The Polyhedral Model

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Agenda

Motivation: Dependency Graphs and Transformations

Polydral Analysis: Iteration Domain, Access Relations, Schedule

Presburger Sets, Relations and Associated Operations

Data-Flow Analysis and Dependency Graph

Examples: Checking Validity of Code Transformations

Assignment Exercises

Acknowledgments

The material presented in these slides was taken from the tutorial ”Presburger Formulas and Polyhedral Compilation” by Sven Verdoolaege, and associated slides, found online for example at

`http://labexcompilation.ens-lyon.fr/wp-content/uploads/2013/02/Sven-slides.pdf`

and

`https://www.researchgate.net/publication/291352331_Presburger_Formulas_and_Polyhedral_Compilation`

Additionally, we have used material from Andreas Kloeckner’s ”Languages and Abstractions for High-Performance Scientific Computing” Course, CS598 APK, available online at:

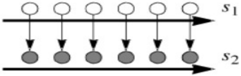
`https://andreask.cs.illinois.edu/cs598apk-f18/notes.pdf#page=214`

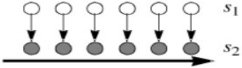
Motivation

Polyhedral model provides a useful framework for reasoning about certain loop-based transformations. Questions to answer:

- How to compute the dependency graph of a loop nest?
- How to represent a code transformation?
- How to prove the legality of such a transformation?

Transformations: Fusion and Fission

<table>
<tr><th>Source Code</th><th>Partition</th><th>Transformed Code</th></tr>
<tr>
<td><pre>
for (i=1; i<=N; i++)
  Y[i] = Z[i]; /*s1*/
for (j=1; j<=N; j++)
  X[j] = Y[j]; /*s2*/
</pre>


</td>
<td>Fusion
$s_1 : p = i$
$s_2 : p = j$</td>
<td><pre>
for (p=1; p<=N; p++){
  Y[p] = Z[p];
  X[p] = Y[p];
}
</pre>


</td>
</tr>
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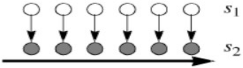
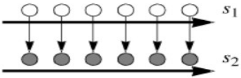
  X[p] = Y[p];
}
</pre>


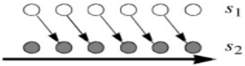
</td>
<td>Fission
$s_1 : i = p$
$s_2 : j = p$</td>
<td><pre>
for (i=1; i<=N; i++)
  Y[i] = Z[i]; /*s1*/
for (j=1; j<=N; j++)
  X[j] = Y[j]; /*s2*/
</pre>


</td>
</tr>
</table>

[Aho/Ullman/Sethi ’07]

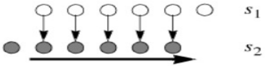
Transformations: Reindexing and Scaling

```
for (i=1; i<=N; i++) {
  Y[i] = Z[i];    /*s1*/
  X[i] = Y[i-1]; /*s2*/
}
```

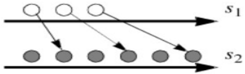


Re-indexing
$s_1 : p = i$
$s_2 : p = i - 1$

```
if (N>=1) X[1]=Y[0];
for (p=1; p<=N-1; p++){
  Y[p]=Z[p];
  X[p+1]=Y[p];
}
if (N>=1) Y[N]=Z[N];
```

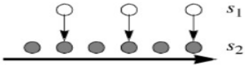



```
for (i=1; i<=N; i++)
  Y[2*i] = Z[2*i]; /*s1*/
for (j=1; j<=2N; j++)
  X[j]=Y[j];        /*s2*/
```



Scaling
$s_1 : p = 2 * i$
$(s_2 : p = j)$

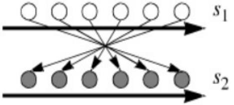
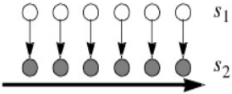
```
for (p=1; p<=2*N; p++){
  if (p mod 2 == 0)
    Y[p] = Z[p];
  X[p] = Y[p];
}
```



[Aho/Ullman/Sethi ’07]

Transformations: Partition

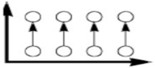
SOURCE CODE	PARTITION	TRANSFORMED CODE

`for (i=0; i>=N; i++)`
`  Y[N-i] = Z[i];   /*s1*/`
`for (j=0; j<=N; j++)`
`  X[j] = Y[j];   /*s2*/`

 	
	Reversal
	$s_1 : p = N - i$
	$(s_2 : p = j)$
		`for (p=0; p<=N; p++){`
		`  Y[p] = Z[N-p];`
		`  X[p] = Y[p];`
		`}`
		

[Aho/Ullman/Sethi '07]

Transformations: Permutation

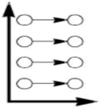
```
for (i=1; i<=N; i++)
  for (j=0; j<=M; j++)
    Z[i,j] =
        Z[i-1,j];
```



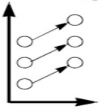
Permutation

$$\begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} i \\ j \end{bmatrix}$$

```
for (p=0; p<=M; p++)
  for (q=1; q<=N; i++)
    Z[q,p] = Z[q-1,p]
```



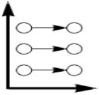
```
for (i=1; i<=N+M-1; i++)
  for (j=max(1,i+N);
          j<=min(i,M); j++)
    Z[i,j] =
      Z[i-1,j-1];
```



Skewing

$$\begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} i \\ j \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

```
for (p=1; p<=N; p++)
  for (q=1; q<=M; q++)
    Z[p,q-p] =
        Z[p-1,q-p-1]
```



[Aho/Ullman/Sethi ’07]

Loop skewing example does not seem quite right (next slide)!

Transformations: Loop Skewing

```
float X[N][N];
for(int i=1; i<N; i++) {
  for(int j=1; j < min(i+2, N); j++) {
    X[i][j] = X[i-1][j] + X[i][j-1];
  }
}
```

Change of variables: $p \leftarrow$ `i+j`, $q \leftarrow$ `j`

```
for(int p=2; p < 2*N-1; p++) {
  int up_bd = ((p+2)/2) + (p%2);
  for(int q=max(1,p-N+1); q<min(up_bd,N); q++) {
    X[p-q][q] = X[p-q-1][q] + X[p-q][q-1];
  }
}
```

Motivation

Polyhedral model provides a useful framework for reasoning about certain loop-based transformations. Questions to answer:

- How to compute the dependency graph of a loop nest?
- How to represent a code transformation?
- How to prove the legality of such a transformation?

Main Components of Polyhedral Analysis

Key features:

- instance based:
  - statement instances
  - array elements
- compact representation
  - Presburger set and relations ...

Program Representation Uses:

- Iteration Domain: the set of all statement instances
- Access Relations: maps each statement instance to the array elements accessed (read/written) by that statement instance.
- Schedule: maps each statement instance to its execution time. (Execution time is abstractly represented by the total order of iterations in the target loop nest).

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- Schedule: maps each statement instance to its execution time. (Execution time is abstractly represented by the total order of iterations in the target loop nest).

$\Rightarrow$ Compute automatically the Dependency Graph: maps the source (statement instance) of a dependence to its sink.

$\Rightarrow$ Check automatically the Validity of a desired transformation.

Illustrative Example (Naive)

```
R:   h(A[2]);
     for(int i=0; i<2; i++)
         for(int j=0; j<2; j++)
S:           A[i+j] = f(i,j);
     for(int k=0; k<2; k++)
T:           g(A[k], A[0]);
```

- Iteration domain: set of all statement instances:

  $I = \{\texttt{R[]; S[0,0];S[0,1];S[1,0];S[1,1]; T[0];T[1]}\}$

- Access relation (statement instance accesses array elements):

  $W = \{\texttt{S[0,0]}\to\texttt{A[0]; S[0,1]}\to\texttt{A[1]; S[1,0]}\to\texttt{A[1]; S[1,1]}\to\texttt{A[2]}\}$

  $R = \{\texttt{R[]}\to\texttt{A[2]; T[0]}\to\texttt{A[0]; T[1]}\to\texttt{A[1]; T[1]}\to\texttt{A[0]}\}$

- Schedule (total ordering of stmts modeling execution time):

  $S = \{\texttt{R[]}\to\texttt{0; S[0,0]}\to\texttt{1; S[0,1]}\to\texttt{2; S[1,0]}\to\texttt{3; S[1,1]}\to\texttt{4;}$
  $\texttt{T[0]}\to\texttt{5; T[1]}\to\texttt{6}\}$

Illustrative Example (Compact)

```
R:   h(A[2]);
     for(int i=0; i<2; i++)
         for(int j=0; j<2; j++)
S:           A[i+j] = f(i,j);
     for(int k=0; k<2; k++)
T:           g(A[k], A[0]);
```

- Iteration domain: set of all statement instances:

  $I = \{\texttt{R[]};\ \texttt{S[i,j]}:\ 0 \le \texttt{i} < 2\ \wedge\ 0 \le \texttt{j} < 2;\ \texttt{T[k]}:\ 0 \le \texttt{k} < 2\}$

- Access relation (statement instance accesses array elements):

  $W = \{\texttt{S[i,j]} \to \texttt{A[i+j]}:\ 0 \le \texttt{i} < 2\ \wedge\ 0 \le \texttt{j} < 2\}$

  $R = \{\texttt{R[]} \to \texttt{A[2]};\ \texttt{T[k]} \to \texttt{A[0]}:\ 0 \le \texttt{k} < 2;\ \texttt{T[k]} \to \texttt{A[k]}:\ 0 \le \texttt{k} < 2\}$

- Schedule (total ordering of stmts modeling execution time):

  $S = \{\texttt{R[]} \to \texttt{[0,0,0]};\ \texttt{S[i,j]} \to \texttt{[1,i,j]}:\ 0 \le \texttt{i} < 2\ \wedge\ 0 \le \texttt{j} < 2;$

  $\texttt{T[k]} \to \texttt{[2,k,0]}:\quad 0 \le \texttt{k} < 2;\}$

Scratch

Parametric Example: Matrix Matrix Multiplication

```
    for(int i=0; i<M; i++)
      for(int j=0; j<N; j++) {
S1:     C[i,j] = 0.0;

        for(int k=0; k<K; k++)
S2:       C[i,j] = C[i,j] + A[i,k] * B[k,j];
      }
```

- Iteration domain (set of all statement instances):

Parametric Example: Matrix Matrix Multiplication

```
    for(int i=0; i<M; i++)
      for(int j=0; j<N; j++) {
S1:     C[i,j] = 0.0;

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S2:       C[i,j] = C[i,j] + A[i,k] * B[k,j];
      }
```

- Iteration domain (set of all statement instances):

$$
I = \left\{ \begin{array}{lcl} S1[i,j] & : & 0 \leq i < M \ \wedge \ 0 \leq j < N; \\ S2[i,j,k] & : & 0 \leq i < M \ \wedge \ 0 \leq j < N \ \wedge \ 0 \leq k < K \end{array} \right\}
$$

- Access relation (R = Read, W = Write):

Parametric Example: Matrix Matrix Multiplication

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    for(int i=0; i<M; i++)
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- Access relation (R = Read, W = Write):

$W = \{\texttt{S1[i,j]} \rightarrow \texttt{C[i,j]};\ \texttt{S2[i,j,k]} \rightarrow \texttt{C[i,j]}\}$

$R = \{\texttt{S2[i,j,k]}\rightarrow\texttt{C[i,j]};\ \texttt{S2[i,j,k]}\rightarrow\texttt{A[i,k]};\ \texttt{S2[i,j,k]}\rightarrow\texttt{B[k,j]}\}$

- Schedule (total ordering of stmts modeling execution time):

Parametric Example: Matrix Matrix Multiplication

```
     for(int i=0; i<M; i++)
       for(int j=0; j<N; j++) {
S1:      C[i,j] = 0.0;

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       }
```

- Iteration domain (set of all statement instances):

$$I = \{ \quad S1[i,j] \quad : \quad 0 \le i < M \ \wedge \ 0 \le j < N; \\ S2[i,j,k] \quad : \quad 0 \le i < M \ \wedge \ 0 \le j < N \ \wedge \ 0 \le k < K \quad \}$$

- Access relation (R = Read, W = Write):

$W = \{$`S1[i,j]` $\rightarrow$ `C[i,j];` `S2[i,j,k]` $\rightarrow$ `C[i,j]`$\}$

$R = \{$`S2[i,j,k]`$\rightarrow$`C[i,j];` `S2[i,j,k]`$\rightarrow$`A[i,k];` `S2[i,j,k]`$\rightarrow$`B[k,j]`$\}$

- Schedule (total ordering of stmts modeling execution time):

$S = \{$ `S1[i,j]` $\rightarrow$ `[i,j,0,0];` `S2[i,j,k]` $\rightarrow$ `[i,j,1,k]`$\}$

Presburger Sets and Relations

```
R:   h(A[2]);
     for(int i=0; i<2; i++)
         for(int j=0; j<2; j++)
S:           A[i+j] = f(i,j);
     for(int k=0; k<2; k++)
T:           g(A[k], A[0]);
```

Examples:

$I = \{\texttt{R[]};\ \texttt{S[i,j]}:\ 0 \le \texttt{i} < 2\ \wedge\ 0 \le \texttt{j} < 2;\ \texttt{T[k]}:\ 0 \le \texttt{k} < 2\}$

$R = \{\texttt{R[]}\to\texttt{A[2]};\ \texttt{T[k]}\to\texttt{A[0]}:\ 0 \le \texttt{k} < 2;\ \texttt{T[k]}\to\texttt{A[k]}:\ 0 \le \texttt{k} < 2\}$

General Form:

- Sets: $\{\ S_1[i] : f_1(i);\ S_2[i] : f_2(i);\ \ldots\ \}$
  with f_k Preseburger formulas
  $\Rightarrow$ set of elements of form $S_k[i]$, one for each i satisfying $f_k(i)$.
- Relations: $\{\ S_1[i] \to T_1[j] : f_1(i,j);\ S_2[i] \to T_2[j] : f_2(i,j);\ \ldots\ \}$
  $\Rightarrow$ set of pairs of elements of the form $S_k[i] \to T_k[j]$.
  (Not necessarily single-valued functions.)

Presburger Formulas

Presburger arithmetic allows (quasi-)exact answers/solutions.

- Language $\mathcal{L} = \{f_1/r_1,\ f_2/r_2,\ \ldots,\ P_1/S_1,\ P_2/S_2, \ldots\}$
  f_i function symbol with arity $r_i \geq 0$:
  - addition, subtraction: $+/2$, $-/2$
  - constant $d/0$, for each integer d
  - integer division: $\lfloor \frac{\cdot}{d} \rfloor/1$, for a fixed integer $d > 0$
  - set of symbolic constant $c_i/0$

  P_i predicate symbol with arity $s_i \geq 0$, e.g., $\leq /2$.
- Terms (inductive definition)
  - v is a term if v is a variable
  - $f_i(t_1, \ldots, t_{r_i})$ is a term if $t_1, \ldots, t_{r_i}$ are terms
- Formulas (inductive definition)

`true`	$F_1 \wedge F_2$ (conjunction)	quantification:
$P_i(t_1, \ldots, t_{s_i})$	$F_1 \vee F_2$ (disjunction)	$\exists v :\ F_1(v)$ (existential)
$t_1 = t_2$	$\neg F_1$ (negation)	$\forall v :\ F_1(v)$ (universal)

  P_i/s_i are predicates, t_j are terms, v variable, F_k are formulas.

Interpretation of Presburger Formulas

- Domain of Discourse (Universe): sets of integers in $\mathbb{Z}$
- Interpretation function/predicate symbols $\rightarrow$ functions/predicates
  - $+/2, -/2$ map to addition and subtraction on integers ...
  - symbolic constants c_i are "uninterpreted", i.e., consider all possible interpretations as integers
- Truth Values
  - true is true; $P_i(t_1, \ldots, t_{r_i})$ is true if interpretation is true
  - ...
  - $\exists v : F(v)$ is true *iff* $F(d)$ is true for some integer d in the universe ($\mathbb{Z}$).
  - $\forall v : F(v)$ is true *iff* $F(d)$ is true for every integers d in the universe ($\mathbb{Z}$).

Syntactic Sugar

Notation: $\bar{i}^n \equiv i_1, \ldots, i_n$, and n can be left unspecified.

- $false$ is equal to $\neg true$
- $a \Rightarrow b$ is equal to $\neg a \vee b$
- $S[\bar{i}]$ is equal to $S[\bar{i}] : true$
- $S[i_1, \ldots, i_{k-1}, g(i_1, \ldots, i_{k-1}), i_{k+1}, \ldots, i_n] : f(\bar{i})$ is equal to $S[i_1, \ldots, i_{k-1}, i_k, i_{k+1}, \ldots, i_n] : i_k = g(i_1, \ldots, i_{k-1}) \wedge f(\bar{i})$
  e.g., $\{S[i] \rightarrow T(i+1)\}$ is equal to $\{S[i] \rightarrow T(j) : j = i + 1\}$
- $a < b$ is equal to $a \leq b - 1 \ldots$
- $\{ S[i,j] : i,j \geq 0 \}$ is equal to $\{ S[i,j] : i \geq 0 \wedge j \geq 0 \}$
- $\{ S[i] : 0 \leq i \leq 10 \}$ is equal to $\{ S[i] : 0 \leq i \wedge i \leq 10 \}$
- $-e$ is equal to $0 - e$
- $n \cdot e$ is equal to $e + \ldots + e$, with n positive integer constant
- $a_1, \ldots, a_n \prec b_1, \ldots, b_n$ equals $\vee_{i=1}^{n}((\wedge_{j=1}^{i-1} a_j = b_j) \wedge a_i < b_i)$
  e.g., $\{S[i_1, i_2] \rightarrow T[j_1, j_2] : i_1, i_2 \prec j_1, j_2\}$ equals $\{S[i_1, i_2] \rightarrow T[j_1, j_2] : i_1 < j_1 \vee (i_1 = j_1 \wedge i_2 < j_2)\} \ldots$

Examples

- $\{S[i,j] \to [1,i,j] : \; 0 \le i,j < 2; \; T[k] \to [2,k,0] : \; 0 \le k < 2\}$ is equal to $\{S[0,0] \to [1,0,0]; \; S[0,1] \to [1,0,1]; \; S[1,0] \to [1,1,0]; \; S[1,1] \to [1,1,1]; \; T[0] \to [2,0,0]; \; T[1] \to [2,1,0]\}$
- $\{[i] : \; 0 \le i \le 10 \; \wedge \; \exists \alpha : \; i = 2 \cdot \alpha\}$ is equal to $\{[0]; \; [2]; \; [4]; \; [6]; \; [8]; \; [10]\}$
- $\{[i] : \; 0 \le i \le 10 \; \wedge \; i = 2 \cdot \alpha\}$ is equal to

$$\begin{cases} \{[2 \cdot \alpha]\} & \texttt{if } 0 \le \alpha \le 5 \\ \emptyset & \texttt{otherwise} \end{cases}$$

- $\{[i] : \; \forall j : i + j \le 10\}$ is equal to $\emptyset$.

Spaces

Recall:

- Sets: $\{\ S_1[\bar{i}] : f_1(\bar{i});\ S_2[\bar{i}] : f_2(\bar{i});\ \ldots\ \}$
- Relations: $\{\ S_1[\bar{i}] \to T_1[\bar{j}] : f_1(\bar{i},\bar{j});\ S_2[\bar{i}] \to T_2[\bar{j}] : f_2(\bar{i},\bar{j});\ \ldots\ \}$

The identifier (e.g., S_1, S_2, T_1, T_2) together with the dimension, i.e., the number of elements in the subsequent tuple (e.g., $\bar{i},\bar{j}$) will be called a space.

When we say $S_2[\bar{i}] = T_1[\bar{j}]$ we mean:

- the identifiers S_2 and T_1 are the same, and
- the dimensions of $\bar{i}$ and $\bar{j}$ are the same.

For example: $S[] \neq S[i];\ S[a] = S[b];\ S[] \neq T[]$.

Operations on Relations

Union: $\{S_1[\bar{i}] \to T_1[\bar{j}] : f_1(\bar{i},\bar{j});\ \ldots\} \cup \{S_2[\bar{i}] \to T_2[\bar{j}] : f_2(\bar{i},\bar{j});\ \ldots\}$
$\Rightarrow \{S_1[\bar{i}] \to T_1[\bar{j}] : f_1(\bar{i},\bar{j});\ \ldots;\ S_2[\bar{i}] \to T_2[\bar{j}] : f_2(\bar{i},\bar{j});\ \ldots\}$

Inverse: $R = \{S[\bar{i}] \to T[\bar{j}] : f(\bar{i},\bar{j})\} \Rightarrow R^{-1} = \{T[\bar{j}] \to S[\bar{i}] : f(\bar{i},\bar{j})\}$

Dom: $R = \{S[\bar{i}] \to T[\bar{j}] : f(\bar{i},\bar{j})\} \Rightarrow \mathrm{dom}\, R = \{S[\bar{i}] : \exists \bar{j} : f(\bar{i},\bar{j})\}$

Range: $R = \{S[\bar{i}] \to T[\bar{j}] : f(\bar{i},\bar{j})\} \Rightarrow \mathtt{ran}\, R = \{T[\bar{j}] : \exists \bar{i} : f(\bar{i},\bar{j})\}$

UnivRel: $A = \{S[\bar{i}] : f(\bar{i})\}$ and $B = \{T[\bar{j}] : g(\bar{j})\}$
$\Rightarrow A \to B = \{S[\bar{i}] \to T[\bar{j}] : f(\bar{i}) \wedge g(\bar{j})\}$

Intersect $\{S_1[\bar{i}_1] \to T_1[\bar{j}_1] : f_1(\bar{i}_1,\bar{j}_1)\} \cap \{S_2[\bar{i}_2] \to T_2[\bar{j}_2] : f_2(\bar{i}_2,\bar{j}_2)\} \Rightarrow$

$$\begin{cases} \{S_1[\bar{i}] \to T_1[\bar{j}] : f_1(\bar{i},\bar{j}) \wedge f_2(\bar{i},\bar{j})\} & \texttt{if } S_1[\bar{i}_1] = S_2[\bar{i}_2] \text{ and} \\ & \quad T_1[\bar{j}_1] = T_2[\bar{j}_2] \\ \emptyset & \texttt{otherwise} \end{cases}$$

Examples: Operations on Relations

```
R:   h(A[2]);
     for(int i=0; i<2; i++)
         for(int j=0; j<2; j++)
S:           A[i+j] = f(i,j);
     for(int k=0; k<2; k++)
T:           g(A[k], A[0]);
```

- Access relation (statement instance accesses array elements):
  $W = \{S[i,j] \to A[i+j] : 0 \le i < 2 \wedge 0 \le j < 2\}$
  $R = \{R[] \to A[2]; T[k] \to A[0] : 0 \le k < 2; T[k] \to A[k] : 0 \le k < 2\}$
- $R \cup W = \{ R[] \to A[2]; T[k] \to A[0] : 0 \le k < 2; T[k] \to A[k] : 0 \le k < 2; S[i,j] \to A[i+j] : 0 \le i < 2 \wedge 0 \le j < 2 \}$
- $R^{-1} = \{A[2] \to R[]; A[0] \to T[k] : 0 \le k < 2; A[k] \to T[k] : 0 \le k < 2\}$
- $\mathrm{dom}\, R = \{R[]; T[k] : 0 \le k < 2\}$
- $\mathtt{ran}\, R = \{A[k] : 0 \le k \le 2\}$
- $\mathrm{dom}\, R \to \mathtt{ran}\, R = \{R[] \to A[j] : 0 \le j \le 2; T[k] \to A[j] : 0 \le k < 2 \wedge 0 \le j \le 2\}$
- $\{T[k] \to A[k] : 0 \le k < 2\} \cap \{T[k] \to A[0] : 0 \le k < 2\} = \{T[0] \to A[0]\}$

Domain/Range Restrictions

Assume $A = \{S_1[i_1] : f(i_1)\}, \quad B = \{S_2[i_2] \to T_2[j_2] : g(i_2, j_2)\}$

- Domain Restrictions: $R \cap_{dom} S \; = \; R \cap (S \to \mathtt{ran}\ R)$

$$A \cap_{\mathtt{dom}} B \; = \; \begin{cases} \{S_2[i] \to T_2[j] : f(i) \wedge g(i,j)\}, & \mathtt{if}\ \ S_1(i_1) = S_2(i_2) \\ \emptyset & \mathtt{otherwise} \end{cases}$$

- Range Restrictions: $R \cap_{\mathtt{ran}} S \; = \; R \cap ((\mathrm{dom}\, R) \to S)$

$$B \cap_{\mathtt{ran}} A \; = \; \begin{cases} \{S_2[i] \to T_2[j] : f(i) \wedge g(i,j)\}, & \mathtt{if}\ \ S_1(i_1) = T_2(j_2) \\ \emptyset & \mathtt{otherwise} \end{cases}$$

Example:

- $I = \{R[]; S[i,j] : \; 0 \le i < 2 \; \wedge \; 0 \le j < 2; \; T[k] : \; 0 \le k < 2\}$
  $S_0 = \{R[] \to [0,0,0]; \; S[i,j] \to [1,i,j]; \; T[k] \to [2,k,0];\}$

  $$S = I \cap_{\mathtt{dom}} S_0 \; = \; \{\, R[] \to [0,0,0]; \; T[k] \to [2,k,0] : 0 \le k < 2; \; S[i,j] \to [1,i,j] : 0 \le i < 2 \wedge 0 \le j < 2 \,\}$$

Relation Difference/Subtraction and Comparisons

$$A = \{S_1[i_1] \to T_1[j_1] : f(i_1, j_1)\},$$
$$B = \{S_2[i_2] \to T_2[j_2] : g(i_2, j_2)\}$$

$$A \setminus B = \begin{cases} \{S_1[i] \to T_1[j] : f(i,j) \land \neg g(i,j)\}, & \texttt{if } S_1(i_1) = S_2(i_2) \texttt{ and} \\ & \quad T_1(j_1) = T_2(j_2) \\ \{S_1[i] \to T_1[j] : f(i,j)\} & \texttt{otherwise} \end{cases}$$

Example:

$$\{T[k] \to A[k] : 0 \le k < 2\} \setminus \{T[k] \to A[0] : 0 \le k < 2\} = \{T[1] \to A[1]\}$$

Comparisons:

- emptiness check (if the Preseburger formula reduces to false)
- $A \subseteq B$ is defined as $A \setminus B = \emptyset$
- $A \supseteq B$ is defined as $B \subseteq A$
- $A = B$ is defined as $B \subseteq A \ \land \ A \subseteq B$
- $A \subset B$ is defined as $A \subseteq B \ \land \ \neg(A = B)$
- $A \supset B$ is defined as $B \subset A$

Composition of Relations

Composition:
$A = \{S_1[i_1] \to T_1[j_1] :\ f(i_1, j_1)\}, \quad B = \{S_2[i_2] \to T_2[j_2] :\ g(i_2, j_2)\}$

$$B \circ A = \begin{cases} \{S_1[i] \to T_2[j] :\ \exists k :\ f(i,k) \wedge g(k,j)\}, & \texttt{if } T_1(j_1) = S_2(i_2) \\ \emptyset & \texttt{otherwise} \end{cases}$$

Example:

Write Set: $W = \{S[i,j] \ \to\ A[i+j] :\ 0 \le i < 2 \ \wedge\ 0 \le j < 2\}$

Inverse of Write set (i.e., written array elements to statements):
$W^{-1} = \{A[a] \to S[i,j] :\ a = i+j \ \wedge\ 0 \le i < 2 \ \wedge\ 0 \le j < 2\}$

Pairs of statement instances that write the same array element:
$W^{-1} \circ W = \{S[i,j] \to S[i',j'] : 0 \le i, j, i', j' < 2 \ \wedge\ i + j = i' + j'\} =$
$\{S[0,0] \to S[0,0]; S[0,1] \to S[0,1]; S[1,0] \to S[1,0]; S[1,1] \to S[1,1]; S[0,1] \to S[1,0]; S[1,0] \to S[0,1]; \}$

Scratch

Application of a Relation to a Set

Application:

$A = \{S_1[i_1] : f(i_1)\}, \quad B = \{S_2[i_2] \to T_2[j_2] : g(i_2, j_2)\}$

$$B(A) = \begin{cases} \{T_2[j] : \exists i : f(i) \wedge g(i,j)\}, & \texttt{if } S_1(i_1) = S_2(i_2) \\ \emptyset & \texttt{otherwise} \end{cases}$$

Example:

Read Set R (Statement instances reading array elements):
$\{R[] \to A[2]; T[k] \to A[0] : 0 \leq k < 2; T[k] \to A[k] : 0 \leq k < 2\}$

Instances of T statements:
$S = \{T[k] : 0 \leq k < 2\}$

Array elements read by S:
$R(S) = \{A[k] : 0 \leq k < 2\}$

Lexicographic Order on Sets

$$A = \{S[\bar{i}] : f(\bar{i})\}, \quad B = \{T[\bar{j}] : g(\bar{j})\}$$

$$A \prec B = \begin{cases} \{S[\bar{i}] \to S[\bar{j}] : f(\bar{i}) \wedge g(\bar{j}) \wedge \bar{i} \prec \bar{j}\}, & \texttt{if } S(\bar{i}) = T(\bar{j}) \\ \emptyset & \texttt{otherwise} \end{cases}$$

Example:

Iteration Domain:
$I = \{R[]; S[i,j] : 0 \leq i < 2 \wedge 0 \leq j < 2; T[k] : 0 \leq k < 2\}$

$I \prec I$ lexicographic order on pairs of statement instances:
$\{S[i,j] \to S[i',j'] : 0 \leq i,j,i',j' < 2 \wedge i,j \prec i',j'; T[0] \to T[1]\} =$

$\{S[0,0] \to S[0,1];\ S[0,0] \to S[1,0];\ S[0,0] \to S[1,1];\ S[0,1] \to S[1,0];\ S[0,1] \to S[1,1];\ S[1,0] \to S[1,1];\ T[0] \to T[1]\}$

Lexicographic Order on Relations

Binary relation on domains reflect lexicographic order of images:

$$A = \{S_1[\bar{i_1}] \to T_1[\bar{j_1}] : \; f(\bar{i_1}, \bar{j_1})\},$$
$$B = \{S_2[\bar{i_2}] \to T_2[\bar{j_2}] : \; g(\bar{i_2}, \bar{j_2})\}$$

$$A \prec B = \begin{cases} \{S_1[\bar{i_1}] \to S_2[\bar{i_2}] : \; \exists j_1, j_2 : f(\bar{i_1}, \bar{j_1}) \wedge \\ \qquad g(\bar{i_2}, \bar{j_2}) \wedge \bar{j_1} \prec \bar{j_2}\}, & \texttt{if } \; T_1(\bar{j_1}) = T_2(\bar{j_2}) \\ \emptyset & \texttt{otherwise} \end{cases}$$

Lexicographic Optimizations: Last Write

Binary relation on domains reflect lexicographic order of images:

$R = \{S[\bar{i}] \to T[\bar{j}] :\ f(\bar{i},\bar{j})\}$,

$\texttt{lexmax}\, R = \{S[\bar{i}] \to T[\bar{j}] :\ f(\bar{i},\bar{j})\ \wedge\ \forall j' : f(i,j') \Rightarrow j \succeq j'\}$

Example:

$(R \cup W)^{-1}$: statement instances accessing array element
$\{A[2] \to R[];\ A[a] \to S[i,j] :\ a = i + j \wedge 0 \le i,j < 2;$
$\ A[0] \to T[k] :\ 0 \le k < 2;\ A[k] \to\ T[k] :\ 0 \le k < 2\}$

lexmax $(R \cup W)^{-1}$: last instance of statement accessing element
$\{A[2] \to R[]; A[a] \to S[a,0] : 0 \le a < 2; A[2] \to S[1,1];$
$\ A[k] \to T[1] : 0 \le k < 2\ \}$

Last-Write Analysis (see file last-write.py)

Given a read from an array element, what was the last write to the same array element before the read?

```
    for(int i=0; i<N; i++)
      for(int j=0; j<N-i; j++)
F:        A[i+j] = f(A[i+j]);

    for(int i=0; i<N; i++)
S:   X[i] = g(A[i]);
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```

- Access relations:
  $W_1 = \{F[i,j] \to A[i+j] : 0 \le i < N \land 0 \le j < N-i\}$
  $R_2 = \{S[i] \to A[i] : 0 \le i < N\}$

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  $R_2 = \{S[i] \ \rightarrow \ A[i] : \ 0 \leq i < N\}$

- Map each statement instance reading an element to all the statements that have written that element:
  $R = W_1{}^{-1} \circ R_2 \ = \ \{S[i] \rightarrow F[i', i - i'] : \ 0 \leq i' \leq i < N\}$

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- Map each statement instance reading an element to all the statements that have written that element:
  $R = {W_1}^{-1} \circ R_2 = \{S[i] \rightarrow F[i', i-i'] : 0 \leq i' \leq i < N\}$
- Last Write: `lexmax` $R = \{S[i] \rightarrow F[i,0] : 0 \leq i < N\}$

Dependency Graph and Code Transformations

Recall: iteration $\bar{j}$ depends on iteration $\bar{i}$ *iff*:

- $\bar{j}$ is executed before $\bar{i}$ in the original program,
- $\bar{i}$ and $\bar{j}$ may access the same memory location, and
- at least one of those two accesses is a write!

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Dependency Graph Computation:

$$D = ((W^{-1} \circ R) \cup (W^{-1} \circ W) \cup (R^{-1} \circ W)) \cap (S \prec S)$$

W: write-access relation, R: read-access relation,
S: original schedule.

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W: write-access relation, R: read-access relation,
S: original schedule.

A code transformation corresponds to computing a new schedule S' that executes the same statements in a different order. The transformation is valid if S' respects the dependencies of D:

$$\bar{i} \rightarrow \bar{j} \in D \quad \Rightarrow \quad S'(\bar{i}) \prec S'(\bar{j})$$

Validating New Schedules (see file common.py)

Dependency Graph Computation:

$$D = ((W^{-1} \circ R) \cup (W^{-1} \circ W) \cup (R^{-1} \circ W)) \cap (S \prec S)$$

Safe rescheduling S' *iff*

$$\bar{i} \rightarrow \bar{j} \in D \quad \Rightarrow \quad S'(\bar{i}) \prec S'(\bar{j})$$

How to implement the test above? Assume S' a new schedule (mapping original statement instances to new time abstraction).

$$T_{src \rightarrow sink} = (S' \circ D) \circ S'^{-1}$$

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$T_{src \rightarrow sink} = (S' \circ D) \circ S'^{-1}$

- Maps the source stmt to the time of the dependence sink;
- Maps the time of the source stmt to the time of the sink.

$S'_{desc} = (\mathtt{ran}\ S') \succeq (\mathtt{ran}\ S')$

(S'_{desc} denotes all illegal re-orderings)

Code Transformation is valid if $T_{src \rightarrow sink} \cap S'_{desc} = \emptyset$

Validating New Schedules (see file common.py)

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$D \ = \ ((W^{-1} \circ R) \ \cup \ (W^{-1} \circ W) \ \cup \ (R^{-1} \circ W)) \ \cap \ (S \ \prec \ S)$

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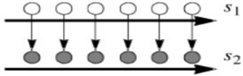
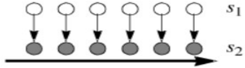
(S'_{desc} denotes all illegal re-orderings)

Code Transformation is valid if $\quad T_{src \rightarrow sink} \cap S'_{desc} \ = \ \emptyset$
(if for all dependencies, in the new schedule, the time of the source is still smaller than the time of the sink statement!)

Transformations: Fusion and Fission

Source Code	Partition	Transformed Code

`for (i=1; i<=N; i++)`
`  Y[i] = Z[i]; /*s1*/`
`for (j=1; j<=N; j++)`
`  X[j] = Y[j]; /*s2*/`

	Fusion
	$s_1 : p = i$
	$s_2 : p = j$
		`for (p=1; p<=N; p++){`
		`  Y[p] = Z[p];`
		`  X[p] = Y[p];`
		`}`
		

`for (p=1; p<=N; p++){`
`  Y[p] = Z[p];`
`  X[p] = Y[p];`
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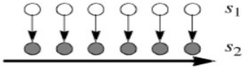
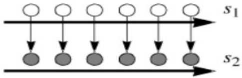

	Fission
	$s_1 : i = p$
	$s_2 : j = p$
		`for (i=1; i<=N; i++)`
		`  Y[i] = Z[i]; /*s1*/`
		`for (j=1; j<=N; j++)`
		`  X[j] = Y[j]; /*s2*/`
		

[Aho/Ullman/Sethi ’07]

Fusion Encoding (see fusion.py)

- Iteration Domain:
  $I = \{S1[i] : \ 1 \leq i \leq N; \ S2[j] : 1 \leq j \leq N\}$

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  $S = I \ \cap_{dom} \ \{S1[i] \to [1, i]; \ S2[j] \to [2, j]\}$
- Read and Write Access Relations:
  $W^{rel}_{access} = I \ \cap_{dom} \ \{S1[i] \to Y[i]; \ S2[j] \to X[j]\}$
  $R^{rel}_{access} = I \ \cap_{dom} \ \{S1[i] \to Z[i]; \ S2[j] \to Y[j]\}$

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  $D = \texttt{mkDepGraph}(S, R^{rel}_{access}, W^{rel}_{access})$

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- Make Dependence Graph:
  $D = \texttt{mkDepGraph}(S, R^{rel}_{access}, W^{rel}_{access})$
- Fused Schedule:
  $S' = I \ \cap_{dom} \ \{S1[i] \rightarrow [i, 1]; \ S2[i] \rightarrow [i, 2]\}$
- Check Fusion Safety:
  $\texttt{checkTimeDepsPreserved}(S', D)$

Parallelism

Is the fused loop parallel?

Fused and Parallel Schedule:

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Fused and Parallel Schedule:
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Fission Encoding (see fission-wrong.py)

Is it safe to distribute the loop across statements *S*1 and *S*2?

```
    for(p=1; p<=N; p++)  {
S1:      Y[p] = f(Z[p]);
S2:      X[p] = g(Y[p+1]) }
```

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- Read and Write Access Relations:

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```

- Iteration Domain:
  $I = \{S1[p] : \; 1 \le p \le N; \; S2[p] : 1 \le p \le N\}$
- Original Schedule:
  $S = I \cap_{dom} \{S1[p] \rightarrow [p, 1]; \; S2[p] \rightarrow [p, 2]\}$
- Read and Write Access Relations:
  $W^{rel}_{access} = I \cap_{dom} \{S1[p] \rightarrow Y[p]; \; S2[p] \rightarrow X[p]\}$
  $R^{rel}_{access} = I \cap_{dom} \{S1[p] \rightarrow Z[p]; \; S2[p] \rightarrow Y[p+1]\}$
- $D = \mathtt{mkDepGraph}(S, R^{rel}_{access}, W^{rel}_{access})$
- Fissed Schedule:

Fission Encoding (see fission-wrong.py)

Is it safe to distribute the loop across statements $S1$ and $S2$?

```
     for(p=1; p<=N; p++)  {
S1:       Y[p] = f(Z[p]);
S2:       X[p] = g(Y[p+1]) }
```

- Iteration Domain:
  $I = \{S1[p] :\ 1 \le p \le N;\ S2[p] : 1 \le p \le N\}$
- Original Schedule:
  $S = I \cap_{dom} \{S1[p] \to [p, 1];\ S2[p] \to [p, 2]\}$
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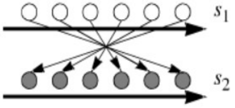
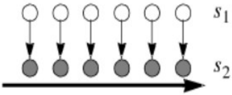
  $W^{rel}_{access} = I \cap_{dom} \{S1[p] \to Y[p];\ S2[p] \to X[p]\}$
  $R^{rel}_{access} = I \cap_{dom} \{S1[p] \to Z[p];\ S2[p] \to Y[p+1]\}$
- $D = \mathtt{mkDepGraph}(S, R^{rel}_{access}, W^{rel}_{access})$
- Fissed Schedule:
  $S' = I \cap_{dom} \{S1[i] \to [1, i];\ S2[j] \to [2, j]\}$
- Is Fission Safe? $\mathtt{checkTimeDepsPreserved}(S', D)$

Transformation: Reversal + Fusion

SOURCE CODE	PARTITION	TRANSFORMED CODE

`for (i=0; i>=N; i++)`
`  Y[N-i] = Z[i];   /*s1*/`
`for (j=0; j<=N; j++)`
`  X[j] = Y[j];   /*s2*/`

	Reversal
	$s_1 : p = N - i$
	$(s_2 : p = j)$
		`for (p=0; p<=N; p++){`
		`  Y[p] = Z[N-p];`
		`  X[p] = Y[p];`
		`}`
		

BUG: should be `for (i=0; i<=N; i++)`!

[Aho/Ullman/Sethi ’07]

Reversal + Fusion Encoding (see fused-rev.py)

- Iteration Domain:

Reversal + Fusion Encoding (see fused-rev.py)

- Iteration Domain:
  $I = \{S1[i] : \; 1 \leq i \leq N; \; S2[j] : 1 \leq j \leq N\}$
- Original Schedule:

Reversal + Fusion Encoding (see fused-rev.py)

- Iteration Domain:
  $I = \{S1[i] : \ 1 \leq i \leq N; \ S2[j] : 1 \leq j \leq N\}$
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  $S = I \cap_{dom} \{S1[i] \rightarrow [1, i]; \ S2[j] \rightarrow [2, j]\}$
- Read and Write Access Relations:

Reversal + Fusion Encoding (see fused-rev.py)

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- Original Schedule:
  $S = I \;\cap_{dom}\; \{S1[i] \to [1, i]; \; S2[j] \to [2, j]\}$
- Read and Write Access Relations:
  $W^{rel}_{access} = I \;\cap_{dom}\; \{S1[i] \to Y[N - i]; \; S2[j] \to X[j]\}$
  $R^{rel}_{access} = I \;\cap_{dom}\; \{S1[i] \to Z[i]; \; S2[j] \to Y[j]\}$
- $D = \texttt{mkDepGraph}(S, R^{rel}_{access}, W^{rel}_{access})$
- Transformed Schedule:

Reversal + Fusion Encoding (see fused-rev.py)

- Iteration Domain:
  $I = \{S1[i] : \ 1 \leq i \leq N; \ S2[j] : 1 \leq j \leq N\}$
- Original Schedule:
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- Read and Write Access Relations:
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  $R^{rel}_{access} = I \ \cap_{dom} \ \{S1[i] \rightarrow Z[i]; \ S2[j] \rightarrow Y[j]\}$
- $D = \texttt{mkDepGraph}(S, R^{rel}_{access}, W^{rel}_{access})$
- Transformed Schedule:
  - the statements of the first loop are reversed;
  - the two loops are fused, hence $S2[j] \rightarrow [j, 2]$ instead of $S2[j] \rightarrow [2, j]$

  $S' = I \ \cap_{dom} \ \{S1[p] \rightarrow [N - p, 1]; \ S2[j] \rightarrow [j, 2]\}$
- Is Fission Safe? $\texttt{checkTimeDepsPreserved}(S', D)$

Transformation: Loop Skewing

```
     float X[N][N];
     for(int i=1; i<N; i++) {
       for(int j=1; j < min(i+2, N); j++) {
S1:      X[i][j] = X[i-1][j] + X[i][j-1];
       }
     }
```

Change of variables: p $\leftarrow$ `i+j`, q $\leftarrow$ `j`

```
     for(int p=2; p < 2*N-1; p++) {
       int up_bd = ((p+2)/2) + (p%2);
       for(int q=max(1,p-N+1); q<min(up_bd,N); q++)
S1:      X[p-q][q] = X[p-q-1][q] + X[p-q][q-1];
       }
     }
```

Encoding Loop Skewing (see loop-skewing.py)

- Iteration Domain:
  $I = \{S1[i, j] : \; 1 \leq i < N \; \wedge \; 1 \leq j < min(i + 2, N)\}$
- Original Schedule: $S = I \; \cap_{dom} \; \{ \; S1[i, j] \rightarrow [i, j] \; \}$
- Read and Write Access Relations:

Encoding Loop Skewing (see loop-skewing.py)

- Iteration Domain:
  $I = \{S1[i,j] : \ 1 \leq i < N \ \wedge \ 1 \leq j < min(i+2, N)\}$
- Original Schedule: $S = I \ \cap_{dom} \ \{ \ S1[i,j] \rightarrow [i,j] \ \}$
- Read and Write Access Relations:
  $W^{rel}_{access} = I \ \cap_{dom} \ \{S1[i,j] \rightarrow X[i,j]; \}$
  $R^{rel}_{access} = I \ \cap_{dom} \ \{S1[i,j] \rightarrow X[i-1,j]; \ S1[i,j] \rightarrow X[i,j-1]\}$
- $D = \mathtt{mkDepGraph}(S, R^{rel}_{access}, W^{rel}_{access})$
- Transformed Schedule: `p` $\leftarrow$ `i+j`, `q` $\leftarrow$ `j`
  - Original stmt $S1[i,j] = S1[p-q,q]$ is rescheduled to iter $[p,q]$;

Encoding Loop Skewing (see loop-skewing.py)

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  $W^{rel}_{access} = I \ \cap_{dom} \ \{S1[i,j] \rightarrow X[i,j]; \}$
  $R^{rel}_{access} = I \ \cap_{dom} \ \{S1[i,j] \rightarrow X[i-1,j]; \ S1[i,j] \rightarrow X[i,j-1]\}$
- $D = \texttt{mkDepGraph}(S, R^{rel}_{access}, W^{rel}_{access})$
- Transformed Schedule: $\texttt{p} \leftarrow \texttt{i+j}, \ \texttt{q} \leftarrow \texttt{j}$
  - Original stmt $S1[i,j] = S1[p-q,q]$ is rescheduled to iter $[p,q]$;
  - Hence, $S1[x,q] \rightarrow [x+q,q]$

  $S' = I \ \cap_{dom} \ \{ \ S1[x,q] \rightarrow [x+q,q] \ \}$
- Is Loop-Skewing Safe? $\texttt{checkTimeDepsPreserved}(S', D)$
- Inner loop parallel? Try $S'' = I \ \cap_{dom} \ \{ \ S1[x,q] \rightarrow [x+q,1] \ \}$

Motivation: Dependency Graphs and Transformations

Polydral Analysis: Iteration Domain, Access Relations, Schedule

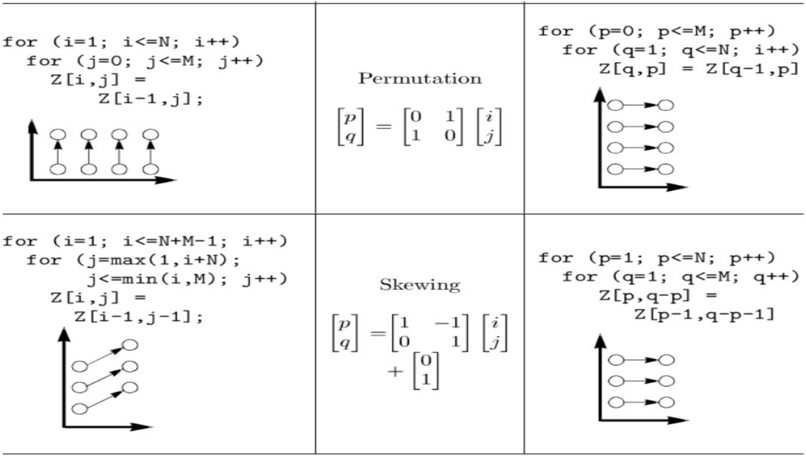
Presburger Sets, Relations and Associated Operations

Data-Flow Analysis and Dependency Graph

Examples: Checking Validity of Code Transformations

Assignment Exercises

Exercise: Permutation (Ignore Loop Skewing)

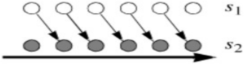


[Aho/Ullman/Sethi ’07]

Ignore Loop Skewing

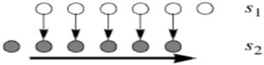
Exercise: Reindexing and Scaling

```
for (i=1; i<=N; i++) {
  Y[i] = Z[i];    /*s1*/
  X[i] = Y[i-1]; /*s2*/
}
```

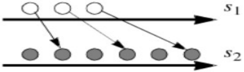


Re-indexing
$s_1 : p = i$
$s_2 : p = i - 1$

```
if (N>=1) X[1]=Y[0];
for (p=1; p<=N-1; p++){
  Y[p]=Z[p];
  X[p+1]=Y[p];
}
if (N>=1) Y[N]=Z[N];
```

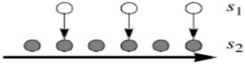



```
for (i=1; i<=N; i++)
  Y[2*i] = Z[2*i]; /*s1*/
for (j=1; j<=2N; j++)
  X[j]=Y[j];         /*s2*/
```



Scaling
$s_1 : p = 2 * i$
$(s_2 : p = j)$

```
for (p=1; p<=2*N; p++){
  if (p mod 2 == 0)
    Y[p] = Z[p];
  X[p] = Y[p];
}
```



[Aho/Ullman/Sethi ’07]