# Cost models and advanced Futhark programming

Troels Henriksen (athas@sigkill.dk)
Some material by Martin Elsman

DIKU
University of Copenhagen

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Parallel cost models

Prefix sums (scans)

Using scans

Auxiliary

# The need for cost models

Which is better?

```
import numpy as np

def inc_scalar(x):
  for i in range(len(x)):
    x[i] = x[i] + 1

def inc_par(x):
  return x + np.ones(x.shape)
```

# The need for cost models

Which is better?

```
import numpy as np

def inc_scalar(x):
  for i in range(len(x)):
    x[i] = x[i] + 1

def inc_par(x):
  return x + np.ones(x.shape)
```

Intuitively, `inc_par` is better because it is “more parallel”.

**Parallel cost models make this notion precise.**

# Dependency DAG for `inc_scalar`

```
           len(x)
             |
         range(...)
        /          \
  x[0]               1
        \          /
   x[0] = ...  + ...
             |
            ...
             |
  x[n-1] = x[n-1] + 1
```

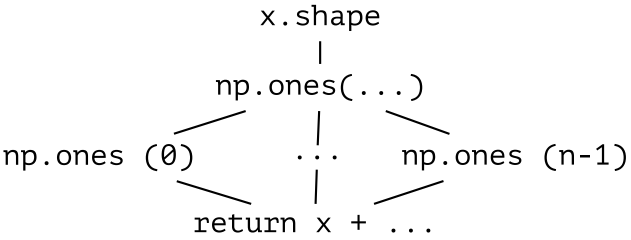
- Total count of nodes is the *work*, $W(p)$.
- Length of longest path from a leaf to the root is the *span*.
- **With an infinite number of processors, if a program $p$ has span $k$, written $S(p) = k$, the program can execute in $O(k)$ time.**
- Here, $W(p) = O(n)$, $S(p) = O(n)$.

# Dependency DAG for `inc_par`

```
                x.shape
                   |
             np.ones(...)
           /       |       \
np.ones (0)       ...       np.ones (n-1)
           \       |       /
            return x + ...
```

**What is the work and span complexity?**

# Dependency DAG for `inc_par`




**What is the work and span complexity?**

- $W(p) = O(n)$
- $S(p) = O(1)$

# Parallel cost model based on work and span

Instead of giving just a simple cost-model based on the total notion of work carried out by a program, we give instead a *refined* cost model, which aims at providing both:

- a notion of how much total work ($W$) the program does;
- a notion of the *span*[1] ($S$) of the program, specifying the maximum depth required by the computation.

**Notice:**

- The span is the length of the longest sequence of operations that must be performed sequentially due to data dependencies.
- With an infinite number of processors, if a program $p$ has span $k$, written $S(p) = k$, the program can execute in $O(k)$ time.

[1] Sometimes also called *depth*.

# Brent's Theorem (1974) (or Lemma, or Law...)

Writing $T_i$ for the time taken to execute an algorithm on $i$ processors, Brent's Theorem states that

$$\frac{T_1}{p} \leq T_p \leq T_\infty + \frac{T_1}{p}$$

**Proof sketch:** At level $j$ of the DAG there are $M_j$ independent operations, which can clearly be executed by $p$ processors in time

$$\left\lceil \frac{M_j}{p} \right\rceil$$

Sum these for each level of the DAG. $\square$

## Ramification

We can simulate an "infinitely parallel" machine on a real machine at an overhead proportional to the amount of "missing" hardware parallelism.

# Language-based cost models

- Tallying up levels in an infinite DAG is impractical for real programs. Instead we prefer a *language-based cost model*
- E.g. $W(x + y)$ is defined as $W(x) + W(y)$.
- The following slides define work and span cost for a small subset of Futhark.
- Write $[\![ e ]\!]$ for the result of evaluating expression $e$ (we are being intuitive about scopes and such).

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## Cost model must be implementable

*A provable time and space efficient implementation of NESL*—Guy Blelloch and John Greiner, 1996

# Simple cases

$$
\begin{aligned}
W(v) &= \\
S(v) &= \\
W(e_1 \ \oplus \ e_2) &= \\
S(e_1 \ \oplus \ e_2) &= \\
W(\backslash x \texttt{ -> } e) &= \\
S(\backslash x \texttt{ -> } e) &=
\end{aligned}
$$

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$$\begin{aligned}
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W(e_1 \ \oplus \ e_2) &= W(e_1) + W(e_2) + 1 \\
S(e_1 \ \oplus \ e_2) &= S(e_1) + S(e_2) + 1 \\
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$$
\begin{aligned}
W([e_1, \cdots, e_n]) &= \\
S([e_1, \cdots, e_n]) &= \\
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\begin{aligned}
W([e_1, \cdots, e_n]) &= W(e_1) + \ldots + W(e_n) + 1 \\
S([e_1, \cdots, e_n]) &= S(e_1) + \ldots + S(e_n) + 1 \\
W((e_1, \cdots, e_n)) &= W(e_1) + \ldots + W(e_n) + 1 \\
S((e_1, \cdots, e_n)) &= S(e_1) + \ldots + S(e_n) + 1
\end{aligned}
$$

# Interesting cases

$$W(\texttt{iota}\ e) =$$
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$$\text{where } [\![ e_1 ]\!] = \backslash x\ \texttt{->}\ e'$$
$$S(e_1\ e_2) = S(e_1) + S(e'[x \mapsto [\![ e_2 ]\!]]) + 1$$
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# Work and span of map

$$W(\texttt{map}\ e_1\ e_2) =$$

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&W(\texttt{map}\ e_1\ e_2) = \\
&\quad W(e_1) + W(e_2) + W(e'[x \mapsto v_1]) + \ldots + W(e'[x \mapsto v_n]) \\
&\quad \text{where } [\![\, e_1 \,]\!] = \backslash x\ \texttt{->}\ e' \\
&\quad \text{where } [\![\, e_2 \,]\!] = [v_1, \cdots, v_n]
\end{aligned}
$$

$$
\begin{aligned}
&S(\texttt{map}\ e_1\ e_2) = \\
&\quad S(e_1) + S(e_2) + max(S(e'[x \mapsto v_1]), \ldots, S(e'[x \mapsto v_n])) + 1 \\
&\quad \text{where } [\![\, e_1 \,]\!] = \backslash x\ \texttt{->}\ e' \\
&\quad \text{where } [\![\, e_2 \,]\!] = [v_1, \cdots, v_n]
\end{aligned}
$$

# Reduction by contraction

```
let npow2 (n:i64) : i64 =
  loop a = 2 while a < n do 2*a

-- Pad a vector to make its size a power of two
let padpow2 [n] (ne: i32) (v:[n]i32) : []i32 =
  concat v (replicate (npow2 n - n) ne)

-- Reduce by contraction
let red (xs : []i32) : i32 =
  let xs =
    loop xs = padpow2 0 xs
    while length xs > 1 do
      let n = length xs / 2
      in map2 (+) xs[0:n] xs[n:2*n]
  in xs[0]
```

# Work and span of `loop`

$$W(\texttt{loop}\ x = e_1\ \texttt{while}\ e_2\ \texttt{do}\ e_3) =$$

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# Work and Span for `npow2 n`

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By inspection, we have

$$W(\texttt{npow2 n}) = S(\texttt{npow2 n}) = O(\log \texttt{n})$$

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Because $\texttt{npow2 n} \leq 2\,\texttt{n}$, we have (where $\texttt{n} = \texttt{length v}$)

$$\begin{aligned} W(\texttt{padpow2 ne v}) &= W(\texttt{concat}\ v\ (\texttt{replicate}\ (\texttt{npow2 n - n})\ \texttt{ne})) \\ &= O(\texttt{n}) \end{aligned}$$

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### Work and Span for `red`

Each loop iteration in `red` has span $O(1)$. Because the loop is iterated at-most $\log(2\,\texttt{n})$ times, we have (where $\texttt{n} = \texttt{length v}$)

$$W(\texttt{red v}) = O(\texttt{n}) + O(\texttt{n}/2) + O(\texttt{n}/4) + \cdots + O(1) =$$

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# Work efficiency

**A parallel algorithm is said to be *work efficient* if it has at most the same work as the best sequential algorithm.**

Is `red` work efficient?

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**A parallel algorithm is said to be *work efficient* if it has at most the same work as the best sequential algorithm.**

Is `red` work efficient?

**Yes**, because it does $O(n)$ work, which is as good as a sequential summation.

Is it also *efficient*?

# Performance Compared to the Built-in Reduction SOAC

```
-- ==
-- entry: test_red test_reduce
-- random input { [10000000]i32 }
entry test_red = red
entry test_reduce = reduce (+) 0
```

# Performance Compared to the Built-in Reduction SOAC

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```
$ futhark bench --backend=opencl reduce.fut
Compiling reduce.fut...
Results for reduce.fut:test_red:
dataset [10000000]i32:    4675.40μs
Results for reduce.fut:test_reduce:
dataset [10000000]i32:     273.80μs
```

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**If you are not using `futhark bench`, then you are probably doing it wrong.**

Parallel cost models

Prefix sums (scans)

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# Inclusive and exclusive prefix sum

## Exclusive prefix sum (“prescan”)

Given

$$[1, 2, 3, 4]$$

produce

$$[0, 1, 3, 6]$$

## Inclusive prefix sum

Given

$$[1, 2, 3, 4]$$

produce

$$[1, 3, 6, 10]$$

# Prefix sums are scans

Generalising the addition and zero used by a prefix sum to an arbitrary associative operator $\oplus$ and neutral element $0_{\oplus}$, we get *scan*.

```
-- The scan in Futhark is inclusive.
> scan (+) 0 [1,2,3,4]
[1, 3, 6, 10]
```

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-- The scan in Futhark is inclusive.
> scan (+) 0 [1,2,3,4]
[1, 3, 6, 10]
```

- Scans are a fundamental tool for parallelising seemingly-sequential algorithms.
- Let us see how scans can be computed in parallel.

# Sequential prefix sum

```
acc = 0
for i < n:
  acc = acc + input[i]
  scanned[i] = acc
```

# Sequential prefix sum

```
acc = 0
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```

Work: $O(n)$

Span: $O(n)$

# Brute force

To calculate the prefix sum of $[x_0, \ldots, x_{n-1}]$, compute

$$
\begin{aligned}
[&sum([x_0]) \\
&sum([x_0, x_1]) \\
&\vdots \\
&sum([x_0, x_1, \ldots, x_{n-1}])]
\end{aligned}
$$

Assume $S(sum([x_0, \ldots, x_{n-1}])) = log_2(n)$.

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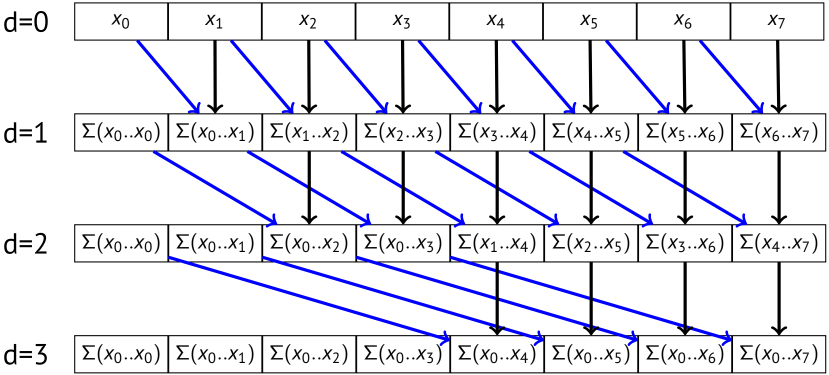
$$\begin{aligned}[&sum([x_0]) \\ &sum([x_0, x_1]) \\ &\vdots \\ &sum([x_0, x_1, \ldots, x_{n-1}])]\end{aligned}$$

Assume $S(sum([x_0, \ldots, x_{n-1}])) = log_2(n)$.

Work: $O(\sum_{i<n} i) = O(n^2)$
Span: $O(max(S(sum([x_0])), \ldots S(sum([x_0, \ldots, x_{n-1}])))) = O(log_2(n))$

**Terrible.** The sequential implementation is faster for large $n$!

# Hillis–Steele scan (1986)




For each $d$, element $x_i^d$ is updated by $x_{i-2^i}^{d-1} + x_i^{d-1}$.

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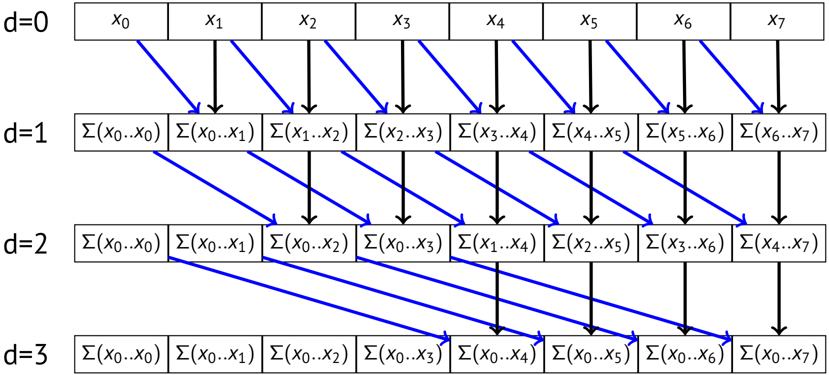



For each $d$, element $x_i^d$ is updated by $x_{i-2^i}^{d-1} + x_i^{d-1}$.

Work: For $n = 2^m$, $O(\sum_{i<m} 2^m - 2^i) = O(n \log(n))$

Span: $\log(n)$

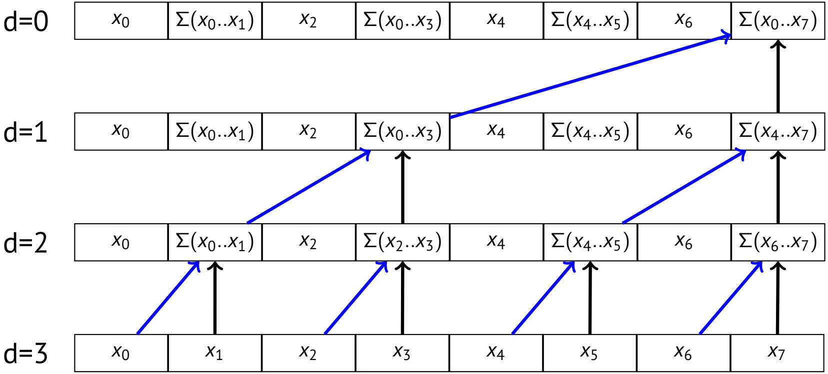
# Work-efficient scan

## Two passes

| | |
|---|---|
| Upsweep | Build a balanced binary tree of partial sums stored in every other cell. |
| Downsweep | Use the partial sums to fill out the missing parts. |

**The binary tree does not actually exist as a recursive pointer structure, but is just a communications concept.**

# Upsweep ("reduction phase")




$$x_i^d = x_{i-2^{m-d-1}}^{d+1} + x_i^{d+1}$$

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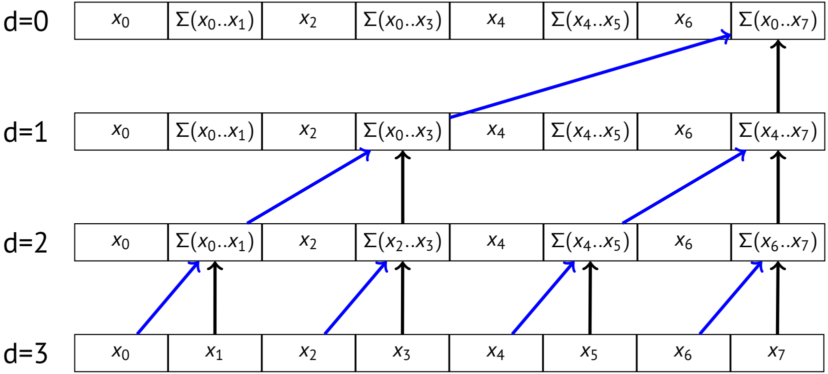


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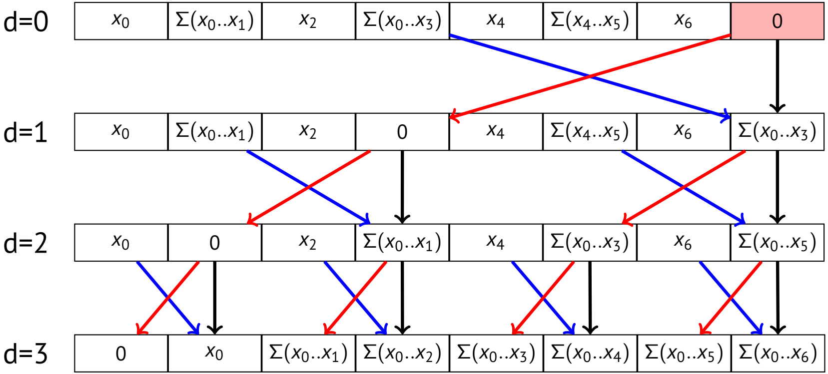


Inverse indexing of the upsweep phase.

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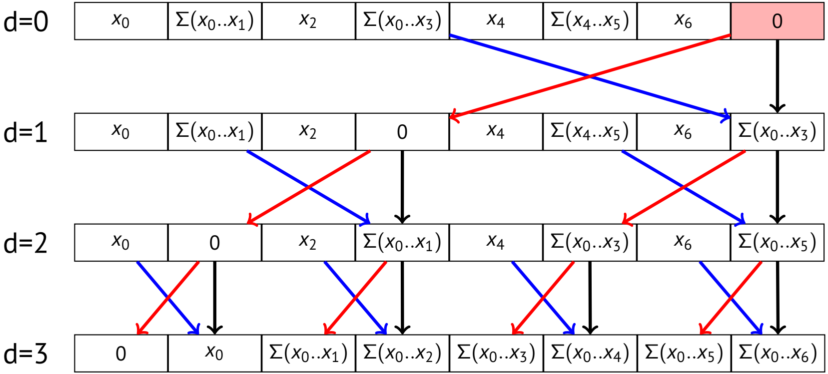


Inverse indexing of the upsweep phase.

Work: For $n = 2^m$, $O(\sum_{i \leq m} 2^i) = O(n)$

Span: $\log(n)$

# Work efficient scan

**Complexity of *scan* on size-$n$ input**

Work: $O(n)$

Span: $\log(n)$

- Optimal, as *reduce* is the same.
- Can now depend on scan as a relatively cheap building block.

**Real-world scan implementations are often very different for technical reasons, but we can depend on these asymptotics when analysing and designing parallel algorithms.**

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**Using scans**

Auxiliary

# Filtering

Suppose we wish to remove negative elements from the list

```
let as = [-1, 2, -3, 4, 5, -6]
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let keep = map (\a -> if a >= 0 then 1 else 0) as
-- [ 0, 1, 0, 1, 1, 0]
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let offsets1 = scan (+) 0 keep
-- [ 0, 1, 1, 2, 3, 3]
```

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-- [ 0, 1, 0, 1, 1, 0]

let offsets1 = scan (+) 0 keep
-- [ 0, 1, 1, 2, 3, 3]

let offsets = map (\x -> x - 1) offsets1
-- [-1, 0, 0, 1, 2, 2]
```

# Filtering

Suppose we wish to remove negative elements from the list

```
let as = [-1, 2, -3, 4, 5, -6]
```

For each element, see if we want to keep it:

```
let keep = map (\a -> if a >= 0 then 1 else 0) as
-- [ 0, 1, 0, 1, 1, 0]

let offsets1 = scan (+) 0 keep
-- [ 0, 1, 1, 2, 3, 3]

let offsets = map (\x -> x - 1) offsets1
-- [-1, 0, 0, 1, 2, 2]
```

`offsets[i]` now indicates position in filtered list iff

`keep[1] == 1`

# `scatter`

`scatter xs is vs` computes equivalent of the imperative pseudocode

```
for j < n:
  xs[is[j]] = vs[j]
```

- Out-of-bound writes are ignored
- Writing different values to same index is *undefined*[2]
- Work $O(n)$, span $O(1)$

**Just what we need for filtering!**

[2] `reduce_by_index` handles conflicts with provided operator.

# scatter

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- Out-of-bound writes are ignored
- Writing different values to same index is *undefined*[2]
- Work $O(n)$, span $O(1)$

**Just what we need for filtering!**

```
scatter (replicate (last offsets1) 0)
        (map2 (\i k -> if k == 1 then i else -1)
              offsets keep)
        as
```

[2] `reduce_by_index` handles conflicts with provided operator.

# Implementing `filter`

```
let filter 'a (p: a -> bool) (as: []a): []a =
  let keep = map (\a -> if p a then 1 else 0) as
  let offsets1 = scan (+) 0 keep
  let num_to_keep = reduce (+) 0 keep
  in if num_to_keep == 0
     then []
     else scatter (replicate num_to_keep as[0])
                  (map2 (\i k -> if k == 1
                                 then i-1
                                 else -1)
                        offsets1 keep)
                  as
```

# Radix sort

- Many classical sorting algorithms are a poor fit for data parallelism, but *radix sort* works well.
- Radix-2 sort works by repeatedly partitioning elements according to one bit at a time, while preserving the ordering of the previous steps.

# Example with radix-10

| | | | | | | |
|---|---|---|---|---|---|---|
| 3 2 6 | | 6 9 **0** | | 7 **0** 4 | | **3** 2 4 |
| 4 5 3 | | 7 5 **1** | | 6 **0** 8 | | **4** 3 8 |
| 6 0 8 | | 4 5 **3** | | 3 **2** 6 | | **4** 5 6 |
| 8 3 5 | | 7 0 **4** | | 8 **3** 5 | | **6** 0 5 |
| 7 5 1 | $\Rightarrow$ | 8 3 **5** | $\Rightarrow$ | 4 **3** 5 | $\Rightarrow$ | **6** 9 5 |
| 4 3 5 | | 4 3 **5** | | 7 **5** 1 | | **7** 0 1 |
| 7 0 4 | | 3 2 **6** | | 4 **5** 3 | | **7** 5 3 |
| 6 9 0 | | 6 0 **8** | | 6 **9** 0 | | **8** 3 0 |

# Example with radix-10

| | | | | | | |
|---|---|---|---|---|---|---|
| 3 2 6 | | 6 9 0 | | 7 0 4 | | 3 2 4 |
| 4 5 3 | | 7 5 1 | | 6 0 8 | | 4 3 8 |
| 6 0 8 | | 4 5 3 | | 3 2 6 | | 4 5 6 |
| 8 3 5 | $\Rightarrow$ | 7 0 4 | $\Rightarrow$ | 8 3 5 | $\Rightarrow$ | 6 0 5 |
| 7 5 1 | | 8 3 5 | | 4 3 5 | | 6 9 5 |
| 4 3 5 | | 4 3 5 | | 7 5 1 | | 7 0 1 |
| 7 0 4 | | 3 2 6 | | 4 5 3 | | 7 5 3 |
| 6 9 0 | | 6 0 8 | | 6 9 0 | | 8 3 0 |

- **Radix sort is not as general as a comparison-based sort.**
- Assumes sorting key can be decomposed into "digits".

# Sorting xs : [n]u32 by bit b

```
-- 1 if bit b set.
let check_bit b x =
  (i64.u32 (x >> u32.i32 b)) & 1
```

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let bits_neg = map (1-) bits
let offs = reduce (+) 0 bits_neg
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let bits_neg = map (1-) bits
let offs = reduce (+) 0 bits_neg
```

## Example

```
b        = 0
xs       = [0, 1, 2, 3, 4]
bits     = [0, 1, 0, 1, 0]
bits_neg = [1, 0, 1, 0, 1]
offs     = 3
```

```
let idxs0 = map2 (*)
                 bits_neg
                 (scan (+) 0 bits_neg)
let idxs1 = map2 (*)
                 bits
                 (map (+offs) (scan (+) 0 bits))
```

```
let idxs0 = map2 (*)
                 bits_neg
                 (scan (+) 0 bits_neg)
let idxs1 = map2 (*)
                 bits
                 (map (+offs) (scan (+) 0 bits))
```

### Example

```
bits                    = [0, 1, 0, 1, 0]
bits_neg                = [1, 0, 1, 0, 1]
offs                    = 3
idxs0                   = [1, 0, 2, 0, 3]
idxs1                   = [0, 4, 0, 5, 0]
map2 (+) idxs0 idxs1 = [1, 4, 2, 5, 3]
```

**Then `scatter` as when filtering.**

# The whole step

```
let check_bit b x = (i64.u32 (x >> u32.i32 b)) & 1

let radix_sort_step [n] (xs: [n]u32) (b: i32): [n]u32 =
  let bits = map (check_bit b) xs
  let bits_neg = map (1-) bits
  let offs = reduce (+) 0 bits_neg
  let idxs0 = map2 (*) bits_neg
                   (scan (+) 0 bits_neg)
  let idxs1 = map2 (*) bits
                   (map (+offs) (scan (+) 0 bits))
  let idxs2 = map2 (+) idxs0 idxs1
  let idxs  = map (\x->x-1) idxs2
  let xs' = scatter (copy xs) idxs xs
  in xs'
```

# Radix sort in Futhark

```
let radix_sort [n] (xs: [n]u32): [n]u32 =
  loop xs for i < 32 do radix_sort_step xs i
```

See worked example at
`https://futhark-lang.org/examples/radix-sort.html`

# Segmented scan

```
val segmented_scan [n] 't
    :  (op: t -> t -> t) -> (ne: t)
    -> (flags: [n]bool) -> (as: [n]t)
    -> [n]t
```

`true` starts a segment and `false` continues a segment.

## Example

```
segmented_scan (+) 0
  [true, false, true, false, false, true]
  [0, 1, 2, 3, 4, 5]
== scan (+) 0 [0,1] ++
   scan (+) 0 [2,3,4] ++
   scan (+) 0 [5]
== [0, 1, 2, 5, 9, 5]
```

# Segmented reduction

```
val segmented_reduce [n] 't
    :  (op: t -> t -> t) -> (ne: t)
    -> (flags: [n]bool) -> (as: [n]t)
    -> []t
```

## Example

```
segmented_reduce (+) 0
  [true, false, true, false, false, true]
  [0, 1, 2, 3, 4, 5]
== reduce (+) 0 [0,1] ++
   reduce (+) 0 [2,3,4] ++
   reduce (+) 0 [5]
== [1, 9, 5]
```

# Generalised histograms

Like `scatter`, but uses a provided `reduce`-like operator to handle multiple writes to same index.

### Type

```
val reduce_by_index [k] [n] 'a :
     (dest: *[k]a)
  -> (f: a -> a -> a) -> (ne: a)
  -> (is: [n]i64) -> (vs: [n]a) -> *[k]a
```

### Semantics

```
for index in 0..k-1:
  i = is[index]
  v = vs[index]
  dest[i] = f(as[i], v)
```

Futhark uses parallel implementation with GPU *atomics*.

# Proving associativity and neutral elements

```
let op (x, i) (y, j) : (i32, i32) =
  if x < y then (y, j) else (x, i)

let argmax [n] (xs: [n]i32) =
  reduce op
         (i32.smallest, -1)
         (zip xs (iota n))
```

- Is op associative?
- Is `(i32.smallest, -1)` a neutral element?

# argmax: associativity

First, inline definitions:

```
   (a `op` b) `op` c
== ((ax, ai) `op` (bx, bi)) `op` (cx, ci)
== let (x, i) = if ax < bx then (bx, bi)
                           else (ax, ai)
   in if x < cx then (cx, ci)
                else (x, i)
```

---

```
   a `op` (b `op` c)
== (ax, ai) `op` ((bx, bi) `op` (cx, ci))
== let (x, i) = if bx < cx then (cx, ci)
                           else (bx, bi)
   in if ax < x then (x, i)
                else (ax, ai)
```

Then enumerate all possible comparisons between `ax`, `bx`, and `cx` and show that these two expressions are equivalent.

# E.g. for `!(ax < bx) && bx < cx && cx < ax`

```
   let (x, i) = if ax < bx then (bx, bi)
                            else (ax, bx)
   in if x < cx then (cx, ci)
                 else (x, i)
== if ax < cx then (cx, ci)
               else (ax, ai)
== (ax, ai)
```

---

```
   let (x, i) = if bx < cx then (cx, ci)
                            else (bx, bi)
   in if ax < x then (x, i)
                 else (ax, ai)
== if ax < cx then (cx, ci)
               else (ax, ai)
== (ax, ai)
```

□

# argmax: neutral element

Similarly, by equational reasoning.

```
   (a `op` (i32.smallest, -1))
== ((x, i) `op` (i32.smallest, -1))
== if x < i32.smallest then (i32.smallest, -1)
                       else (x, i)
== (x, i)
```

---

```
   ((i32.smallest, -1) `op` a)
== ((i32.smallest, -1) `op` (x, i))
== if i32.smallest < x then (x, i)
                       else (i32.smallest, -1)
== (x, i)
```

□

# A more calculational approach

`https://byorgey.wordpress.com/2020/02/23/what-would-dijkstra-do-proving-the-associativity-of-min/`

- Worth a read!
- More elegant and concise, but requires more creative thinking to characterise a useful property of the operator.

# Commutativity?

**Exercise for home:** The `argmax` operator is not commutative. Try to come up with a counterexample, and see if you can change its definition such that it becomes commutative.

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**Exercise for home:** The `argmax` operator is not commutative. Try to come up with a counterexample, and see if you can change its definition such that it becomes commutative.

### Commutative reductions

Futhark has a `reduce_comm` function that can be used for commutative operators. This runs faster than normal `reduce`. Not necessary for built-in operators.

# Summary

- *Work* measures the total number of operations, *span* measures the longest chain of dependencies.
- Language-based cost models let us reason about program performance in a hardware-agnostic and composable way.
- Scans are a useful building block in advanced data parallel algorithms, but an efficient implementation is not straightforward.