

Flattening Irregular Nested Parallelism

Cosmin E. Oancea
`cosmin.oancea@diku.dk`

Department of Computer Science (DIKU)
University of Copenhagen

December 2022 DPP Lecture Slides

Parallel Basic Blocks

Flattening Nested and Irregular Parallelism

- How To Flatten: Intuition

- Irregular Multi-Dimensional Array Representation

- Flattening at A High-Level

- Rules For Flattening

- Flattening Quicksort

- Flattening Prime-Number (Sieve) Computation

Zip, Unzip, iota, replicate

- $\text{zip} : [n]_{\alpha_1} \rightarrow [n]_{\alpha_2} \rightarrow [n](\alpha_1, \alpha_2)$
- $\text{zip } [a_1, \dots, a_n] [b_1, \dots, b_n] \equiv [(a_1, b_1), \dots, (a_n, b_n)],$

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- $\text{unzip} : [n](\alpha_1, \alpha_2) \rightarrow ([n]_{\alpha_1}, [n]_{\alpha_2})$
- $\text{unzip } [(a_1, b_1), \dots, (a_n, b_n)] \equiv ([a_1, \dots, a_n], [b_1, \dots, b_n]),$
- In some sense zip/unzip are syntactic sugar

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- $\text{unzip} : [n]_{(\alpha_1, \alpha_2)} \rightarrow ([n]_{\alpha_1}, [n]_{\alpha_2})$
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- In some sense zip/unzip are syntactic sugar
- $\text{replicate} : (n: \text{int}) \rightarrow \alpha \rightarrow [n]_{\alpha}$
- $\text{replicate } n \ a \equiv [a, a, \dots, a],$

Zip, Unzip, iota, replicate

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- In some sense zip/unzip are syntactic sugar
- $\text{replicate} : (n: \text{int}) \rightarrow \alpha \rightarrow [n]_{\alpha}$
- $\text{replicate } n \ a \equiv [a, a, \dots, a],$
- $\text{iota} : (n: \text{int}) \rightarrow [n]_{\text{int}}$
- $\text{iota } n \equiv [0, 1, \dots, n-1]$

Note: in Haskell zip does not expect same-length arrays;
in Futhark it does!

Map, Reduce, and Scan Types and Semantics

- $[n]\alpha$ denotes the type of an array of n elements of type α .
- $\text{map} : (\alpha \rightarrow \beta) \rightarrow [n]\alpha \rightarrow [n]\beta$
 $\text{map } f \ [x_1, \dots, x_n] = [f \ x_1, \dots, f \ x_n],$
i.e., $x_i : \alpha, \forall i$, and $f : \alpha \rightarrow \beta$.
- $\text{reduce} : (\alpha \rightarrow \alpha \rightarrow \alpha) \rightarrow \alpha \rightarrow [n]\alpha \rightarrow \alpha$
 $\text{reduce } \odot \ e \ [x_1, x_2, \dots, x_n] = e \odot x_1 \odot x_2 \odot \dots \odot x_n,$
i.e., $e : \alpha,$ $x_i : \alpha, \forall i$, and $\odot : \alpha \rightarrow \alpha \rightarrow \alpha$.
- $\text{scan}^{\text{exc}} : (\alpha \rightarrow \alpha \rightarrow \alpha) \rightarrow \alpha \rightarrow [n]\alpha \rightarrow [n]\alpha$
 $\text{scan}^{\text{exc}} \odot \ e \ [x_1, \dots, x_n] = [e, e \odot x_1, \dots, e \odot x_1 \odot \dots \odot x_{n-1}]$
i.e., $e : \alpha,$ $x_i : \alpha, \forall i$, and $\odot : \alpha \rightarrow \alpha \rightarrow \alpha$.
- $\text{scan}^{\text{inc}} : (\alpha \rightarrow \alpha \rightarrow \alpha) \rightarrow \alpha \rightarrow [n]\alpha \rightarrow [n]\alpha$
 $\text{scan}^{\text{inc}} \odot \ e \ [x_1, \dots, x_n] = [e \odot x_1, \dots, e \odot x_1 \odot \dots \odot x_n]$
i.e., $e : \alpha,$ $x_i : \alpha, \forall i$, and $\odot : \alpha \rightarrow \alpha \rightarrow \alpha$.

Map2, Filter

- $\text{map2} : (\alpha_1 \rightarrow \alpha_2 \rightarrow \beta) \rightarrow [n]\alpha_1 \rightarrow [n]\alpha_2 \rightarrow [n]\beta$
- $\text{map2} \odot [a_1, \dots, a_n] [b_1, \dots, b_n] \equiv [a_1 \odot b_1, \dots, a_n \odot b_n]$
- $\text{map3} \dots$

Map2, Filter

- $\text{map2} : (\alpha_1 \rightarrow \alpha_2 \rightarrow \beta) \rightarrow [n]\alpha_1 \rightarrow [n]\alpha_2 \rightarrow [n]\beta$
- $\text{map2} \odot [a_1, \dots, a_n] [b_1, \dots, b_n] \equiv [a_1 \odot b_1, \dots, a_n \odot b_n]$
- $\text{map3} \dots$
- $\text{filter} : (\alpha \rightarrow \text{Bool}) \rightarrow [n]\alpha \rightarrow [m]\alpha \ (m \leq n)$
- $\text{filter } p [a_1, \dots, a_n] = [a_{k_1}, \dots, a_{k_m}]$ such that $k_1 < k_2 < \dots < k_m$, and denoting $\bar{k} = k_1, \dots, k_m$, we have $(p \ a_j == \text{true}) \ \forall j \in \bar{k}$, **and** $(p \ a_j == \text{false}) \ \forall j \notin \bar{k}$.

Note: in Haskell `map2`, `map3` do not expect same-length arrays; in Futhark they do!

Scatter: A Parallel Write Operator

Scatter **updates in parallel** a base array with a set of values at specified indices:

$\text{scatter} : *[m]\alpha \rightarrow [n]\text{int} \rightarrow [n]\alpha \rightarrow *[m]\alpha$

A (data vector) = [b0, b1, b2, b3]

I (index vector) = [2, 4, 1, -1]

X (input array) = [a0, a1, a2, a3, a4, a5]

$\text{scatter } X \mid A = [a0, b2, b0, a3, b1, a5]$

Scatter: A Parallel Write Operator

Scatter **updates in parallel** a base array with a set of values at specified indices:

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I (index vector) = [2, 4, 1, -1]

X (input array) = [a0, a1, a2, a3, a4, a5]

scatter X I A = [a0, b2, b0, a3, b1, a5]

scatter has $D(n) = \Theta(1)$ and $W(n) = \Theta(n)$,
i.e., requires n update operations (n is the size of I or A , not of X !).

- 1 Array X is consumed by **scatter**; following uses of X are illegal!
- 2 Similarly, X can alias neither I nor A !

In Futhark, **scatter** check and ignores the indices that are out of bounds (no update is performed on those). This is useful for padding the iteration space in order to obtain regular parallelism.

Partition2/Filter Implementation

`partition2: ($\alpha \rightarrow \text{Bool}$) \rightarrow $[n]\alpha \rightarrow (\text{i32}, [n]\alpha)$`

In result, the elements satisfying the predicate occur before the others. **Can be implemented by means of map, scan, scatter.**

```
let partition2 't [n] (dummy: t)
  (cond: t -> bool) (X: [n]t) :
  (i32, [n]t) =
```

Assume $X = [5, 4, 2, 3, 7, 8]$, and
cond is T(rue) for even nums.

```
let cs = map cond X
let tfs = map (\ f -> if f then 1
                      else 0) cs

let isT = scan (+) 0 tfs
let i = isT[n-1]

let ffs = map (\ f -> if f then 0
                      else 1) cs
let isF = map (+ i) <| scan (+) 0 ffs
let inds = map (\ (c, iT, iF) ->
                  if c then iT-1
                  else iF-1
                ) (zip3 cs isT isF)

let tmp = replicate n dummy
in (i, scatter tmp inds X)
```

Partition2/Filter Implementation

`partition2: ($\alpha \rightarrow \text{Bool}$) \rightarrow $[n]\alpha \rightarrow (\text{i32}, [n]\alpha)$`

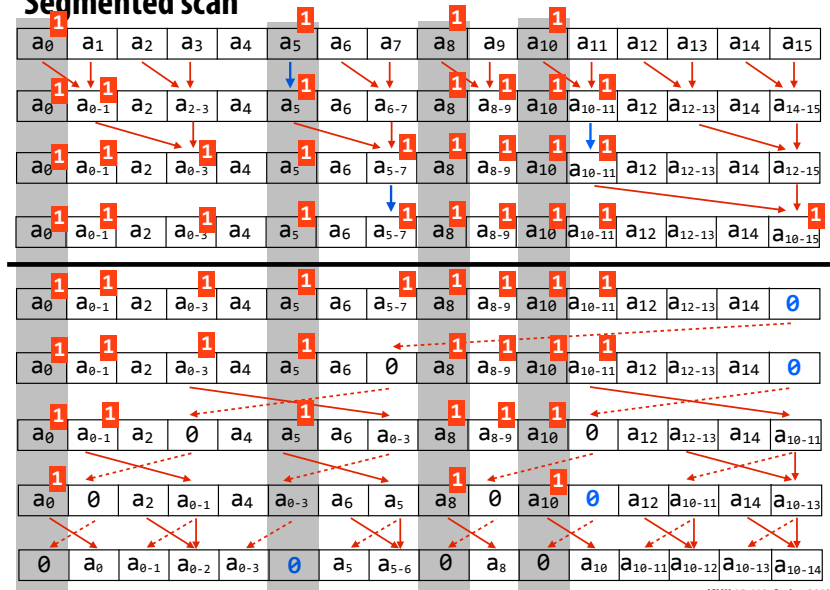
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```
let partition2 't [n] (dummy: t)
  (cond: t  $\rightarrow$  bool) (X: [n]t) :
  (i32, [n]t) =
  let cs = map cond X
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  let inds = map (\(c, iT, iF)  $\rightarrow$ 
                      if c then iT-1
                      else iF-1
                  ) (zip3 cs isT isF)
  let tmp = replicate n dummy
  in (i, scatter tmp inds X)
```

Assume $X = [5, 4, 2, 3, 7, 8]$, and
cond is $T(\text{rue})$ for even nums.
 $n = 6$
 $cs = [F, T, T, F, F, T]$
 $tfs = [0, 1, 1, 0, 0, 1]$
 $isT = [0, 1, 2, 2, 2, 3]$
 $i = 3$
 $ffs = [1, 0, 0, 1, 1, 0]$
 $isF = [4, 4, 4, 5, 6, 6]$
 $inds = [3, 0, 1, 4, 5, 2]$
 $flags = [3, 0, 0, 3, 0, 0]$
 $Result = [4, 2, 8, 5, 3, 7]$

Segmented scan



Segmented Scan Is a Sort of Scan

Futhark Implementation:

```
let sgmscan 't [n] (op: t->t->t) (ne: t)
    (flg : [n]i32) (arr : [n]t) : [n]t =
  let flgs_vals =
    scan ( \ (f1, x1) (f2,x2) ->
      let f = f1 | f2 in
      if f2 != 0 then (f, x2)
      else (f, op x1 x2) )
    (0,ne) (zip flg arr)
  let (_, vals) = unzip flgs_vals
  in vals
```

```
sgmscan (+) 0 [1,0,0,1,0, 0, 0]
               [1,2,3,4,5, 6, 7]
               = = = = = = =
               [1,3,6,4,9,15,22]
```

```
map ( \ row -> scan (+) 0 row)
    [[1,2,3], [4,5, 6, 7]]
    = = = = = = =
    [[1,3,6], [4,9,15,22]]
```

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- Flattening Prime-Number (Sieve) Computation

How to Flatten? A Relatively Simple Case

```
let arr = [1, 2, 3, 4] in  
map (\i -> map (+(i+1)) (iota i)) arr  
-- Result: [[2],[3,4],[4,5,6],[5,6,7,8]]
```

Normalize the code:

How to Flatten? A Relatively Simple Case

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let arr = [1, 2, 3, 4] in
map (\i -> map (+(i+1)) (iota i)) arr
-- Result: [[2],[3,4],[4,5,6],[5,6,7,8]]
```

Normalize the code:

```
map (\i -> let ip1 = i+1 in
            let iot = (iota i) in
            let ip1r= (replicate i ip1) in
            map2 (+) ip1r iot
          ) arr
```

Distribute the map over every instruction in the body

(bottom-up if nest > 2), where \mathcal{F} denotes the flattening transf, and modify the inputs (results) accordingly.

How to Flatten? A Relatively Simple Case

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        ) arr
```

Distribute the map over every instruction in the body

(bottom-up if nest > 2), where \mathcal{F} denotes the flattening transf, and modify the inputs (results) accordingly.

```
 $\mathcal{F}(\text{map } (\lambda i \rightarrow \text{map } (+i) (\text{iota } i)) [0..n-1]) \equiv$ 
1. let ip1s = map (\i -> i+1) arr in -- [2, 3, 4, 5]
2. let iots =  $\mathcal{F}(\text{map } (\lambda i \rightarrow (\text{iota } i)) \text{arr})$  in
3. let ip1rs=  $\mathcal{F}(\text{map2 } (\lambda i \text{ ip1} \rightarrow (\text{replicate } i \text{ ip1})) \text{arr } \text{ip1s})$ 
4. in  $\mathcal{F}(\text{map2 } (\lambda \text{ ip1r } \text{iot} \rightarrow \text{map2 } (+) \text{ip1r } \text{iot}) \text{ip1rs } \text{iots})$ 
```

How to Flatten? A Relatively Simple Case

According to rule (4) iota nested inside a map

(assuming `arr = [1,2,3,4]`):

```
2. let iots =  $\mathcal{F}$ (map (\i -> iota i) arr)
```

≡

```
inds = scanexc (+) 0 arr -- [0,1,3,6]
size = (last inds) + (last arr) -- 6 + 4 = 10
flag = scatter (replicate size 0)
      inds arr
--      [1, 2, 0, 3, 0, 0, 4, 0, 0, 0]
tmp  = replicate size 1
iots = sgmScanexc (+) 0 flag tmp -- [0, 0, 1, 0, 1, 2, 0, 1, 2, 3]
```

How to Flatten? A Relatively Simple Case

According to rule (3) replicate nested inside a map

(assuming `arr = [1,2,3,4]`):

```
3. let ip1rs=  $\mathcal{F}$ (map2 (\ i ip1 -> replicate i ip1) arr ip1s)
 $\equiv$ 
vals = scatter (replicate size 0) inds ip1s -- [2,3,0,4,0,0,5,0,0,0]
ip1rs= sgmScaninc (+) 0 flag vals -- [2,3,3,4,4,4,5,5,5,5]
```

According to rule (2) map nested inside a map

```
 $\mathcal{F}$ (map2 (\ ip1r iot -> map2 (+) ip1r iot) ip1rs iots)
 $\equiv$ 
4. result = map (+) ip1rs iots
-- [2, 3, 3, 4, 4, 4, 5, 5, 5, 5]
-- [0, 0, 1, 0, 1, 2, 0, 1, 2, 3]
-- + + + + + + + + +
-----
-- [2, 3, 4, 4, 5, 6, 5, 6, 7, 8] values
-- [1, 2, 0, 3, 0, 0, 4, 0, 0, 0] flags
```

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Shape-Based Representation

- Two dimensional arrays:

`arr = [[1,2,3], [4], [], [5,6]]`

\Rightarrow

$S_{arr}^0 = [4]$

$S_{arr}^1 = [3, 1, 0, 2]$

$D_{arr} = [1, 2, 3, 4, 5, 6]$

Shape-Based Representation

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\Rightarrow

$S_{arr}^0 = [4]$

$S_{arr}^1 = [3, 1, 0, 2]$

$D_{arr} = [1, 2, 3, 4, 5, 6]$

- Three dimensional arrays:

`arr = [[], [[1,2,3], [4], [], [5,6]], [[7], [], [8,9,10]]]`

\Rightarrow

Shape-Based Representation

- Two dimensional arrays:

$arr = [[1,2,3], [4], [], [5,6]]$

\Rightarrow

$S_{arr}^0 = [4]$

$S_{arr}^1 = [3, 1, 0, 2]$

$D_{arr} = [1, 2, 3, 4, 5, 6]$

- Three dimensional arrays:

$arr = [[], [[1,2,3], [4], [], [5,6]], [[7], [], [8,9,10]]]$

\Rightarrow

$S_{arr}^0 = [3]$

$S_{arr}^1 = [0, 4, 3]$

$S_{arr}^2 = [3, 1, 0, 2, 1, 0, 3]$

$flen_{arr} = 10$

$D_{arr} = [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]$

Assume a n-dimensional array; The following invariant holds:

$length\ S_{arr}^i = reduce\ (+)\ 0\ S_{arr}^{i-1}, \forall 1 \leq i < n$

$length\ D_{arr} = reduce\ (+)\ 0\ S_{arr}^{n-1}$

Flat Representation: Auxiliary Structures

arr = [[], [[1,2,3], [4], [], [5,6]], [[7], [], [8,9,10]]

⇒

$S_{arr}^0 = [3]$

$S_{arr}^1 = [0, 4, 3]$

$S_{arr}^2 = [3, 1, 0, 2, 1, 0, 3]$

$D_{arr} = [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]$

- **Offset Indices (B):** segment-start offset in the flat data:

$B_{arr}^1 = [0, 0, 6]$

$B_{arr}^2 = [0, 3, 4, 4, 6, 7, 7]$

- **Flag Array (F):** start of a segment indicated by a value !=0
for example used for segmented scan operations:

$F_{arr}^1 = [1, 0, 0, 0, 0, 0, 1, 0, 0, 0]$

$F_{arr}^2 = [1, 0, 0, 1, 1, 0, 1, 1, 0, 0]$

- **Segment and Inner indices (II):**

$II_{arr}^1 = [1, 1, 1, 1, 1, 1, 2, 2, 2, 2]$

$II_{arr}^2 = [0, 0, 0, 1, 3, 3, 0, 2, 2, 2]$

$II_{arr}^3 = [0, 1, 2, 0, 0, 1, 0, 0, 1, 2]$

Auxiliary Structures: Intuitive Motivation

Auxiliary structures are useful to optimize the replication of values.

Nested-Execution Example:

```
let xss = [ [1,2,3], [], [5,7] ]  
let ys  = [ 4, 2, 1 ]  
let rss = map2 (\ xs y -> map (+y) xs ) xss ys
```

⇒

```
rss = [ map (+4) [1,2,3], map (+2) [], map (+1) [5,7] ]  
rss = [ [5,6,7], [], [6,8] ]
```

Traditional flattening would replicate the values of y:

```
let (Syss1, Dyss) =  $\mathcal{F}$ (map2 (\ xs y -> replicate (length xs) y) xss ys)  
let Drss = map2 (\ x y -> x + y) Dxss Dyss
```

⇒

```
Dxss = [1, 2, 3, 5, 7]  
      + + + + +  
Dyss = [4, 4, 4, 1, 1]  
      = = = = =  
Drss = [5, 6, 7, 6, 8]
```

Auxiliary Structures: Intuitive Motivation

Auxiliary structures are useful to optimize the replication of values.

Nested-Execution Example:

```
let xss = [ [1,2,3], [], [5,7] ]  
let ys  = [ 4, 2, 1 ]  
let rss = map2 (\ xs y -> map (+y) xs ) xss ys  
=>  
rss = [ map (+4) [1,2,3], map (+2) [], map (+1) [5,7] ]  
rss = [ [5,6,7], [], [6,8] ]
```

Using the auxiliary structures we indirectly access other arrays:

```
let Drss = map2 (\ x sgmind -> x + ys[sgmind]) Dxss II1rss  
=>  
S1rss = [3, 0, 2]  
II1rss = [0, 0, 0, 2, 2]  
Dxss = [1, 2, 3, 5, 7]  
Drss = [1+4, 2+4, 3+4, 5+1, 7+1] = [5, 6, 7, 6, 8]
```

But what have we gained? Creating II_{rss}^1 is as expensive as xss (or better said the expanded yss from the other slide) ...

Auxiliary Structures: Intuitive Motivation

Auxiliary structures are useful to optimize replication:

- they depend only on the shape of the result (created once)
- can indirectly access several lower-dimensional arrays, sharing parallel dimensions!

Nested-Execution Example:

```
let xss = [ [1,2,3], [], [5,7] ]  
let ys  = [ 4, 2, 1 ]  
let zs  = [ 1, 2, 3 ]  
let rss = map3 (\ xs y z -> map (\ x -> x*y + z ) xs ) xss ys zs  
⇒  
rss = [ [5,9,13], [], [8,10] ]
```

Using the auxiliary structures we indirectly access other arrays:

```
let Drss = map2 (\ y sgmind -> x*ys[sgmind] + zs[sgmind]) Dxss II1rss  
⇒  
II1rss = [0, 0, 0, 2, 2]  
Dxss = [1, 2, 3, 5, 7]  
Drss = [1*4+1, 2*4+1, 3*4+1, 5*1+3, 7*1+3] = [5, 9, 13, 8, 10]
```

We build II_{rss}^1 once and reuse it twice; also improves locality!

Auxiliary Structures: Intuitive Motivation

Nested-Execution Example:

```
let xss = [ [1,3], [2] ]
let yss = [ [2], [4,5] ]
let rss = map2 (\xs ys -> map (\x -> map (+x) ys) xs) xss yss
⇒
rss = [ [[3],[5]], [[6,7]] ]
```

Using the auxiliary structures we indirectly access other arrays:

```
let Drss = map3(\ s1 s2 s3 -> let ind_x = B1xss[s1] + s2
                                let ind_y = B1yss[s1] + s3
                                in X[ind_x] + Y[ind_y]
                                ) ||1rss ||2rss ||3rss
```

⇒

```
B1xss = [0, 2]
B1yss = [0, 1]
||1rss = [0, 0, 1, 1]
||2rss = [0, 1, 0, 0]
||3rss = [0, 0, 0, 1]
Drss = [ Dxss[0+0]+Dyss[0+0], Dxss[0+1]+Dyss[0+0]
          , Dxss[2+0]+Dyss[1+0], Dxss[2+0]+Dyss[1+1] ]
Drss = [ 1+2, 3+2, 2+4, 2+5 ] = [ 3, 5, 6, 7 ]
```

Constructing the Offset Indices (B)

```
arr = [ [], [ [1,2,3], [4], [], [5,6] ], [ [7], [], [8,9,10] ]
```

⇒

```
S0arr = [3]
```

```
S1arr = [0, 4, 3]
```

```
S2arr = [3, 1, 0, 2, 1, 0, 3]
```

```
Darr = [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]
```

Offset Indices (B): segment-start offset in the flat data:

```
B1arr = [0, 0, 6]
```

```
B2arr = [0, 3, 4, 4, 6, 7, 7]
```

How to construct Offset Indices (B)?

Constructing the Offset Indices (B)

```
arr = [ [], [ [1,2,3], [4], [], [5,6] ], [ [7], [], [8,9,10] ]
```

⇒

```
S0arr = [3]
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```
S1arr = [0, 4, 3]
```

```
S2arr = [3, 1, 0, 2, 1, 0, 3]
```

```
Darr = [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]
```

Offset Indices (B): segment-start offset in the flat data:

```
B1arr = [0, 0, 6]
```

```
B2arr = [0, 3, 4, 4, 6, 7, 7]
```

How to construct Offset Indices (B)?

By using exclusive scan on the shape!

```
B2arr = scanexc (+) 0 S2arr — [0, 3, 4, 4, 6, 7, 7]
```

```
B1arr = scanexc (+) 0 S1arr — [0, 0, 4]  
      |> map (\i -> B2arr[i]) — [0, 0, 6]
```


Constructing the Flag Array

From now on, we discuss only TWO-dimensional irregular arrays!

```
mkFlagArray 't [m]
  (aoa_shp: [m]i32) (zero: t)    —aoa_shp=[0,3,1,0,4,2,0]
  (aoa_val: [m]t ) : []i32 =    —aoa_val=[1,1,1,1,1,1,1]
  let shp_rot = map (\i->if i==0 then zero—shp_rot=[0,0,3,1,0,4,2]
                      else aoa_shp[i-1]
                      ) (iota m)
  let shp_scn = scan (+) 0 shp_rot    —shp_scn=[0,0,3,4,4,8,10]
  let aoa_len = shp_scn[m-1]+aoa_shp[m-1]—aoa_len= 10
  let shp_ind = map2 (\shp ind ->      —shp_ind=
                      if shp==0 then -1 — [-1,0,3,-1,4,8,-1]
                      else ind         —scatter
                      ) aoa_shp shp_scn — [0,0,0,0,0,0,0,0,0,0,0]
  in scatter (replicate aoa_len zero) — [-1,0,3,-1,4,8,-1]
             shp_ind aoa_val          — [1,1,1,1,1,1,1]
                                     — F = [1,0,0,1,1,0,0,0,1,0]
```

Why do we need aoa_val?

Constructing the Flag Array

From now on, we discuss only TWO-dimensional irregular arrays!

```
mkFlagArray 't [m]
  (aoa_shp: [m]i32) (zero: t)    —aoa_shp=[0,3,1,0,4,2,0]
  (aoa_val: [m]t ) : []i32 =    —aoa_val=[1,1,1,1,1,1,1]
  let shp_rot = map (\i->if i==0 then zero—shp_rot=[0,0,3,1,0,4,2]
                     else aoa_shp[i-1]
                     ) (iota m)
  let shp_scn = scan (+) 0 shp_rot    —shp_scn=[0,0,3,4,4,8,10]
  let aoa_len = shp_scn[m-1]+aoa_shp[m-1]—aoa_len= 10
  let shp_ind = map2 (\shp ind ->      —shp_ind=
                     if shp==0 then -1 — [-1,0,3,-1,4,8,-1]
                     else ind         —scatter
                     ) aoa_shp shp_scn — [0,0,0,0,0,0,0,0,0,0,0]
  in scatter (replicate aoa_len zero) — [-1,0,3,-1,4,8,-1]
             shp_ind aoa_val          — [1,1,1,1,1,1,1]
                                     — F = [1,0,0,1,1,0,0,0,1,0]
```

Why do we need `aoa_val`?

Because there are many valid flag arrays, i.e., the start of the segment can be denoted by any value different than zero!

Constructing the Segment and Inner Indices

From now on, we discuss only TWO-dimensional irregular arrays!

```
arr = [ [1,2,3], [4], [], [5,6], [7], [], [8,9,10] ]
```

⇒

```
Sarr0 = [7]
```

```
Sarr1 = [3, 1, 0, 2, 1, 0, 3]
```

```
flenarr = reduce (+) 0 Sarr1 = 10
```

```
Darr = [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]
```

Segment and Inner indices (II):

```
IIarr1 = [0, 0, 0, 1, 3, 3, 4, 6, 6, 6]
```

```
IIarr2 = [0, 1, 2, 0, 0, 1, 0, 0, 1, 2]
```

Constructing Segment and Inner indices (II):

```
Farr = mkFlagArray Sarr1 0 [1...length Sarr1]  
      — [1, 0, 0, 2, 4, 0, 5, 7, 0, 0]
```

```
IIarr1 = ???
```

```
IIarr2 = ???
```

Constructing the Segment and Inner Indices

I need to get this:

$II_{arr}^1 = [0, 0, 0, 1, 3, 3, 4, 6, 6, 6]$

from this:

$F_{arr} = \text{mkFlagArray } S_{arr}^1 \ 0 \ [1 \dots \text{length } S_{arr}^1]$
— $[1, 0, 0, 2, 4, 0, 5, 7, 0, 0]$

How?

Constructing the Segment and Inner Indices

I need to get this:

$$II_{arr}^2 = [0, 1, 2, 0, 0, 1, 0, 0, 1, 2]$$

from this:

$$F_{arr} = \text{mkFlagArray } S_{arr}^1 \ 0 \ [1 \dots \text{length } S_{arr}^1]$$

— $[1, 0, 0, 2, 4, 0, 5, 7, 0, 0]$

How?

Constructing the Segment and Inner Indices

From now on, we discuss only TWO-dimensional irregular arrays!

```
arr = [ [1,2,3], [4], [], [5,6], [7], [], [8,9,10] ]
```

⇒

```
S0arr = [7]
```

```
S1arr = [3, 1, 0, 2, 1, 0, 3]
```

```
flenarr = reduce (+) 0 S1arr = 10
```

```
Darr = [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]
```

Segment and Inner indices (II):

```
II1arr = [0, 0, 0, 1, 3, 3, 4, 6, 6, 6]
```

```
II2arr = [0, 1, 2, 0, 0, 1, 0, 0, 1, 2]
```

Constructing Segment and Inner indices (II):

```
Farr = mkFlagArray S1arr 0 [1...length S1arr]  
      — [1, 0, 0, 2, 4, 0, 5, 7, 0, 0]
```

```
II1arr = sgmscan (+) 0 Farr Farr |> map (-1)
```

```
II2arr = sgmscan (+) 0 Farr (replicate flen 1) |> map (-1)
```

Parallel Basic Blocks

Flattening Nested and Irregular Parallelism

How To Flatten: Intuition

Irregular Multi-Dimensional Array Representation

Flattening at A High-Level

Rules For Flattening

Flattening Quicksort

Flattening Prime-Number (Sieve) Computation

Flattening by Function Lifting: Basic Idea

Assume a simple function f :

```
let f (x: i32) : i32 = x + 1
```

f lifted, denoted f^L semantically corresponds to `map f`, where the arguments have been expanded to an extra array dimension, and the inner operators/functions have also been lifted:

```
let  $+^L$  [n] (as: [n]i32) (bs: [n]i32) : [n]i32 =  
    map2 (+) as bs
```

```
let  $f^L$  [n] (xs: [n]i32) : [n]i32 =  
    xs  $+^L$  (replicate n 1)
```


Flattening by Function Lifting: Basic Idea

Assume a simple function f :

```
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```

f lifted, denoted f^L semantically corresponds to `map f`, where the arguments have been expanded to an extra array dimension, and the inner operators/functions have also been lifted:

```
let +L [n] (as: [n]i32) (bs: [n]i32) : [n]i32 =  
  map2 (+) as bs
```

```
let fL [n] (xs: [n]i32) : [n]i32 =  
  xs +L (replicate n 1)
```

- Locals such as $x \Rightarrow$ left alone
- Global such as $+$ \Rightarrow lifted ($+^L$)
- Constants such as $k \Rightarrow \text{replicate (length xs) } k$
 - ▶ good for vectorization, bad for locality, asymptotics
 - ▶ for GPU better to indirectly index into a smaller array, rather than `replicate`.

Flattening by Function Lifting: Key Insight!

```
let f (xs: []i32) : []i32 = map g xs -- = gL xs  
let fL (xss: [][]i32) : [][]i32 = (gL)L -- ???
```

How do we stop lifting? g and g^L are enough: no need for (g^L)^L!

Flattening by Function Lifting: Key Insight!

```
let f (xs: []i32) : []i32 = map g xs -- = gL xs  
let fL (xss: [][]i32) : [][]i32 = (gL)L -- ???
```

How do we stop lifting? g and g^L are enough: no need for (g^L)^L!

```
let f (xs: []i32) : []i32 = map g xs -- = gL xs  
-- in nested parallel form  
let fL (xss: [][]i32) : [][]i32 =  
    segment xss (gL (concat xss))  
-- in flatten form  
let fL (S1xss: []i32, Dxss: []i32) : ([]i32, []i32) =  
    (S1xss, gL Dxss)
```

In Haskell Notation:

concat	::	[[a]]	->	[a]	
segment	::	[[a]]	->	[b]	-> [[b]]
		shape		flat data	nested data

How to Flatten? A Relatively Simple Case

```
let arr = [1, 2, 3, 4] in  
map (\i -> map (+(i+1)) (iota i)) arr  
-- Result: [[2],[3,4],[4,5,6],[5,6,7,8]]
```

How to Flatten? A Relatively Simple Case

```
let arr = [1, 2, 3, 4] in
map (\i -> map (+(i+1)) (iota i)) arr
-- Result: [[2],[3,4],[4,5,6],[5,6,7,8]]
```

Normalize the code:

```
map (\i -> let ip1 = i+1 in
            let iot = (iota i) in
            let ip1r= (replicate i ip1) in
            map2 (+) ip1r iot                ) arr
```

Distribute the map over every instruction in the body

(bottom-up if $\text{nest} > 2$), where \mathcal{F} denotes the flattening transf, and modify the inputs (results) accordingly.

How to Flatten? A Relatively Simple Case

```
let arr = [1, 2, 3, 4] in
map (\i -> map (+(i+1)) (iota i)) arr
-- Result: [[2],[3,4],[4,5,6],[5,6,7,8]]
```

Normalize the code:

```
map (\i -> let ip1 = i+1 in
            let iot = (iota i) in
            let ip1r= (replicate i ip1) in
            map2 (+) ip1r iot                ) arr
```

Distribute the map over every instruction in the body

(bottom-up if nest > 2), where \mathcal{F} denotes the flattening transf, and modify the inputs (results) accordingly.

```
 $\mathcal{F}(\text{map } (\lambda i \rightarrow \text{map } (+(i+1)) (\text{iota } i)) [0..n-1]) \equiv$ 
1. let ip1s = map (\i -> i+1) arr in -- [2, 3, 4, 5]
2. let iots =  $\mathcal{F}(\text{map } (\lambda i \rightarrow (\text{iota } i)) \text{ arr})$  in
3. let ip1rs=  $\mathcal{F}(\text{map2 } (\lambda i \text{ ip1} \rightarrow (\text{replicate } i \text{ ip1})) \text{ arr } \text{ip1s})$ 
4. in  $\mathcal{F}(\text{map2 } (\lambda \text{ ip1r } \text{iot} \rightarrow \text{map2 } (+) \text{ ip1r } \text{iot}) \text{ ip1rs } \text{iots})$ 
```

How to Flatten? A Relatively Simple Case

According to rule (4) iota nested inside a map

(assuming `arr = [1,2,3,4]`):

```
2. let iots =  $\mathcal{F}$ (map (\i -> iota i) arr)
```

≡

```
inds = scanexc (+) 0 arr -- [0,1,3,6]
size = (last inds) + (last arr) -- 6 + 4 = 10
flag = scatter (replicate size 0)
      inds arr
--      [1, 2, 0, 3, 0, 0, 4, 0, 0, 0]
tmp  = replicate size 1
iots = sgmScanexc (+) 0 flag tmp -- [0, 0, 1, 0, 1, 2, 0, 1, 2, 3]
```

How to Flatten? A Relatively Simple Case

According to rule (3) replicate nested inside a map

(assuming `arr = [1,2,3,4]`):

```
3. let ip1rs=  $\mathcal{F}$ (map2 (\ i ip1 -> replicate i ip1) arr ip1s)
 $\equiv$ 
vals = scatter (replicate size 0) inds ip1s -- [2,3,0,4,0,0,5,0,0,0]
ip1rs= sgmScaninc (+) 0 flag vals          -- [2,3,3,4,4,4,5,5,5,5]
```

According to rule (2) map nested inside a map

```
 $\mathcal{F}$ (map2 (\ ip1r iot -> map2 (+) ip1r iot) ip1rs iots)
 $\equiv$ 
4. result = map (+) ip1rs iots
-- [2, 3, 3, 4, 4, 4, 5, 5, 5, 5]
-- [0, 0, 1, 0, 1, 2, 0, 1, 2, 3]
-- + + + + + + + + +
-----
-- [2, 3, 4, 4, 5, 6, 5, 6, 7, 8] values
-- [1, 2, 0, 3, 0, 0, 4, 0, 0, 0] flags
```


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Nested vs Flattened Parallelism: Scan inside a Map

(1) Scan inside a nested map:

```
res = map (\row->scaninc (+) 0 row) [[1,3], [2,4,6]]  
≡  
res = [ scaninc (+) 0 [1,3],      scaninc (+) 0 [2,4,6] ]  
≡  
res = [ [ 1, 4],                  [2, 6, 12] ]
```

Nested vs Flattened Parallelism: Scan inside a Map

(1) Scan inside a nested map:

```
res = map (\row->scaninc (+) 0 row) [[1,3], [2,4,6]]  
≡  
res = [ scaninc (+) 0 [1,3],      scaninc (+) 0 [2,4,6] ]  
≡  
res = [ [ 1, 4],                  [2, 6, 12] ]
```

becomes a segmented scan, which requires a flag array as arg:

```
sgmScaninc (+) 0 [1, 0, 1, 0, 0] [1, 3, 2, 4, 6] ≡ [ 1, 4, 2, 6, 12 ]
```

Flattening a scan directly nested inside a map:

- the flat-data array is obtained by a segmented scan;
- the shape of the result array is the same as the input array.

$\mathcal{F}(\text{res} = \text{map } (\backslash \text{row} \rightarrow \text{scan } (\odot) 0_{\odot} \text{ row}) \text{ arr}) \Rightarrow$
 $S_{\text{res}}^1 = S_{\text{arr}}^1$
 $D_{\text{res}} = \text{sgmScan } (\odot) 0_{\odot} F_{\text{arr}} D_{\text{arr}}$

Nested vs Flattened Parallelism: Map inside a Map

(2) Map nested inside a map:

```
res = map (\row->map f row) [[1,3], [2,4,6]]
```

≡

```
res = [ map f [1, 3],      map f [2, 4, 6] ]
```

≡

```
res = [ [f(1),f(3)], [f(2),f(4),f(6)] ]
```

Nested vs Flattened Parallelism: Map inside a Map

(2) Map nested inside a map:

```
res = map (\row->map f row) [[1,3], [2,4,6]]  
≡  
res = [ map f [1, 3],      map f [2, 4, 6] ]  
≡  
res = [ [f(1),f(3)], [f(2),f(4),f(6)] ]
```

Flattening a map directly nested inside a map:

- the flat-data array is obtained by a map on the flat input;
- the shape of the result array is the same as the input array.

$\mathcal{F}(\text{res} = \text{map } (\lambda \text{row} \rightarrow \text{map } f \text{ row}) \text{ arr}) \Rightarrow$

$$S_{\text{res}}^1 = S_{\text{arr}}^1$$

$$D_{\text{res}} = \text{map } f \ D_{\text{arr}}$$

Nested vs Flattened Parallelism: Replicate in a Map

(3) Replicate nested inside a map:

```
res = map2 (\ n m -> replicate n m) [1,0,3,2] [7,3,8,9] ≡  
res = [ replicate 1 7, replicate 0 3, replicate 3 8, replicate 2 9 ] ≡  
res = [ [7], [], [8,8,8], [9,9] ]
```

Nested vs Flattened Parallelism: Replicate in a Map

(3) Replicate nested inside a map:

```
res = map2 (\ n m -> replicate n m) [1,0,3,2] [7,3,8,9] ≡  
res = [ replicate 1 7, replicate 0 3, replicate 3 8, replicate 2 9 ] ≡  
res = [ [7], [], [8,8,8], [9,9] ]
```

```
res = map2(\n m-> replicate n m) ns ms
```

becomes a scan-scatter composition:

Nested vs Flattened Parallelism: Replicate in a Map

(3) Replicate nested inside a map:

```
res = map2 (\ n m -> replicate n m) [1,0,3,2] [7,3,8,9] ≡  
res = [ replicate 1 7, replicate 0 3, replicate 3 8, replicate 2 9 ] ≡  
res = [ [7], [], [8,8,8], [9,9] ]
```

```
res = map2(\n m-> replicate n m) ns ms
```

becomes a scan-scatter composition:

1. the shape of the result array is ns
- 2-3. builds the indices at which segment start (-1 for null shape)
4. get the size of the flat array (summing ns)
- 5-6. write the ms and ns values at the start of their segments
7. propagate the ms values throughout their segments.

■ **Implementation shortcomings:**

Nested vs Flattened Parallelism: Replicate in a Map

(3) Replicate nested inside a map:

```
res = map2 (\ n m -> replicate n m) [1,0,3,2] [7,3,8,9] ≡  
res = [ replicate 1 7, replicate 0 3, replicate 3 8, replicate 2 9 ] ≡  
res = [ [7], [], [8,8,8], [9,9] ]
```

```
res = map2(\n m-> replicate n m) ns ms
```

becomes a scan-scatter composition:

1. the shape of the result array is ns
- 2-3. builds the indices at which segment start (-1 for null shape)
4. get the size of the flat array (summing ns)
- 5-6. write the ms and ns values at the start of their segments
7. propagate the ms values throughout their segments.
 - **Implementation shortcomings:** $\text{sgmScan}^{inc} (+)$?

$\mathcal{F}(\text{res} = \text{map2} (\backslash n m \rightarrow \text{replicate } n m) \text{ ns ms}) \Rightarrow$

1. $S_{res}^1 = \text{ns}$
2. $\text{inds} = \text{scan}^{exc} (+) 0 \text{ ns}$
3. $|> \text{map2} (\backslash n i \rightarrow \text{if } n > 0 \text{ then } i \text{ else } -1) \text{ ns}$
4. $\text{size} = (\text{last inds}) + (\text{last ns})$
5. $\text{vls} = \text{scatter} (\text{replicate size } 0) \text{ inds ms}$
6. $F_{arr} = \text{scatter} (\text{replicate size } 0) \text{ inds ns}$
6. $D_{res} = \text{sgmScan}^{inc} (+) 0 F_{res} \text{ vls}$

```
— ms = [7,3,8,9]  
— ns = [1,0,3,2]  
— [0,1,1,4]  
— inds = [0,-1,1,4]  
— 4 + 2 = 6  
— [7, 8, 0, 0, 9, 0]  
— [1, 3, 0, 0, 2, 0]  
— [7, 8, 8, 8, 9, 9]
```

Nested vs Flattened Parallelism: Iota in a Map

(4) Iota nested inside a map $((\text{iota } n) \equiv [0, \dots, n-1])$:

```
res = map (\i -> iota i) [1,3,2]  $\equiv$   
res = [iota 1, iota 3, iota 2]  $\equiv$  [ [0], [0,1,2], [0,1] ]
```

Nested vs Flattened Parallelism: Iota in a Map

(4) Iota nested inside a map ($\text{iota } n \equiv [0, \dots, n-1]$):

```
res = map (\i -> iota i) [1,3,2]  $\equiv$   
res = [iota 1, iota 3, iota 2]  $\equiv$  [ [0], [0,1,2], [0,1] ]
```

boils down to a segmented scan applied to an array of ones:

1. by definition of `iota`, `ns` contains the size of each subarray, hence the shape of the result is `ns`;
2. the flag-array of the result, F_{res} , is constructed from `ns`;
3. the result is obtained by an exclusive segmented scan operation applied to an array of ones.

$\mathcal{F}(\text{res} = \text{map } (\backslash n \rightarrow \text{iota } n) \text{ ns}) \Rightarrow$

- | | |
|---|--|
| 1. $S^1_{res} = ns$ | $\text{--- } ns = [1, 3, 2]$ |
| 2. $F_{res} = \text{mkFlagArray } ns \ 0 \ ns$ | $\text{--- } F_{res} = [1, 1, 0, 0, 1, 0]$ |
| 3. $D_{res} = \text{sgmScan}^{exc} (+) \ 0 \ F_{res} \ (\text{replicate } \text{flen}_{res} \ 1)$ | $\text{--- } [0, 0, 1, 2, 0, 1]$ |

Note that $\text{iota } n \equiv \text{scan}^{exc} (+) \ 0 \ (\text{replicate } n \ 1)$.

Nested vs Flattened Parallelism: Reduce Inside Map

(5) Reduce Inside a Map or Segmented Reduce:

```
let arr = [[1, 3, 4], [6, 7]] in
let res = map (\x -> reduce (+) 0 x) arr
-- should result in [8, 13]
```

Nested vs Flattened Parallelism: Reduce Inside Map

(5) Reduce Inside a Map or Segmented Reduce:

```
let arr = [[1, 3, 4], [6, 7]] in
let res = map (\x -> reduce (+) 0 x) arr
-- should result in [8, 13]
```

translates to a **scan-pack** composition:

1. the length of `res` equals the number of subarrays of `arr`;
2. the shape of `arr` is scanned: the result records the position of the last element in a segment plus one;
3. segmented scan is applied on the input array: the last elem in a segment holds the reduced value of the segment;
4. segment's last element is extracted by a map operation.

$\mathcal{F}(\text{res} = \text{map } (\backslash \text{row} \rightarrow \text{reduce } \odot 0 \odot \text{row}) \text{ arr}) \Rightarrow$

— $S_{arr}^0 = [2]$, $S_{arr}^1 = [3, 2]$, $F_{arr} = [1, 0, 0, 1, 0]$, $D_{arr} = [1, 3, 4, 6, 7]$

1. $S_{res}^0 = S_{arr}^0$

— $S_{res}^0 = [2]$

2. $\text{indsp1} = \text{scan } (+) 0 S_{arr}^1$

— $\text{indsp1} = [3, 5]$

3. $\text{tmp} = \text{sgmScan } (\odot) 0 \odot F_{arr} D_{arr}$

— $\text{tmp} = [1, 4, 8, 6, 13]$

4. $D_{res} = \text{map2}(\backslash s \text{ ip1} \rightarrow \text{if } s \leq 0 \text{ then } 0 \odot$

$\text{else tmp[ip1 - 1]]) } S_{arr}^1 \text{ indsp1} \text{ — } D_{res} = [8, 13]$

Treating a Scalar Variant to the Outer Map

(6) The inner construct uses a scalar variant to the outer map:

```
let res = map2 (\x ys -> map (+x) ys) [1,3] [[4,5,6], [9,7]] ≡  
let res = [map (+1) [4,5,6], map (+3) [9,7]]  
let res = [ [5,6,7], [12,10] ]
```

Treating a Scalar Variant to the Outer Map

(6) The inner construct uses a scalar variant to the outer map:

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let res = map2 (\x ys -> map (+x) ys) [1,3] [[4,5,6], [9,7]] ≡  
let res = [map (+1) [4,5,6], map (+3) [9,7]]  
let res = [ [5,6,7], [12,10] ]
```

Traditionally, this is handled by expanding (replicating) each x across the whole segment

```
let Dxss = [1, 1, 1, 3, 3]  
let res = map2 (+) [1, 1, 1, 3, 3 ]  
                  [4, 5, 6, 9, 7 ]  
                  = = = = =  
                  [5, 6, 7, 12, 10]
```

Instead, we use II_{arr}^1 to indirectly access in the xs array:

```
 $\mathcal{F}$ (res = map2 (\x ys -> map (f x) ys) xs yss)  $\Rightarrow$   
—  $xs = [1,3], S_{yss}^1 = [3,2], F_{yss} = [1,0,0,1,0], D_{yss} = [4,5,6,9,7]$   
1.  $S_{res}^1 = S_{yss}^1$   
2.  $D_{res} = \text{map2 } (\backslash y \text{ sgmind} \rightarrow f \text{ xs[sgmind] } y) \ D_{yss} \ II_{yss}^1$   
—  $II_{yss}^1 = [0,0,0,1,1], D_{res} = [5,6,7,12,10]$ 
```

Treating Indexing Variant to the Outer Map

(7) Indexing Operations Variant to the Outer Map:

```
let res = map2 (\i xs -> xs[i]) [2,0] [[4,5,6], [9,7]] ≡  
let res = [ 6, 9 ]
```


Treating Indexing Variant to the Outer Map

(7) Indexing Operations Variant to the Outer Map:

```
let res = map2 (\i xs -> xs[i]) [2,0] [[4,5,6], [9,7]] ≡  
let res = [ 6, 9 ]
```

To corresponding flat index in $D_{y_{ss}}$ is obtained by summing up

- the start offset of every segment, which we get from $B_{y_{ss}}^1$, and
- the index inside the segment, which we get from i_s

$\mathcal{F}(\text{res} = \text{map2 } (\lambda i \text{ xs} \rightarrow \text{xs}[i]) \text{ is } x_{ss}) \Rightarrow$

— $i_s = [2,0]$, $S_{x_{ss}}^1 = [3,2]$, $B_{x_{ss}}^1 = [0,3]$, $D_{x_{ss}} = [4,5,6,9,7]$

1. $S_{res}^0 = S_{i_s}^0$ — = $S_{i_s}^0 = [2]$

2. $D_{res} = \text{map2 } (\lambda \text{ off } i \rightarrow D_{x_{ss}}[\text{off} + i]) B_{x_{ss}}^1 \text{ is } \text{— } D_{res} = [6, 9]$

Nested vs Flattened Parallelism: If Inside a Map 2D Case

(8) If-Then-Else with inner parallelism nested inside a map:

```
bs  = [F,T,F,T]
xss = [[1,2,3],[4,5,6,7],[8,9],[10]]
res = map(\b xs -> if b then map (+1) xs else map (*2) xs) bs xss
res = [ map(*2)[1,2,3], map(+1)[4,5,6,7], map(*2)[8,9], map(+1)[10] ]
res = [ [2,4,6], [5,6,7,8], [16,18], [11] ]
```

Nested vs Flattened Parallelism: If Inside a Map 2D Case

(8) If-Then-Else with inner parallelism nested inside a map:

```
bs = [F,T,F,T]
xss = [[1,2,3],[4,5,6,7],[8,9],[10]]
res = map(\b xs -> if b then map (+1) xs else map (*2) xs) bs xss
res = [ map(*2)[1,2,3], map(+1)[4,5,6,7], map(*2)[8,9], map(+1)[10] ]
res = [ [2,4,6], [5,6,7,8], [16,18], [11] ]
```

translates to a **scatter-map-gather** composition. Intuition:

1. compute *iinds*, the permutation of segments w.r.t. *bs*;
- 2-3. partition the *xss* array based on *bs*;
- 4-5. **flatten outer map and/on top of the parallel code of the then and else branches**;
6. inverse permute the resulted segments according to *iinds*.

```
1. iinds = partition2 (\i -> bs[i]) (iota (length b)) -- [1,3,0,2]
2. xssthen = gatherThen iinds xss -- ([4,1], [4,5,6,7, 10])
3. xsselse = gatherElse iinds xss -- ([3,2], [1,2,3, 8,9])
4. resthen = map (+1) xssthen -- ([4,1], [5,6,7,8, 11]) Flattened rec!
5. reselse = map (*2) xsselse -- ([3,2], [2,4,6,16,18]) Flattened rec!
6. res = inversePermute iinds (resthen++reselse)
-- ([3,4,2,1], [2,4,6, 5,6,7,8, 16,18, 11])
```

Nested vs Flattened Parallelism: If Inside a Map 2D Case

(8) If-Then-Else with inner parallelism nested inside a map:

```
bs = [F,T,F,T], xss = [[1,2,3],[4,5,6,7],[8,9],[10]], S1xss=[3,4,2,1], f=map (+1), g=map (*2)

F(res = map2 (\b xs -> if b then f xs else g xs) bs xss) =>
(spl, iinds) = partition2 bs (iota (length bs)) — (2, [1,3,0,2])
(S1xssthen, S1xsselse) = split spl (map (\ii -> S1xss[ii]) iinds) — ([4,1],[3,2])
maskxss = map (\sgmind -> bs[sgmind]) ||1xss — [F,F,F,T,T,T,T,F,F,T]
(brk, Dpxss) = partition2 maskxss Dxss
(Dxssthen, Dxsselse) = split brk Dpxss — ([4,5,6,7,10],[1,2,3,8,9])
(S1resthen, Dresthen) = F(map f) (S1xssthen, Dxssthen) — ([4,1], [5,6,7,8,11])
(S1reselse, Dreselse) = F(map g) (S1xsselse, Dxsselse) — ([3,2], [2,4,6,16,18])
S1pres = S1resthen ++ S1reselse — [4,1,3,2]
S1res = scatter (replicate (length bs) 0) iinds S1pres — [3,4,2,1]
B1res = scanexc (+) 0 S1res — [0,3,7,9]
Fpres = mkFlagArray S1pres 0 (map (+1) iinds) — [2,0,0,0,4,1,0,0,3,0]
||1pres = sgmscan (+) 0 Fpres Fpres |> map (-1) — [1,1,1,1,3,0,0,0,2,2]
||2pres = ||2resthen ++ ||2reselse — [0,1,2,3,0, 0,1,2,0,1]
sindsres = map2 (\sgm iin -> B1res[sgm] + iin) ||1pres ||2pres
— [3+0,3+1,3+2,3+3, 9+0, 0+0,0+1,0+2, 7+0,7+1]=[3,4,5,6,9,0,1,2,7,8]
Dres = scatter (replicate flenres 0) sindsres (Dresthen ++ Dreselse)
— [2,4,6, 5,6,7,8, 16,18, 11]
(S1res, Dres)
```

Nested vs Flattened Parallelism: Do Loop Inside a Map

(9) Flattening a Do Loop Nested Inside a Map:

- compute the maximal loop count n_{max}
- interchange the loop and the map:
 - ▶ loop count becomes n_{max}
 - ▶ the loop body is wrapped inside a `if i < n` condition, and
 - ▶ the new loop body is flattened!

$\mathcal{F}(\text{res} = \text{map2}(\backslash n \text{ xs} \rightarrow \text{loop}(\text{xs}) \text{ for } i < n \text{ do } f \text{ xs}) \text{ ns } \text{xss}) \Rightarrow$

1. $n_{max} = \text{reduce max } 0 \text{ i32 } \text{ ns}$
2. `g i m arr = if i < m then f arr else arr`
3. `loop(S_{xss}^1 , D_{xss}) for i < n_{max} do`
4. $\mathcal{F}(\text{map2 } (g \text{ i})) \text{ ns } (S_{xss}^1, D_{xss})$
5. $\text{--- } (g \text{ i})^L \text{ ns } (S_{xss}^1, D_{xss})$

Parallel Basic Blocks

Flattening Nested and Irregular Parallelism

How To Flatten: Intuition

Irregular Multi-Dimensional Array Representation

Flattening at A High-Level

Rules For Flattening

Flattening Quicksort

Flattening Prime-Number (Sieve) Computation

Recounting Quicksort

Recount the classic nested-parallel definition:

```
let quicksort [n] (arr : [n]f32) : [n]f32 =  
  if n < 2 then arr else  
    let i = getRand (0, (length arr) - 1)  
    let a = arr[i]  
    let s1 = filter (< a) arr  
    let s2 = filter (== a) arr  
    let s3 = filter (> a) arr  
    in (quicksort s1) ++ s2 ++ (quicksort s3)  
— can be re-written as:  
— rs = map nestedQuicksort [s1, s3]  
— in (rs[0]) ++ s2 ++ (rs[1])
```

Note: Futhark does not support recursive calls, hence not valid code!

Nested-Parallel Quicksort Simplified

For simplicity we will rewrite it in terms of `partition2`:

```
let isSorted [n] (as: [n]f32) : bool =  
  map (\i -> if i==0 then true else as[i-1] < as[i]) (iota n)  
  |> reduce (&&) true  
  
let quicksort [n] (arr: [n]f32) : [n]f32 =  
  if isSorted arr then arr else  
    let i = getRand (0, (length arr) - 1)  
    let a = arr[i]  
    let bs = map (< a) arr  
    let (q, arr') = partition2 bs 0.0f32 arr  
    let (arr<, arr≥) = split q arr'  
    in concat <| map quicksort [arr<, arr≥]
```

Note: Futhark does not support recursive calls, irregular map operation, or concat!

Partition2

Reorders the elements of an array such that those that correspond to a true mask come before those corresponding to false.

```
let partition2 [n] 't (conds: [n]bool) (dummy: t) (arr: [n]t)
  : (i32, [n]t) =
  let tflgs = map (\ c => if c then 1 else 0) conds
  let fflgs = map (\ b => 1 - b) tflgs

  let indsT = scan (+) 0 tflgs
  let tmp    = scan (+) 0 fflgs
  let lst    = if n > 0 then indsT[n-1] else -1
  let indsF = map (+lst) tmp

  let inds = map3 (\ c indT indF => if c then indT-1 else indF-1)
    conds indsT indsF

  let fltarr = scatter (replicate n dummy) inds arr
  in (lst, fltarr)
```

For example:

```
conds = [F,T,F,T,F,F,T]
xss = [1,2,3,4,5,6,7]
partition2 conds 0 xss => (3, [2,4,7,1,3,5,6])
```

Lifting Quicksort

Key Idea: write a function with the semantics of

`map nestedQuicksort`, i.e., it operates on array of arrays.

```
let isSorted [n] (as: [n]f32) : bool =  
  map (\i -> if i==0 then true else as[i-1] < as[i]) (iota n)  
  |> reduce (&&) true  
  
let quicksortL (xss: [][]f32) : [][]f32 =  
  map (\xs ->  
    if isSorted xs then xs else  
    let i = getRand (0, (length xs) - 1)  
    let a = xs[i]  
    let bs = map (< a) xs  
    let (q, xsp) = partition2 bs 0.0f32 xss  
    let (xs<, xs≥) = split q xsp  
    in concat <| map quicksort [xs<, xs≥]  
  ) xss
```

Important observations:

- `map quicksort` \equiv `quicksortL`
- `map(map quicksort)` \equiv `quicksortL` \equiv `segment o quicksortL o concat`
- the flat data of `[xs<, xs≥]` \equiv `xsp`, the result of `partition2`

Lifting Quicksort

Let us treat the last three lines from the previous implem.:

```
let quicksortL (S1xss:[ ] i32 , Dxss:[ ] f32): ([ ] i32 , [ ] f32) =—(xss: [ ] [ ] f32)
  let (S1bss , Dbss) =  $\mathcal{F}$  (
    map (\ xs →
      if isSorted xs then xs else
        let i = getRand (0, (length xs) - 1)
        let a = xs[i]
        let bs = map (< a) xs
        in bs
    ) xss )
  let (ps, (S1xssp , Dxssp)) = partition2L Dbss 0.0f32 (S1xss , Dxss)
  — Invariant: S1xssp == S1bss == S1xss
  let S1[xss<,xss≥] = filter (!=0) <| flatten <|
    map2 (λ p s → if s==0 then [0,0] else [p,s-p]) ps S1xss
  in quicksortL (S1[xss<,xss≥] , Dxssp)
```

- S¹_[xss<a,xss≥a] is the shape of [xs<, xs≥]
- (concat <| quicksort^L)^L xsss ≡ concat <| segment xsss <| quicksort^L (concat xsss) ≡ quicksort^L (concat xsss)
- The function looks tail recursive now: let's replace it with a loop!

Lifting Quicksort: Final Implementation

```
let quicksortL [m][n] (SXSS1:[m]i32, DXSS:[n]f32): [n]f32 =
  let (stop, count) = (isSorted DXSS, 0i32)
  let (_, res, _, _) =
    loop (SXSS1, DXSS, stop, count) while (!stop) do
      — compute helper-representation structures
      let BXSS1 = scanexc (+) 0 SXSS1
      let FXSS1 = mkFlagArray SXSS1 0i32 <| map (+1) <| iota m
      let IIXSS1 = sgmscan (+) 0 FXSS1 <|
        map (\f → if f==0 then 0 else f-1) FXSS1
      — flattening quicksort:
      let rL = map (\u → randomInd (0,u-1) count) SXSS1
      let aL = map3 (\r l i → if l <= 0 then 0.0 else DXSS[BXSS1[i]+r]
        ) rL SXSS1 (iota m)
      let Dbss = map2 (\x sgmind → aL[sgmind] > x) DXSS IIXSS1
      let (ps, (SXSSp1, DXSSper)) = partition2L Dbss 0.0f32 (SXSS1, DXSS)
      let S[XSS<,XSS≥]1 = filter (!=0) <| flatten <|
        map2 (\p s → if s==0 then [0,0] else [p,s-p]) ps SXSS1
      in (S[XSS<,XSS≥]1, DXSSper, isSorted DXSSper, count+1)
  in res
```

PFP Weekly 2 Exercise: Implement partition2^L

Parallel Basic Blocks

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- Irregular Multi-Dimensional Array Representation

- Flattening at A High-Level

- Rules For Flattening

- Flattening Quicksort

- Flattening Prime-Number (Sieve) Computation

How Does One Flatten Prime Numbers?

The important bit with nested parallelism:

```
sqrt_primes = primesOpt (sqrt (fromIntegral n))
nested = map (\p -> let m = (n `div` p)
                  in map (\j -> j*p) [2..m]
                ) sqrt_primes
not_primes = reduce (++) [] nested
```

How Does One Flatten Prime Numbers?

The important bit with nested parallelism:

```
sqrt_primes = primesOpt (sqrt (fromIntegral n))
nested = map (\p -> let m = (n `div` p)
                    in map (\j -> j*p) [2..m]
                    ) sqrt_primes
not_primes  = reduce (++) [] nested
```

Normalize the nested map:

```
sqrt_primes = primesOpt (sqrt (fromIntegral n))
nested = map (\p ->
              let m    = n `div` p          in      -- distribute map
              let mm1  = m - 1              in      -- distribute map
              let iot   = iota mm1          in      --  $\mathcal{F}$  rule 4
              let twom = map (+2) iot       in      --  $\mathcal{F}$  rule 2
              let rp    = replicate mm1 p   in      --  $\mathcal{F}$  rule 3
              in map (\(j,p) -> j*p) (zip twom rp) --  $\mathcal{F}$  rule 2
              ) sqrt_primes
not_primes  = reduce (++) [] nested           -- ignore, already flat
```

Flattening PrimeOpt was part of PMPH's Weekly Assignment 2!