Foundations of Computing – Discrete Mathematics
Solutions to exercises for week 12

Agata Murawska (agmu@itu.dk)

November 13, 2013

Exercise (6.1.2). A multiple-choice test contains 10 questions. There are four possible answers for each question[1].

a) In how many ways can a student answer the questions on the test if the student answers every question?
For each question a student has 4 options from which he chooses one. That makes the number of ways to be equal to 4^{10} by product rule.

b) In how many ways can a student answer the questions on the test if the student can leave answers blank?
Now student has 5 options as he can choose any of the existing options or leave no answer. Again by product rule, the number of ways is equal to 5^{10}.

Exercise (6.1.12). How many positive integers between 100 and 999 inclusive

a) are divisible by 7?
Every 7th number is divisible by 7, 1000 `div` 7 $= 142$. We have to exclude numbers smaller than 100, there are 100 `div` 7 $= 14$ if them. The final answer is therefore $142 - 14 = 128$.

b) are odd?
Every 2nd number is odd. 1000 `div` 2 $-$ 100 `div` 2 $= 500 - 50 = 450$

c) have the same three decimal digits?
It is simplest to just list these numbers: 111, 222, 333, 444, 555, 666, 777, 888 and 999 – there are 9 of them.

---

[1]We are assuming there is exactly one correct answer to each question

d) are not divisible by 4?
There are 1000 `div` $4 - 100$ `div` $4 = 250 - 25 = 225$ numbers divisible by 4 and 900 numbers between 100 and 999 in general, so there are $900 - 225 = 675$ numbers not divisible by 4.

e) are divisible by 3 or 4?
There are 1000 `div` $4 - 100$ `div` $4 = 225$ numbers divisible by 4 and 1000 `div` $3 - 100$ `div` $3 = 333 - 33 = 300$ numbers divisible by 3. However, we have counted numbers divisible by 12 in both groups. There are 1000 `div` $12 - 100$ `div` $12 = 83 - 8 = 75$ numbers like that. Finally, there are $225 + 300 - 75 = 450$ numbers divisible by 3 or 4.

f) are not divisible by either 3 or 4?
We will use the solution from previous step. We know that 450 numbers between 100 and 999 are divisible by 3 or by 4. Therefore, there are 900-450=450 numbers not divisible by either 3 or 4.

g) are divisible by 3 but not by 4?
There are 1000 `div` $3 - 100$ `div` $3 = 333 - 33 = 300$ numbers divisible by 3. Out of those, 1000 `div` $12 - 100$ `div` $12 = 83 - 8 = 75$ are also divisible by 4 (as they are divisible by $3 * 4 = 12$. So there are $300 - 75 = 225$ numbers divisible by 3 but not by 4.

h) are divisible by 3 and 4?
Numbers divisible by 3 and 4 are divisible by 12. There are 1000 `div` $12 - 100$ `div` $12 = 83 - 8 = 75$ such numbers.

Exercise (6.2.4). The bowl contains 10 red balls and 10 blue balls. A woman selects balls at random without looking at them.

a) How many balls must she select to be sure of having at least three balls of the same color?
She must select 5 balls. A justification for this is the following: having only 4 balls, you either have at least 3 of one color or you are in a situation where you have 2 blue balls and 2 red ones. In the latter case, selecting any additional ball gives you three balls of the same color. Therefore, selecting 5 balls is sufficient.
Formally, we can use the generalized pigeonhole principle, where balls are objects and colors are boxes. If we put 5 objects into 2 boxes, there is at least one box (so, color of the balls) containing $\lceil \frac{5}{2} \rceil = 3$ objects.

b) How many balls must she select to be sure of having at least three blue balls?

She must select 13 balls. To see this, consider the (unlikely) scenario that first ten balls she selects are all red. Then she still needs to pick 3 more balls to have three blue ones.

Exercise (6.3.8). A group contains n men and n women. How many ways are there to arrange these people in a row if the men and women alternate? We have to fill a row of length $2n$ in such a way that men and women alternate. Note that we can select order of men and order of women separately. If W is the number of ways in which to order women and M is the number of ways to order men, then the final answer will be $2 * W * M$ as the only choice we have once men and women are "ordered" is whether to start the final row with a man or a woman.
Now, it remains to calculate W and M. Notice, that in fact $W = M$ as there are n men and n women to arrange (and we have to "use" them all). In how many ways can we arrange men? It will be a permutation of length n, therefore by definition, $M = \mathrm{P}(n, n) = n(n-1)(n-2)\ldots(n-n+1) = n!$. Finally, the number of ways to arrange these people is $2 * W * M = 2 * n! * n!$.

Exercise (6.3.12). A coin is flipped 10 times where each flip comes up either heads or tails. How many possible outcomes

a) are there in total?
Each of 10 coins can come up heads (H) or tails (T), There are 2 possible outcomes when flipping one coin (H or T), 4 possible outcomes when flipping 2 coins (HH, HT, TH and TT) and so on. In particular, for 10 coins there are $2^{10} = 1024$ possible outcomes.

b) contain exactly two heads?
If there are exactly two Hs, there are also exactly 8 Ts when 10 coins are flipped. This means that different possible outcomes come from the exact order in which these two heads are obtained. We need to choose 2 positions out of 10 for the heads, so by definition of combination we have $\mathrm{C}(10, 2) = \frac{10!}{2!*(10-2)!} = 45$ ways to get exactly 2 heads.[2]

c) contain at most three tails?
We will solve this by summing up number of possible outcomes with exactly 0, 1, 2 and 3 tails. For 0 tails, there is only one possible outcome – all heads. For 1 tails there are 10 possible outcomes, as the tails can occur in any of the 10 flips. For 2 tails there are 45 possible outcomes (by the same argument as we have used in part b) for exactly 2 heads). For

---

[2]Notice, that if we want to instead choose 8 positions out of 10 for the tails, we get the exact same result.

3 tails there are C$(10, 2)$ = $\frac{10!}{3!*(10-3)!}$ = 120 possible outcomes. Finally, summing it all up we get $1 + 10 + 45 + 120 = 176$ possible outcomes with three or less tails.

d) contain the same number of heads and tails?
There must be exactly 5 heads and 5 tails. Similarly to the case in part b) we can only choose the 5 positions where the result will be heads. Therefore there are C$(10, 5)$ = $\frac{10!}{5!*(10-5)!}$ = 252 possible outcomes.

Exercise (6.3.16). Thirteen people on a softball team show up for a game.

a) How many ways are there to choose 10 players to take the field?
We choose 10 out of 13 players and we do not care about the order, in which they are chosen. Therefore, we use definition of a combination. There are C$(13, 10)$ = $\frac{13!}{10!*3!}$ = 286 ways to choose 10 players.

b) How many ways there are to assign the 10 positions by selecting players from the 13 people who show up?
We can solve this in two ways:

- We can use the previous solution and notice, that after the 10 players were chosen, we can assign them positions in P$(10, 10)$ = $10!$ ways, resulting in $286 * 10! = 1037836800$ possibilities

- We can just choose 10 out of 13 people and assign them positions. This will result in P$(13, 10)$ = $\frac{13!}{3!}$ = 1037836800, so the exact same result.

c) Of the 13 people who show up, three are women. How many ways are there to choose 10 players to take the field if at least one of these players must be a woman?
Again, this part can be solved in two ways:

- There can be exactly 3, 2 or just 1 woman on the team. The choice of men on the team is independent from the choice of women. For 3 women on the team, we choose men in C$(10, 7)$ = 120 ways (as there are 10 men and 7 slots for them) and there is only one way to choose the women (which is consistent with C$(3, 3)$ = $\frac{3!}{0!*3!}$ = 1). For 2 women on the team, we choose men in C$(10, 8)$ = 45 ways and women in C$(3, 2)$ = 3 ways, resulting in a total of $45*3 = 135$ options. For 1 women on the team, we choose men in C$(10, 9)$ = 10 ways and woman in C$(3, 1)$ = 3 way, resulting in a total of $10*3 = 30$ options. Summing up these possibilities, there are $120 + 135 + 30 = 285$ ways to have at least one women on the team.

- From part a) we know that there were a total of 286 ways to have 10 people selected. In the current setting, with 10 men and 3 women there is only one way to get a team with no women in it – it is to take all the men. Therefore the remaining 285 ways must be ones where at least one woman is on the team.

Exercise (6.5.4)**.** How many ways are there to select five unordered elements from a set with three elements when repetition is allowed?

We are asked to find a number of combinations where repetition is allowed. The set contains $n = 3$ elements and we select $r = 5$ of them, therefore the number of possible ways of doing that is $C(n + r - 1, r) = C(7, 5) = 21$.

Exercise (6.5.14)**.** How many solutions are there to the inequality

$$x_1 + x_2 + x_3 \leq 11$$

where x_1, x_2 and x_3 are nonnegative integers?

Using the hint, we will introduce nonnegative integer x_4 such that $x_1 + x_2 + x_3 + x_4 = 11$.[3] We need to find a way of selecting 11 items from a set of four elements so that there are x_1 items of type one, x_2 items of type two, x_3 items of type three and x_4 items of type four. This is again a number of combinations where repetition is allowed: $C(4 + 11 - 1, 11) = C(14, 11) = \frac{14*13*12}{1*2*3} = 308$.

Exercise (6.5.26)**.** How many different bit strings can be formed using six 1s and eight 0s?

We have 14 "boxes" to use up in total. We have to select places for objects of one type (e.g. for 1s), then fill the rest of boxes with the 0s. This results in the following number of different bit strings: $C(14, 6) = 3003$.

---

[3]The reasoning follows the solution for example 5.