

List homomorphisms

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29-11-2021

Preliminaries

Material

- *The Third Homomorphism Theorem* (Gibbons; 1995)
- *Construction of List Homomorphisms by Tupling and Fusion* (Hu, Iwasaki, Takeichi; 1996)

Approach

- **This material is dense.**
- Do not get dissuaded!
 - ▶ Ask about notation that is unclear.
- Proofs in the papers, not the lecture.
- We focus on intuition about working algebraically with (parallel!) programs.
- My notation is slightly different than in the papers.
 - ▶ I write *hom* for homomorphisms (and leave out neutral elements).
 - ▶ I write *map f* instead of $f*$.

List homomorphisms

- Definitions and properties

- Homomorphism theorems

- Applying the third homomorphism theorem

Deriving efficient almost homomorphisms

- Almost homomorphisms

- Deriving initial almost homomorphisms by tupling

- Fusion theorems for homomorphisms

- Fusion of *mss*

Conclusions

List homomorphisms

Definition: list homomorphism

A function h is called a list homomorphism if there exists a binary operator \odot such that for all finite lists x, y

$$h (x \mathbin{++} y) = h x \odot h y$$

where $\mathbin{++}$ means list concatenation.

- Intuitively, homomorphisms are functions where we can split the input into chunks arbitrarily, recursively process the chunks, and combine the partial results.
- Each “chunk” can be processed in parallel.

Examples of list homomorphisms

Summation

$$\mathit{sum} [] = 0$$

$$\mathit{sum} [x] = x$$

$$\mathit{sum} (x \mathbin{++} y) = \mathit{sum} x + \mathit{sum} y$$

Length

$$\mathit{len} [] = 0$$

$$\mathit{len} [x] = 1$$

$$\mathit{len} (x \mathbin{++} y) = \mathit{len} x + \mathit{len} y$$

Observations

$$h (x \oplus y) = h x \odot h y$$

- \odot is necessarily associative because \oplus is associative.
- $h []$ is necessarily a neutral element for \odot .

For concision, we write

$$\text{hom } (\odot) f$$

for a list homomorphism h , where

$$f x = h [x]$$

$$0_{\odot} = h []$$

Examples

$$\begin{aligned} \text{sum} &= \text{hom } (+) \text{ id} \\ \text{length} &= \text{hom } (+) (\text{const } 1) \end{aligned}$$

Reductions

Reductions are homomorphisms

A function of the form

$$\text{hom } (\odot) \text{ id}$$

is called a *reduction* with operator \odot .

- Equivalent to what we would write as `reduce \odot 0 \odot` in Futhark.
- For concision we will abbreviate as follows:

$$\text{red } \odot = \text{hom } (\odot) \text{ id}$$

- ▶ Given \odot , the neutral element 0_{\odot} is always unique (think about why).

Maps

Maps are homomorphisms

A function of the form

$$\text{hom } (+) ([\cdot] \circ f)$$

is called a *map* with function f .

- $[\cdot]$ is the function $\lambda x \rightarrow [x]$ that turns x into $[x]$.
- We'll write this as *map* f .
- Really is just what you expect, no surprises here.

Does this mean you could define `map` with `reduce` in Futhark?

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Does this mean you could define map with reduce in Futhark?

- **No**, `reduce (++) []` is not well-typed.
- Futhark disallows irregular arrays like `[[1, 2], [3]]`.
- We will see many things you cannot express in most parallel languages.

Leftwards and rightwards functions

A function on lists h is \oplus -leftwards if and only if for all elements a and lists y ,

$$h ([a] \uplus y) = a \oplus h y$$

- \oplus need not be associative.
- **Intuitively, what does this say about h ?**

Leftwards and rightwards functions

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- \oplus need not be associative.
- **Intuitively, what does this say about h ?**
- Suppose

$$h [] = e$$

then

$$h [a, b, c] = a \oplus (b \oplus (c \oplus e))$$

- A \oplus -leftwards function is a right-to-left *fold*, denoted by *foldr* $(\oplus) e$.
- Symmetrically, for an \otimes -rightwards function h ,

$$h (y \mathbin{++} [a]) = h y \otimes a$$

which we denote by *foldl* $(\otimes) e$.

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First Homomorphism Theorem

Every monomorphism can be written as the composition of a reduction and a map.

$$\text{hom } (\odot) f = \text{red } (\odot) \circ \text{map } f$$

Conversely, every such composition is a homomorphism.

- **This is why *map – reduce* compositions are so crucial.**
- Sometimes it is easiest to invent parallel algorithms by thinking “homomorphism-style” in terms of recursively splitting and combining subresults.
 - ▶ Can systematically turn this into data-parallel map-reduce code.

Second Homomorphism Theorem

If \odot is associative, then

$$\begin{aligned} \text{hom } (\odot) f &= \text{foldr } (\oplus) 0_{\odot} \quad \text{where } x \oplus y = f \ x \odot y \\ &= \text{foldl } (\otimes) 0_{\otimes} \quad \text{where } x \otimes y = x \odot f \ y \end{aligned}$$

That is, *every homomorphism is both a leftwards and a rightwards function.*

- **Intuition:** as a homomorphism allows us to split “however we want”, we can also split consistently with a left (or right) bias.
- **What about the other direction?**

Third Homomorphism Theorem

If h is leftwards and rightwards, then h is a homomorphism.

- **Why this is useful:** If we can come up with left-folds and right-folds that both solve the problem, then a homomorphism must also exist.
- **Put another way:** If a problem can be solved with both a *foldr* and a *foldl*, then a “parallel” algorithm for the problem also exists.

Third Homomorphism Theorem

If h is leftwards and rightwards, then h is a homomorphism.

- **Why this is useful:** If we can come up with left-folds and right-folds that both solve the problem, then a homomorphism must also exist.
- **Put another way:** If a problem can be solved with both a *foldr* and a *foldl*, then a “parallel” algorithm for the problem also exists.
- **Unfortunately:**
 - ▶ Not all homomorphisms are usefully parallel.
 - ▶ The proof does not tell us how to *construct* a practical homomorphism from the left and right folds.

Proof sketch (full details in Gibbons' paper)

The statement to prove

If h is leftwards and rightwards, then h is a homomorphism.

Steps:

1. Prove that

$$h\ v = h\ x \wedge h\ w = h\ y \Rightarrow h\ (v \uplus w) = h\ (x \uplus y)$$

► Rough steps:

$$\begin{aligned} h\ (v \uplus w) &= \dots \\ &= h\ (v \uplus y) && \text{(By treating } h \text{ as leftwards function.)} \\ &= \dots \\ &= h\ (x \uplus y) && \text{(By treating } h \text{ as rightwards function.)} \end{aligned}$$

2. Prove that this means h is a homomorphism.

The function h is a homomorphism if and only if the implication

$$h\ v = h\ x \wedge h\ w = h\ y \Rightarrow h\ (v \# w) = h\ (x \# y)$$

holds for all lists v, w, x, y .

“Only if” part

If h is a homomorphism then for all a, b there is a \oplus such that,

$$h\ (a \# b) = h\ a \oplus h\ b$$

and then

$$h\ (v \# w) = h\ v \oplus h\ w \quad (\text{Def. of homomorphism})$$

$$= h\ x \oplus h\ y \quad (\text{LHS of implication})$$

$$= h\ (x \# y) \quad (\text{Def. of homomorphism})$$

The function h is a homomorphism if and only if the implication

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holds for all lists v, w, x, y .

“If” part

- Choose g such that $h \circ g \circ h = h$ and define

$$a \odot b = h\ (g\ a \uplus g\ b)$$

- (We'll show that g exists in a moment.)
- We have $h\ x = h\ (g\ (h\ x))$ and $h\ y = h\ (g\ (h\ y))$.
- With $v = g\ (h\ x)$ and $w = g\ (h\ y)$ then

$$\begin{aligned} h\ (x \uplus y) &= h\ (v \uplus w) \\ &= h\ (g\ (h\ x) \uplus g\ (h\ y)) \\ &= h\ x \odot h\ y \end{aligned}$$

Existence of g

Given h , we must construct g such that

$$h \circ g \circ h = h$$

- Can we be sure such a g exists for any h ?
 - ▶ **Yes:** To compute $g x$, enumerate domain of h and produce first y such that $h y = x$.
 - ▶ **Theoretically sound, but not really practical.**
 - ▶ When we apply the Third Homomorphism Theorem we better come up with a more reasonable g than this.

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Consider sorting

We can define

$$\text{sort} = \text{foldr ins } []$$

where

$$\begin{aligned} \text{ins } a [] &= [a] \\ \text{ins } a (b : x) &= \begin{cases} a : b : x & \text{if } a \leq b \\ b : \text{ins } a x & \text{otherwise} \end{cases} \end{aligned}$$

That is, $\text{ins } a l$ inserts a at the appropriate location in the sorted list l .

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That is, $\text{ins } a l$ inserts a at the appropriate location in the sorted list l .

$$\text{sort} = \text{foldl ins}' []$$

where

$$\text{ins}' x a = \text{ins } a x$$

As sort is both leftwards and rightwards, there is a homomorphic sorting algorithm.

Homomorphic sorting

By the Third Homomorphism Theorem, we get a homomorphism $hom \odot [\cdot]$ where

$$u \odot v = sort (unsort\ u \uplus unsort\ v)$$

for some function *unsort* such that

$$sort \circ unsort \circ sort = sort$$

Can you think of such a function?

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Can you think of such a function?

$$unsort = id$$

This gives

$$u \odot v = sort (u \uplus v)$$

Is this a good sorting algorithm?

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Homomorphic sorting in words

“To sort the concatenation of two sorted lists u and v , concatenate u and v and then sort the result using some other sorting algorithm.”

Improving the homomorphic sort

- In the paper, Gibbons goes on to mechanically derive a more efficient \odot , based on the property that u will be sorted.

$$\begin{aligned}u \odot v &= \text{sort } (u \uplus v) \\&= \text{foldl } \text{ins}' \ [] \ (u \uplus v) \\&= \text{foldl } \text{ins}' \ (\text{foldl } \text{ins}' \ [] \ u) \ v \\&= \text{foldl } \text{ins}' \ (\text{sort } u) \ v \\&= \text{foldl } \text{ins}' \ u \ v\end{aligned}$$

- Now $u \odot v$ merges two sorted lists in $O(n^2)$ time.
- By also assuming v is sorted, Gibbons eventually derives mergesort.
 - ▶ Read the paper!
- **Not parallel**, but we could in principle have derived a parallel \odot instead.

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Almost homomorphisms

Some functions are not homomorphisms.

- Knowing the means of arrays x and y does not let us compute the mean of $x \uplus y$.

But they can be turned into homomorphisms if we compute a little extra information.

- If we know the means x_{mean}, y_{mean} and sizes x_n, y_n of x, y , we can compute the mean and size of $x \uplus y$ as

$$\frac{x_{mean} \cdot x_n + y_{mean} \cdot y_n}{x_n + y_n}$$

and

$$x_n + y_n$$

This is called an *almost homomorphism*.

mean as an almost homomorphism

$$\begin{aligned} \text{mean } [] &= (0, 0) \\ \text{mean } [x] &= (x, 1) \\ \text{mean } (x \uplus y) &= \text{mean } x \odot \text{mean } y \end{aligned}$$

where

$$(x_{\text{mean}}, x_n) \odot (y_{\text{mean}}, y_n) = \left(\frac{x_{\text{mean}} \cdot x_n + y_{\text{mean}} \cdot y_n}{x_n + y_n}, x_n + y_n \right)$$

To compute the actual mean we then project out the first component:

$$\pi_1 \circ \text{mean}$$

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The maximum segment sum problem

Problem statement

Given a list A , find the largest sum of a subsequence $A[i : j]$.

Examples

$$mss [0, 3, -2] = 3$$

$$mss [4, -1, 0] = 4$$

$$mss [0, 3, -2, 4, -1, 0] = 5$$

Not a homomorphism

- Knowing $mss\ x$ and $mss\ y$ is not enough to give us $mss(x \# y)$.
- E.g. $x = [0, 3, -2]$, $y = [4, -1, 0]$ as above.

But is an almost homomorphism

- With human creativity we can come up with an associative operator on 4-tuples that lets us solve MSS as an almost homomorphism.
- But can we also derive it systematically?

Specification of *mss*

$$mss = max^s \circ (map^s \text{ sum}) \circ segs$$

Note that max^s and map^s are set operations.

- We will try to *systematically* derive an efficient almost homomorphism from the specification.
- **Well-known technique in (academic) functional programming**, often called things like “constructive algorithmics” or “program calculation”.

$$mss = max^s \circ (map^s \text{ sum}) \circ segs$$

$$segs [] = \{\}$$

$$segs [x] = \{[x]\}$$

$$segs (xs \# ys) = segs\ xs \cup segs\ ys \cup (tails\ xs\ \mathcal{X}_{\#}\ inits\ ys)$$

$$inits [] = []$$

$$inits [x] = [[x]]$$

$$inits (xs \# ys) = inits\ xs \# map\ (xs \#)\ (inits\ ys)$$

$$tails [] = []$$

$$tails [x] = [[x]]$$

$$tails (xs \# ys) = map\ (\#ys)\ (tails\ xs) \# tails\ ys$$

$$[x_1, \dots, x_n] \mathcal{X}_{\oplus} [y_1, \dots, y_m] = \{x_1 \oplus y_1, \dots, x_1 \oplus y_m, \dots, x_n \oplus y_1, \dots, x_n \oplus y_m\}$$

$$(f \triangle g) x = (f x, g x)$$

So $f \triangle g$ is a function that applies both f and g to its argument.

$$a(\oplus \triangle \otimes)b = (a \oplus b, a \otimes b)$$

Example

$$(+1) \triangle (\cdot 2) \triangle (-3) = \lambda x \rightarrow (x + 1, x \cdot 2, x - 3)$$

$$(+) \triangle (\cdot) \triangle (-) = \lambda(x, y) \rightarrow (x + y, x \cdot y, x - y)$$

$$(f \triangle g) x = (f x, g x)$$

So $f \triangle g$ is a function that applies both f and g to its argument.

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$$(+) \triangle (\cdot) \triangle (-) = \lambda(x, y) \rightarrow (x + y, x \cdot y, x - y)$$

$$\Delta_1^n f_i = f_1 \triangle \cdots \triangle f_n$$

So Δ_1^n tuples the n functions f_i :

$$(\Delta_1^n f_i) x = (f_1 x, \cdots, f_n x)$$

Tupling of homomorphisms

Let h_1, \dots, h_n be mutually defined as follows:

$$\begin{aligned}h_j [] &= 0_{\oplus_j} \\h_j [x] &= f_j x \\h_j (xs \# ys) &= ((\Delta_1^n h_i) xs) \oplus_j ((\Delta_i^n h_i) ys)\end{aligned}$$

Then

$$\Delta_1^n h_i = \text{hom } (\Delta_1^n \oplus_i) (\Delta_1^n f_i)$$

Implication: tupling mutually recursive functions that *traverse the same list in a specific way* will produce a list homomorphism.

Notation reminder

$$(\Delta_1^n f_i) x = (f_1 x, \dots, f_n x)$$

Writing *segs* in the required form

We have

$$\begin{aligned} \text{segs } [] &= \{\} \\ \text{segs } [x] &= \{[x]\} \\ \text{segs } (xs \mathbin{++} ys) &= \underline{\text{segs } xs} \cup \underline{\text{segs } ys} \cup (\underline{\text{tails } xs} \mathcal{X}_{++} \underline{\text{inits } ys}) \end{aligned}$$

and want n mutually recursive functions

$$\begin{aligned} h_j [] &= 0_{\oplus_j} \\ h_j [x] &= f_j x \\ h_j (xs \mathbin{++} ys) &= ((\Delta_1^n h_i) xs) \oplus_j ((\Delta_i^n h_i) ys) \end{aligned}$$

- *segs* must be tupled with *tails* because *segs* and *tails* traverse the same list *xs*.
- *segs* must be tupled with *inits* because *segs* and *inits* traverse the same list *xs*.

Considering *inits* and *tails*

$$\begin{aligned} \text{inits } [] &= [] \\ \text{inits } [x] &= [[x]] \\ \text{inits } (xs ++ ys) &= \underline{\text{inits } xs} ++ \text{map } (\underline{xs}++) (\text{inits } ys) \end{aligned}$$

$$\begin{aligned} \text{tails } [] &= [] \\ \text{tails } [x] &= [[x]] \\ \text{tails } (xs ++ ys) &= \text{map } (++) \underline{ys} (\text{tails } xs) ++ \underline{\text{tails } ys} \end{aligned}$$

We need to tuple with *id*:

$$\begin{aligned} \text{id } [] &= [] \\ \text{id } [x] &= [x] \\ \text{id } (xs ++ ys) &= \text{id } xs ++ \text{id } ys \end{aligned}$$

- To summarise, our tupled function will be

$$\text{segs } \triangle \text{ inits } \triangle \text{ tails } \triangle \text{ id}$$

Tupling $segs \triangle inits \triangle tails \triangle id$

Still need to phrase this in the form

$$\begin{aligned}h_j [] &= 0_{\oplus_j} \\h_j [x] &= f_j x \\h_j (xs \# ys) &= ((\Delta_1^n h_i) xs) \oplus_j ((\Delta_i^n h_i) ys)\end{aligned}$$

i.e. derive f_1, \oplus_1 for $segs$, f_2, \oplus_2 for $inits$, f_3, \oplus_3 for $tails$, f_4, \oplus_4 for id .

- Actually straightforward: pick corresponding recursive calls from the tuples.

Deriving f_1 and \oplus_1

Original

$$\begin{aligned} \text{segs } [] &= \{\} \\ \text{segs } [x] &= \{[x]\} \\ \text{segs } (xs \text{ ++ } ys) &= \underline{\text{segs } xs} \cup \underline{\text{segs } ys} \cup (\underline{\text{tails } xs} \mathcal{X}_{\text{++}} \underline{\text{inits } ys}) \end{aligned}$$

Goal

$$\begin{aligned} h_1 [] &= 0_{\oplus_1} \\ h_1 [x] &= f_1 x \\ h_1 (xs \text{ ++ } ys) &= ((\Delta_1^n h_i) xs) \oplus_1 ((\Delta_i^n h_i) ys) \end{aligned}$$

Derivation

$$\begin{aligned} f_1 x &= \{[x]\} \\ (s_1, i_1, t_1, d_1) \oplus_1 (s_2, i_2, t_2, d_2) &= s_1 \cup s_2 \cup (t_1 \mathcal{X}_{\text{++}} i_2) \end{aligned}$$

Intuition s_1 to $\text{segs } xs$, i_1 to $\text{inits } xs$, t_1 to $\text{tails } xs$, d_1 to $\text{id } xs$, s_2 to $\text{segs } ys$, i_2 to $\text{inits } ys$, t_2 to $\text{tails } ys$, d_2 to $\text{id } ys$.

All derivations

Skeleton as a reminder

$$\begin{aligned}h_j [] &= 0_{\oplus_j} \\h_j [x] &= f_j x \\h_j (xs \text{ ++ } ys) &= ((\Delta_j^n h_i) xs) \oplus_j ((\Delta_i^n h_i) ys)\end{aligned}$$

Derivation

$$\begin{aligned}f_1 x &= \{[x]\} \\(s_1, i_1, t_1, d_1) \oplus_1 (s_2, i_2, t_2, d_2) &= s_1 \cup s_2 \cup (t_1 \mathcal{X}_{\text{++}} i_2) \\f_2 x &= [[x]] \\(s_1, i_1, t_1, d_1) \oplus_2 (s_2, i_2, t_2, d_2) &= i_1 \text{ ++ } \text{map } (d_1 \text{ ++}) i_2 \\f_3 x &= [[x]] \\(s_1, i_1, t_1, d_1) \oplus_2 (s_2, i_2, t_2, d_2) &= \text{map } (d_2 \text{ ++}) t_1 \text{ ++ } t_2 \\f_4 x &= [x] \\(s_1, i_1, t_1, d_1) \oplus_2 (s_2, i_2, t_2, d_2) &= d_1 \text{ ++ } d_2\end{aligned}$$

The almost homomorphism

$$\text{segs} = \pi_1 \circ \text{hom } (\Delta_1^n \oplus_i) (\Delta_1^n f_i)$$

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Fusion

Reminder of where we are

$$mss = max^s \circ (map^s sum) \circ segs$$

where

$$segs = \pi_1 \circ hom (\Delta_1^4 \oplus i) (\Delta_1^4 f_i)$$

Fusion

Merging adjacent operations to avoid intermediate structures or to allow further simplification.

Example of fusion

$$map f \circ map g = map (f \circ g)$$

Can we fuse $segs$ into $(map^s sum)$?

Fusion theorem for homomorphisms

Let h and $\text{hom}(\oplus) f$ be given. If there exists \otimes such that

$$\forall x, y. h(x \oplus y) = h x \otimes h y$$

then

$$h \circ \text{hom}(\oplus) f = \text{hom}(\otimes) (h \circ f)$$

Example of applying fusion theorem

- Suppose $h = (2 \cdot), (\oplus) = (\uparrow), f = \text{abs}$.

(\uparrow is the “max” operator)

- ▶ What does $(2 \cdot) \circ \text{hom}(\uparrow) \text{abs}$ compute?

- Must find \otimes such that

$$\forall x, y. 2 \cdot (x \uparrow y) = (2 \cdot x) \otimes (2 \cdot y)$$

Example of applying fusion theorem

- Suppose $h = (2 \cdot)$, $(\oplus) = (\uparrow)$, $f = \text{abs}$. (\uparrow is the “max” operator)
 - What does $(2 \cdot) \circ \text{hom } (\uparrow) \text{ abs}$ compute?
- Must find \otimes such that

$$\forall x, y. 2 \cdot (x \uparrow y) = (2 \cdot x) \otimes (2 \cdot y)$$

We fiddle a bit and find

$$x \otimes y = x \uparrow y$$

- Which means we can fuse as

$$\begin{aligned}(2 \cdot) \circ \text{hom } (\uparrow) \text{ abs} &= \text{hom } (\uparrow) ((2 \cdot) \circ \text{abs}) \\ &= \text{red } (\uparrow) \circ \text{map } ((2 \cdot) \circ \text{abs})\end{aligned}$$

Sadly this is not yet enough

Reminder:

$$mss = max^s \circ (map^s \text{ sum}) \circ segs$$

where

$$segs = \pi_1 \circ hom (\Delta_1^4 \oplus_i) (\Delta_1^4 f_i)$$

- We want to fuse $segs$ into $(map^s \text{ sum})$
- Since $segs$ is not a homomorphism but an *almost homomorphism*, we cannot apply the fusion theorem.

Fortunately:

- There is also a fusion theorem for almost homomorphisms.
- Idea is to create a function H that “contains” h but also transforms the other tuple elements as necessary.

Fusion theorem for almost homomorphisms

Let h and $hom(\Delta_1^n \oplus_i)(\Delta_1^n f_i)$ be given. If there exists \otimes_i ($i = 1, \dots, n$) and a function $H = h_1 \times \dots \times h_n$ where $h_1 = h$ such that for all i ,

$$\forall x, y. h_i(x \oplus_i y) = H x \otimes_i H y$$

then

$$h \circ \pi_1 \circ hom(\Delta_1^n \oplus_i)(\Delta_1^n f_i) = \pi_1 \circ hom(\Delta_1^n \otimes_i)(\Delta_1^n (h_i \circ f_i))$$

Notation note: crossing functions, sometimes called *maps*

$$(h_1 \times \dots \times h_n)(x_1, \dots, x_n) = (h_1 x_1, \dots, h_n x_n)$$

Example of applying almost fusion theorem

Suppose we want to fuse

$$\log \circ \pi_1 \circ \text{hom} (\Delta_1^2 \oplus_i) (\Delta_1^2 f_i)$$

where

$$\begin{aligned} f_1 x &= x & (x_{\text{mean}}, x_n) \oplus_1 (y_{\text{mean}}, y_n) &= \frac{x_{\text{mean}} \times x_n + y_{\text{mean}} \times y_n}{x_n + y_n} \\ f_2 x &= 1 & (x_{\text{mean}}, x_n) \oplus_2 (y_{\text{mean}}, y_n) &= x_n + y_n \end{aligned}$$

We must define

$$\begin{aligned} h_1 &= \log \\ h_2 &=? \\ \otimes_1 &=? \\ \otimes_2 &=? \end{aligned}$$

such that

$$\begin{aligned} h_1 (x \oplus_1 y) &= (h_1 \times h_2) x \otimes_1 (h_1 \times h_2) y \\ h_2 (x \oplus_2 y) &= (h_1 \times h_2) x \otimes_2 (h_1 \times h_2) y \end{aligned}$$

Finding h_2, \otimes_2

With

$$\begin{aligned}(x_{mean}, x_n) \otimes_2 (y_{mean}, y_n) &= x_n + y_n \\ h_2 x &= x\end{aligned}$$

then

$$\begin{aligned}h_2 (x \oplus_2 y) &= x_n + y_n \\ &= (\log x_{mean}, h_2 x_n) \otimes_2 (\log y_{mean}, h_2 y_n) \\ &= (\log \times id) x \otimes_2 (\log \times id) y \\ &= H x \otimes_2 H y\end{aligned}$$

With

$$\begin{aligned}(x_{mean}, x_n) \otimes_1 (y_{mean}, y_n) &= \log \left(\frac{\exp x_{mean} \times x_n + \exp y_{mean} \times y_n}{x_n + y_n} \right) \\ h_1 x &= \log x \\ h_2 x &= x\end{aligned}$$

then

$$\begin{aligned}h_1 (x \oplus_1 y) &= h_1 ((x_{mean}, x_n) \oplus_1 (y_{mean}, y_n)) \\ &= \log \left(\frac{x_{mean} \times x_n + y_{mean} \times y_n}{x_n + y_n} \right) \\ &= (\log x_{mean}, x_n) \otimes_1 (\log y_{mean}, y_n) \\ &= (\log \times id) x \otimes_1 (\log \times id) y \\ &= H x \otimes_1 H y\end{aligned}$$

Applying almost fusion

So we can fuse

$$\log \circ \pi_1 \circ \text{hom} (\Delta_1^2 \oplus_i) (\Delta_1^2 f_i)$$

to

$$\pi_1 \circ \text{hom} (\Delta_1^2 \otimes_i) (\Delta_1^2 (h_i \circ f_i))$$

where

$$h_1 x = \log x$$

$$h_2 x = x$$

$$(x_{\text{mean}}, x_n) \otimes_1 (y_{\text{mean}}, y_n) = \log \left(\frac{\exp x_{\text{mean}} \times x_n + \exp y_{\text{mean}} \times y_n}{x_n + y_n} \right)$$

$$(x_{\text{mean}}, x_n) \otimes_2 (y_{\text{mean}}, y_n) = x_n + y_n$$

$$mss = max^s \circ map^s \, sum \circ \pi_1 \circ hom \, (\Delta_1^4 \oplus_i) \, (\Delta_1^4 f_i)$$

We need to find $h_1, h_2, h_3, h_4, \otimes_1, \otimes_2, \otimes_3, \otimes_4$ that solve these equations ($j = 1, \dots, 4$):

$$\begin{aligned} h_j ((s_1, i_1, t_1, d_1) \oplus_j (s_2, i_2, t_2, d_2)) = \\ (h_1 s_1, h_2 i_1, h_3 t_1, h_4 d_1) \otimes_j (h_1 s_2, h_2 i_2, h_3 t_2, h_4 d_2) \end{aligned}$$

Procedure: calculate LHS of equation j by promoting h_j into result and determine h_j, \otimes_j by matching with its RHS.

Example for $j = 1$

Equation

$$\begin{aligned} \text{map}^s \text{sum} ((s_1, i_1, t_1, d_1) \oplus_1 (s_2, i_2, t_2, d_2)) = \\ (\text{map}^s \text{sum } s_1, h_2 \ i_1, h_3 \ t_1, h_4 \ d_1) \otimes_1 (\text{map}^s \text{sum } s_2, h_2 \ i_2, h_3 \ t_2, h_4 \ d_2) \end{aligned}$$

Calculation

$$\begin{aligned} & \text{map}^s \text{sum} ((s_1, i_1, t_1, d_1) \oplus_1 (s_2, i_2, t_2, d_2)) \\ &= (\text{map}^s \text{sum}) (s_1 \cup s_2 \cup (t_1 \ \mathcal{X}_+ \ i_2)) \\ &= \text{map}^s \text{sum } s_1 \cup \text{map}^s \text{sum } s_2 \cup \text{map}^s \text{sum}(t_1 \ \mathcal{X}_+ \ i_2) \\ &= \text{map}^s \text{sum } s_1 \cup \text{map}^s \text{sum } s_2 \cup (t_1 \ \mathcal{X}_{\text{map}^s \text{sum} \circ +} \ i_2) \\ &= \text{map}^s \text{sum } s_1 \cup \text{map}^s \text{sum } s_2 \cup (\text{map sum } t_1 \ \mathcal{X}_+ \ \text{map sum } i_2) \end{aligned}$$

Matching last expression with RHS of the equation gives

$$\begin{aligned} h_2 = h_3 = \text{map sum} \\ (s_1, i_1, t_1, d_1) \otimes_1 (s_2, i_2, t_2, d_2) = s_1 \cup s_2 \cup (t_1 \ \mathcal{X}_+ \ i_2) \end{aligned}$$

Example for $j = 2$

Equation

$$\begin{aligned} \text{map sum } ((s_1, i_1, t_1, d_1) \oplus_2 (s_2, i_2, t_2, d_2)) = \\ (\text{map}^s \text{sum } s_1, h_2 \ i_1, h_3 \ t_1, h_4 \ d_1) \otimes_2 (\text{map}^s \text{sum } s_2, h_2 \ i_2, h_3 \ t_2, h_4 \ d_2) \end{aligned}$$

Calculation

$$\begin{aligned} & \text{map sum } ((s_1, i_1, t_1, d_1) \oplus_2 (s_2, i_2, t_2, d_2)) \\ &= \text{map sum } (i_1 \# \text{map } (d_1 \#) \ i_2) \\ &= \text{map sum } i_1 \# \text{map sum } (\text{map } (d_1 \#) \ i_2) \\ &= \text{map sum } i_1 \# \text{map } (\text{sum} \circ (d_1 \#)) \ i_2 \\ &= \text{map sum } i_1 \# \text{map } (\text{sum } d_1 \#) \ (\text{map sum } i_2) \end{aligned}$$

Matching last expression with RHS of the equation gives

$$\begin{aligned} h_4 &= \text{sum} \\ (s_1, i_1, t_1, d_1) \otimes_2 (s_2, i_2, t_2, d_2) &= i_1 \# \text{map } (d_1 \#) \ i_2 \end{aligned}$$

If we continue

$$h_1 = \text{map}^s \text{ sum}$$

$$h_2 = \text{map sum}$$

$$h_3 = \text{map sum}$$

$$h_4 = \text{sum}$$

$$(s_1, i_1, t_1, d_1) \otimes_1 (s_2, i_2, t_2, d_2) = s_1 \cup s_2 \cup (t_1 \mathcal{X}_+ i_2)$$

$$(s_1, i_1, t_1, d_1) \otimes_2 (s_2, i_2, t_2, d_2) = i_1 \uplus \text{map } (d_1 +) i_2$$

$$(s_1, i_1, t_1, d_1) \otimes_3 (s_2, i_2, t_2, d_2) = \text{map } (+d_2) t_1 \uplus t_2$$

$$(s_1, i_1, t_1, d_1) \otimes_4 (s_2, i_2, t_2, d_2) = d_1 + d_2$$

The f parts

To apply the Almost Homomorphism Fusion Theorem we also need fused f functions:

$$f_1 x = \{[x]\}$$

$$f_2 x = [[x]]$$

$$f_3 x = [[x]]$$

$$f_4 x = [x]$$

$$f'_1 x = ((map^s sum) \circ f_1) x = \{x\}$$

$$f'_2 x = ((map sum) \circ f_2) x = [x]$$

$$f'_3 x = ((map sum) \circ f_3) x = [x]$$

$$f'_4 x = (sum \circ f_4) x = x$$

And finally we obtain:

$$map^s sum \circ \pi_1 \circ hom (\Delta_1^4 \oplus_i) (\Delta_1^4 f_i) = \pi_1 \circ hom (\Delta_1^4 \otimes_i) (\Delta_1^4 f'_i)$$

Finishing up

$$mss = max^s \circ \pi_1 \circ hom (\Delta_1^4 \otimes_i) (\Delta_1^4 f'_i)$$

We can fuse max^s with the remaining almost homomorphism with

$$H = max^s \times max \times max \times id$$

and get the final almost homomorphism

$$mss = \pi_1 \circ hom (\Delta_1^4 \otimes'_i) id$$

where

$$(s_1, i_1, t_1, d_1) \otimes'_1 (s_2, i_2, t_2, d_2) = s_1 \uparrow s_2 \uparrow (t_1 + i_2)$$

$$(s_1, i_1, t_1, d_1) \otimes'_2 (s_2, i_2, t_2, d_2) = i_1 \uparrow (d_1 + i_2)$$

$$(s_1, i_1, t_1, d_1) \otimes'_3 (s_2, i_2, t_2, d_2) = (t_1 + d_2) \uparrow t_2$$

$$(s_1, i_1, t_1, d_1) \otimes'_4 (s_2, i_2, t_2, d_2) = d_1 + d_2$$

What has been accomplished?

We went from

$$mss = max^s \circ (map^s sum) \circ segs$$

to

$$mss = \pi_1 \circ hom (\Delta_1^4 \otimes'_i) id$$

- Specification was an inefficient $O(n^2)$ algorithm involving computationally difficult objects (sets).
- Final derivation has only (tuples of) scalars as intermediate values.
 - ▶ You could easily translate the final bit into Futhark or any other parallel language.

What do we make of all this?

- Easy to get lost in the details of formalisms.
- Only a subset of the programs described can be implemented properly in real languages.
- These theorems and techniques **do not give us algorithms for free**.
- They **help us structure our search for efficient implementations**.
- Usually easier to find small functions that satisfy specific properties than to come up with algorithms from scratch using sheer creativity.
- (But maybe some of this could be mechanised? Maybe a fun (research) project!)