Math Preliminaries

A Short Introduction to Tensor Algebra and Analysis

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A Vector

Let $\mathbf{v}$ be a first-order tensor this means

$$\mathbf{v} = \sum_{i=1}^{3} v_i \mathbf{e}_i$$

where $\mathbf{e}_i$'s are some base vectors. We call this a vector. The components are given by

$$\mathbf{v}_i = \mathbf{v} \cdot \mathbf{e}_i$$

In matrix-vector notation, we write the components as a 3-by-1 (column) vector

$$\left[\mathbf{v}\right] = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{bmatrix}$$

Dot Product of Vectors

From the definition of a vector, we may now work out what the dot product of two vectors $\mathbf{v}$ and $\mathbf{u}$ is:

$$\mathbf{u} \cdot \mathbf{v} = \left(\sum_{i=1}^{3} u_i \mathbf{e}_i\right) \cdot \left(\sum_{j=1}^{3} v_j \mathbf{e}_j\right)$$

$$= \sum_{i=1}^{3} \sum_{j=1}^{3} u_i v_j \mathbf{e}_i \cdot \mathbf{e}_j$$

$$= \sum_{i=1}^{3} u_i v_i$$

A 2nd Order Tensor

Let $\mathbf{A}$ be a second-order tensor means $\mathbf{A}$ is a linear map that associates a given vector $\mathbf{u}$ with a second vector $\mathbf{v}$

$$\mathbf{v} = \mathbf{A}\mathbf{u}$$

The components are given by

$$\mathbf{A}_{ij} = \mathbf{e}_i \cdot \mathbf{A}\mathbf{e}_j$$

where $\mathbf{e}_i$, $\mathbf{e}_j$ are base vectors. Or

$$\mathbf{A} = \sum_{i,j=1}^{3} A_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$$

where $(\mathbf{u} \otimes \mathbf{v})\,\mathbf{w} = (\mathbf{w} \cdot \mathbf{v})\,\mathbf{u}$.

Tensor applied to vector

With this expression, we may have a second look at the definition of a second-order tensor as a linear mapping of vectors

$$
\begin{aligned}
\mathbf{v} &= \mathbf{A}\mathbf{u} \\
&= \left(\sum_{i=1}^{3} \sum_{j=1}^{3} A_{ij} \mathbf{e}_i \otimes \mathbf{e}_j \right) \left(\sum_{k=1}^{3} u_k \mathbf{e}_k \right) \\
&= \sum_{i=1}^{3} \sum_{j=1}^{3} A_{ij} u_j \mathbf{e}_i
\end{aligned}
$$

We observe the similarity between the definition of the second-order tensor with a matrix-vector product.

The Relation to a Matrix

When dealing with a tensor then it is “independent” of the basis, hence

$$\mathbf{A} = \sum_{i,j=1}^{3} A_{ij} \mathbf{e}_i \otimes \mathbf{e}_j = \sum_{i,j=1}^{3} A'_{ij} \mathbf{e}'_i \otimes \mathbf{e}'_j$$

where the primed basis is some other basis. If we pick a specific basis then we may write the components of the second-order tensor as entries of a 3-by-3 matrix

$$[\mathbf{A}] = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \mathbf{A}_{13} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \mathbf{A}_{23} \\ \mathbf{A}_{31} & \mathbf{A}_{32} & \mathbf{A}_{33} \end{bmatrix}$$

Sometimes subscripts on the bracket notation can be used to indicate the basis wrt matrix representation.

The ∇ operator – The Gradient of a scalar (1/2)

Let

$$\nabla(\cdot) = \sum_i \partial_i (\cdot)\, \mathbf{e}_i$$

be the differential vector operator where

$$\partial_i \equiv \frac{\partial}{\partial x_i}$$

If we used some other set of coordinates $\mathbf{y}$ then we write

$$\nabla_{\mathbf{y}}(\cdot) = \sum_i \frac{\partial}{\partial y_i} (\cdot)\, \mathbf{e}_i$$

The ∇ operator – The Gradient of a scalar (2/2)

In matrix-vector notation, we may write the ∇-operator as

$$[\nabla] = \begin{bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{bmatrix}$$

The Gradient of a Vector

The ∇ notation may be defined when applied to a first-order tensor $\mathbf{b}$ to mean

$$(\nabla \mathbf{b})_{ij} = \partial_j b_i$$

Our definition is precisely the partial derivative of the vector function $\mathbf{b}$ or what is also termed the Jacobian of $\mathbf{b}$. Observe that when ∇ is applied to a first-order tensor the result is a second-order tensor.

Divergence of a Vector

Another useful differential operation is the divergence operator of a vector which we define as

$$\nabla \cdot \mathbf{b} \equiv \sum_i \partial_i b_i$$

Notice this maps a first-order tensor into a zero-order tensor as we expect.

Divergence of a Tensor

We write the divergence of $\mathbf{A}$ as

$$\nabla \cdot \mathbf{A} = \sum_j \sum_i \partial_i \mathbf{A}_{ij} \mathbf{e}_j$$

where $\mathbf{e}_j$ is the j^{th} unit vector. In terms of matrix notation, this corresponds to

$$\underbrace{\left[\partial_1 \partial_2 \partial_3\right]}_{\nabla^T} \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \mathbf{A}_{13} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \mathbf{A}_{23} \\ \mathbf{A}_{31} & \mathbf{A}_{32} & \mathbf{A}_{33} \end{bmatrix} = \left[\left(\sum_i \partial_i \mathbf{A}_{i1}\right) \quad \left(\sum_i \partial_i \mathbf{A}_{i2}\right) \quad \left(\sum_i \partial_i \mathbf{A}_{i3}\right) \right]$$

Each component is the divergence of the corresponding column vector.

Product Rule for Divergence

Let $\mathbf{A}$ be a second order tensor and $\mathbf{b}$ a vector and $\nabla = \begin{bmatrix} \partial_1 & \partial_2 & \partial_3 \end{bmatrix}^T$ Then

$$\nabla \cdot (\mathbf{A}\mathbf{b}) = (\nabla \cdot \mathbf{A}) \cdot \mathbf{b} + \mathbf{A} : (\nabla \mathbf{b})^T$$

where

$$\nabla \mathbf{b} = \begin{bmatrix} \partial_1 \mathbf{b}_1 & \partial_2 \mathbf{b}_1 & \partial_3 \mathbf{b}_1 \\ \partial_1 \mathbf{b}_2 & \partial_2 \mathbf{b}_2 & \partial_3 \mathbf{b}_2 \\ \partial_1 \mathbf{b}_3 & \partial_2 \mathbf{b}_3 & \partial_3 \mathbf{b}_3 \end{bmatrix}.$$

The double contraction between two second-order tensors $\mathbf{A}, \mathbf{B}$ is defined as

$$\mathbf{A} : \mathbf{B} = \sum_i \sum_j A_{ij} B_{ij}$$

Symmetric Product Rule for Divergence

If $\mathbf{A}$ is a symmetric second-order tensor then we may write $\mathbf{A}^T = \mathbf{A}$. This means

$$\mathbf{e}_i \cdot \mathbf{A}\mathbf{e}_j = \mathbf{e}_j \cdot \mathbf{A}\mathbf{e}_i$$

Or using index notation $\mathbf{A}_{ij} = \mathbf{A}_{ji}$.

Now we assume that $\mathbf{A}$ is a symmetric tensor then the product rule from the previous page can be rewritten as

$$\nabla \cdot (\mathbf{A}\mathbf{b}) = (\nabla \cdot \mathbf{A}) \cdot \mathbf{b} + \mathbf{A} : (\nabla \mathbf{b})$$

Motivation for Change of Variables

TODO: Motivate the need for change of variables when doing integration

TODO: make a drawing explaining notation of $\mathbf{x}$, $\mathbf{X}$, ϕ etc.

The Rule for Change of Variables

If $\mathbf{x} = \Phi(\mathbf{X})$ and the domain is given by $\mathbf{X} \in V$ and the range by $\mathbf{x} \in v$ then change of variables is given by

$$\int_v f(\mathbf{x}) dv = \int_V f(\Phi(\mathbf{X})) \left| \det \left(\frac{\partial \Phi}{\partial \mathbf{X}} \right) \right| dV$$

Usually we write $j = \left| \det \left(\frac{\partial \Phi}{\partial \mathbf{X}} \right) \right|$ and then we have

$$dv = j dV$$

Motivation for Nansen's Relation

TODO: Motivate the need for Nansen's relation
TODO: make a drawing explaining what goes on... what is the difference between Nansen's relation and the rule for change of variables.

Nanson's Relation (1/2)

Let the volume elements in spatial (lower case) and material (upper case) coordinates be written as

$$dv = d\mathbf{x} \cdot d\mathbf{s}$$
$$dV = d\mathbf{X} \cdot d\mathbf{S}$$

where $d\mathbf{x} = \frac{\partial \Phi}{\partial \mathbf{X}} d\mathbf{X}$ and $d\mathbf{s} = ds\, \mathbf{n}$ and $d\mathbf{S} = dS\, \mathbf{N}$. From change of variables and using $\mathbf{F} \equiv \frac{\partial \Phi}{\partial \mathbf{X}}$ we may write up the relations,

$$dv = j\, dV$$
$$d\mathbf{x} \cdot d\mathbf{s} = j\, d\mathbf{X} \cdot d\mathbf{S}$$
$$d\mathbf{X} \cdot \mathbf{F}^T\, ds\, \mathbf{n} = d\mathbf{X} \cdot j\, dS\, \mathbf{N}$$

Nanson's Relation (2/2)

From the relation,

$$d\mathbf{X} \cdot \mathbf{F}^T \, ds \, \mathbf{n} = d\mathbf{X} \cdot j \, dS \, \mathbf{N}$$

We have Nanson's Relation

$$ds \, \mathbf{n} = j \, \mathbf{F}^{-T} \, dS \, \mathbf{N}$$

The Contraction and Trace Relation

$$\mathrm{tr}\left(\mathbf{A}\mathbf{B}^T\right) = \mathrm{tr}\left(\mathbf{A}^T\mathbf{B}\right) = \mathbf{A} : \mathbf{B}$$

Proof, by definition

$$\begin{aligned}
\mathrm{tr}\left(\mathbf{A}^T\mathbf{B}\right) &= \sum_j \left(\mathbf{A}^T\mathbf{B}\right)_{jj} \\
&= \sum_j \left(\sum_i \mathbf{A}^T_{ji}\mathbf{B}_{ij}\right) \\
&= \sum_j \sum_i \mathbf{A}_{ij}\mathbf{B}_{ij}
\end{aligned}$$

The proof follows by comparison with $\mathbf{A} : \mathbf{B} = \sum_j \sum_i \mathbf{A}_{ij}\mathbf{B}_{ij}$

Trace

It can be shown for second-order tensors $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$, that

$$a \operatorname{tr}(\mathbf{A}) = \operatorname{tr}(a\,\mathbf{A})$$

and

$$\operatorname{tr}(\mathbf{AB}) = \operatorname{tr}(\mathbf{BA})$$

also

$$\operatorname{tr}(\mathbf{ABC}) = \operatorname{tr}(\mathbf{CAB})$$

The first two “rules” follow straightforward definitions. The third rule comes from applying the second rule, $\operatorname{tr}((\mathbf{AB})\,\mathbf{C}) = \operatorname{tr}(\mathbf{C}\,(\mathbf{AB}))$.

Derivatives wrt. Tensors

Let $f(\mathbf{A})$ be a scalar function of the tensor $\mathbf{A}$ then by definition we have

$$\left[\frac{\partial f(\mathbf{A})}{\partial \mathbf{A}}\right]_{ij} = \frac{\partial f(\mathbf{A})}{\partial \mathbf{A}_{ij}}$$

Differentiation Rules

Let $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{3\times 3}$

$$\frac{\partial\left(\mathbf{A} : \mathbf{B}\right)}{\partial \mathbf{A}} = \mathbf{B}$$

$$\frac{\partial\left(\mathbf{A} : \mathbf{B}\right)}{\partial \mathbf{B}} = \mathbf{A}$$

$$\frac{\partial\left(\mathbf{A} : \mathbf{A}\right)}{\partial \mathbf{A}} = 2\,\mathbf{A}$$

$$\frac{\partial \mathrm{tr}\left(\mathbf{A}\right)}{\partial \mathbf{A}} = \mathbf{I}$$

$$\frac{\partial \mathrm{tr}\left(\mathbf{A}^2\right)}{\partial \mathbf{A}} = \mathbf{A} + \mathbf{A}^T$$

$$\frac{\partial \mathrm{tr}\left(\mathbf{A}\right)^2}{\partial \mathbf{A}} = 2\,\mathrm{tr}\left(\mathbf{A}\right)\mathbf{I}$$

More Differentiation

Let **A** be a second-order tensor then

$$\frac{\partial \det(\mathbf{A})}{\partial \mathbf{A}} = \det(\mathbf{A})\mathbf{A}^{-T}$$

Assignment

- Prove the Product Rule formula on Page 12.

Assignment

- Prove the symmetric form of the vector divergence formula on Page 13.

Assignment

- Derive the change of variables formula on Page 15 for specialized cases of 2D and 3D.

Assignment

- TODO: Apply the rule of changes of variables from Page 15 to 3-4 simple examples

Assignment

- TODO: Make some simple examples to play with Nansen's relation from Page 18

Assignment

- Prove the differentiation rules on Page 22.

Assignment

- Prove the determinant differentiation rule on Page 23 for a symmetric tensor.

Hints: First prove $\frac{\partial \lambda_k}{\partial \mathbf{A}_{ij}} = (\mathbf{n}_k \cdot \mathbf{e}_i)(\mathbf{n}_k \cdot \mathbf{e}_j) = (\mathbf{n}_k \otimes \mathbf{n}_k)_{ij}$ where λ_k's are eigenvalues and $\mathbf{n}_k$ are eigenvectors. Then use chain rule $\frac{\partial \det \mathbf{A}}{\partial \mathbf{A}_{ij}} = \sum_{m=1}^{3} \frac{\partial \det \mathbf{A}}{\partial \lambda_m} \frac{\partial \lambda_m}{\partial \mathbf{A}_{ij}}$.