **assuming** $\log 3 = 0.47$.

We have

$$\begin{aligned} \log 0.27 &= \log \frac{27}{100} \\ &= \log 27 - \log 100 \\ &= 3\log 3 - 2 \\ &= 1.41 - 2 \\ &= -0.59. \end{aligned}$$

5. $\log_a b = \dfrac{\log_c b}{\log_c a}$

EXAMPLE

Find the value of $\log_{32} 2$.

We have

$$\begin{aligned} \log_{32} 2 &= \frac{\log_2 2}{\log_2 32} \\ &= \frac{1}{5\log_2 2} \\ &= \frac{1}{5} \\ &= 0.2. \end{aligned}$$

Properties of Logarithms - Intermediate

EXAMPLE

What is the value of $\log_3 15 + \log_3 81 - \log_3 5$?

Using properties of logarithms, we can rewrite in the following way:

$$\begin{aligned} \log_3 15 + \log_3 81 - \log_3 5 &= \log_3 15 - \log_3 5 + \log_3 3^4 \\ &= \log_3 \frac{15}{5} + \log_3 3^4 \\ &= \log_3 3 + 4\log_3 3 \\ &= 5 \end{aligned}$$

Take quiz: Properties of Logarithms - Intermediate

Problem Solving - Basic

EXAMPLE

What is the solution(s) of the quadratic equation

$$\log 2x + \log(x-1) = \log(x^2+3)?$$

We have

$$\begin{aligned}\log 2x + \log(x-1) &= \log(x^2+3)\\ \log 2x(x-1) &= \log(x^2+3)\\ \Rightarrow 2x(x-1) &= x^2+3\\ x^2-2x-3 &= 0\\ (x+1)(x-3) &= 0\\ x &= -1, 3.\end{aligned}$$

Since the logarithm functions $\log(x-1)$ and $\log 2x$ are defined over positive numbers, the values of $x-1$ and $\log 2x$ are positive. Thus, -1 is can not be the value of x, implying that the value of x satisfying $\log 2x + \log(x-1) = \log(x^2+3)$ is $x = 3$. $\square$

EXAMPLE

What is the solution(s) of the quadratic equation

$$2(\log x)^2 = 7\log x - 3?$$

We have

$$\begin{aligned}2(\log x)^2 &= 7\log x - 3\\ 2(\log x)^2 - 7\log x + 3 &= 0\\ (\log x - 3)(2\log x - 1) &= 0\\ \Rightarrow \log x &= 3, \frac{1}{2}\\ x &= 1000, \sqrt{10}. \ \square\end{aligned}$$

Take quiz: Logarithmic Functions Problem Solving - Basic

Problem Solving - Intermediate

EXAMPLE

If the solutions of the quadratic equation $x^{\log_3 x - 2} = 27$ are a and b, what is $\log_a b + \log_b a$?

Taking logs with base 3 on both sides, we have

$$\begin{aligned}x^{\log_3 x - 2} &= 27\\ \Rightarrow (\log_3 x - 2)\log_3 x &= \log_3 27\\ (\log_3 x)^2 - 2\log_3 x - 3 &= 0\\ (\log_3 x + 1)(\log_3 x - 3) &= 0\\ \log_3 x &= -1, 3.\end{aligned}$$

Since $\log_a b + \log_b a$ can be expressed as $\frac{\log_3 b}{\log_3 a} + \frac{\log_3 a}{\log_3 b}$ using log base 3, thus

$$\begin{aligned}\log_a b + \log_b a &= \frac{\log_3 b}{\log_3 a} + \frac{\log_3 a}{\log_3 b} \\ &= \frac{-1}{3} + \frac{3}{-1} \\ &= -\frac{10}{3}. \ \square\end{aligned}$$

EXAMPLE

If the solutions of the equation

$$\log_2 x + a\log_x 8 = b$$

are 2 and $\frac{1}{8}$, what are a and b?

We have

$$\begin{aligned}\log_2 x + a\log_x 8 &= b \\ \log_2 x + a\log_x 2^3 &= b \\ \log_2 x + \frac{3a}{\log_2 x} &= b \\ (\log_2 x)^2 - b\log_2 x + 3a &= 0. \qquad (1)\end{aligned}$$

Since the solutions of the equation